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Evolutionary game theory (EGT) has become a powerful mathematical framework for the modelling and analysis of complex biological/economical systems whenever there is frequency dependent selection – the fitness of an individual does not only depend on its strategy, but also on the composition of the population in relation with (multiple) other strategies (Maynard Smith and Price, 1973; Hofbauer and Sigmund, 1998). The payoff from the games is interpreted as individual fitness, naturally leading to a dynamical approach. Random evolutionary games in which the payoff entries are random variables form an important subclass of EGT. They are necessary to model social and biological systems in which very limited information is available, or where the environment changes so rapidly and frequently that one cannot describe the payoffs of their inhabitants' interactions (Fudenberg and Harris, 1992; Gross et al., 2009). As in classical game theory with the Nash equilibrium, see e.g. (McLennan, 2005), the analysis of properties of equilibrium points in EGT has been of special interest, see e.g. (Gokhale and Traulsen, 2010). These equilibrium points predict the composition of strategy frequencies where all the strategies have the same average fitness. In random games, due to the randomness of the payoff entries, it is essential to study statistical properties of equilibria. How to determine the distribution of internal equilibria in random evolutionary games is an intensely investigated subject with numerous practical ramifications in ecology, population genetics, social sciences, economics and computer science providing essential understanding of complexity in a dynamical system, such as its behavioural, cultural or biological diversity and the maintenance of polymorphism. Properties of equilibrium points, particularly the probability of observing the maximal number of equilibrium points, the attainability and stability of the patterns of evolutionarily stable strategies have been studied recently (Gokhale and Traulsen, 2010; Han et al., 2012; Gokhale and Traulsen, 2014). However, as these papers used a direct approach that consists of solving a system of polynomial equations, the mathematical analysis was mostly restricted to evolutionary games with a small number of players, due to the impossibility of solving general polynomial equations of a high degree.

In this extended abstract, we present a summary of our recent works (Duong and Han, 2015, 2016; Duong et al., 2018b,a), in which we analyze random evolutionary games with an arbitrary number of players. The key technique that we develop is to connect the number of equilibria in an evolutionary game to the number of real roots of a system of multi-variate random polynomials (Bharucha-Reid and Sambandham, 1986; Edelman and Kostlan, 1995). Assuming that we consider d-player n-strategy evolutionary games, then the system consists of $n-1$ polynomial equations of degree $d-1$:

$$\sum_{\substack{0\leq k_1,\ldots,k_{n-1}\leq d-1,\\ \sum\limits_{i=1}^{n-1}k_i\leq d-1}} \beta^i_{k_1,\ldots,k_{n-1}}\binom{d-1}{k_1,\ldots,k_n}\prod_{i=1}^{n-1}y_i^{k_i}=0,$$

for $i=1,\ldots,n-1$. Here $\beta^i_{k_1,\ldots,k_{n-1}} := \alpha^i_{k_1,\ldots,k_n} - \alpha^n_{k_1,\ldots,k_n}$ where $\alpha^{i_0}_{k_1,\ldots,k_n} := \alpha^{i_0}_{i_1,\ldots,i_{d-1}}$ is the payoff of the focal player and k_i, $1\leq i\leq n$, with $\sum_{i=1}^n k_i = d-1$, is the number of players using strategy i in $\{i_1,\ldots,i_{d-1}\}$. In (Duong and Han, 2015, 2016), we analyze the mean number $E(n,d)$ and the expected density $f(n,d)$ of internal equilibria in a general d-player n-strategy evolutionary game when the individuals' payoffs are *independent, normally distributed*. We provide computationally implementable formulas of these quantities for the general case and characterize their asymptotic behaviour for the two-strategy games (i.e. $E(2,d)$ and $f(2,d)$), estimating their lower and upper bounds as d increases. For instance, under certain assumptions on the payoffs, we obtain

- Asymptotic behaviour of $E(2,d)$:

$$\sqrt{d-1}\lesssim E(2,d)\lesssim\sqrt{d-1}\ln(d-1).$$

As a consequence,

$$\lim_{d\to\infty}\frac{\ln E(2,d)}{\ln(d-1)}=\frac{1}{2}.$$

- Explicit formula of $E(n,2)$: $E(n,2) = \frac{1}{2^{n-1}}$.

For a general d-player n-strategy game, as supported by extensive numerical results, we describe a conjecture regarding the asymptotic behaviours of $E(n,d)$ and $f(n,d)$. We also show that the probability of seeing the maximal possible number of equilibria tends to zero when d or n respectively goes to infinity and that the expected number of stable equilibria is bounded within a certain interval.

In (Duong et al., 2018b) we generalize our analysis for random evolutionary games where the payoff matrix entries are *correlated* random variables. In social and biological contexts, correlations may arise in various scenarios particularly when there are environmental randomness and interaction uncertainty such as in games of cyclic dominance, coevolutionary multi-games or when individual contributions are correlated to the surrounding contexts (e.g. due to limited resource)(Szolnoki and Perc, 2014; Santos et al., 2012). We establish a closed formula for the mean numbers of internal (stable) equilibria and characterize the asymptotic behaviour of this important quantity for large group sizes and study the effect of the correlation. The results show that decreasing the correlation among payoffs (namely, of a strategist for different group compositions) leads to larger mean numbers of (stable) equilibrium points, suggesting that the system or population behavioral diversity can be promoted by increasing independence of the payoff entries.

As a further development, in (Duong et al., 2018a) we derive a closed formula for the distribution of internal equilibria, for both normal and uniform distributions of the game payoff entries. We also provide several universal upper and lower bound estimates, which are independent of the underlying payoff distribution, for the probability of obtaining a certain number of internal equilibria. In addition, the asymptotic behaviour of the probability of having no internal equilibria is then obtained (Can et al., 2018). The distribution of equilibria provides more elaborate information about the level of complexity or the number of different states of biodiversity that will occur in a dynamical system, compared to what obtained with the expected number of internal equilibria.

In short, by connecting EGT to random polynomial theory, we have achieved new results on the expected number and distribution of internal equilibria in multi-player multi-strategy games. Our studies provide new insights into the overall complexity of dynamical systems, including biological, social and Artificial Life ones, as the numbers of players and strategies in an interaction within the systems increase. As the theory of random polynomials is rich, we expect that our novel approach can be extended to obtain results for other more complex models in population dynamics such as the replicator-mutator equation and evolutionary games with environmental feedback (Weitz et al., 2016).

References

Bharucha-Reid, A. T. and Sambandham, M. (1986). *Random polynomials*. Probability and Mathematical Statistics. Academic Press, Inc., Orlando, FL.

Can, V. H., Duong, M. H., and Pham, V. H. (2018). Persistence probability of a random polynomial arising from evolutionary game theory. arXiv:1804.05908.

Duong, M. H. and Han, T. A. (2015). On the expected number of equilibria in a multi-player multi-strategy evolutionary game. *Dynamic Games and Applications*, pages 1–23.

Duong, M. H. and Han, T. A. (2016). Analysis of the expected density of internal equilibria in random evolutionary multi-player multi-strategy games. *Journal of Mathematical Biology*, 73(6):1727–1760.

Duong, M. H., Tran, H. M., and Han, T. A. (2018a). On the distribution of the number of internal equilibria in random evolutionary games. *Journal of Mathematical Biology*.

Duong, M. H., Tran, H. M., and Han, T. A. (2018b). On the expected number of internal equilibria in random evolutionary games with correlated payoff matrix. *Dynamic Games and Applications*.

Edelman, A. and Kostlan, E. (1995). How many zeros of a random polynomial are real? *Bull. Amer. Math. Soc. (N.S.)*, 32(1):1–37.

Fudenberg, D. and Harris, C. (1992). Evolutionary dynamics with aggregate shocks. *J. Econ. Theory*, 57:420–441.

Gokhale, C. S. and Traulsen, A. (2010). Evolutionary games in the multiverse. *Proc. Natl. Acad. Sci. U.S.A.*, 107(12):5500–5504.

Gokhale, C. S. and Traulsen, A. (2014). Evolutionary multiplayer games. *Dynamic Games and Applications*, 4(4):468–488.

Gross, T., Rudolf, L., Levin, S. A., and Dieckmann, U. (2009). Generalized models reveal stabilizing factors in food webs. *Science*, 325(5941):747–750.

Han, T. A., Traulsen, A., and Gokhale, C. S. (2012). On equilibrium properties of evolutionary multi-player games with random payoff matrices. *Theoretical Population Biology*, 81(4):264 – 272.

Hofbauer, J. and Sigmund, K. (1998). *Evolutionary Games and Population Dynamics*. Cambridge University Press, Cambridge.

Maynard Smith, J. and Price, G. R. (1973). The logic of animal conflict. *Nature*, 246:15–18.

McLennan, A. (2005). The expected number of nash equilibria of a normal form game. *Econometrica*, 73(1):141–174.

Santos, F. C., Pinheiro, F. L., Lenaerts, T., and Pacheco, J. M. (2012). The role of diversity in the evolution of cooperation. *Journal of theoretical biology*, 299:88–96.

Szolnoki, A. and Perc, M. (2014). Coevolutionary success-driven multigames. *EPL (Europhysics Letters)*, 108(2):28004.

Weitz, J. S., Eksin, C., Paarporn, K., Brown, S. P., and Ratcliff, W. C. (2016). An oscillating tragedy of the commons in replicator dynamics with game-environment feedback. *Proceedings of the National Academy of Sciences*, 113(47):E7518–E7525.