

Expansivity and unique shadowing

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1

EXPANSIVITY AND UNIQUE SHADOWING

2

CHRIS GOOD, SERGIO MACÍAS, JONATHAN MEDDAUGH, JOEL MITCHELL,
3 AND JOE THOMAS

3

ABSTRACT. Let $f: X \rightarrow X$ be a continuous function on a compact metric space. We show that shadowing is equivalent to backwards shadowing and two-sided shadowing when the map f is onto. Using this we go on to show that, for expansive surjective maps the properties shadowing, two-sided shadowing, s-limit shadowing and two-sided s-limit shadowing are equivalent. We show that f is positively expansive and has shadowing if and only if it has unique shadowing (i.e. each pseudo-orbit is shadowed by a unique point), extending a result implicit in Walter's proof that positively expansive maps with shadowing are topologically stable. We use the aforementioned result on two-sided shadowing to find an equivalent characterisation of shadowing and expansivity and extend these results to the notion of n -expansivity due to Morales.

4

1. INTRODUCTION

5 Let $f: X \rightarrow X$ be a continuous function on a compact metric space X . A δ -
6 *pseudo-orbit* is a sequence $(x_i)_{i \in \mathbb{N}_0}$ such that $d(f(x_i), x_{i+1}) < \delta$. Pseudo-orbits
7 are of importance when calculating an orbit numerically, as rounding errors mean
8 a computed orbit will be a pseudo-orbit. The sequence (y_i) from X is said to ε -
9 shadow the sequence (x_i) provided $d(y_i, x_i) < \varepsilon$ for all i . We then say that the
10 system has *shadowing*, or *the pseudo-orbit tracing property*, if pseudo-orbits are
11 shadowed by true orbits (see below for precise definitions). Motivating this paper
12 is Walters [48] result that if h is an expansive homeomorphism with shadowing,
13 then for every $\varepsilon > 0$ there is a $\delta > 0$ such that every δ -pseudo-orbit is ε -shadowed
14 by a unique point from X . We show that the converse is true; a system is shadowing
15 and expansive if and only if it has unique shadowing. We go on to obtain results
16 of a similar flavour using the notion of n -expansivity due to Morales [37].

17 Shadowing is important when modelling a system numerically (for example see
18 [16, 41]). However, it is also important theoretically. For example, Bowen [6]
19 used shadowing implicitly as a key step in his proof that the nonwandering set
20 of an Axiom A diffeomorphism is a factor of a shift of finite type. Since then it
21 has been studied extensively, in the setting of numerical analysis [16, 17, 41], as
22 a key factor in stability theory [44, 46, 48], in understanding the structure of ω -
23 limit sets and Julia sets, [1, 3, 4, 5, 7, 27, 35], and as a property in and of itself
24 [18, 23, 26, 33, 38, 42, 44, 47].

25 Many other notions of shadowing have been studied including, for example,
26 ergodic, thick and Ramsey shadowing [8, 9, 10, 20, 22, 40], limit shadowing [2, 30,
27 45], s-limit shadowing [2, 30, 33], orbital shadowing [25, 36, 43, 45], and inverse

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1 shadowing [17, 28, 32]. In this paper we focus on shadowing, s-limit shadowing,
 2 h-shadowing and limit shadowing.

3 In Section 3, we observe (Theorem 3.2) that if f is surjective then it has shad-
 4 owing if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every backwards
 5 δ -pseudo orbit is ε -shadowed by some backwards orbit of a point: thus shadowing
 6 is equivalent to *backwards shadowing*. We additionally show that it is equivalent
 7 to *two-sided shadowing* (i.e. two-sided pseudo-orbits are shadowed by a two-sided
 8 trajectory of a point). We then strengthen a result in [3] (Corollary 3.7), by demon-
 9 strating that for expansive maps, the properties shadowing, two-sided shadowing,
 10 s-limit shadowing and two-sided s-limit shadowing are equivalent. In Section 4, we
 11 turn our attention to the notion of n -expansivity due to Morales [37]. We show
 12 (Theorem 4.3) that pseudo-orbits are shadowed by at most n points if and only
 13 if f has shadowing and is n -expansive. We then construct an example of a posi-
 14 tively n -expansive system with shadowing which is not positively $(n-1)$ -expansive.
 15 We close by examining the consequences of uniqueness in three other shadowing
 16 properties, namely s-limit shadowing, limit shadowing and h-shadowing.

17

2. PRELIMINARIES

18 This section serves to outline the preliminary background definitions and notions
 19 for the remainder of this paper and are standard across the literature. Throughout,
 20 we will assume that a discrete *dynamical system* is a pair (X, f) consisting of a
 21 compact metric space X and a continuous map $f: X \rightarrow X$. Note that we do not
 22 assume, in general, that the map f is onto. However, since surjective dynamical
 23 systems are usually the more interesting from a dynamics viewpoint, we ensure that
 24 every example we construct in this paper is surjective (unless it is the property of
 25 surjectivity itself which is under examination). We say that the *orbit* of x under f
 26 is the set of points $\{x, f(x), f^2(x), \dots\}$; we denote this set by $\text{Orb}_f(x)$. A (finite or
 27 infinite) sequence $(x_i)_{0 \leq i \leq n}$ for some $n \in \mathbb{N} \cup \{\infty\}$ is said to be a δ -*pseudo-orbit*
 28 for some $\delta > 0$ if $d(f(x_i), x_{i+1}) < \delta$ for each $i \leq n$. The infinite sequence $(x_i)_{i \in \mathbb{N}_0}$
 29 is an *asymptotic pseudo-orbit* provided that $\lim_{i \rightarrow \infty} d(f^i(x_i), x_{i+1}) = 0$ and we say
 30 that $(x_i)_{i \in \mathbb{N}_0}$ is an *asymptotic δ -pseudo-orbit* if it is both a δ -pseudo-orbit and an
 31 asymptotic pseudo-orbit. The point $z \in X$ is said to ε -*shadow* $(x_i)_{0 \leq i \leq n}$ for some
 32 $\varepsilon > 0$ if $d(x_i, f^i(z)) < \varepsilon$ for each $i \leq n$. It *asymptotically shadows* the sequence
 33 $(x_i)_{i \in \mathbb{N}_0}$ if $\lim_{i \rightarrow \infty} d(x_i, f^i(z)) = 0$ and *asymptotically ε -shadows* the sequence if it
 34 both ε -shadows and asymptotically shadows it.

35 The classical notion of shadowing states that (X, f) has *shadowing* provided for
 36 any $\varepsilon > 0$ there exists $\delta > 0$ such that every (infinite) δ -pseudo-orbit is ε -shadowed.
 37 The system has *limit shadowing*, a property first introduced in [21] with reference
 38 to hyperbolic sets, if every asymptotic pseudo-orbit is asymptotically shadowed.
 39 The notion of limit shadowing was extended in [33] to a property the authors
 40 called s-limit shadowing to accommodate the fact that many systems exhibit limit
 41 shadowing but not shadowing [31, 44]. The system (X, f) has *s-limit shadowing* if,
 42 in addition to having shadowing¹, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for
 43 any asymptotic δ -pseudo orbit $(x_i)_{i \in \mathbb{N}_0}$ there exists $z \in X$ which asymptotically
 44 ε -shadows $(x_i)_{i \in \mathbb{N}_0}$. Finally, the system (X, f) has *h-shadowing*, or *shadowing with*

¹We note that postulating shadowing as part of the definition of s-limit shadowing is actually unnecessary when the phase space is compact (see by [29, Theorem 11.0.1]).

1 *exact hit*, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite δ -pseudo orbit
 2 (x_0, x_1, \dots, x_m) there exists $z \in X$ which ε -shadows it and for which $f^m(z) = x_m$.

3 We remark that h -shadowing was introduced in [3] and was motivated by the
 4 fact that an important class of shift systems, called shifts of finite type, which
 5 are fundamental in the study of shadowing (see [26]) exhibit this stronger form of
 6 shadowing and that it coincide with the usual form for shift systems but is distinct
 7 in general (see [2, Example 6.4]). Moreover, it is known from results in [2] that
 8 h -shadowing implies s -limit shadowing which further implies limit shadowing.

9 3. TWO-SIDED SHADOWING

10 We start with the following simple observation, which we nevertheless believe
 11 to be new for functions in general. The classical notion of shadowing states that
 12 (X, f) has shadowing provided for any $\varepsilon > 0$ there exists $\delta > 0$ such that every
 13 δ -pseudo-orbit is ε -shadowed. It is a standard result in the theory of shadowing [44]
 14 that a compact dynamical system (X, f) has shadowing if and only if for any $\varepsilon > 0$
 15 there is a $\delta > 0$ such that every finite δ -pseudo orbit (x_0, \dots, x_n) is ε -shadowed by
 16 some $x \in X$. It is shown here that in a compact space, one obtains an equivalent
 17 notion of shadowing in terms of backwards and two-sided (pseudo-)orbits.

18 **Definition 3.1.** Suppose that (X, f) is a dynamical system.

- 19 (1) A *backwards orbit* of the point $x \in X$ is a sequence $(x_i)_{i \leq 0} \subseteq X$ for which
 20 $f(x_i) = x_{i+1}$ for all $i \leq -1$ and $x_0 = x$.
- 21 (2) A *two-sided orbit* of the point $x \in X$ is a sequence $(x_i)_{i \in \mathbb{Z}} \subseteq X$ for which
 22 $f(x_i) = x_{i+1}$ for all $i \in \mathbb{Z}$ and $x_0 = x$.
- 23 (3) The sequence $(x_i)_{i \leq 0} \subseteq X$ is a *backwards δ -pseudo-orbit* if $d(f(x_i), x_{i+1}) <$
 24 δ for each $i \leq -1$.
- 25 (4) The sequence $(x_i)_{i \in \mathbb{Z}} \subseteq X$ is a *two-sided δ -pseudo-orbit* if $d(f(x_i), x_{i+1}) <$
 26 δ for each $i \in \mathbb{Z}$.
- 27 (5) (X, f) is said to have the *backwards shadowing property* if for any $\varepsilon > 0$,
 28 there exists $\delta > 0$ for which every backwards δ -pseudo-orbit in X is ε -
 29 shadowed by some backwards orbit of a point in X .
- 30 (6) (X, f) is said to have the *two-sided shadowing property* if for any $\varepsilon > 0$,
 31 there exists $\delta > 0$ for which every two-sided δ -pseudo-orbit in X is ε -
 32 shadowed by some two-sided orbit of a point in X .

33 Obviously if f is not a homeomorphism, backwards and two-sided orbits need
 34 not be unique.

35 **Theorem 3.2.** *Let (X, f) be a dynamical system with X compact. Then, of the*
 36 *following, (1) implies (2) which implies (3). Furthermore, if f is onto then (3)*
 37 *implies (1).*

- 38 (1) *f has shadowing;*
- 39 (2) *f has two-sided shadowing;*
- 40 (3) *f has backwards shadowing.*

41 *Proof.* (1) \implies (2): Suppose that (X, f) has shadowing. Let $\varepsilon > 0$ and choose
 42 $\delta > 0$ such that every δ -pseudo-orbit is $\varepsilon/2$ -shadowed. Suppose that $(x_n)_{n \in \mathbb{Z}}$ is a
 43 two-sided δ -pseudo-orbit. For each $n > 0$, let y_{-n} be a point which $\varepsilon/2$ -shadows
 44 the δ -pseudo-orbit $(x_{-n}, x_{-n+1}, x_{-n+2}, \dots)$. There exists a point $z_0 \in X$ and an
 45 infinite subset N_0 of \mathbb{N}_0 such that $f^n(y_{-n}) \rightarrow z_0$ as $n \rightarrow \infty$ and $n \in N_0$. Clearly

1 the forward orbit of z_0 ε -shadows (x_0, x_1, x_2, \dots) . Given z_{-k} and an infinite subset
 2 N_k of \mathbb{N}_0 such that $f^{n-k}(y_{-n}) \rightarrow z_{-k}$ as $n \rightarrow \infty$ and $n \in N_k \cap \{k+1, k+2, \dots\}$, we
 3 can find a point z_{-k-1} and an infinite $N_{k+1} \subseteq N_k$ such that $f^{n-k-1}(y_{-n}) \rightarrow z_{-k-1}$
 4 as $n \rightarrow \infty$ and $n \in N_{k+1} \cap \{k+2, k+3, \dots\}$. Note that $d(x_{-k}, z_{-k}) < \varepsilon$ and that,
 5 by continuity, $f(z_{-k-1}) = z_{-k}$ for all $k \geq 0$. Hence z_0 has a two-sided orbit that
 6 ε -shadows $(x_n)_{n \in \mathbb{Z}}$.

7 It is clear that (2) implies (3). Finally (3) implies (1) because, given that every
 8 point has a pre-image, (3) implies that finite pseudo-orbits are shadowed, which is
 9 equivalent to shadowing in compact metric spaces (see [44, Lemma 1.1.1]). \square

10 One can also extend the notion of s-limit shadowing to the two-sided and back-
 11 ward varieties. For this, one requires the notions of two-sided asymptotic pseudo-
 12 orbits and backward asymptotic pseudo-orbits. These are defined analogously to
 13 the normal (forward) asymptotic pseudo-orbits but in the spirit of Definition 3.1.

14 **Definition 3.3.** Suppose that (X, f) is a dynamical system.

- 15 (1) A *backwards asymptotic δ -pseudo-orbit* is a backwards δ -pseudo orbit $(x_i)_{i \leq 0}$
 16 which in addition satisfies $d(f(x_i), x_{i+1}) \rightarrow 0$ as $i \rightarrow \infty$.
- 17 (2) A *two-sided asymptotic δ -pseudo-orbit* is a two-sided δ -pseudo orbit $(x_i)_{i \in \mathbb{Z}}$
 18 which in addition satisfies $d(f(x_i), x_{i+1}) \rightarrow 0$ as $i \rightarrow \pm\infty$.
- 19 (3) (X, f) is said to have the *backwards s-limit shadowing property* if it has the
 20 backwards shadowing property and for any $\varepsilon > 0$ there exists a $\delta > 0$ such
 21 that every backwards asymptotic δ -pseudo-orbit in X is asymptotically ε -
 22 shadowed by some backwards orbit in X .
- 23 (4) (X, f) is said to have the *two-sided s-limit shadowing property* if it has
 24 the two-sided shadowing property and for any $\varepsilon > 0$ there exists a $\delta > 0$
 25 such that every two-sided asymptotic δ -pseudo-orbit in X is asymptotically
 26 ε -shadowed by some two-sided orbit in X .

27 We then obtain a connection between the different varieties of s-limit shadowing.

28 **Proposition 3.4.** *If f has two-sided s-limit shadowing then it has backward s-limit*
 29 *shadowing. If, in addition, f is a surjection then f has s-limit shadowing.*

30 *Proof.* Let $\varepsilon > 0$ be given and let $\delta > 0$ correspond to this for two-sided s-limit
 31 shadowing.

32 Let $(x_i)_{i \leq 0}$ be a backward asymptotic δ -pseudo-orbit. Extend this into a two-
 33 sided asymptotic δ -pseudo-orbit by letting $x_i = f^i(x_0)$ for all $i > 0$. By two-sided
 34 s-limit shadowing there exists $z \in X$ which asymptotically ε -shadows $(x_i)_{i \in \mathbb{Z}}$. In
 35 particular, z backwards asymptotically ε -shadows $(x_i)_{i \leq 0}$. This part of the result
 36 now follows by the fact that two-sided shadowing implies backward shadowing (see
 37 the proof of Theorem 3.2).

38 Now let $(x_i)_{i \geq 0}$ be an asymptotic δ -pseudo-orbit and suppose f is onto: for each
 39 $i < 0$ let x_i be such that $f(x_i) = x_{i+1}$. By two-sided s-limit shadowing there exists
 40 $z \in X$ which asymptotically ε -shadows $(x_i)_{i \in \mathbb{Z}}$. In particular, z asymptotically
 41 ε -shadows $(x_i)_{i \leq 0}$. It remains to note that f has shadowing by Theorem 3.2. \square

42 The following example shows the necessity of surjectivity in the previous result.
 43 Indeed, one can exhibit a non-surjective system with two-sided s-limit shadowing
 44 but not s-limit shadowing.

1 **Example 3.5.** Let $X = \{-1, -1/2, 0, 1/2^n \mid n \in \mathbb{N}_0\}$ with the induced metric from
 2 the real line. Let

$$f(x) = \begin{cases} x + 1/2 & \text{if } x \in \{-1, 1/2\}, \\ x & \text{if } x \in \{0, 1\}, \\ 1/2^{n-1} & \text{if } x = 1/2^n \text{ for } n \geq 1. \end{cases}$$

3 For any δ , we can construct a δ -pseudo-orbit starting from -1 and ending with a
 4 sequence of 1s. This cannot be, for example, $1/3$ -shadowed, thus the system does
 5 not have s-limit shadowing. However, for $\delta < 1/3$ every backward asymptotic δ -
 6 pseudo orbit lies in $[0, 1]$. Similarly, every two-sided asymptotic δ -pseudo-orbit lies
 7 in $[0, 1]$, and it is clear that the subsystem $X \cap [0, 1]$ has backward and two-sided
 8 s-limit shadowing.

9 Recall that a system (X, f) is *c-expansive* if there exists some $\eta > 0$ such that
 10 for any $x, y \in X$ and two-sided orbits $(x_i)_{i \in \mathbb{Z}}$ and $(y_i)_{i \in \mathbb{Z}}$ in X with $x_0 = x, y_0 = y$
 11 and $d(x_i, y_i) < \eta$ for all $i \in \mathbb{Z}$ one has $x = y$. It is often seen that systems with
 12 expansivity properties guarantee that certain characteristic properties of shadowing
 13 varieties hold. For example, in [3] the first named author *et al* show that an
 14 expansive map has shadowing if and only if it has s-limit shadowing. The next
 15 result extends this further by providing an equivalence between s-limit shadowing
 16 and two-sided s-limit shadowing.

17 **Theorem 3.6.** *Let (X, f) be a dynamical system. If f is c-expansive then f has*
 18 *two-sided shadowing if and only if f has two-sided s-limit shadowing.*

19 *Proof.* If f has two-sided s-limit shadowing then it has two-sided shadowing by
 20 definition. Therefore, suppose that f has two-sided shadowing. Let $\eta > 0$ be the
 21 c-expansivity constant for f and take $\varepsilon > 0$ with $\varepsilon < \eta/2$: let $\delta > 0$ correspond
 22 to this ε in the definition of two-sided shadowing (without loss of generality we
 23 assume $\delta < \varepsilon/2$). Let $(x_i)_{i \in \mathbb{Z}}$ be a two-sided asymptotic δ -pseudo-orbit. By two-
 24 sided shadowing, there exists a full orbit $(z_i)_{i \in \mathbb{Z}}$ such that $d(x_i, z_i) < \varepsilon$ for all $i \in \mathbb{Z}$.
 25 The proof of Theorem 3.7 in [3] shows that under these conditions, $d(z_i, x_i) \rightarrow 0$
 26 as $i \rightarrow \infty$ and thus it suffices to show that $d(z_i, x_i) \rightarrow 0$ as $i \rightarrow -\infty$. This can be
 27 done by a similar argument to that of [3].

28 Suppose that $d(z_i, x_i)$ does not converge to 0 as $i \rightarrow -\infty$. Then by compactness
 29 of X there exists $a_0, b_0 \in X$ and an infinite set of negative integers, N_0 , such that

- 30 i). $\lim_{i \rightarrow -\infty, i \in N_0} x_i = a_0$,
- 31 ii). $\lim_{i \rightarrow -\infty, i \in N_0} z_i = b_0$;
- 32 iii). $d(a_0, b_0) = r > 0$.

33 Note that by the fact that $(z_i)_{i \in \mathbb{Z}}$ ε -shadows $(x_i)_{i \in \mathbb{Z}}$, it follows that $r = d(a_0, b_0) \leq$
 34 ε . By continuity, for any $k \in \mathbb{N}$, $\lim_{i \rightarrow -\infty, i \in N_0} z_{i+k} = f^k(b_0) =: b_k$. Furthermore,
 35 since $(x_i)_{i \in \mathbb{Z}}$ is a two-sided asymptotic pseudo-orbit it is in particular a backward
 36 asymptotic pseudo-orbit when restricted to $i \leq 0$. Thus, by continuity for any
 37 $k \in \mathbb{N}$, $\lim_{i \rightarrow -\infty, i \in N_0} x_{i+k} = f^k(a_0) =: a_k$. By shadowing, $d(a_k, b_k) \leq \varepsilon$ for all
 38 $k \in \mathbb{N}$.

39 By the compactness of X , there exist points a_{-1} and b_{-1} and an infinite subset
 40 $N_{-1} \subseteq N_0$ such that

- 41 i). $\lim_{i \rightarrow -\infty, i \in N_{-1}} x_{i-1} = a_{-1}$,
- 42 ii). $\lim_{i \rightarrow -\infty, i \in N_{-1}} z_{i-1} = b_{-1}$.

1 By the continuity of f , combined with the fact that $(x_i)_{i \in \mathbb{Z}}$ is a backward asymptotic
 2 pseudo-orbit, we have $f(a_{-1}) = a_0$ and $f(b_{-1}) = b_0$. Notice that, once again,
 3 by shadowing $d(a_{-1}, b_{-1}) \leq \varepsilon$. Continuing in this manner we can obtain two
 4 sequences of points, $a_0, a_{-1}, a_{-2} \dots$, and $b_0, b_{-1}, b_{-2} \dots$ as well as a sequence of
 5 subsets $N_0 \supseteq N_{-1} \supseteq N_{-2} \dots$, such that for any $k \in \mathbb{N}$

- 6 i). $i - k \leq 0$ for all $i \in N_{-k}$,
- 7 ii). $\lim_{i \rightarrow -\infty, i \in N_k} x_{i-k} = a_{-k}$ and $f(a_{-k}) = a_{-k+1}$,
- 8 iii). $\lim_{i \rightarrow -\infty, i \in N_k} z_{i-k} = b_{-k}$ and $f(b_{-k}) = b_{-k+1}$.

9 Therefore we have full orbits $(a_i)_{i \in \mathbb{Z}}$ and $(b_i)_{i \in \mathbb{Z}}$ for which $d(a_i, b_i) \leq \varepsilon < \eta/2$ (once
 10 again, using the fact that $(z_i)_{i \in \mathbb{Z}}$ ε -shadows $(x_i)_{i \in \mathbb{Z}}$). But this is a contradiction; c-
 11 expansivity there exists $k \in \mathbb{Z}$ such that $d(a_k, b_k) \geq \eta$. Thus our initial assumption
 12 was false: we have that $d(z_i, x_i) \rightarrow 0$ as $i \rightarrow -\infty$. \square

13 The following is then immediate.

14 **Corollary 3.7.** *Let (X, f) be a dynamical system. If f is an expansive surjection*
 15 *then the following are equivalent:*

- 16 (1) f has shadowing;
- 17 (2) f has two-sided shadowing;
- 18 (3) f has s-limit shadowing;
- 19 (4) f has two-sided s-limit shadowing.

20 We note that the second property in the definition of two-sided s-limit shadowing
 21 (see Definition 3.3(4)), namely that for every $\varepsilon > 0$, there exists $\delta > 0$ such that
 22 each asymptotic δ -pseudo-orbit in X is ε -shadowed by a two-sided orbit in X , has
 23 been previously studied in [14]. The authors of that work coined this property
 24 as the *L-shadowing property* and studied it in the context of dynamical systems
 25 whose mapping is a homeomorphism. We next show that under surjectivity, the
 26 *L-shadowing property* is sufficient to show two-sided shadowing. In other words,
 27 when the mapping is surjective, two-sided s-limit shadowing reduces simply to *L*-
 28 shadowing. This result is similar to that of the first, fourth and fifth named authors
 29 in [29] where it is shown that s-limit shadowing is equivalent to the second property
 30 in the definition of s-limit shadowing when the phase space is compact metric.

31 **Proposition 3.8.** *When (X, f) is a surjective dynamical system, then (X, f) has*
 32 *two-sided s-limit shadowing if and only if it has L-shadowing.*

33 *Proof.* By the proof of Proposition 3.4, as f is onto, (X, f) satisfies the first con-
 34 dition in s-limit shadowing (i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any
 35 asymptotic δ -pseudo orbit $(x_i)_{i \in \mathbb{N}}$ there exists a point $z \in X$ which asymptotically
 36 ε -shadows $(x_i)_{i \in \mathbb{N}}$). Therefore by the aforementioned result in [29], (X, f) has s-
 37 limit shadowing and, in particular, shadowing. The result now follows by applying
 38 Theorem 3.2. \square

39

4. UNIQUE SHADOWING

40 In his study of shadowing and stability, Walters [48] proves that if h is an ex-
 41 pansive homeomorphism with shadowing, then for every $\varepsilon > 0$ there is a $\delta > 0$ such
 42 that every δ -pseudo-orbit is ε -shadowed by a unique point from X . It turns out
 43 that the converse is true; a system is shadowing and expansive if and only if it has
 44 unique shadowing. By using a natural generalisation as seen in the work of Morales

1 [37] of the notions of expansivity and positive expansivity, one can obtain results
2 of a similar flavour which we exhibit in this section.

3 **Definition 4.1.** Let (X, f) be a dynamical system.

4 (1) (X, f) is said to be *positively n -expansive*, for $n \in \mathbb{N}$, if there exists $r > 0$
5 such that for any $x \in X$, the set

$$\Gamma_+(x, r) = \{y \in X \mid \forall k \in \mathbb{N}_0 \, d(f^k(x), f^k(y)) < r\},$$

6 contains at most n points.

7 (2) (X, f) is said to be *n -expansive*, for $n \in \mathbb{N}$, if there exists $r > 0$ such that
8 for any x_0 and any two-sided orbit $(x_n)_{n \in \mathbb{Z}}$ of x_0 the set of y_0 such that
9 y_0 has a two-sided orbit $(y_n)_{n \in \mathbb{Z}}$ with $d(x_i, y_i) < r$ for all $i \in \mathbb{Z}$ contains at
10 most n points.

11 Hence, a system is (positively) 1-expansive precisely when it is (positively) ex-
12 pansive.

13 **Definition 4.2.** A dynamical system (X, f) is said to have (*two-sided*) *n -shadowing*
14 if there exists $\eta > 0$ such that for any $\varepsilon > 0$ with $\varepsilon < \eta$ there exists $\delta > 0$ such that
15 given any (two-sided) δ -pseudo-orbit there exists at least one point and at most n
16 points which ε -shadow it.

17 We refer to the property of 1-shadowing as *unique shadowing*. We first demon-
18 strate a basic characterisation of these shadowing properties using the expansivity
19 notions introduced above.

20 **Theorem 4.3.** *Let X be a metric space. For any $n \in \mathbb{N}$, a dynamical system (X, f)*

- 21 (1) *has n -shadowing if and only if it has shadowing and is positively n -expansive.*
22 (2) *has two-sided n -shadowing if and only if it has two-sided shadowing and is*
23 *n -expansive.*

24 *Proof.* Clearly if (X, f) has n -shadowing then it has shadowing. Suppose for a
25 contradiction that it is not positively n -expansive. Let $\eta > 0$ be as in the definition
26 of n -shadowing and suppose that $\varepsilon > 0$ is such that $\varepsilon < \frac{\eta}{2}$. Then there exists $\delta > 0$
27 ($\delta < \varepsilon$) such that every δ -pseudo-orbit is ε -shadowed and by at most n points. Let
28 x_0 be a point such that $\Gamma_+(x_0, \varepsilon)$ contains $n+1$ distinct points x_0, x_1, \dots, x_n . Then
29 $d(f^k(x_0), f^k(x_j)) < \varepsilon$ for all $k \geq 0$ and all $0 \leq j \leq n$. Thus since $\{f^k(x_0)\}_{k \in \mathbb{N}_0}$ is a
30 δ -pseudo-orbit, and is ε -shadowed by every such x_j , one obtains a contradiction to
31 n -shadowing.

32 Now suppose (X, f) has shadowing and is positively n -expansive. Let $r > 0$ be
33 a constant of the positive n -expansivity. We claim that (X, f) has n -shadowing
34 with $\eta = \frac{r}{2}$. Pick $\varepsilon < \frac{r}{2}$ and let $\delta > 0$ correspond to $\varepsilon > 0$ in the definition of
35 shadowing. Suppose there exists a δ -pseudo-orbit $(y_i)_{i \in \mathbb{N}_0}$ which is ε -shadowed by
36 $n+1$ distinct points $x_0, \dots, x_n \in X$. Then by the triangle inequality, for all $n \in \mathbb{N}_0$
37 and any $i, j \in \{0, \dots, n\}$, $d(f^n(x_i), f^n(x_j)) < 2\varepsilon < r$, a contradiction.

38 The proof of (2) can be argued similarly. □

39 **Corollary 4.4.** *If (X, f) has n -shadowing then it has two-sided n -shadowing.*

40 *Proof.* This follows immediately by combining Theorems 3.2 and 4.3. □

41 The converse of this is not true in general. Indeed, on infinite spaces there are
42 no positively expansive homeomorphisms [19] but there are expansive ones; the full
43 shift on two symbols is such an example.

1 *Remark 4.5.* Since there are no positively expansive maps of the interval, no interval
2 map has unique shadowing.

3 *Remark 4.6.* Note that a positively n -expansive map on a compact metric is finite-
4 to-one; if $f^{-1}(x)$ is infinite, then it has a limit point z so that for any $r > 0$, $\Gamma_+(z, r)$
5 is infinite.

6 One can also investigate how these versions of shadowing and expansivity interact
7 with the h -shadowing property. Recall that a system (X, f) has h -shadowing if for
8 every $\varepsilon > 0$, there is a $\delta > 0$ such that every finite δ -pseudo orbit (x_0, \dots, x_n)
9 is ε -shadowed by a point z such that $f^n(z) = x_n$. It is known that h -shadowing
10 implies s -limit shadowing which in turn implies limit shadowing [2]. In [2] it is
11 also shown that a positively expansive map has shadowing if and only if it has h -
12 shadowing. Carvalho and Cordiero [11] prove that an n -expansive homeomorphism
13 with shadowing has limit shadowing. Using these results, together with Theorem
14 3.6, the following is almost immediate. The proof of (3) follows directly from the
15 proof of Theorem C in [11] given Remark 4.6 above.

16 **Corollary 4.7.** *Let $f: X \rightarrow X$ be a continuous map on the compact metric space*
17 *X .*

- 18 (1) *f has unique shadowing if and only if it has h -shadowing and is positively*
19 *expansive.*
20 (2) *f has two-sided unique shadowing if and only if it has two-sided s -limit*
21 *shadowing and is expansive.*
22 (3) *If f is a positively n -expansive surjection and has shadowing, then it has*
23 *limit shadowing.*

24 One may question how distinct the different notions of n (positive) expansivity
25 are for different values of n . This has been investigated previously in the context
26 of homeomorphism systems. For example, Li and Zhang [34] construct homeomor-
27 phisms that are (positively) n -expansive but not (positively) $(n - 1)$ -expansive for
28 any $n \geq 2$. In [11], Carvalho and Cordiero show that for any $n \geq 2$ there exists
29 a homeomorphism with shadowing that is n -expansive but not $(n - 1)$ -expansive.
30 Here we provide an example of a surjective system with shadowing that is positively
31 n -expansive but not positively $(n - 1)$ -expansive on the forward orbits as per the
32 definition above. When $n = 2$, our example does not have h -shadowing, meaning
33 that this serves as a counterexample to the would-be natural generalisation of (1)
34 in Corollary 4.7; that is, n -shadowing is not necessarily equivalent to h -shadowing
35 and positive n -expansivity. We note, however, that the system does have two-sided
36 s -limit shadowing.

Example 4.8. Fix $n \geq 2$. Firstly, we define a subset X_0 of \mathbb{R}^2 recursively in the
following manner. Let $Y_0 = \{(3, 0)\}$. Given sets Y_0, \dots, Y_k , one obtains Y_{k+1} by
considering the point $(x, 0) \in Y_k$ with the smallest first coordinate. Let Y_{k+1} consist
of the points $(x - 2^{-k}, 0)$ and $(x - 2^{-k} - 2^{-(k+1)}, 0)$ along with $n - 2$ points on the
straight line segment whose endpoints are $(x - 2^{-k}, 0)$ and $(x - 2^{-k} - 2^{-(k+1)}, 0)$ such
that all n of the points are equidistant. Thus, each Y_k for $k \geq 1$ contains exactly
 n points that are equally spaced along the x -axis. Moreover, by construction, all
points in each Y_k have positive first coordinate and are distinct and in addition,
 $Y_i \cap Y_j = \emptyset$ for all $i \neq j$. Let $X_0 = \overline{\bigcup_{k=0}^{\infty} Y_k} = \bigcup_{k=0}^{\infty} Y_k \cup \{(0, 0)\}$. One then defines
the sets X_i recursively. Given X_0, \dots, X_k , let $(p, q) \in X_k$ be the point such that p

is maximal over the collection of all first coordinates of points in X_k (the second coordinate will be the same for all points in X_k by construction). Then, define X_{k+1} as

$$X_{k+1} = \{(x, y) \in \mathbb{R}^2 \mid (x, y + 2^{-k}) \in X_k \setminus \{(p, q)\}\}.$$

1 One then defines $X = \bigcup_{k=0}^{\infty} X_k \cup \{(0, -2)\}$ so that X is closed in \mathbb{R}^2 . One may
 2 then endow X with the standard metric from \mathbb{R}^2 to form a metric space.

Next, one defines a map $f : X \rightarrow X$ in the following manner. Let $(0, 0)$, $(3, 0)$
 and $(0, -2)$ each be fixed points. For each $k \geq 1$, let f map each point in Y_k to the
 point in Y_{k-1} with minimal first coordinate so that this defines f on the entirety of
 X_0 . For each $k \geq 1$, define f on X_k to map the point $(x, y) \in X_k \setminus \{(0, -\sum_{i=0}^{k-1} 2^{-i})\}$
 to the point $(z, y + 2^{-(k-1)}) \in X_{k-1}$ where

$$z = \min\{x' \mid x < x' \text{ and } (x', y + 2^{-(k-1)}) \in X_{k-1}\},$$

3 and then let $f : (0, -\sum_{i=0}^{k-1} 2^{-i}) \mapsto (0, -\sum_{i=0}^{k-2} 2^{-i})$ for $k \geq 2$, and $f : (0, -1) \mapsto$
 4 $(0, 0)$ for $k = 1$. By construction, f is then a continuous surjection.

5 Moreover, it is positively n -expansive. Indeed, take $r = 1/4$ and suppose firstly
 6 that $x \in X$ is not one of the fixed points. By construction, the set $\Gamma_+(x, r)$ can
 7 contain no points from an X_k different to that containing x . Indeed, if $x \in X_0$
 8 this is clear since points in X_k for $k \geq 1$ are at least a distance of 1 away from x .
 9 Moreover, if $x \in X_k$ for some $k \geq 1$ then there is an iterate of x that is in X_1
 10 and so the corresponding iterate of any point that began on X_j for some $j \neq k$ will not
 11 be on X_1 and hence must be at least a distance of $1/2$ from the iterate of x . So,
 12 consider firstly the case when $x \in X_0$. Suppose that $j \geq 1$ is such that $x \in Y_j$ then
 13 by construction, no point in X_0 that is not in Y_j can be in $\Gamma_+(x, r)$ since there
 14 will be an iterate of x that is in Y_1 , and the corresponding iterate of the points not
 15 in Y_j will not be in Y_1 and hence will be at least a distance of $1/2$ away. Thus,
 16 $\Gamma_+(x, r) \subseteq Y_j$ and so since Y_j consists of n points, $|\Gamma_+(x, r)| \leq n$. Suppose now
 17 then that $x \in X_k$ for some $k \geq 1$ then by construction, there exists an $\ell \in \mathbb{N}_0$ such
 18 that all points in $\Gamma_+(x, r)$ map onto X_0 for the first time under the ℓ th iterate.
 19 Since f is injective on the points in $X \setminus X_0$, each of these ℓ th iterates must be
 20 distinct in X_0 and so from the case described previously where x originated in X_0 ,
 21 this means that there can be at most n points in $\Gamma_+(x, r)$. It remains to check the
 22 fixed points. If $x = (3, 0)$, then there is no other point that lies within a distance
 23 of $1/4$ from it so $\Gamma_+(x, r) = \{x\}$. If $x = (0, 0)$ or $(0, -2)$, then every point that lies
 24 with a distance of $1/4$ from it has some iterate that is equal to $(3, 0)$ and hence has
 25 distance greater than $1/4$ from it so that in these cases also, $\Gamma_+(x, r) = \{x\}$. Thus,
 26 (X, f) is positively n -expansive.

27 Conversely, (X, f) is not $(n - 1)$ -expansive. Indeed, suppose there were such an
 28 $r > 0$ that exhibited this type of expansivity. Select $k > 0$ such that $2^{-k} < r$. Let
 29 $x \in Y_k$, then note that by construction each point in Y_k has distance less than r
 30 from x and has the same image under f . Thus $Y_k \subseteq \Gamma_+(x, r)$, so that $n \leq |\Gamma_+(x, r)|$
 31 (in fact it is equal by n -expansivity). Hence, (X, f) is not $(n - 1)$ expansive.

32 *Remark 4.9.* It is known that if a system on a compact space has shadowing and
 33 is expansive then it is topologically stable (see [48]). By Theorem 4.3 it can be
 34 equivalently said that compact systems with unique shadowing are topologically
 35 stable.

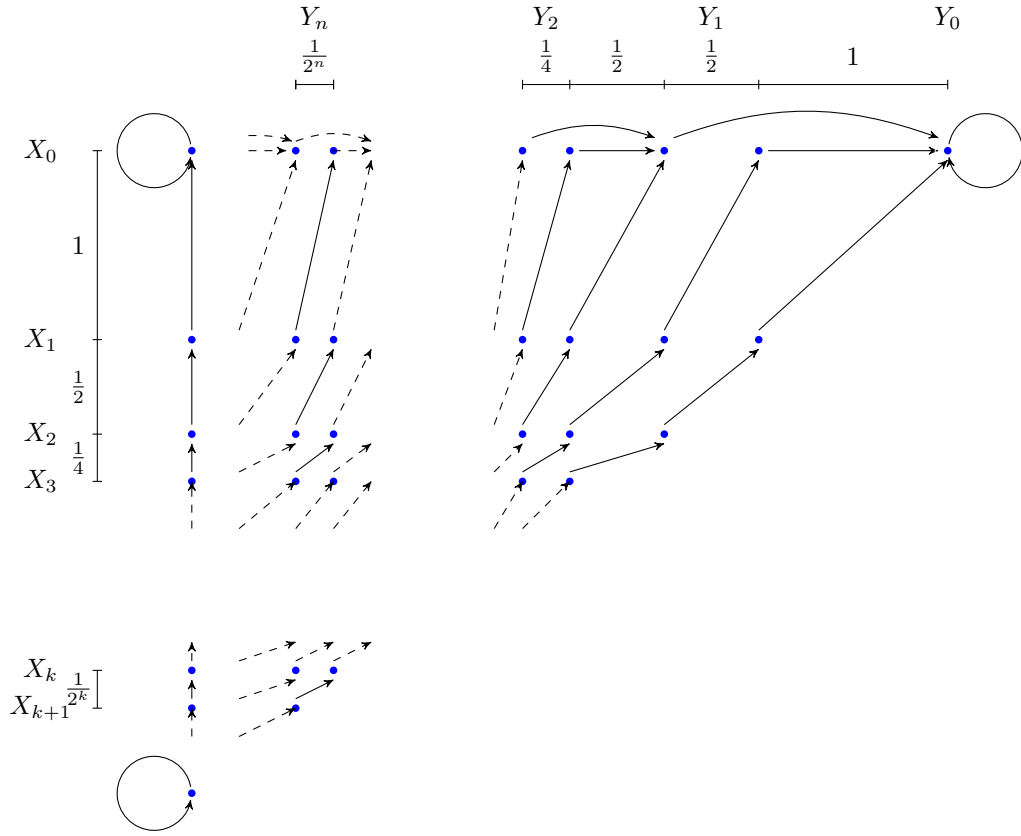


FIGURE 1. The construction from Example 4.8 for a 2-expansive map

1 Clearly there is more to be said on uniqueness and how it modifies other forms
 2 of shadowing. For example, one may define suitably ‘unique’ variants of the other
 3 shadowing types mentioned in this paper, i.e. s-limit shadowing, h-shadowing and
 4 limit shadowing.

5 **Definition 4.10.** Let (X, f) be a dynamical system. The map f has *unique s-limit*
 6 *shadowing* if

- 7 (1) it has unique shadowing; and,
 8 (2) there exists $\eta > 0$ such that for any $\varepsilon > 0$ with $\varepsilon < \eta$ there exists $\delta > 0$
 9 such that every asymptotic δ -pseudo-orbit is asymptotically ε -shadowed by
 10 a unique point.

11 We note that postulating uniqueness in condition (2) is of course unnecessary
 12 in virtue of condition (1). On the other hand, unlike the situation for s-limit
 13 shadowing, it is not clear that condition (2) implies condition (1).

14 **Definition 4.11.** Let (X, f) be a dynamical system. The map f has *unique h-*
 15 *shadowing* if there exists $\eta > 0$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that
 16 for any finite δ -pseudo-orbit (x_0, \dots, x_n) there exists a unique point z such that
 17 $d(f^i(z), x_i) < \varepsilon$ for each $i \in \{0, \dots, n\}$ and $f^n(z) = x_n$.

1 **Definition 4.12.** Let (X, f) be a dynamical system. The map f has *unique limit*
 2 *shadowing* if every asymptotic pseudo-orbit is asymptotically shadowed by a unique
 3 point.

4 The proofs of Theorems 4.13, 4.14 and 4.16 come easily given our previous dis-
 5 cussions and are thereby omitted. It is worth remarking upon how ‘uniqueness’
 6 modifies the various properties: for example, on a compact space shadowing is
 7 strictly weaker than h-shadowing [2] but Theorem 4.14 and Example 4.15 together
 8 entail that unique shadowing is strictly stronger than unique h-shadowing.

9 **Theorem 4.13.** *Let (X, f) be a dynamical system. Then the map f has*
 10 *(1) unique shadowing if and only if it has unique s-limit shadowing.*
 11 *(2) two-sided unique shadowing if and only if it has two-sided unique s-limit*
 12 *shadowing.*

13 **Theorem 4.14.** *Let (X, f) be a dynamical system. If f has unique shadowing then*
 14 *it has unique h-shadowing.*

15 Example 4.15 shows that the converse to Theorem 4.14 is false.

16 **Example 4.15.** Consider $X = \{1/2^n \mid n \in \mathbb{N}_0\} \cup \{0\}$ and let f be the identity map
 17 on X . Then f has unique h-shadowing but, by Corollary 4.7, not unique shadowing
 18 because it is not positively expansive.

19 **Theorem 4.16.** *Let (X, f) be a dynamical system. The map f has unique limit*
 20 *shadowing if and only if it has limit shadowing and no asymptotic pairs. Moreover,*
 21 *if f has unique shadowing, then f is injective.*

22 As with classical shadowing and s-limit shadowing, limit shadowing has a two-
 23 sided analogue: A system (X, f) has *(unique) two-sided limit shadowing* if for any
 24 two-sided asymptotic pseudo-orbit $(x_i)_{i \in \mathbb{Z}}$ there exists a (unique) two-sided orbit
 25 $(z_i)_{i \in \mathbb{Z}}$ which asymptotically shadows it (i.e. $d(z_i, x_i) \rightarrow 0$ as $i \rightarrow \pm\infty$). Two-sided
 26 limit shadowing has recently attracted an array of interest (e.g. [12, 13, 15, 39]).
 27 Of particular note, is its strength as a condition: it is among the strongest of
 28 the pseudo-orbit tracing properties. For homeomorphisms, it has been shown to
 29 imply shadowing, mixing and the specification property [15]. We close this paper
 30 by examining how uniqueness modifies two-sided limit shadowing. Since our map
 31 is not necessarily a homeomorphism, we first require some additional terminology.
 32 Given a continuous self-map $f: X \rightarrow X$ on a compact metric space X , the set
 33 $K_f = \bigcap_{n \in \mathbb{N}} f^n(X)$, which might be termed the surjective core of f , is a nonempty
 34 set on which f is surjective (see, for example [24]). We may then define the *induced*
 35 *core system* $(K_f, f \upharpoonright_{K_f})$, which is easily seen to be a surjective dynamical system.
 36 We omit the proof of the following lemma.

37 **Lemma 4.17.** *If (X, f) has two-sided limit shadowing then the induced core system*
 38 *has two-sided limit shadowing.*

39 Recall that a system (X, f) is *transitive* if for any pair of nonempty open sets U
 40 and V there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. It is *mixing* if for any such pair
 41 there exists $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$. It is well-known that if
 42 f is a transitive surjection then the system (X, f) either consists of a single periodic
 43 orbit or X contains at least continuum many points (with none being isolated). In
 44 similar fashion, it is easily observed that if f is a mixing surjection then the system

1 (X, f) either consists of a single fixed point or X contains at least continuum many
2 points (with none being isolated).

3 **Theorem 4.18.** *Let (X, f) be a dynamical system, where f is an injective map
4 with two-sided shadowing. If f is mixing and X contains more than one point then
5 it is not positively n -expansive for any n .*

6 *Proof.* Note first that a transitive system on a compact space is onto, so f is a
7 homeomorphism. Let $n > 1$ be given. Since f is mixing and X consists of more
8 than one point, X is infinite. Let $A = \{x_0, \dots, x_n, z\}$ be a set of $n + 2$ distinct
9 points in X . Let $\varepsilon > 0$ be such that the ε -balls around points in A are pairwise
10 disjoint and let $\delta > 0$ satisfy the shadowing condition for $\varepsilon/2$. Without loss of
11 generality $\delta < \varepsilon/2$. By mixing, there exists $n_{-1} > 1$ and, for each $i \in \{0, \dots, n\}$,
12 $x_i^{(-1)} \in B_{\frac{\delta}{2}}(x_i)$ such that

$$f^{n-1} \left(x_i^{(-1)} \right) \in B_{\frac{\delta}{2}} \left(x_{i+1 \pmod{n+1}} \right).$$

13 Recursively by mixing there exists, for each $j < -1$, $n_j > 1$ and there exists, for
14 each $i \in \{0, \dots, n\}$, $x_i^{(j)} \in B_{2^j \delta}(x_i)$ such that

$$f^{n_j} \left(x_i^{(j)} \right) \in B_{2^j \delta} \left(x_{i+1 \pmod{n+1}} \right).$$

Finally, for each $i \in \{0, \dots, n\}$ there exists $x'_i \in B_{\frac{\delta}{2}}(x_i)$ and $m \in \mathbb{N}$ such that
 $f^m(x'_i) \in B_\delta(z)$. For each $i \in \{0, \dots, n\}$ we now have a two-sided (asymptotic)
 δ -pseudo orbit:

$$\left(\begin{aligned} & \dots, x_{i+j \pmod{n+1}}^{(j)}, f \left(x_{i+j \pmod{n+1}}^{(j)} \right), f^2 \left(x_{i+j \pmod{n+1}}^{(j)} \right), \dots, \\ & f^{n_j-1} \left(x_{i+j \pmod{n+1}}^{(j)} \right), x_{i+j+1 \pmod{n+1}}^{(j+1)}, f \left(x_{i+j+1 \pmod{n+1}}^{(j+1)} \right), \dots, \\ & f^{n_{j+1}-1} \left(x_{i+j+1 \pmod{n+1}}^{(j+1)} \right), x_{i+j+2 \pmod{n+1}}^{(j+2)}, \dots, \\ & \dots, x_{i-1 \pmod{n+1}}^{(-1)}, f \left(x_{i-1 \pmod{n+1}}^{(-1)} \right), \dots, f^{n_{-1}-1} \left(x_{i-1 \pmod{n+1}}^{(-1)} \right), \\ & x'_i, f(x'_i), f^2(x'_i), \dots, f^{m-1}(x'_i), z, f(z), f^2(z), \dots \end{aligned} \right)$$

where the 0th term is given by x'_i for each such i . Note that these pseudo-orbits are
distinct. By two-sided shadowing these pseudo-orbits are $\varepsilon/2$ -shadowed. Notice
that each one is shadowed by a distinct point since, for each distinct pair $i, j \in$
 $\{0, \dots, n\}$,

$$\begin{aligned} d(x'_i, x'_j) &> d(x_i, x_j) - \delta \\ &> d(x_i, x_j) - \varepsilon/2 \\ &> 3\varepsilon/2. \end{aligned}$$

15 Let $y_i \in X$ be a point which $\varepsilon/2$ -shadows the pseudo-orbit through x'_i (so that
16 $y_i \in B_{\frac{\varepsilon}{2}}(x'_i)$). Since f is injective $f^k(y_i) \neq f^k(y_j)$ for any $k \in \mathbb{N}$ and distinct i and
17 j . It remains now to observe that for each $k \geq m$ and all $i \in \{0, \dots, n\}$ we have
18 $f^k(y_i) \in B_{\frac{\varepsilon}{2}}(f^{k-m}(z))$. In particular $f^m(y_i) \in \Gamma_+(z, \varepsilon)$ for each $i \in \{0, \dots, n\}$ and
19 so $|\Gamma_+(z, \varepsilon)| \geq n + 1$. Since we could have chosen ε arbitrarily small it follows that
20 (X, f) is not positively n -expansive for any n . \square

1 **Corollary 4.19.** *Let (X, f) be a dynamical system, where f is an injective map*
 2 *with two-sided limit shadowing. If the surjective core contains more than one point*
 3 *then f is not positively n -expansive for any n .*

4 *Proof.* By Lemma 4.17 the induced core system has two-sided limit shadowing and
 5 therefore, by [15, Theorem B], is mixing. By [15, Theorem A] the induced core
 6 system has two-sided shadowing. Therefore, by Theorem 4.18 the induced core
 7 system is not positively n -expansive for any $n \in \mathbb{N}$. It immediately follows that
 8 neither is (X, f) . \square

9 **Corollary 4.20.** *An injective map with unique two-sided limit shadowing does not*
 10 *have n -shadowing for any $n \in \mathbb{N}$. In particular, it does not have unique shadowing.*

11 We close this paper with a question suggested to us by an anonymous referee.
 12 A positively expansive map f on a compact metric space X has shadowing if, and
 13 only if, it is open (i.e. $f(U)$ is open for every open set U) [47, Theorem 1]. With
 14 this in mind, the following question is natural. For each $n \geq 2$ we ask:

15 *Question 4.21.* Is a positively n -expansive map with shadowing necessarily open?

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