Mixed-norm estimates for a class of nonisotropic directional maximal operators and Hilbert transforms

Bez, Richard

DOI:
10.1016/j.jfa.2008.07.026

MIXED-NORM ESTIMATES FOR A CLASS OF NONISOTROPIC DIRECTIONAL MAXIMAL OPERATORS AND HILBERT TRANSFORMS

NEAL BEZ

Abstract. For all $d \geq 2$ and $p \in (1, \max(2, (d+1)/2)]$, we prove sharp L^p to $L^p(L^q)$ estimates (modulo an endpoint) for a directional maximal operator associated to curves generated by the dilation matrices $\exp((\log t)P)$, where P has real entries and eigenvalues with positive real part. For the corresponding Hilbert transform we prove an analogous result for all $d \geq 2$ and $p \in (1, 2]$. As corollaries, we prove L^p bounds for variable kernel singular integral operators and Nikodym-type maximal operators taking averages over certain families of curved sets in $\mathbb{R}^d$.

1. Introduction

Given an integer $d \geq 2$, fix a real d by d matrix P with the property that each eigenvalue of P has positive real part. Define the dilations $\{\delta_t : t \in (0, \infty)\}$ by

$$\delta_t := \exp((\log t)P), \tag{1.1}$$

and for $t \in (-\infty, 0)$ we set $\delta_t := -\delta_{-t}$. For Schwartz functions f on $\mathbb{R}^d$ and $(x, \omega) \in \mathbb{R}^d \times S^{d-1}$, define operators M and H by

$$Mf(x, \omega) := \sup_{h \in (0,\infty)} \frac{1}{h} \left| \int_0^h f(x - \delta_t \omega)\, dt \right|,$$

$$Hf(x, \omega) := p.v. \int_{\mathbb{R}} f(x - \delta_t \omega)\, \frac{dt}{t}.$$

For fixed $\omega \in S^{d-1}$, it was shown by Stein and Wainger [19] that $f \mapsto Mf(\cdot, \omega)$ is bounded on L^p for all $p \in (1, \infty]$ and $f \mapsto Hf(\cdot, \omega)$ is bounded on L^p for each $p \in (1, \infty)$. The purpose of this paper is to address the question of whether M and H are bounded as operators from L^p to the mixed-norm space, $L^p(L^q)$, consisting of all measurable functions $F : \mathbb{R}^d \times S^{d-1} \to \mathbb{C}$ such that $\|F\|_{L^p(L^q)}$ is finite, where

$$\|F\|_{L^p(L^q)} := \left(\int_{\mathbb{R}^d} \left(\int_{S^{d-1}} |F(x, \omega)|^q \, d\Omega(\omega) \right)^{p/q} dx \right)^{1/p}.$$

Here, $d\Omega$ denotes the induced Lebesgue measure on S^{d-1}. In order to describe the earlier developments on this problem, it will be very convenient to introduce the following.

The author acknowledges support from EPSRC with partial support from grant EP/E022340/1.

Notation If $(p,q) \in [1,\infty] \times [1,\infty]$ and $p < q$, let $\Delta_{(p,q)} \subset [0,1] \times [0,1]$ denote the interior of the triangle with vertices $(0,0), (1,1)$, and $(1/p, 1/q)$. It will also be beneficial to set $p_d := \max(2, (d+1)/2)$ and

$$q_d(p) := \frac{p(d-1)}{d-p}, \quad \text{for } p \in (1,\infty)$$

(with the agreement that $q_d(p) = \infty$ when $p \geq d$).

We remark that the arguments of Stein and Wainger in [19] imply that, for each $p \in (1,\infty]$, the L^p bound for $f \mapsto Mf(\cdot,\omega)$ is uniformly bounded over $\omega \in S^{d-1}$. It follows immediately from Minkowski's inequality that when $1 \leq q \leq p \leq \infty$ (and $p > 1$), M is bounded from L^p to $L^p(L^q)$. A similar remark holds for H provided we exclude the case $p = \infty$. Consequently, we are only concerned with the case $p < q$ in this paper and the regions $\Delta_{(p,q)}$ will be used to represent the progress beyond the "trivial" region where $p \geq q$.

When P is (a positive multiple of) the identity matrix observe that the dilations δ_t are the standard isotropic dilations of $\mathbb{R}^d$. The following is the natural conjecture in the isotropic case.

Conjecture 1.1. *When P is a positive multiple of the identity matrix, the operators M and H are bounded from L^p to $L^p(L^q)$ for all $(1/p, 1/q) \in \Delta_{(d,\infty)}$.*

Indeed, by testing on the characteristic function of the unit ball in $\mathbb{R}^d$, it is easy to verify that if M or H are bounded from L^p to $L^p(L^q)$ then necessarily $q \in [1, q_d(p))$. For H, Calderón and Zygmund [3] verified Conjecture 1.1 for all $(1/p, 1/q) \in \Delta_{(2,q_d(2))}$. R. Fefferman [11] proved that Conjecture 1.1 for M is true whenever $(1/p, 1/q) \in \Delta_{(2d/(d+1),2)}$ (note that $q_d(2d/(d+1)) = 2$). Later came the following improvement due to Christ, Duoandikoetxea, and Rubio de Francia.

Theorem 1.2. [7] *Suppose P is a positive multiple of the identity matrix. For any $d \geq 2$, M and H are bounded from L^p to $L^p(L^q)$ for any $(1/p, 1/q) \in \Delta_{(p_d, q_d(p_d))}$.*

Theorem 1.2 thus completely resolves Conjecture 1.1 when $d = 2$, and we believe represents the most progress for $d \geq 3$.

The known results in the nonisotropic setting concern the case that P is a diagonal matrix with distinct, real, and thus positive, diagonal entries. We begin with the maximal operator M and the following results of Sato [17] and Chen [6].

Theorem 1.3. *Suppose $P = \operatorname{diag}(\alpha_1, \ldots, \alpha_d)$ where the α_j are distinct and positive real numbers.*

(1) [17] *For any $d \geq 2$, M is bounded from L^p to $L^p(L^q)$ for any $(1/p, 1/q) \in \Delta_{(p_s,2)}$, where*

$$p_s := \frac{2(d\sum_j \alpha_j - (d-2)\min_j(\alpha_j))}{d\sum_j \alpha_j - (d-4)\min_j(\alpha_j)}.$$

(2) [6] *For any $d \geq 2$, M is bounded from L^p to $L^p(L^q)$ for any $(1/p, 1/q) \in \Delta_{(p_c,q_c)}$, where*

$$p_c := \frac{2(d-1+1/d)}{d-1+2/d}, \quad \text{and} \quad q_c := \frac{2(d-1+1/d)}{d-1}.$$

For H, we believe that the best known result is the following theorem of Chen, which is restricted to the plane.

Theorem 1.4. [5] *Suppose* $P = \mathrm{diag}(\alpha_1, \alpha_2)$ *where* $1 < \alpha_2/\alpha_1 < 4/3$. *Then* H *is bounded from* L^p *to* $L^p(L^q)$ *for any* $(1/p, 1/q) \in \Delta_{(2,4)}$.

We should mention that Theorem 1.2 and Theorem 1.4 in fact hold for the associated maximal Hilbert transform, as was shown in [7] and [5] respectively. We also remark that either Theorem 1.3(1) or Theorem 1.3(2) can subsume the other, depending on certain relationships between the numbers d, $\min_j(\alpha_j)$, and $\sum_j \alpha_j$.

The main results in this paper extend the known results in the nonisotropic setting of diagonal matrices P and, in fact, hold whenever the eigenvalues of P have positive real part. For the maximal operator M, we have the following.

Theorem 1.5. *Suppose* $d \geq 2$ *and* P *is a* d *by* d *matrix whose eigenvalues each have positive real part.*

(1) For any $p \in (1, \infty)$, *a necessary condition that* M *is a bounded operator from* L^p *to* $L^p(L^q)$ *is that* $q \in [1, q_d(p)]$.
(2) For any $(1/p, 1/q) \in \Delta_{(p_d, q_d(p_d))}$, M *is bounded from* L^p *to* $L^p(L^q)$.

Modulo the endpoint $q = q_d(p)$ in the necessary condition, Theorem 1.5 coincides with Theorem 1.2 for M. In particular, modulo this endpoint, Theorem 1.5 is sharp in all dimensions for $p \in (1, p_d]$, and when $d = 2$, sharp for all $p \in (1, \infty)$.

For each $d \geq 2$, one may verify that $\Delta_{(p_s,2)}$ and $\Delta_{(p_c,q_c)}$, the regions obtained in Theorem 1.3(1) and Theorem 1.3(2) respectively, are strict subsets of $\Delta_{(p_d,q_d(p))}$.

Our analysis of the singular integral operator H has been less successful. At the moment, the following is known to us.

Theorem 1.6. *Suppose* $d \geq 2$ *and* P *is a* d *by* d *matrix whose eigenvalues each have positive real part.*

(1) For any $p \in (1, \infty)$, *a necessary condition that* H *is a bounded operator from* L^p *to* $L^p(L^q)$ *is that* $q \in [1, q_d(p)]$.
(2) For any $(1/p, 1/q) \in \Delta_{(2, q_d(2))}$, H *is bounded from* L^p *to* $L^p(L^q)$.

It follows from Theorem 1.6 that we have a sharp result for H in all dimensions for $p \in (1, 2]$, and, when $d = 2$, for all $p \in (1, \infty)$ (modulo an endpoint). When $d = 2$, the region $\Delta_{(2,4)}$ obtained in Theorem 1.4 for H is strictly smaller than the (essentially) optimal region $\Delta_{(2,\infty)}$.

In the coming section, we present some preliminary results concerning the dilations δ_t. In Section 3 we prove the necessity parts of Theorem 1.5 and Theorem 1.6. Section 4 and Section 5 are devoted to the sufficiency parts of Theorem 1.5 and Theorem 1.6 respectively. In Section 6 we prove the main oscillatory integral estimate used for these results. Finally,

in Section 7 we exhibit some applications of our results to variable kernel singular integral operators and a Nikodym-type maximal operator associated to certain families of curved sets.

Notation Let A and B be nonnegative real numbers. We write $A \lesssim B$ and $B \gtrsim A$ for $A \leq CB$, where the constant C may depend only on d, the matrix P, and any index p or q that may be present. If $A \lesssim B \lesssim A$ then we write $A \sim B$, and if $A \sim 1$ then we may say that A is $O(1)$.

For $(x, r) \in \mathbb{R}^d \times (0, \infty)$, we define $B(x, r) := \{y \in \mathbb{R}^d : |x - y| < r\}$.

This work formed part of the author's PhD thesis at the University of Edinburgh and was supported by an EPSRC award. The author would very much like to thank Jim Wright for his guidance on this work.

2. Preliminaries

Particularly with the applications in Section 7 in mind, we desire a parametrisation of $\mathbb{R}^d$ by polar coordinates adapted to our nonisotropic setting. For this we will follow the approach of Stein and Wainger in [19], and thus refer the reader to their paper for further details.

Choose a real symmetric positive definite matrix Q such that, for fixed $x \in \mathbb{R}^d \setminus \{0\}$,

$$t \longmapsto \langle Q\delta_t x, \delta_t x\rangle^{1/2} \quad \text{is strictly increasing on } (0, \infty) \tag{2.1}$$

(here $\langle \cdot, \cdot \rangle$ is the usual Euclidean inner product on $\mathbb{R}^d$). Assuming that such a matrix Q exists for the moment, for each $x \in \mathbb{R}^d \setminus \{0\}$, we (uniquely) define $\varrho(x) \in (0, \infty)$ by the following:

$$\langle Q\delta_{\varrho(x)^{-1}} x, \delta_{\varrho(x)^{-1}} x\rangle = 1. \tag{2.2}$$

When $x = 0$ we set $\varrho(x) = 0$. The function ϱ is $C^\infty(\mathbb{R}^d \setminus \{0\})$ and is homogeneous in the sense that

$$\varrho(\delta_t x) = t\varrho(x) \quad \text{for all } t \in (0, \infty) \text{ and all } x \in \mathbb{R}^d.$$

Furthermore, we let Σ_Q^{d-1} be the ellipsoid given by

$$\Sigma_Q^{d-1} := \{\omega \in \mathbb{R}^d : \langle Q\omega, \omega\rangle = 1\} = \{\omega \in \mathbb{R}^d : \varrho(\omega) = 1\}. \tag{2.3}$$

Remarks (1) On the existence of a matrix Q satisfying (2.1), one may take

$$Q = \int_0^\infty \exp(-tP^*) \exp(-tP)\, dt. \tag{2.4}$$

It is straightforward to check that this has the requisite property; this rather cute choice can be found in [19]. Note that the choice of Q is certainly not unique.

(2) In general, $t \mapsto \langle \delta_t x, \delta_t x\rangle^{1/2}$ is not strictly increasing on $(0, \infty)$, and thus the identity matrix is not always an appropriate choice of Q. The identity matrix is, however, appropriate when P is a diagonal matrix, for example.

(3) In light of the above, a question arises about what is the most "natural" domain of the angular variable ω in the definition of our operators M and H. Our proofs of Theorem 1.5 and Theorem 1.6 are sufficiently robust in the sense that one may (appropriately) replace S^{d-1} with the ellipsoid Σ_Q^{d-1} without altering the conclusions. This is a pertinent point for our applications in Section 7 and we refer the reader there for more details.

For each nonzero $x \in \mathbb{R}^d$ there exists a unique pair $(r, \omega) \in (0, \infty) \times \Sigma_Q^{d-1}$ such that

$$x = \delta_r \omega;$$

of course, $r = \varrho(x)$ and $\omega = \delta_{\varrho(x)}^{-1} x$. The volume element in $\mathbb{R}^d$ is

$$dx = r^{\tau-1} dr d\Omega_Q(\omega), \tag{2.5}$$

where dr is Lebesgue measure on the positive real line, $d\Omega_Q$ is a smooth measure on Σ_Q^{d-1}, and τ is the trace of P.

We conclude this section with a brief presentation of some properties of the dilations δ_t that we rely upon throughout. Firstly, it is a triviality to check that the group property, $\delta_s \delta_t = \delta_{st}$, holds for all $s, t \in (0, \infty)$. The following estimates will also be useful.

$$t^{\alpha_1} |x| \lesssim |\delta_t x| \lesssim t^{\alpha_2} |x| \quad \text{for all } t < 1, \text{ and} \tag{2.6}$$

$$t^{\alpha_3} |x| \lesssim |\delta_t x| \lesssim t^{\alpha_4} |x| \quad \text{for all } t \geq 1, \tag{2.7}$$

where each α_j is a positive real number depending only on P. Finally, we note that, although the triangle inequality for our nonisotropic distance function ϱ may fail in general,

$$\varrho(x + y) \lesssim \varrho(x) + \varrho(y) \quad \text{for all } x, y \in \mathbb{R}^d. \tag{2.8}$$

3. Necessity

As far as we know, no necessary conditions have been given in the nonisotropic case. The necessary condition in Theorem 1.5(1) and Theorem 1.6(1) follows by testing on an arbitrarily small Euclidean ball centred at the origin. Notice that because of scaling in the isotropic case, this is equivalent to testing on the unit Euclidean ball and this is in fact sufficient to generate Conjecture 1.1.

Suppose first that M is bounded from L^p to $L^p(L^q)$ and set f_N to be the characteristic function of $B(0, C/N)$. Here, $C \sim 1$ will be determined later in the proof, and N is an arbitrarily large positive number. Thus,

$$N^{-d} \sim \|f_N\|_p^p \gtrsim \int_{\mathbb{R}^d} \left(\int_{S^{d-1}} \left(\sup_{h \in (0,\infty)} \frac{1}{h} \int_0^h f_N(x - \delta_t \omega)\, dt \right)^q d\Omega(\omega) \right)^{p/q} dx. \tag{3.1}$$

Next, we change variables to polar coordinates, $x = \delta_r \theta$ for $(r, \theta) \in (0, \infty) \times \Sigma_Q^{d-1}$, where Q is, say, given by (2.4).

For each $\theta \in \Sigma_Q^{d-1}$ let R_θ be any positive number such that $\delta_{R_\theta}\theta \in S^{d-1}$. By (2.6) and (2.7) it is easy to see that at least one choice of R_θ exists and moreover $R_\theta \sim 1$. For fixed $r \in (1,2)$ and $\theta \in \Sigma_Q^{d-1}$ let $r_\theta := r/R_\theta$ and $\theta' := \delta_{R_\theta}\theta$. We claim that, if $t \in (r_\theta, r_\theta + 1/N)$ and $\omega \in S^{d-1}$ satisfies $|\omega - \theta'| < 1/N$ then $\delta_r\theta - \delta_t\omega \in B(0, C/N)$. This claim granted, it follows from (3.1) that

$$\begin{aligned} N^{-d} &\gtrsim \int_1^2 \int_{\Sigma_Q^{d-1}} \left(\int_{\substack{w \in S^{d-1} \\ |\omega - \theta'| < 1/N}} \left(\int_{r_\theta}^{r_\theta + 1/N} dt \right)^q d\Omega(\omega) \right)^{p/q} d\Omega_Q(\theta) r^{\tau - 1}\, dr \\ &\sim N^{-p-(d-1)p/q}, \end{aligned}$$

and this implies that $q \in [1, q_d(p)]$, as required.

To prove the claim, first write $t = r_\theta + h$ where $h \in (0, 1/N)$ and

$$\delta_r\theta - \delta_t\omega = \delta_{r_\theta}(I - \delta_{1+h/r_\theta})\theta' - \delta_{r_\theta + h}(\omega - \theta'),$$

from which the Euclidean triangle inequality, and the estimates in (2.6) and (2.7) imply that

$$|\delta_r\theta - \delta_t\omega| \lesssim \|I - \delta_{1+h/r_\theta}\| + 1/N.$$

Using the trivial estimate $\log(1 + h/r_\theta) \leq h/r_\theta$ and the matrix-norm triangle inequality, it follows that

$$\|I - \delta_{1+h/r_\theta}\| = \left\| \sum_{k=1}^{\infty} \frac{\log(1 + h/r_\theta)^k}{k!} P^k \right\| \lesssim h$$

and therefore $|\delta_r\theta - \delta_t\omega| \lesssim 1/N$. By making an appropriate choice of C in the definition of f_N, our claim now follows; this completes our proof of Theorem 1.5(1).

To prove Theorem 1.6(1), we also test H on the function f_N, for arbitrarily large positive N. The only difference to the above argument given for M is that one should restrict the θ-integral to some smaller subset of Σ_Q^{d-1} of size $O(1)$ to remove the cancellation in the t-integral. We omit the details.

4. Proof of Theorem 1.5(2)

By interpolation with $(1/p, 1/q)$ near $(0,0)$ and $(1,1)$ it suffices to prove Theorem 1.5(2) when $p = p_d$. Unlike previous approaches in the nonisotropic setting, we shall use the successful techniques used for the isotropic case in [7]. First, we setup a square function type argument using a fixed number $\sigma \in (1, \infty)$ which we do not specify at all but emphasise that it only depends on P. Select $\varsigma \in (1, \sigma)$ for which $(\varsigma, \varsigma^2) \subsetneq (1, \sigma)$. Then choose $\psi \in \mathcal{S}(\mathbb{R})$ such that ψ vanishes outside $(1, \sigma)$, ψ is equal to 1 on (ς, ς^2), and $0 \leq \psi \leq 1$. Now choose a positive function $\phi \in \mathcal{S}(\mathbb{R}^d)$ for which $\int \phi = \int \psi$ and $\phi = \widetilde{\phi}(\varrho(\cdot))$ for some decreasing function $\widetilde{\phi}$ on $[0, \infty)$. Here, ϱ is the P-homogeneous distance function given by (2.2) associated to the matrix Q, which we now choose to be given by (2.4).

Define, for each $k \in \mathbb{Z}$,

$$A_k f(x, \omega) := \int_{\mathbb{R}} f(x - \delta_t\omega)\psi_k(t)\, dt - \int_{\mathbb{R}^d} f(x - y)\phi_k(y)\, dy, \tag{4.1}$$

where

$$\psi_k(t) := \varsigma^{-k}\psi(\varsigma^{-k}t) \quad \text{and} \quad \phi_k(x) := \det \delta_{\varsigma^{-k}}\phi(\delta_{\varsigma^{-k}}x).$$

It is clear that

$$Mf(x,\omega) \lesssim \sup_{k\in\mathbb{Z}} |A_k f(x,\omega)| + \sup_{k\in\mathbb{Z}} \left| \int_{\mathbb{R}^d} f(x-y)\phi_k(y)\,dy \right|. \tag{4.2}$$

Furthermore, the operator

$$f \longmapsto \sup_{r\in(0,\infty)} |\delta_r B(0,1)|^{-1} \int_{\delta_r B(0,1)} |f(\cdot - z)|\,dz$$

is bounded on L^p for each $p \in (1,\infty)$ (see, for example, Chapter 1 of [18]) and pointwise dominates the second term on the right hand side of (4.2) up to an $O(1)$ constant. Therefore, it suffices to prove Theorem 1.5(2) with M replaced by $\mathcal{M}$, where

$$\mathcal{M}f(x,\omega) := \sup_{k\in\mathbb{Z}} |A_k f(x,\omega)|.$$

The first step is to invoke some known Littlewood-Paley theory: Begin with a function $\eta \in \mathcal{S}(\mathbb{R}^d)$ such that η vanishes outside $B(0,2)$, η equals 1 on $B(0,1)$, and $0 \le \eta \le 1$. It can be shown (see, for example, [4]) that there exists a natural number D depending on P such that if

$$\lambda_k := \eta_{k+D} - \eta_{k-D} \quad \text{and} \quad \widehat{\Lambda_k} := \lambda_k,$$

where $\eta_k(\xi) := \eta(\delta^*_{\varsigma^k}\xi)$, then the following is true.

Theorem 4.1. *(1) The Λ_k decompose the identity operator in the sense that $\sum_{k\in\mathbb{Z}} \lambda_k(\xi) = 2D$ for each $\xi \neq 0$.*

(2) For any $\xi \in \mathbb{R}^d$, the number of $k \in \mathbb{Z}$ for which $\lambda_k(\xi) \neq 0$ is $O(1)$.

(3) If either $|\delta^*_{\varsigma^{k-D}}\xi| \ge 2$ or $|\delta^*_{\varsigma^{k+D}}\xi| \le 1$ then $\lambda_k(\xi) = 0$.

(4) For all $p \in (1,\infty)$, $\left\| \left(\sum_{k\in\mathbb{Z}} |\Lambda_k * f|^2\right)^{1/2} \right\|_p \lesssim \|f\|_p$.

For any Schwartz function f we have

$$\mathcal{M}f(x,\omega) \sim \sup_{k\in\mathbb{Z}} \left| A_k \left(\sum_{j\in\mathbb{Z}} \Lambda_{j+k} * f \right)(x,\omega) \right| \le \sum_{j\in\mathbb{Z}} B_j f(x,\omega),$$

where

$$B_j f(x,\omega) := \sup_{k\in\mathbb{Z}} |A_k(\Lambda_{j+k} * f)(x,\omega)|.$$

We claim that it suffices to prove the following inequalities for each Schwartz function f and each $j \in \mathbb{Z}$.

$$\|B_j f\|_{L^2(L^q)} \lesssim \varsigma^{-\alpha_q|j|}\|f\|_2 \quad \text{for some } \alpha_q > 0 \text{ and } q < q_d(2); \tag{4.3}$$

$$\|B_j f\|_{L^p(L^q)} \lesssim \|f\|_p \quad \text{for each } (1/p, 1/q) \in \Delta_{((d+1)/2,d+1)}. \tag{4.4}$$

Given the claim, (4.3) immediately implies Theorem 1.5(2) when $d = 2$. For $d \ge 3$, interpolation between (4.3) and (4.4) implies that

$$\|B_j f\|_{L^p(L^q)} \lesssim \varsigma^{-\alpha_{p,q}|j|}\|f\|_p$$

for each $p \in (2, p_d)$ and $q \in [1, q_d(p))$. Hence, for such p and q,

$$\|\mathcal{M}f\|_{L^p(L^q)} \lesssim \sum_{j\in\mathbb{Z}} \|B_j f\|_{L^p(L^q)} \lesssim \|f\|_p. \tag{4.5}$$

We can now use this estimate and interpolation to achieve the same conclusion when $p = p_d$ and $q \in [1, q_d(p_d))$. Indeed, one should interpolate (4.5) for p sufficiently close to p_d and an appropriate $q \in [1, q_d(p))$, with the trivial estimate $\|\mathcal{M}f\|_{L^\infty(L^\infty)} \lesssim \|f\|_\infty$. The rest of this section is thus dedicated to the proof of (4.3) and (4.4).

Proof of (4.3). Fix $q \in (2, q_d(2))$ and choose $\nu \in (0, 1/2)$ such that

$$q^{-1} = 1/2 - \nu/(d-1). \tag{4.6}$$

By Minkowski's inequality and Sobolev's embedding theorem for manifolds,

$$\left\| \left(\sum_{k\in\mathbb{Z}} |A_k(\Lambda_{j+k} * f)(x,\cdot)|^2 \right)^{1/2} \right\|_q \lesssim \left(\sum_{k\in\mathbb{Z}} \|A_k(\Lambda_{j+k} * f)(x,\cdot)\|_{L^2_\nu}^2 \right)^{1/2},$$

where L^2_ν denotes the Sobolev space $L^2_\nu(S^{d-1})$. Hence, by Plancherel's theorem,

$$\|B_j f\|_{L^2(L^q)}^2 \lesssim \sum_{k\in\mathbb{Z}} \int_{\mathcal{A}_k} |\widehat{f}(\xi)|^2 \|m(\delta^*_{\varsigma^k}\xi, \cdot)\|_{L^2_\nu}^2 \, d\xi,$$

where

$$m(\xi,\omega) := \int_{\mathbb{R}} \psi(t) e^{i\langle \delta_t\omega, \xi\rangle} \, dt - \int_{\mathbb{R}^d} \phi(x) e^{i\langle x,\xi\rangle} \, dx, \quad \text{and}$$

$$\mathcal{A}_k := \{\xi \in \mathbb{R}^d : |\delta^*_{\varsigma^{j+k+D}}\xi| > 1 \text{ and } |\delta^*_{\varsigma^{j+k-D}}\xi| < 2\}.$$

We claim that for almost all $\xi \in \mathbb{R}^d$,

$$\|m(\xi,\cdot)\|_{L^2_\nu} \lesssim \min(|\xi|, |\xi|^{-\varepsilon}),$$

where $\varepsilon := 1/2(1/2 - \nu)$. The claim granted, it is not difficult to verify that (4.3) follows from (2.6), (2.7), and Theorem 4.1(2). To prove our claim, we shall show that the following estimates hold almost everywhere:

$$\|m(\xi,\cdot)\|_{L^2_0} \lesssim \min(|\xi|, |\xi|^{\varepsilon - 1/2}); \tag{4.7}$$

$$\|m(\xi,\cdot)\|_{L^2_1} \lesssim \min(|\xi|, |\xi|^{\varepsilon + 1/2}); \tag{4.8}$$

and then interpolate between L^2_0 and L^2_1. Firstly, for small $|\xi|$, we use the fact that $\int \psi = \int \phi$ to get

$$m(\xi,\omega) = \int_{\mathbb{R}} \psi(t)(e^{i\langle \delta_t\omega,\xi\rangle} - 1) \, dt - \int_{\mathbb{R}^d} \phi(x)(e^{i\langle x,\xi\rangle} - 1) \, dx,$$

and hence $|m(\xi,\omega)| \lesssim |\xi|$ by the mean value theorem. Since the modulus of any first order derivative of $\omega \mapsto \langle \delta_t\omega, \xi\rangle$ on S^{d-1} is majorised by $|\xi|$, the estimates for small $|\xi|$ in (4.7) and (4.8) follow. The estimates in (4.7) and (4.8) for large $|\xi|$ are implied by the following lemma, whose proof is delayed until Section 6.

Lemma 4.2. *Fix $a \in \{0,1\}$. Suppose that for each fixed $(\xi,\omega) \in \mathbb{R}^d \setminus \{0\} \times S^{d-1}$, the function $\Psi_{(\xi,\omega)}$ is supported in $[1,\sigma]$, smooth on $(1,\sigma)$, and*

$$|\Psi_{(\xi,\omega)}(t)| + |\Psi'_{(\xi,\omega)}(t)| \lesssim |\xi|^a \quad \text{for all } t \in [1,\sigma]. \tag{4.9}$$

Then,

$$\int_{S^{d-1}} \left| \int_{\mathbb{R}} \Psi_{(\xi,\omega)}(t) e^{i\langle \delta_t\omega,\xi\rangle} \, dt \right|^2 d\Omega(\omega) \lesssim |\xi|^{-1+2a+2\varepsilon}.$$

Proof of (4.4). If $(1/p, 1/q) \in \Delta_{((d+1)/2,d+1)}$ then

$$\|B_j f\|^p_{L^p(L^q)} \le \int_{\mathbb{R}^d} \left(\sum_{k\in\mathbb{Z}} \int_{S^{d-1}} |A_k(\Lambda_{j+k} * f)(x,\omega)|^q \, d\Omega(\omega) \right)^{p/q} dx$$
$$\le \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^d} \left(\int_{S^{d-1}} |A_k(\Lambda_{j+k} * f)(x,\omega)|^q \, d\Omega(\omega) \right)^{p/q} dx.$$

Thus, by Theorem 4.1(4), it suffices to prove the following.

Lemma 4.3. *There exists a constant $C_{p,q} < \infty$ such that for all $k \in \mathbb{Z}$,*

$$\|A_k f\|_{L^p(L^q)} \le C_{p,q} \|f\|_p \tag{4.10}$$

whenever $(1/p, 1/q) \in \Delta_{((d+1)/2,d+1)}$.

Proof. Fix $(1/p, 1/q) \in \Delta_{((d+1)/2,d+1)}$. Since $f \mapsto \sup_{k\in\mathbb{Z}} |f * \phi_k|$ is bounded on L^p, it is immediate from the definition of A_k in (4.1) that it suffices to prove (4.10) with A_k replaced by T_k, where

$$T_k f(x,\omega) := \int_{\mathbb{R}} f(x - \delta_t \omega) \psi_k(t) \, dt.$$

Since $\|T_k f\|^p_{L^p(L^q)} = \det \delta_{\varsigma^k} \|T_0(f(\delta_{\varsigma^k} \cdot))\|^p_{L^p(L^q)}$, we may restrict our attention to $k = 0$. Furthermore, since T_0 is a local operator and ψ is nonnegative, it suffices to prove (4.10) for T_0 and nonnegative functions f with support inside the unit cube centred at the origin, $\mathfrak{Q}_0$. For such f, Hölder's inequality implies that $\|T_0 f\|_{L^p(L^q)} \lesssim \|T_0 f\|_{L^q(L^q)}$, which means it suffices to show

$$\int_{C\mathfrak{Q}_0} \int_{S^{d-1}} \left(\int_1^\sigma f(x - \delta_t \omega) \, dt \right)^q d\Omega(\omega) dx \lesssim \|f\|_p^q \quad \text{where } C \sim 1, \tag{4.11}$$

or, by duality,

$$\left| \int f(x - \delta_t \omega) g(x,\omega) \, dt d\Omega(\omega) dx \right| \lesssim \|f\|_p \left(\int g(x,\omega)^{q'} \, d\Omega(\omega) dx \right)^{1/q'}. \tag{4.12}$$

Here, we are, of course, suppressing the regions of integration which are determined by (4.11). To show (4.12) we use a recent theorem of Gressman in [12]. For completeness, we now describe the general setup and main theorem in [12], and then demonstrate that (4.12) follows immediately as a special case.

Let X and Y be smooth manifolds equipped with measures of smooth density and assume $\dim X < \dim Y$. Let $\mathfrak{M}$ be a smooth $(\dim Y + 1)$-dimensional submanifold of $X \times Y$, also equipped with a measure, and such that the natural projections $\pi_X : \mathfrak{M} \to X$ and $\pi_Y : \mathfrak{M} \to Y$ have everywhere surjective differential maps. Furthermore, let $\mathfrak{X}_1$ and $\mathfrak{N}_1$ be those vector fields on $\mathfrak{M}$ which are annihilated by $d\pi_X$ and $d\pi_Y$, respectively. Now choose a nonvanishing representative $Y_1 \in \mathfrak{N}_1$ and define $T(V) := [V, Y_1]$, where $[\cdot,\cdot]$ denotes the Lie bracket. Define $\mathfrak{X}_j$ to be the collection of all vector fields in $\mathfrak{X}_{j-1}$ such that $T(V) \in \mathfrak{X}_{j-1} + \mathfrak{N}_1$.

Definition 4.4. The quintuplet $(\mathfrak{M}, X, Y, \pi_X, \pi_Y)$ is said to be nondegenerate through order k at $m \in \mathfrak{M}$ if there are $\dim X - 1$ vector fields $X_j \in \mathfrak{X}_k$ such that $\{\mathfrak{X}_1|_m, \mathfrak{N}_1|_m, T^k(X_j)|_m : j = 1, \dots, \dim X - 1\}$ spans the tangent space of $\mathfrak{M}$ at m.

Let $\mathcal{R}_k \subset [0,1]\times[0,1]$ be the interior of the convex hull of the points $(0,1)$, $(1,0)$, $(0,0)$, and

$$\left(\frac{2}{j\dim X - j + 2}, 1 - \frac{2}{(j+1)(j\dim X - j + 2)}\right) \quad \text{for } j = 1,\dots,k.$$

Then the following is the main result in [12].

Theorem 4.5. [12] *Let $(\mathfrak{M}, X, Y, \pi_X, \pi_Y)$ be nondegenerate through order k at $m \in \mathfrak{M}$. Then there exists an open set $U \subset \mathfrak{M}$ containing m and a constant $C_{p,q'} < \infty$ such that, for any functions f_X and f_Y on X and Y respectively,*

$$\left|\int_U f_X(\pi_X(m)) f_Y(\pi_Y(m))\, dm\right| \leq C_{p,q'} \|f_X\|_p \|f_Y\|_{q'}$$

whenever $(1/p, 1/q') \in \mathcal{R}_k$.

To see how (4.12) follows from Theorem 4.5, we take

$$X := \mathbb{R}^d, \quad Y := \mathbb{R}^d \times S^{d-1}, \quad \text{and}$$
$$\mathfrak{M} := \{(x - \delta_t\omega, x, \omega) : (t,x,\omega) \in (1,\sigma)\times C\mathfrak{Q}_0 \times S^{d-1}\},$$

each equipped with their natural Lebesgue measure. Since $\mathfrak{M}$ is compact it is clear that Theorem 4.5 implies (4.12) once we demonstrate that, at each point $m \in \mathfrak{M}$, $(\mathfrak{M}, X, Y, \pi_X, \pi_Y)$ is nondegenerate through order 1 at m. To this end, consider m lying in the piece of $\mathfrak{M}$ parametrised by,

$$\Phi : (1,\sigma)\times C\mathfrak{Q}_0 \times B(0,1) \longrightarrow \mathfrak{M}; \quad (t,x,y) \longmapsto (x - \delta_t\omega, x, \omega),$$

where

$$\omega := (y_1,\dots,y_{d-1},\Gamma(y)) \quad \text{and} \quad \Gamma(y) := (1 - (y_1^2 + \ldots + y_{d-1}^2))^{1/2}.$$

We can parametrise the rest of $\mathfrak{M}$ using (a finite number of) maps which are similar to Φ and it will be apparent that the argument which follows can be modified to get the same outcome for the remaining elements of $\mathfrak{M}$. Our computations of the vector fields $\mathfrak{X}_1$ and $\mathfrak{N}_1$ occur in a Euclidean space and thus appear as $2d$-tuples. Our choice of parametrisation means that it is convenient to write these $2d$-tuples in the form $(t|x|y)$ where $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, and $y \in \mathbb{R}^{d-1}$.

One can easily verify that, if e_j is the jth standard basis vector in $\mathbb{R}^{d-1}$ and $\omega_j := (e_j, \partial_{y_j}\Gamma(y)) \in \mathbb{R}^d$, then the vectors

$$X_j := (0|\delta_t\omega_j|e_j) \quad \text{for } j = 1,\dots,d-1 \text{ and} \quad X_d := (1|t^{-1}P\delta_t\omega|0)$$

lie in $\mathfrak{X}_1$, and the vector $(1|0|0)$ lies in $\mathfrak{N}_1$. It is also straightforward to verify that

$$T(X_j) = (0|t^{-1}P\delta_t\omega_j|0) \quad \text{for } j = 1,\dots,d-1.$$

We claim that, for each fixed $(t,x,y) \in (1,\sigma)\times C\mathfrak{Q}_0 \times B(0,1)$, the set

$$\{Y_1, X_j, X_d, T(X_j) : j = 1\dots,d-1\} \tag{4.13}$$

is linearly independent. Upon a dimension count, this implies that $(\mathfrak{M}, X, Y, \pi_X, \pi_Y)$ is nondegenerate through order 1 at m, as claimed.

To see that the set in (4.13) is linearly independent, suppose that

$$\alpha Y_1 + \sum_{j=1}^{d-1}\beta_j X_j + \beta_d X_d + \sum_{j=1}^{d-1}\gamma_j T(X_j) = 0.$$

The last $d-1$ components force $\beta_j = 0$ for $j = 1, \dots, d-1$. Therefore,

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & t^{-1}P\delta_t\omega & t^{-1}P\delta_t\omega_1 & \cdots & t^{-1}P\delta_t\omega_{d-1} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta_d \\ \gamma_1 \\ \vdots \\ \gamma_{d-1} \end{pmatrix} = 0, \tag{4.14}$$

and it suffices to show that the determinant of the matrix in (4.14) is nonzero. This determinant is clearly equal to

$$\det(t^{-1}P\delta_t)\det(\omega, \omega_1, \dots, \omega_{d-1}) = t^{\tau-1}\det(P)\Gamma(y)^{-1},$$

which is nonzero for each $(t, x, y) \in (1, \sigma) \times C\mathfrak{Q}_0 \times B(0, 1)$. This completes the proof of Lemma 4.3 and thus (4.4). □

Remarks (1) The quintuplet $(\mathfrak{M}, X, Y, \pi_X, \pi_Y)$ is not nondegenerate through order k for any $k \geq 2$, in our particular setup.

(2) In a subsequent paper, Gressman [13] proves completely sharp L^p to L^q mapping properties for a family of operators related to those arising in Theorem 4.5.

5. Proof of Theorem 1.6(2)

We prove Theorem 1.6(2) using a similar approach to our proof of (4.3). Fix $q \in (2, q_d(2))$ and choose $\nu \in (0, 1/2)$ as in (4.6). By Plancherel's theorem and Sobolev's embedding theorem, it suffices to prove

$$\sup_{\xi \in \mathbb{R}^d} \|\widetilde{m}(\xi, \cdot)\|_{L^2_\nu} \lesssim 1, \quad \text{where} \tag{5.1}$$

$$\widetilde{m}(\xi, \omega) := p.v. \int_{\mathbb{R}} e^{i\langle \delta_t\omega, \xi\rangle} \frac{dt}{t}. \tag{5.2}$$

We make a dyadic splitting of the integral in (5.2) using the (unspecified) constant $\sigma \in (1, \infty)$ from Section 4. Thus, we write $\widetilde{m}(\xi, \omega) = \sum_{k \in \mathbb{Z}} \widetilde{m}_k(\xi, \omega)$ where

$$\widetilde{m}_k(\xi, \omega) := \int_{|t| \in [1, \sigma]} e^{i\langle \delta^*_{\sigma^k}\xi, \delta_t\omega\rangle} \frac{dt}{t}.$$

We claim that, if $\varepsilon = 1/2(1/2 - \nu)$ then for almost all ξ,

$$\|\widetilde{m}_0(\xi, \cdot)\|_{L^2_\nu} \lesssim \min(|\xi|, |\xi|^{-\varepsilon}). \tag{5.3}$$

It follows from $\widetilde{m}_k(\xi, \cdot) = \widetilde{m}_0(\delta^*_{\sigma^k}\xi, \cdot)$, along with (2.6) and (2.7), that (5.3) implies (5.1). We prove (5.3) by showing that (4.7) and (4.8) hold with $\widetilde{m}_0$ replacing the m which appears in these equations, and interpolating. The estimates for small $|\xi|$ are again easy to verify. The estimates for large $|\xi|$ follow from Lemma 4.2 and the fact that $\delta_t = -\delta_{-t}$ for negative t. This completes the proof of Theorem 1.6(2).

Remark In the isotropic case, the schema in [7] is to deduce the same estimates for H from those known for M (see Lemma 4.1 of [7] on pages 197-198). The argument there relies on the fact that, for fixed ω, $\{\delta_t\omega : t \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^d$ and thus $\mathbb{R}^d$ has an

associated orthogonal decomposition. In this way, H can be seen to arise from the classical one-dimensional Hilbert transform. This approach is clearly unique to the isotropic case. However, as an aside, the point at which the argument breaks down throws up an interesting question involving weighted inequalities for operators along curves. Specifically, for fixed $\omega \in S^{d-1}$ what values of $r \in (1,\infty)$ and $s \in (0,\infty)$ is it true that the estimate,

$$\int_{\mathbb{R}^d} |Hf(x,\omega)|^r Mf(x,\omega)^{-s}\,dx \leq C(r,s,\omega)\int_{\mathbb{R}^d} |f(x)|^r Mf(x,\omega)^{-s}\,dx$$

holds for some finite constant $C(r,s,\omega)$, and if so, how does $C(r,s,\omega)$ depend on ω?

6. Proof of Lemma 4.2

Firstly, choose $C_\varepsilon > \sigma$ such that $\log|\xi| \leq |\xi|^{2\varepsilon}$ for $|\xi| \geq C_\varepsilon$. Since $C_\varepsilon \sim 1$ it is clear that we only need to consider $|\xi| \geq C_\varepsilon$.

We shall handle the cases $d \geq 3$ and $d = 2$ separately. In the former case, we make use of the well-known fact that

$$|\widehat{d\Omega}(\xi)| \lesssim \min(1, |\xi|^{-(d-1)/2}) \tag{6.1}$$

(see, for example, [18]). The decay exponent $(d-1)/2$ in (6.1) is sharp and we shall see that this is the reason for our dimensional dichotomy.

Firstly, if $d \geq 3$ we multiply out to get,

$$\int_{S^{d-1}} \left|\int_1^\sigma \Psi_{(\xi,\omega)}(t)e^{i\langle \xi,\delta_t\omega\rangle}\,dt\right|^2 d\Omega(\omega) = \int_\square \widehat{d\Omega}((\delta_t^* - \delta_s^*)\xi)\Psi_{(\xi,\omega)}(t)\overline{\Psi_{(\xi,\omega)}(s)}\,dsdt,$$

where $\square := [1,\sigma]\times[1,\sigma]$. Thus, by (6.1) and (4.9),

$$\begin{aligned}\int_{S^{d-1}} \left|\int_1^\sigma \Psi_{(\xi,\omega)}(t)e^{i\langle \xi,\delta_t\omega\rangle}\,dt\right|^2 d\Omega(\omega) &\lesssim |\xi|^{2a}\int_{\substack{(s,t)\in\square\\ 0<(t-s)|\xi|\leq 1}} 1\,dsdt + \\ &\quad |\xi|^{2a}\int_{\substack{(s,t)\in\square\\ 1<(t-s)|\xi|}} |(\delta_t^* - \delta_s^*)\xi|^{-(d-1)/2}\,dsdt \\ &=: |\xi|^{2a}(I + II).\end{aligned}$$

Clearly I is comparable to the measure of a rectangle in $\mathbb{R}^2$ with sidelengths $|\xi|^{-1}$ and 1. Hence $I \lesssim |\xi|^{-1}$, and the contribution from this term is suitably under control.

We claim that for all $|\xi| \geq C_\varepsilon$, and all $(s,t) \in \square$ with $t > s$ we have,

$$|(\delta_t^* - \delta_s^*)\xi| \gtrsim (t-s)|\xi|. \tag{6.2}$$

Equipped with (6.2), it is straightforward to verify that

$$II \lesssim \begin{cases} |\xi|^{-1} & \text{for } d \geq 4, \\ |\xi|^{-1+2\varepsilon} & \text{for } d = 3, \end{cases}$$

and this completes the proof of Lemma 4.2 for $d \geq 3$. A simple computation shows that when $d = 2$ the best one can hope from term II is the weaker estimate $|\xi|^{-1/2}$. When $d = 2$ we instead capitalise on the decay from the t-integral for fixed ω. Before moving on to this

case, we prove our claim in (6.2). For this, it clearly suffices to prove that for all $(s,t) \in \square$ with $t > s$,

$$\|(\delta_t - \delta_s)^{-1}\| \lesssim (t-s)^{-1}. \tag{6.3}$$

So we fix such (s,t), and by writing $\delta_t - \delta_s = \delta_s(\delta_{t/s} - I)$ we seek to get a bound on the norm of the inverse of $\delta_{t/s} - I$. Setting $u = t/s$ for notational convenience, we have $u \in [1,\sigma]$ and

$$\delta_u - I = (\log u)P\left(I + \sum_{j=2}^{\infty} \frac{(\log u)^{j-1}}{j!} P^{j-1}\right).$$

Setting $B(u) := -\sum_{j=2}^{\infty}(j!)^{-1}(\log u)^{j-1}P^{j-1}$ and ensuring $\sigma < 2$, we have

$$\begin{aligned}\|B'(v)\| = v^{-1}\left\|\sum_{j=2}^{\infty}\frac{(j-1)(\log v)^{j-2}}{j!}P^{j-1}\right\| &\lesssim \sum_{j=2}^{\infty}\frac{(j-1)(\log 2)^{j-2}}{j!}\|P\|^{j-1} \\ &=: C_P < \infty,\end{aligned}$$

for each $v \in (1,\sigma)$. Hence, if we choose $\sigma \in (1, \min(2, 1 + (2C_P)^{-1}))$ then $\|B(u)\| \leq C_P(u-1) \leq 1/2$ by the mean value theorem. This implies $I - B(u)$ is invertible and moreover $\|(I - B(u))^{-1}\| \leq (1 - \|B(u)\|)^{-1} \leq 2$. Whence,

$$\|(\delta_u - I)^{-1}\| \leq (\log u)^{-1}\|P^{-1}\|\|(I - B(u))^{-1}\| \lesssim (u-1)^{-1}.$$

Since $s \sim 1$, this immediately implies (6.3) and consequently completes the proof of Lemma 4.2 in the case $d \geq 3$.

For $d = 2$, note that

$$\int_1^{\sigma} \Psi_{(\xi,\omega)}(t)e^{i\langle\delta_t\omega,\xi\rangle}\,dt = \int_0^{\log\sigma} e^t\Psi_{(\xi,\omega)}(e^t)e^{i\Theta(t)}\,dt \tag{6.4}$$

where

$$\Theta(t) := \langle\xi, \exp(tP)\omega\rangle.$$

We claim that it suffices to show that

$$\left|\int_0^s e^{i\Theta(t)}\,dt\right|^2 \lesssim \frac{1}{|\Theta(0)|} \tag{6.5}$$

uniformly in $s \in [0, \log\sigma]$. To see why (6.5) is sufficient, first observe that

$$\begin{aligned}\int_{|\Theta(0)|\geq 1}\left|\int_1^{\sigma}\Psi_{(\xi,\omega)}(t)e^{i\langle\delta_t\omega,\xi\rangle}\,dt\right|^2 d\omega &\lesssim |\xi|^{2a}\int_{|\Theta(0)|\geq 1}\frac{1}{|\Theta(0)|}\,d\omega \\ &\lesssim |\xi|^{-1+2a}\log|\xi| \lesssim |\xi|^{-1+2a+2\varepsilon};\end{aligned}$$

the first estimate follows by (6.4), (4.9), and a standard integration by parts argument, whilst the second estimate follows by a direct computation. Since we also have the trivial estimate

$$\int_{|\Theta(0)|<1}\left|\int_1^{\sigma}\Psi_{(\xi,\theta)}(t)e^{i\langle\delta_t\omega,\xi\rangle}\,dt\right|^2 d\omega \lesssim \int_{|\Theta(0)|<1}|\xi|^{2a}\,d\omega \lesssim |\xi|^{-1+2a},$$

the proof of Lemma 4.2 will be complete once we demonstrate (6.5).

To this end, our first observation is that the matrix P can be thought of as a mapping on $\mathbb{C}^2$ and therefore there exists an invertible matrix U such that $P = UJU^{-1}$, where J is the

Jordan canonical form of P. Since the eigenvalues of P have positive real part it follows that either

(a) $J = J_1(\lambda) \oplus J_1(\mu)$ for some distinct $\lambda, \mu \in (0, \infty)$; or
(b) $J = J_1(\lambda) \oplus J_1(\overline{\lambda})$ for some $\lambda \in \mathbb{C}$ such that $\Re(\lambda) \in (0, \infty)$; or
(c) $J = J_2(\lambda)$ for some $\lambda \in \mathbb{C}$ such that $\Re(\lambda) \in (0, \infty)$.

Thus, in either case (a) or (b), we have the following representation of the phase Θ.

$$\Theta(t) = \langle \xi, U \exp(tJ) U^{-1} \omega \rangle = Ae^{\lambda t} + Be^{\mu t},$$

for some $A, B \in \mathbb{C}$ (depending on ω and ξ) such that $A + B = \Theta(0)$.

We first consider when (a) holds and, without loss of generality, suppose $\lambda < \mu$. Fix a positive constant C such that

$$\frac{\mu \sigma^{\mu}}{\lambda} < C < \frac{\mu^2}{\lambda^2 \sigma^{\lambda}}; \tag{6.6}$$

the existence of C is guaranteed since $\lambda < \mu$ and upon a choice of σ sufficiently close to 1. For $|A| \geq C|B|$ it follows from (6.6) that $|\Theta'(t)| \gtrsim |A| \gtrsim |\Theta(0)|$ for all $t \in [0, \log \sigma]$. Furthermore, for $|A| \leq C|B|$ it is easy to see that $|\Theta''(t)| \gtrsim |B| \gtrsim |\Theta(0)|$ for all $t \in [0, \log \sigma]$. The estimate in (6.5) now follows from van der Corput's lemma; see, for example, [18] for a statement of this standard result.

Suppose (b) holds and without loss of generality, suppose that $|B| \leq |A|$. Notice that

$$|\Theta'(t)| \geq |\lambda A e^{i\Re(\lambda)t} + \overline{\lambda} B e^{-i\Re(\lambda)t}| \gtrsim |\Lambda e^{2i\Re(\lambda)t} A + B|$$

where $\Lambda := \lambda / \overline{\lambda}$, and, moreover,

$$\begin{aligned} |\Theta''(t)| &\geq |\lambda^2 A e^{i\Re(\lambda)t} + \overline{\lambda}^2 B e^{-i\Re(\lambda)t}| \\ &\gtrsim |\Lambda^2 e^{2i\Re(\lambda)t} A + B| = |\Lambda(\Lambda - 1) e^{2i\Re(\lambda)t} A + \Lambda e^{2i\Re(\lambda)t} A + B|. \end{aligned}$$

We decompose $[0, \log \sigma] = I \cup J$, where

$$I := \{ t \in [0, \log \sigma] : |\Lambda e^{2i\Re(\lambda)t} A + B| \geq |A + B| / C \}$$

where C is a fixed positive constant such that $C > 2/|\Lambda - 1|$ and $J := [0, \log \sigma] \setminus I$. By an application of van der Corput's lemma, it is immediate that the contribution from the integral over I is suitably under control. For $t \in J$ observe that

$$|\Theta''(t)| \geq |\Lambda - 1||A| - |\Lambda e^{2i\Re(\lambda)t} A + B| \gtrsim |A|,$$

where the final bound follows because $|B| \leq |A|$. Another application of van der Corput's lemma concludes the proof of (6.5) when (b) holds.

Finally, if (c) holds it is straightforward to verify that

$$\Theta(t) = e^{\lambda t} (\Theta(0) + t \widetilde{A}),$$

for some $\widetilde{A} \in \mathbb{C}$ (depending on ω and ξ). Let C be a positive constant such that

$$\frac{|\lambda|}{2 - |\lambda| \log \sigma} < C < \frac{|\lambda|}{1 + |\lambda| \log \sigma}$$

(again, such a constant exists if σ is chosen appropriately close to 1). It is now straightforward to verify that if $|\widetilde{A}| \leq C|\Theta(0)|$ then $|\Theta'(t)| \gtrsim |\Theta(0)|$ for all $t \in [0, \log\sigma]$ and if $|\widetilde{A}| \geq C|\Theta(0)|$ then $|\Theta'(t)| \gtrsim |\Theta(0)|$ for all $t \in [0, \log\sigma]$. A final application of van der Corput's lemma implies that (6.5) holds for case (c). This concludes our proof of Lemma 4.2.

Remark A consequence of our proof of Lemma 4.2 is that if $d \geq 4$ and P is a real d by d matrix whose eigenvalues have positive real part, then there exists a number $\sigma \in (1, \infty)$ such that,

$$\int_{S^{d-1}} \left| \int_1^\sigma e^{i\langle \xi, \delta_t \omega\rangle}\, dt \right|^2 d\Omega(\omega) \lesssim |\xi|^{-1}; \tag{6.7}$$

that is, there is certainly no epsilon-loss in this oscillatory integral estimate when $d \geq 4$. It may be of interest to establish whether (6.7) holds when $d \in \{2, 3\}$. We have verified that this is the case when $d = 2$ and $P = \mathrm{diag}(1, 2)$.

7. Some applications

Variable kernel singular integrals. Suppose K, defined on $\mathbb{R}^d \times \mathbb{R}^d$, is such that for each $x \in \mathbb{R}^d$, $K(x, \cdot)$ is an odd function and satisfies the following homogeneity condition with respect to the dilations δ_t.

$$K(x, \delta_t y) = t^{-\tau} K(x, y) \quad \text{for each } (t, y) \in (0, \infty) \times \mathbb{R}^d.$$

Given this homogeneity condition and our discussion in Section 2, it is natural to view the kernel $K(x, \cdot)$ as living on an ellipsoid $\Sigma_Q^{d-1} = \{\omega \in \mathbb{R}^d : \langle Q\omega, \omega\rangle = 1\}$ governed by some real symmetric positive definite matrix Q such that (2.1) holds. The study of singular integral operators associated with this type of kernel goes back to contributions from Mihlin, Calderón, and Zygmund when P is the identity matrix, and to Jones, Fabes, and Rivière in the case that P is a more general diagonal matrix; see, for example, [16], [1], [14], [8], [9], and [10]. Recall that the identity matrix is an appropriate choice for Q satisfying (2.1) when P is diagonal, and the reader may prefer to bear this more concrete set-up in mind in the sequel.

Notation Suppose Q is a real symmetric positive definite matrix satisfying (2.1). Let $L^p(L^q(\Sigma_Q^{d-1}))$ denote the measurable functions $F : \mathbb{R}^d \times \Sigma_Q^{d-1} \to \mathbb{C}$ such that $\|F\|_{L^p(L^q(\Sigma_Q^{d-1}))}$ is finite, where

$$\|F\|_{L^p(L^q(\Sigma_Q^{d-1}))} := \left(\int_{\mathbb{R}^d} \left(\int_{\Sigma_Q^{d-1}} |F(x, \omega)|^q \, d\Omega_Q(\omega) \right)^{p/q} dx \right)^{1/p}.$$

The ellipsoid Σ_Q^{d-1} and the measure $d\Omega_Q$ are given by (2.3) and (2.5), respectively.

Let T be the operator given *a priori* on the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ by $Tf(x) = \lim_{\varepsilon \to 0} T_\varepsilon f(x)$, where

$$T_\varepsilon f(x) := \int_{|y| > \varepsilon} f(x - y) K(x, y)\, dy.$$

Theorem 7.1. *Given the above set-up and if, in addition,* $\|K\|_{L^\infty(L^r(\Sigma_Q^{d-1}))}$ *is finite for some* $r \in (1,\infty]$ *and some real symmetric positive definite matrix* Q *satisfying (2.1), then* T *extends to a bounded operator on* L^p *for all* $(1/p, 1/r') \in \Delta_{(2,q_d(2))}$.

Proof. For $f \in \mathcal{S}(\mathbb{R}^d)$, one can show that

$$\lim_{\varepsilon\to 0} T_\varepsilon f(x) = \lim_{\varepsilon\to 0} \int_{\varrho(y)>\varepsilon} f(x-y)K(x,y)\,dy$$

where ϱ, defined in (2.2), is the P-homogeneous distance function associated to the matrix Q. By changing to polar coordinates $y = \delta_t\omega$ for $(t,\omega) \in (0,\infty)\times\Sigma_Q^{d-1}$, and using the homogeneity and oddness of $K(x,\cdot)$, it is easy to see that we have the following representation:

$$Tf(x) = \frac{1}{2}\int_{\Sigma_Q^{d-1}} Hf(x,\omega)K(x,\omega)\,d\Omega_Q(\omega).$$

Hence, by Hölder's inequality,

$$\|Tf\|_p \le \frac{1}{2}\|K\|_{L^\infty(L^r(\Sigma_Q^{d-1}))}\|Hf\|_{L^p(L^{r'}(\Sigma_Q^{d-1}))}.$$

One can easily verify that the proof of Theorem 1.6(2) can be adapted to show that, for each $d \ge 2$, H is bounded from L^p to $L^p(L^{r'}(\Sigma_Q^{d-1}))$ for all $(1/p,1/r') \in \Delta_{(2,q_d(2))}$. It follows that T extends to a bounded operator on L^p for the claimed range of p. □

If the weaker cancellation condition, $\int_{S^{d-1}} K(x,\omega)\,d\Omega(\omega) = 0$ for each $x \in \mathbb{R}^d$, and the substantially stronger size condition, $\|\partial^{(0,\beta)}K\|_{L^\infty(L^\infty)} \le C_\beta$ for each index β, hold then T is a bounded operator on L^p for each $p \in (1,\infty)$ (see [9]).

In the isotropic case, we remark that a theorem in [7] significantly improves upon Theorem 7.1. In particular, if the same size condition, $\|K\|_{L^\infty(L^r)} < \infty$, and the weaker cancellation, $\int_{S^{d-1}} K(x,\omega)\,d\Omega(\omega) = 0$ for each $x \in \mathbb{R}^d$, hold then T is bounded on L^p for all $(1/p,1/r') \in \Delta_{(p_d,q_d(p))}$. This theorem is a consequence of Theorem 1.2 and improved upon earlier work Calderón and Zygmund (see [2] and [3]). We include our next theorem as a potential first step towards the goal of improving Theorem 7.1 with a view to weakening the cancellation hypothesis. Indeed, in the isotropic case, the result is crucial to the standard argument for handling even kernels; see, for example, [2] or [7] for more details.

Theorem 7.2. *Suppose* $d \ge 2$ *and let* P *be a real* d *by* d *matrix whose eigenvalues each have positive real part. Let* Q *be a real symmetric positive definite matrix such that (2.1) holds, and let* ϱ *be the associated* P*-homogeneous distance function given by (2.2).*

For $\varepsilon > 0$*, define* $K_\varepsilon(x,y) := \varepsilon^{-\tau}N(x,y)\Psi(\delta_{\varepsilon^{-1}}(y))$*, where* N *and* Ψ *satisfy the following.*

1. *For each* $x \in \mathbb{R}^d$*,* $N(x,\cdot)$ *is homogeneous of degree zero with respect to the dilations,* δ_t*, for positive* t*.*
2. *For some* $r \in (1,\infty]$*, the quantity* $\|N\|_{L^\infty(L^r(\Sigma_Q^{d-1}))}$ *is finite.*
3. Ψ *is a nonnegative and nonincreasing* L^1 *function, radial with respect to* ϱ*; that is,* $\Psi = \psi(\varrho(\cdot))$ *for some nonnegative and nonincreasing function* ψ *on* $[0,\infty)$*.*

Then the operator T^* defined by

$$T^*f(x) := \sup_{\varepsilon>0}\left|\int_{\mathbb{R}^d} f(x-y)K_\varepsilon(x,y)\,dy\right|,$$

is bounded on L^p for all $(1/p, 1/r') \in \Delta_{(p_d,q_d(p_d))}$.

Proof. By hypotheses (1) and (3) it follows that, for each $\varepsilon > 0$,

$$\left|\int_{\mathbb{R}^d} K_\varepsilon(x,y)f(y)\,dy\right| \lesssim \int_{\Sigma_Q^{d-1}} |N(x,\omega)|Mf(x,\omega)\,d\Omega_Q(\omega);$$

therefore $\|T^*f\|_p \lesssim \|N\|_{L^\infty(L^r(\Sigma_Q^{d-1}))}\|Mf\|_{L^p(L^{r'}(\Sigma_Q^{d-1}))}$ by Hölder's inequality. Since $(1/p, 1/r') \in \Delta_{(p_d,q_d(p_d))}$, it is a routine exercise to check that the proof of Theorem 1.5(2) may be modified to show that M is bounded from L^p to $L^p(L^{r'}(\Sigma_Q^{d-1}))$. This completes our proof of Theorem 7.2. □

We remark that Theorem 7.2 improves upon a similar result in [17].

A nonisotropic Nikodym-type maximal operator. Let $d \geq 2$. Fix both a real d by d matrix P whose eigenvalues each have positive real part and a real symmetric positive definite matrix Q satisfying (2.1). Let ϱ be given by (2.2).

Notation For nonnegative numbers A and B, $A \lessapprox B$ means that there exists a constant C depending on at most d, P, and Q such that $A \leq CB$. The distinction with the earlier notation $\lesssim$ is that the implicit constant C should not depend on any indices p and q which are present.

Let $M_{\mathfrak{F}_N}$ be the Nikodym-type maximal operator given by

$$M_{\mathfrak{F}_N}f(x) := \sup_{F\in\mathfrak{F}_N}\frac{1}{|F|}\int_F |f(x-y)|\,dy.$$

Here, N is a large parameter and $\mathfrak{F}_N$ is a family of sets in $\mathbb{R}^d$ which have a certain bounded "eccentricity" and are "star-shaped" in the nonisotropic world determined by P and Q. More precisely, for a bounded subset F of $\mathbb{R}^d$ define $\mathcal{E}(F)$, its "eccentricity", by

$$\mathcal{E}(F) := \frac{\operatorname{diam}(F)^\tau}{|F|},$$

where $\operatorname{diam}(F) := \sup\{\varrho(x-y) : x, y \in F\}$ is the "diameter" of F. Thus, the family $\mathfrak{F}_N$ is defined to be the collection of all subsets F of $\mathbb{R}^d$ such that $\mathcal{E}(F) \leq N$ and which admit a parametrisation of the form

$$F = \{\delta_r\omega : \omega \in \Sigma_Q^{d-1} \text{ and } 0 \leq r \leq R(\omega)\},$$

for some nonnegative measurable function R on Σ_Q^{d-1}.

Theorem 7.3. *Let $d \geq 2$ and N be a positive real number. Then there exists a positive constant λ depending on d, P, and Q such that*

$$\|M_{\mathfrak{F}_N}f\|_{p_d} \lessapprox (\log N)^\lambda N^{1/q_d(p_d)}\|f\|_{p_d}$$

Remark When P and Q are both the identity matrix, Theorem 7.3 was proved in [7]. Moreover, an immediate consequence is that the maximal operator form of the famous Nikodym conjecture holds up to $p = p_d$ for any $d \geq 2$. A considerable amount of effort and new arguments have since seen the range of p improved beyond $p = p_d$ for this particular form of the Nikodym conjecture; we refer the interested reader to the survey article [15] for further details.

Proof. The following argument is identical to the analogous result in [7] which handles the isotropic case. We include the details for completeness.

Let $q \in (1, q_d(p_d))$ be given by $q = (1/q_d(p_d) + 1/\log N)^{-1}$. By changing to polar coordinates, $y = \delta_r \omega$ for $\omega \in \Sigma_Q^{d-1}$ and $r \in [0, R(\omega)]$,

$$\frac{1}{|F|}\int_F |f(x-y)|\,dy = \frac{1}{|F|}\int_{\Sigma_Q^{d-1}}\int_0^{R(\omega)} |f(x-\delta_r\omega)| r^{\tau-1}\,dr d\Omega_Q(\omega).$$

By straightforward arguments,

$$\int_0^{R(\omega)} |f(x-\delta_r\omega)| r^{\tau-1}\,dr \lessapprox R(\omega)^\tau Mf(x,\omega) \lessapprox \operatorname{diam}(F)^{\tau/q} R(\omega)^{\tau/q'} Mf(x,\omega);$$

in the second estimate we used the fact that $\sup\{R(\omega) : \omega \in \Sigma_Q^{d-1}\} \lessapprox \operatorname{diam}(F)$ which follows immediately from (2.8). Therefore, using Hölder's inequality, and the hypothesis that $\mathcal{E}(F) \leq N$,

$$\begin{aligned}\frac{1}{|F|}\int_F |f(x-y)|\,dy &\lessapprox \frac{\operatorname{diam}(F)^{\tau/q}}{|F|}\left(\int_{\Sigma_Q^{d-1}} R(\omega)^\tau\,d\Omega_Q(\omega)\right)^{1/q'} \|Mf(x,\cdot)\|_{L^q(\Sigma_Q^{d-1})} \\ &\lessapprox |F|^{-1+1/q'} \operatorname{diam}(F)^{\tau/q} \|Mf(x,\cdot)\|_{L^q(\Sigma_Q^{d-1})} \\ &\leq N^{1/q} \|Mf(x,\cdot)\|_{L^q(\Sigma_Q^{d-1})}.\end{aligned}$$

Hence, $\|M_{\mathfrak{F}_N} f\|_{p_d} \lessapprox N^{1/q} \|Mf\|_{L^{p_d}(L^q(\Sigma_Q^{d-1}))}$. As in [7], it follows from our proof of Theorem 1.5(2) that there exists λ depending on d, P, and Q such that

$$\|Mf\|_{L^{p_d}(L^q(\Sigma_Q^{d-1}))} \lessapprox \left(\frac{1}{q} - \frac{1}{q_d(p_d)}\right)^{-\lambda} \|f\|_{p_d},$$

and by our choice of q, this completes the proof of Theorem 7.3. □

Maximal operators related to the operator $M_{\mathfrak{F}_N}$ concerning averages over curved sets are in the spirit of Wisewell's work in [20] and [21]. Amongst other results, Wisewell successfully adapted recent techniques from the classical Nikodym and Kakeya problems to prove a $(d+2)/2$ result for a certain class of parabolic curves in $\mathbb{R}^d$ ($d \geq 3$). It would be interesting to see whether Theorem 7.3 could possibly hold for some $p > p_d$ (for $d \geq 3$). We do not pursue this matter here.

References

[1] A. Calderón, A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85–139.
[2] A. Calderón, A. Zygmund, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309.

[3] A. Calderón, A. Zygmund, *On singular integrals with variable kernels*, Appl. Anal. **7** (1978), 221–238.
[4] A. Carbery, M. Christ, J. Vance, S. Wainger, D. Watson, *Operators associated to flat plane curves: L^p estimates via dilation methods*, Duke Math. J. **59** (1989), 677–700.
[5] L.-K. Chen, *The singular integrals related to the Calderón-Zygmund method of rotations*, Appl. Anal. **30** (1988), 319–329.
[6] L.-K. Chen, *The maximal operators related to the Calderón-Zygmund method of rotations*, Illinois J. Math. **33** (1989), 268–279.
[7] M. Christ, J. Duoandikoetxea, J. Rubio de Francia, *Maximal operators related to the Radon transform and the Calderón-Zygmund method of rotations*, Duke Math. J. **53** (1986), 189–209.
[8] E. B. Fabes, *Singular integrals and partial differential equations of parabolic type*, Studia Math. **28** (1966), 81–131.
[9] E. B. Fabes, N. M. Rivière, *Singular integrals with mixed homogeneity*, Studia Math. **27** (1966), 19–38.
[10] E. B. Fabes, N. M. Rivière, *Symbolic calculus of kernels with mixed homogeneity*, Proc. Sympos. Pure Math., vol. 10, Amer. Math. Soc., Providence, R.I (1967), 106–127.
[11] R. Fefferman, *On an operator arising in the Calderón-Zygmund method of rotations and the Bramble-Hilbert lemma*, Proc. Nat. Acad. Sci. U.S.A. **80** (1983), 3877–3878.
[12] P. Gressman, *L^p-improving properties of X-ray like transforms*, Math. Res. Lett. **13** (2006), 10001–10017.
[13] P. Gressman, *Sharp L^p-L^q estimates for generalized k-plane transforms*, Adv. Math. **214** (2007), 344–365.
[14] B. F. Jones, *A class of singular integrals*, Amer. J. Math. **86** (1964), 441–462.
[15] N. Katz, T. Tao, *Recent progress on the Kakeya conjecture*, Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorial, 2000), Publ. Mat. (2002), 161–179.
[16] S. Mihlin, *Singular integral equations*, Amer. Math. Soc. Transl. **10** (1962), 84–198.
[17] S. Sato, *Maximal functions associated with curves and the Calderón-Zygmund method of rotations*, Trans. Amer. Math. Soc. **293** (1986), 799–806.
[18] E. M. Stein, *Harmonic Analysis*, Princeton Univ. Press, Princeton, N. J. (1993).
[19] E. M. Stein, S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), 1239–1295.
[20] L. Wisewell, *Oscillatory Integrals and Curved Kakeya Sets*, PhD thesis, University of Edinburgh (2003).
[21] L. Wisewell, *Kakeya sets of curves*, Geom. Funct. Anal. **15** (2005), 1319–1362.

Neal Bez, School of Mathematics, The University of Birmingham, The Watson Building, Edgbaston, Birmingham, B15 2TT, United Kingdom

E-mail address: `bezn@maths.bham.ac.uk`