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An induction principle for consequence in arithmetic universes

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ABSTRACT

Suppose in an arithmetic universe we have two predicates ϕ and ψ for natural numbers, satisfying a base case $\phi(0) \to \psi(0)$ and an induction step that, for generic n, the hypothesis $\phi(n) \to \psi(n)$ allows one to deduce $\phi(n+1) \to \psi(n+1)$. Then it is already true in that arithmetic universe that $(\forall n)(\phi(n) \to \psi(n))$. This is substantially harder than in a topos, where cartesian closedness allows one to form an exponential $\phi(n) \to \psi(n)$.

The principle is applied to the question of locatedness of Dedekind sections.

The development analyses in some detail a notion of "subspace" of an arithmetic universe, including open or closed subspaces and a Boolean algebra generated by them. There is a lattice of subspaces generated by the open and the closed, and it is isomorphic to the free Boolean algebra over the distributive lattice of subobjects of 1 in the arithmetic universe.

1. Introduction

As has often been said, toposes embody two quite different ideas, under which they are considered either as generalized universes of sets or as generalized topological spaces. Our aim here is to explore the same idea when applied to arithmetic universes (see [8,12]) instead of toposes. The geometric structure of Grothendieck toposes – that is to say, the structure that is used to generate them when one builds classifying toposes, that is preserved by inverse image functors for geometric morphisms, and that appears in Giraud's Theorem – is the set-indexed colimits and finite limits. However, this begs the question of what are the sets that index the colimits. The speculation behind our use of arithmetic universes (AUs), mentioned already in [20, section 6.1] and discussed as "Coherent type theory" in [21], is that one might replace the arbitrary set-indexed colimits by (i) finite colimits, and (ii) those colimits that can be expressed internally using free algebra constructions such as the natural number object and existential quantification over them.

The logical heart of the analogy is seen through the classifying toposes of geometric theories. The classifying topos is, for geometric logic, the appropriate notion of classifying category (or theory category). It is built from a generic model of the theory by adjoining colimits and finite limits. (The power of "arbitrary set-indexed colimits" is seen in the fact that, when the theory itself is small, and sets themselves are taken as forming an elementary topos, the classifying topos will then already be both an arithmetic universe and an elementary topos.) The geometric morphisms between classifying toposes – given essentially by their inverse image functors as functors preserving the geometric structure of colimits and finite limits – then correspond to "continuous maps between the spaces of models of the theories", and this can be made precise in spatial cases. The objects of the classifying topos are the sheaves over the space, or, more generally, sheaves over a site.

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In many cases in practice, the geometric theory can be rephrased in the form of an arithmetic type theory, and then there is a corresponding classifying AU. We consider this as the "arithmetic space" of models of the theory. AU-functors, in the reverse direction, are then the maps between the arithmetic spaces. To put it another way, we define the category **AS** of arithmetic spaces as the opposite of the category **AU** of arithmetic universes. This is analogous to the definition of the category **Loc** of locales as the opposite of the category **Fr** of frames.

We should make a remark about strictness. Our development of AUs relies heavily on the fact that they are the models of a cartesian (finite limit) theory, so that we can use techniques of universal algebra derived from the initial model theorem. For this it is natural to assume that AUs come equipped with canonical choice of AU structure (finite limits – or more precisely: pullbacks and the terminal object, pushouts and the initial object, and list objects) and that the morphisms are strict AU-functors, preserving the AU structure on the nose. On the other hand our structure theorems (Section 3) require the use of AU-functors, preserving the AU structure up to isomorphism but not necessarily strict. The AU extensions that we shall use are characterized up to equivalence in terms of AU-functors (Section 2).

The induction principle that forms the main result of the paper (Theorem 46) is for implications $\phi(n) \rightarrow \psi(n)$ where ϕ and ψ are predicates on the natural numbers N. (Categorically, by "predicate" we mean that ϕ and ψ are subobjects of N rather than – as one might expect from a propositions-as-types interpretation – arbitrary morphisms into N.) An induction proof of $\forall n\ (\phi(n) \rightarrow \psi(n))$ would comprise a base case $\phi(0) \rightarrow \psi(0)$ and an induction step that, for generic n, assumes an induction hypothesis $\phi(n) \rightarrow \psi(n)$ and proves that $\phi(n+1) \rightarrow \psi(n+1)$. The problem arises because, since AUs are not cartesian closed in general, $\phi(n) \rightarrow \psi(n)$ cannot be interpreted as a subobject of N. Instead, we adjoin the induction hypothesis (the generic n and the sequent $\phi(n) \rightarrow \psi(n)$) to generate a new AU, and ask for $\phi(n+1) \rightarrow \psi(n+1)$ there as the induction step. The task then is to use this property of the new AU to deduce the conclusion $\phi \leq \psi$ in the old one.

Our solution has two main stages.

Stage 1 analyses the induction step and how to extract information about the original AU from it.

Somewhat remarkably, we can use classical logic and say $\phi(n) \rightarrow \psi(n)$ is equivalent to $\neg\phi(n) \vee \psi(n)$. However, $\neg$ here is not the usual Heyting negation for subobjects (which in any case cannot normally be done in an AU) but represents a passage from $\phi(n)$ considered as an *open subspace* to its corresponding closed subspace; and $\vee$ is a join in a Boolean lattice of "subspaces" (Section 4). This is directly analogous to the use of subspaces (or sublocales) in point-free topology, and a large part of our work here lies in showing analogous structure for AUs. The induction step then becomes

$$\neg\phi(n) \vee \psi(n) \leq \neg\phi(n+1) \vee \psi(n+1),$$

and Boolean algebra manipulations allow us to eliminate the negations. Then non-trivial conservativity theorems, based on analysing (in Section 3) the concrete structure of the AUs for open and closed subspaces, allow us to transfer this conclusion from subspaces to subobjects of N in the original AU.

Stage 2 is then a new induction principle. It says that the conclusion $\forall n\ (\phi(n) \rightarrow \psi(n))$ can be deduced from the base case $\phi(0) \rightarrow \psi(0)$ and two conditions derived from the induction step.

2. Arithmetic universes

Arithmetic universes are very much the creation of André Joyal, in unpublished work from the 1970s — though see [8]. The general notion was at first not clearly defined, and we shall follow [12] (which also discusses their background in some detail) in defining them as *list arithmetic pretoposes*.

We recall that a *pretopos* is a category equipped with finite limits, stable finite disjoint coproducts and stable effective quotients of equivalence relations. (For more detailed discussion, see, e.g., [6, A1.4.8].)

Definition 1. An *arithmetic universe* (or *AU*) [12] is a list arithmetic pretopos (see also [4]), namely a pretopos in which for any object A there is an object $\mathsf{List}(A)$ with maps $\mathsf{r}_0^A : 1 \rightarrow \mathsf{List}(A)$ and $\mathsf{r}_1^A : \mathsf{List}(A) \times A \rightarrow \mathsf{List}(A)$ such that for every $b : B \rightarrow Y$ and $g : Y \times A \rightarrow Y$ there is a unique $\mathsf{rec}(b, g)$ making the following diagrams commute

$$\begin{tikzcd}
B \arrow[r, "\langle \mathrm{Id}_B, \mathsf{r}_0^A \cdot !_B \rangle"] \arrow[dr, "b"'] & B \times \mathsf{List}(A) \arrow[d, "\mathsf{rec}(b,g)"] & B \times (\mathsf{List}(A) \times A) \arrow[l, "\mathrm{Id}_B \times \mathsf{r}_1^A"'] \arrow[d, "(\mathsf{rec}(b,g) \times \mathrm{Id}_A) \cdot \alpha"] \\
 & Y & Y \times A \arrow[l, "g"]
\end{tikzcd}$$

where $\alpha : B \times (\mathsf{List}(A) \times A) \rightarrow (B \times \mathsf{List}(A)) \times A$ is the associativity isomorphism.

We assume that *each arithmetic universe is equipped with a* choice *of its structure*. Hence we assume all the finite limits and colimits are defined by adjoint functors to diagram functors. For example, given two objects A, B we assume we have a functorial choice of their product and of the pairing morphisms of two morphisms. Note that an AU has all coequalizers, not just the quotients of equivalence relations. This is because the list objects allow one to construct the transitive closure of any relation (see [12]).

Definition 2. A functor between AUs is an *AU-functor* if it preserves the AU structure (finite limits, finite colimits, list objects) non-strictly, i.e. up to isomorphism. It is a *strict AU-functor*, if it preserves the AU structure strictly.

We shall be treating AUs as generalized spaces in a way analogous to that understood for Grothendieck toposes: an AU is in some sense the category of sheaves over its space. However, whereas Grothendieck toposes are all large, having all set-indexed colimits, for AUs we may conveniently restrict our attention to the small ones.

Definition 3. We write $\mathbf{AU}$ and $\mathbf{AU}_s$ for the categories of small AUs and (respectively) AU-functors and strict AU-functors.

2.1. AUs as algebras

The theory of arithmetic universes is essentially algebraic (or cartesian). Ref. [15] gives a simple "quasi-equational" formulation of the logic (similar to that of [3]), as well as a simple predicative account of the initial model theorem.

In the case of AUs, we use a quasi-equational theory with two sorts, for objects and morphisms. The category structure is described using a total operator for identity morphisms and a partial operator for composition; then the finite limit structure is described with total operators for the terminal object and unique morphisms and partial operators for pullbacks, projection morphisms and pairing. As a quasi-equational theory, this much is described for cartesian categories in Section 6.1 of [15]. Finite colimits are described dually. (Remember that an AU has all finite colimits even though a pretopos does not in general.) Then the properties relating colimits to limits can be expressed quasi-equationally. Note that the partial operators for pullbacks and pushouts have domains of definition defined as equations involving the operators for the theory of categories: for example, the pullback projections $\mathsf{p}^k_{f_1,f_2}$ $(k = 1, 2)$ are defined if the morphisms f_1 and f_2 have equal codomain.

Finally, we introduce operators to describe the list objects. List, r_0 and r_1 are described with total operators with a single argument A of sort object. rec is a partial operator, with $\mathsf{rec}^A(b, g)$ defined iff $\mathsf{c}(b) = \mathsf{c}(g)$ and $\mathsf{d}(g) = \mathsf{c}(g) \times A$. Finally, an extra partial operator u is needed to express the uniqueness of $\mathsf{rec}^A(b, g)$. $\mathsf{u}^A_{b,g}(r)$ is defined iff r is a possible solution for $\mathsf{rec}^A(b, g)$ in the diagram of Definition 1: in other words, $\mathsf{c}(b) = \mathsf{c}(g)$ and $\mathsf{d}(g) = \mathsf{c}(g) \times A$ (the same as for $\mathsf{rec}^A(b, g)$), $\mathsf{c}(r) = \mathsf{c}(b)$, $\mathsf{d}(r) = \mathsf{d}(b) \times \mathsf{List}(A)$ and $r \cdot (\mathsf{Id}_{\mathsf{d}(b)} \times \mathsf{r}^A_1) = g \cdot ((r \times \mathsf{Id}_A) \cdot \alpha)$. It is then subject to equations $\mathsf{u}^A_{b,g}(r) \risingdotseq r$ and $\mathsf{u}^A_{b,g}(r) \risingdotseq \mathsf{rec}^A(b, g)$ where "$\risingdotseq$" means that if both sides are defined then they are equal.

The algebraic notion of homomorphism, preserving these operators, corresponds to *strict* AU-functors. Note that it suffices to check strict preservation of certain object-valued operators: the terminal object and pullbacks, the initial object and pushouts, and the list objects. Once that is done, preservation of the other operators follows from the uniqueness conditions in Definition 1. This is important, since those object-valued operators are either total or have a definedness that depends only on the category structure.

The initial model theorem now implies that AUs can be presented by generators and relations, and that forgetful functors have left adjoints. In particular, the forgetful functor $G_0 : \mathbf{AU}_s \to \mathbf{Cat}$ has a left adjoint F_0. We write $(T_0 = G_0F_0, \eta_0, \mu_0)$ for the corresponding monad on $\mathbf{Cat}$. We shall also generally write $\sigma : T_0\mathcal{A} \to \mathcal{A}$ for the structure morphism (a strict AU-functor) of an AU $\mathcal{A}$. If $f : \mathcal{C} \to \mathcal{B}$ is a functor to an AU $\mathcal{B}$, we write $\overline{f} : T_0\mathcal{C} \to \mathcal{B}$ for the strict AU-functor lifting it, that is $\overline{f} = \sigma \cdot T_0(f)$.

We shall require various limits and weighted limits, in both $\mathbf{AU}_s$ and $\mathbf{AU}$. In fact G_0 creates finite weighted limits, and a number of the same constructions also serve as weighted limits in $\mathbf{AU}$.

Our starting point is to show that comma objects in $\mathbf{AU}$ are constructed as comma categories. Because the uniqueness clause in part (1) of the following Lemma is relative to *strict* AU-functors, we do not have that the forgetful functor from $\mathbf{AU}$ to $\mathbf{Cat}$ creates comma objects. However, the forgetful functor from $\mathbf{AU}_s$ to $\mathbf{Cat}$ does.

Proposition 4. *Let $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$ be two AU-functors.*

1. *There is a unique AU structure on the comma category $f \downarrow g$ such that the projection functors $\pi_1 : f \downarrow g \to \mathcal{A}$ and $\pi_2 : f \downarrow g \to \mathcal{B}$ are both strict AU-functors.*
2. *Let $\mathcal{D}$ be another AU, and let $h = \langle f', \alpha, g' \rangle : \mathcal{D} \to f \downarrow g$ be a functor. (Here $f' : \mathcal{D} \to \mathcal{A}$, $g' : \mathcal{D} \to \mathcal{B}$ and $\alpha : ff' \to gg'$.) Then h is an AU-functor iff both f' and g' are. Moreover, h is strict iff both f' and g' are.*

Proof. Essentially the result holds because the AU constructions are all covariant – this is related to the positivity of geometric logic and to the central core of the paper, the lack of exponentials in AUs – and characterized uniquely up to isomorphism. Recall that an object of $f \downarrow g$ is a triple (A, u, B) where A and B are objects of $\mathcal{A}$ and $\mathcal{B}$, and $u : f(A) \to g(B)$. Any construction on such triples must be done componentwise on the As and Bs in order to achieve the strictness of π_1 and π_2, and then the morphisms u lift by covariantness.

We prove the two parts simultaneously, showing for each AU construction its uniqueness as structure on $f \downarrow g$ and its preservation by h. Note that in part (2), the $\Rightarrow$ direction is obvious.

First, consider pullbacks or indeed any finite limits. Let $\Gamma = \langle \Gamma_1, \beta, \Gamma_2 \rangle : J \to f \downarrow g$ be a finite diagram in $f \downarrow g$. Then there is a unique $u : f(\lim \Gamma_1) \to g(\lim \Gamma_2)$ in $\mathcal{C}$ making a cone morphism between the limit cones. This gives, uniquely, our $\lim \Gamma = \langle \lim \Gamma_1, u, \lim \Gamma_2 \rangle$. Now consider a diagram $\Delta : J \to \mathcal{D}$ and let $\Gamma = h\Delta$. By naturality of α, $\alpha_{\lim \Delta}$ satisfies the characteristic conditions of u, and it follows that $h(\lim \Delta)$ is a limit of Γ.

Pushouts are similar.

For list objects, if we have $u : fA \to gB$ then there is a unique $\mathsf{List}(u)$ making these squares commute:

$$\begin{tikzcd}
1 \arrow[r, "\cong"] \arrow[d, "="'] & f1 \arrow[r, "fr_0^A"] & f\mathsf{List}(A) \arrow[d, "\mathsf{List}(u)"] & f(\mathsf{List}(A)\times A) \arrow[l, "fr_1^A"'] & f\mathsf{List}(A)\times fA \arrow[l, "\cong"'] \arrow[d, "\mathsf{List}(u)\times u"] \\
1 \arrow[r, "\cong"'] & g1 \arrow[r, "gr_0^B"'] & g\mathsf{List}(B) & g(\mathsf{List}(B)\times B) \arrow[l, "gr_1^B"] & g\mathsf{List}(B)\times gB \arrow[l, "\cong"]
\end{tikzcd}$$

By a similar argument to the above, but now using the fact that h preserves finite products, we see that h preserves list objects.

The remark on the strictness is clear. □

Proposition 5. *Let $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$ be two AU-functors, and let $f \downarrow_{\cong} g$ be the pseudopullback, the full subcategory of $f \downarrow g$ whose objects are those (A, u, B) for which u is an isomorphism. Then $f \downarrow_{\cong} g$ is a strict sub-AU of $f \downarrow g$.*

If f and g are both strict, then the pullback $f \downarrow_{=} g$ is a strict sub-AU of $f \downarrow_{\cong} g$.

Proof. – clear. □

Corollary 6. *The forgetful functor $G_0 : \mathbf{AU}_s \to \mathbf{Cat}$ creates finite limits.*

Proof. By Proposition 5 it creates pullbacks, and it clearly creates terminal objects. □

Theorem 7. *The forgetful functor $G_0 : \mathbf{AU}_s \to \mathbf{Cat}$ creates finite weighted limits.*

Proof. By Corollary 6 G_0 creates finite limits. By Proposition 4 it also creates cotensors by the category with two objects and one non-identity morphism, since that cotensor for $\mathcal{A}$, in other words the arrow category $\mathcal{A}^{\to}$, is the comma category $\mathrm{Id}_{\mathcal{A}} \downarrow \mathrm{Id}_{\mathcal{A}}$. The result now follows from [18]. □

Lemma 8. *Any G_0-split fork of strict AU-functors is a coequalizer in $\mathbf{AU}_s$ and also in $\mathbf{AU}$.*

Proof. Suppose we have

$$\begin{tikzcd}
\mathcal{A} \arrow[r, shift left=2, "f"] \arrow[r, "g"'] & \mathcal{B} \arrow[l, shift left=3, "t"] \arrow[r, shift left, "e"] & \mathcal{C} \arrow[l, shift left, "s"]
\end{tikzcd}$$

where f, g and e are strict AU-functors, $e \cdot s = \mathrm{Id}_{\mathcal{C}}$, $g \cdot t = \mathrm{Id}_{\mathcal{B}}$ and $f \cdot t = s \cdot e$. Let $h : \mathcal{B} \to \mathcal{D}$ be a strict AU-functor with $h \cdot f = h \cdot g$. We must show that $h \cdot s$ is also a strict AU-functor, in other words that it preserves final and initial objects, pullbacks, pushouts, and list objects. Let ω be the operator in the theory of AUs for any one of these, and suppose $\omega(\vec{x})$ is defined in $\mathcal{C}$. Even if ω is partial, its domain of definition is defined solely by the category structure of $\mathcal{C}$ and it follows that $\omega(s\vec{x})$ is defined and $e\omega(s\vec{x}) = \omega(\vec{x})$. Then

$$hs\omega(\vec{x}) = hs\omega(es\vec{x}) = hse\omega(s\vec{x}) = hft\omega(s\vec{x}) = hgt\omega(s\vec{x}) = h\omega(s\vec{x}) = \omega(hs\vec{x}).$$

In $\mathbf{AU}$ the argument is the same except that we have $h\omega(s\vec{x}) \cong \omega(hs\vec{x})$. □

Proposition 9. *The comparison functor $\gamma : \mathbf{AU}_s \to \mathbf{Cat}^{T_0}$ is full and faithful.*

Proof. Faithfulness is obvious. Fulness follows from Lemma 8 by standard results associated with Beck's Monadicity Theorem (see [1, Thm 3.13, Cor. 3.11]). □

Corollary 10. *A functor $f : \mathcal{A} \to \mathcal{B}$ between two AUs is a strict AU-functor iff $f \cdot \sigma = \overline{f}$.*

Proof. Recalling that $\overline{f} = \sigma \cdot T_0(f)$, the equation is the condition for f to be a morphism of T_0-algebras. □

Proposition 11. *Let $f : \mathcal{A} \to \mathcal{B}$ be a functor between two AUs. Then f is an AU-functor iff $f \cdot \sigma \cong \overline{f}$. In this situation there is a unique natural isomorphism whose composite with $(\eta_0)_{\mathcal{A}}$ is the identity on f.*

Proof. ⇐: Since $\overline{f}$ is by definition a strict AU-functor, it follows that $f \cdot \sigma$ is an AU-functor. Now consider Lemma 8 applied to the canonical presentation of $\mathcal{A}$ (where e, f, g in the Lemma are σ, μ_0, $T_0\sigma$ here) with $f \cdot \sigma$ for h. It follows that $f = f \cdot \sigma \cdot \eta_0$ is an AU-functor.

⇒: Let $\mathcal{C} = f \downarrow_{\cong} \mathcal{B}$ as in Proposition 5. The functor $g = \langle \mathcal{A}, =, f \rangle : \mathcal{A} \to \mathcal{C}$ extends uniquely to a strict AU-functor $\overline{g} = \langle \sigma, \cong, \overline{f} \rangle : T_0\mathcal{A} \to \mathcal{C}$ such that $g = \overline{g} \cdot \eta_0$, giving $f \cdot \sigma \cong \overline{f}$. □

Proposition 12. *The subcategory inclusion $G : \mathbf{AU}_s \to \mathbf{AU}$ has a left adjoint.*

Proof. We define its left adjoint $F : \mathbf{AU} \to \mathbf{AU}_s$ as follows. If $\mathcal{A}$ is an AU, then $F\mathcal{A}$ is the AU defined over $T_0\mathcal{A}$ by freely adjoining a natural isomorphism $\upsilon : \eta_0 \cdot \sigma \cong \mathrm{Id}_{T_0\mathcal{A}}$, subject to its composing with η_0 to give the identity on η_0. Let $\eta' : T_0\mathcal{A} \to F\mathcal{A}$ be the canonical strict AU-functor. By Proposition 11, AU-functors $f : \mathcal{A} \to \mathcal{B}$ are in bijection with strict AU-functors $\overline{f} : T_0\mathcal{A} \to \mathcal{B}$ equipped with isomorphisms υ of the kind described, and they are in bijection with strict AU-functors $F\mathcal{A} \to \mathcal{B}$. In fact, $\eta' \cdot \eta_0$ is a universal arrow from $\mathcal{A}$ to G, so we obtain a functor F left adjoint to G. (The fact that $\eta' \cdot \eta_0$ is an AU-functor follows from Proposition 11.) $\square$

We write (T, η, μ) for the corresponding *strictification* monad on **AU**. Note that *every* object is a T-algebra, in a unique way – we write $\beta : T\mathcal{A} \to \mathcal{A}$ for the structure morphism, a strict AU-functor. The point is that not every morphism is a T-homomorphism.

Theorem 13. *If $\mathcal{A}$ is an AU then the functors η and β form an equivalence $\mathcal{A} \simeq T\mathcal{A}$.*

Proof. By definition, $\beta \cdot \eta = \mathrm{Id}_{\mathcal{A}}$. To prove $\eta \cdot \beta \cong \mathrm{Id}_{T\mathcal{A}}$ we use a proof similar to that of Proposition 11. Let $\mathcal{C}$ be the AU $\eta \downarrow_{\cong} T\mathcal{A}$. There is an AU-functor $g : \mathcal{A} \to \mathcal{C}$ given by $g(A) = (A, \eta(A), \mathrm{Id}_{\eta(A)})$. Its extension to a strict AU-functor $\overline{g} : T\mathcal{A} \to \mathcal{C}$ provides the required natural isomorphism. $\square$

2.2. AUs as spaces

Our general philosophy is to view an *arithmetic space* (or *AS*) X as a small AU $\mathcal{A}X$. A *map* $f : X \to Y$ between ASs is an AU-functor $f^* : \mathcal{A}Y \to \mathcal{A}X$. Thus the category **AS** of ASs and maps is equal to $\mathbf{AU}^{op}$.

We shall be interested in understanding **AS** as a "category of (generalized) spaces", and in particular we shall be interested in understanding a slice $\mathbf{AS}/X$ as a category of spaces "fibred over X". Typical questions, for a map $f : Y \to X$, would be: When is f fibrewise discrete (i.e. a local homeomorphism or sheaf, hence corresponding to an object of $\mathcal{A}X$)? When is it a subspace inclusion? When is it an open or closed subspace inclusion? And in all those we should like to know something about the structure of $\mathcal{A}Y$ and the AU-functor f^*. This paper takes a step towards addressing such questions.

In the first question, on fibrewise discreteness, one expects the fibrewise discrete spaces over X to be equivalent to objects of $\mathcal{A}X$ and hence form an AU. Taylor [19] has investigated the analogous question in his system of Abstract Stone Duality and shown that there too the discrete spaces (or, to be precise in his terminology, the overt discrete spaces) form an AU.

Our fundamental construction of spaces over X is to adjoin structure freely to the AU $\mathcal{A}X$, and for this we use universal algebra. In the present work, the kind of structure we shall adjoin is morphisms constrained by commutative diagrams, although it is clear that the techniques could also be extended to adjoining objects, subject to equational constraints in terms of the AU structure.

Definition 14. Let $\mathcal{A}$ be an AU, and let S *specify morphisms and diagrams over* $\mathcal{A}$. That is to say, it comprises an additional set of symbols for morphisms, with specified domain and codomain already in $\mathcal{A}$, and of equations amongst them and the morphisms already in $\mathcal{A}$. We say that a *model* of S in an AU $\mathcal{B}$ is a pair (F, α) where $F : \mathcal{A} \to \mathcal{B}$ is an AU-functor and α assigns to each morphism listed in S an interpretation in $\mathcal{B}$, such that the equations specified in S all hold in $\mathcal{B}$ when interpreted by F and α.

The model is *strict* if F is strict.

Note that any AU-functor $H : \mathcal{B} \to \mathcal{C}$ transforms models (F, α) in $\mathcal{B}$ to models $H \cdot (F, \alpha) = (H \cdot F, H \cdot \alpha)$ in $\mathcal{C}$, and that if H is strict then the transformation preserves strictness.

For example, if U and V are objects in $\mathcal{A}$ and S specifies a morphism $f : U \to V$, then a model in $\mathcal{B}$ is an AU-functor F together with a given morphism $f : F(U) \to F(V)$ in $\mathcal{B}$.

Comma categories can be used to make a category $\mathrm{Mod}_S(\mathcal{B})$ of models of S in $\mathcal{B}$. If (F, α) and (G, β) are two models in $\mathcal{B}$, then a homomorphism between them is a model in $\mathcal{B} \downarrow \mathcal{B}$ whose two projections down to $\mathcal{B}$ give (F, α) and (G, β).

We shall adjoin structure in two styles: strictly and non-strictly.

Definition 15. Let $\mathcal{A}$ be an AU, and let S specify morphisms and diagrams over $\mathcal{A}$. Then $\mathcal{A}[S]_s$ is the AU presented by generators corresponding to the objects and morphisms of $\mathcal{A}$ and the morphisms in S, and relations to require that the AU structure of $\mathcal{A}$ is strictly preserved and the equations of S hold.

It is equipped with a strict model $(\mathcal{I}, \alpha)$ of S and is characterized up to isomorphism by the universal property that for any AU $\mathcal{B}$ and strict S-model (F, β) in $\mathcal{B}$, there is a unique strict AU-functor $\widetilde{F} : \mathcal{A}[S]_s \to \mathcal{B}$ such that $(F, \beta) = (\widetilde{F} \cdot \mathcal{I}, \widetilde{F} \cdot \alpha)$.

If X is an AS, then $X[S]_s$ is defined by $\mathcal{A}(X[S]_s) = (\mathcal{A}X)[S]_s$. Thus we have a map $X[S]_s \to X$.

Definition 16. Let $\mathcal{A}$ be an AU, and let S specify morphisms and diagrams over $\mathcal{A}$. Then $\mathcal{A}[S]$ is defined as $T(\mathcal{A})[\eta S]_s$.

If X is an AS, then $X[S]$ is defined by $\mathcal{A}(X[S]) = (\mathcal{A}X)[S]$. Thus we have a map $X[S] \to X$.

Theorem 17. *$\mathcal{A}[S]$ is equipped with a generic model $(\mathcal{I}, \alpha)$. Given an AU $\mathcal{B}$, let us write $i : \mathbf{AU}_s(\mathcal{A}[S], \mathcal{B}) \to \mathbf{AU}(\mathcal{A}[S], \mathcal{B})$ for the subcategory inclusion (which is full), and $j : \mathbf{AU}(\mathcal{A}[S], \mathcal{B}) \to \mathrm{Mod}_S(\mathcal{B})$ for the functor $F \mapsto F \cdot (\mathcal{I}, \alpha)$.*

1. *$\mathcal{A}[S]$ is characterized up to strict AU isomorphism by the universal property that $j \cdot i$ is an isomorphism of categories for all $\mathcal{B}$.*
2. *j is an equivalence of categories. (Hence: to show that two AU-functors $F, G : \mathcal{A}[S] \to \mathcal{B}$ are isomorphic, one shows that $F \cdot \mathcal{I} \cong G \cdot \mathcal{I}$ and $F(\alpha) \cong G(\alpha)$.)*

Proof. The first part is immediate from the universal characterizations of $T(\mathcal{A})$ and then of $T(\mathcal{A})[S]_s$.

For the second part, we already know that $j \cdot (i \cdot (j \cdot i)^{-1}) = \mathrm{Id}$ and it remains to show that $(i \cdot (j \cdot i)^{-1}) \cdot j \cong \mathrm{Id}$. Let $F : \mathcal{A}[S] \to \mathcal{B}$ be an AU-functor and let $\widetilde{F} : \mathcal{A}[S] \to \mathcal{B}$ be the strict AU-functor $(j \cdot i)^{-1}(j(F))$. We have $\widetilde{F} \cdot (\mathcal{I}, \alpha) = F \cdot (\mathcal{I}, \alpha)$, and these identities make a model of S in $(\widetilde{F} \downarrow_{\cong} F)$, which is an AU by Proposition 5, whose first and second projections both give $(\mathcal{I}, \alpha)$. So there exists a strict AU-functor $\mathcal{A}[S] \to (\widetilde{F} \downarrow_{\cong} F)$ that composed with both projections gives the identity on $\mathcal{A}[S]$ and so an isomorphism $\widetilde{F} \cong F$. □

It is also possible to use the type theoretic methods of [11]. Though much more complex, these deal with structure of general dependent type theory and hence cover much more general kinds of adjoined structure. The basic idea is as follows.

First, given an AU $\mathcal{A}$, one can make a $\mathcal{T}_{au}$-theory $T_{iso}(\mathcal{A})$ (where $\mathcal{T}_{au}$ is the typed calculus of AUs) that has constants (and suitable axioms) for all the category structure of $\mathcal{A}$; and also constants and axioms for coherent isomorphisms between the AU types that can be expressed in the type theory and the constants for the corresponding values in $\mathcal{A}$.

It has the property that, for any AU $\mathcal{B}$, the interpretations of $T_{iso}(\mathcal{A})$ in $\mathcal{B}$ are equivalent to AU-functors $\mathcal{A} \to \mathcal{B}$.

Next, if S expresses extra ingredients of type theory that we wish to adjoin to $\mathcal{A}$, then we can make an extended theory $T_{iso}(\mathcal{A})[S]$. We then write $\mathcal{A}[S]_t$ for the syntactic category $\mathcal{C}_{T_{iso}(\mathcal{A})[S]}$, i.e. the category of ground types for $T_{iso}(\mathcal{A})[S]$. It has the property that for any AU $\mathcal{B}$, pairs (F, α) (where $F : \mathcal{A} \to \mathcal{B}$ is an AU-functor and α is an interpretation of S in $\mathcal{B}$ with respect to the image of F) are equivalent to AU-functors $\mathcal{A}[S]_t \to \mathcal{B}$. The notion of "interpretation of S" is as described in section 5 of [11] and is highly non-trivial.

For a detailed proof of this approach, see [13].

3. Some structure theorems

In this section we discuss some theorems that describe concrete structure of certain AUs presented as $\mathcal{A}[S]$ (Definition 16).

For our induction principle we shall need to analyse $X[n : 1 \to N]$ (discrete space of natural numbers over X), $X[\top \leq \phi]$ (open subspace for ϕ a subobject of 1) and $X[\phi \leq \bot]$ (closed subspace).

The first part (Theorem 19) presents for AUs a categorical construction that is well known for a range of categorical structures: to adjoin to $\mathcal{C}$ an indeterminate global element $c : 1 \to U$ is equivalent to taking the slice $\mathcal{C}/U$ with c represented by the diagonal $\Delta : U \to U \times U$.

However, it is worth considering a topological aspect in the case of Grothendieck toposes, since that also motivates the development for AUs. In general with the slice we construct the local homeomorphism corresponding to the sheaf U. If U is a subobject ϕ of 1, i.e. an open of the topos, then the topos of sheaves for the corresponding open subspace is got by adjoining an element of ϕ — i.e. (since the element is unique) insisting that ϕ is the whole of 1. Open subspaces are got by adjoining global elements.

In topos theory a key result is what we might call the *localic bundle theorem*, which establishes an equivalence between localic maps (i.e. localic geometric morphisms) with codomain a topos $\mathcal{E}$, and internal locales in $\mathcal{E}$ (see [9]). Given an internal frame in $\mathcal{E}$, you then take the topos of internal sheaves over it. For the open subspace ϕ, the frame is presented over Ω by the relation $\top \leq \phi$. However, this general approach is not available in AUs since there we have neither Ω nor internal frames.

Having found a more ad hoc method for open subspaces, we still have to deal with their closed complements, presented by a relation $\phi \leq \bot$ (Section 3.2). The same technique of adjoining an element will certainly not work, since closed embeddings, unlike open ones, are not local homeomorphisms in general. However, they are Stone maps. That is to say, using the equivalence of the localic bundle theorem, the corresponding internal frame in the topos of sheaves over the codomain is the ideal completion of a Boolean algebra. Specifically, it is the initial Boolean algebra 2 but with $\top = \bot$ forced, making inconsistency, over ϕ. In an AU we find that the Boolean algebra exists even though the frame does not, and our construction in effect shows how to describe the sheaves over the frame, or those sheaves one needs for an AU, purely in terms of the Boolean algebra.

3.1. Adjoining a global element to an AU

Lemma 18. *In any arithmetic universe $\mathcal{A}$,* List *preserves equalizers.*

Proof. We give an argument whose essential ingredients may be found in [12]. Let $e : E \hookrightarrow A$ be an equalizer of $f, g : A \to B$, and let $e_L : E_L \hookrightarrow \mathsf{List}(A)$ be an equalizer of $\mathsf{List}(f)$ and $\mathsf{List}(g)$. Clearly $\mathsf{List}(e)$ factors via e_L; we must show the reverse, with a morphism $E_L \to \mathsf{List}(E)$.

First, note that

$$1 \xrightarrow{\mathsf{r}_0} \mathsf{List}(A) \xleftarrow{\mathsf{r}_1} \mathsf{List}(A) \times A$$

is a coproduct diagram. This follows by using the morphism

$$\mathsf{inr} \cdot ([\mathsf{r}_0, \mathsf{r}_1] \times \mathrm{Id_A}) : (1 + \mathsf{List}(A) \times A) \times A \to 1 + \mathsf{List}(A) \times A$$

to define a morphism $\mathsf{rec}(\mathsf{inl}, \mathsf{inr} \cdot ([\mathsf{r}_0, \mathsf{r}_1] \times \mathrm{Id_A})) : \mathsf{List}(A) \to 1 + \mathsf{List}(A) \times A$. The image of r_1 is the object $\mathsf{List}^*(A)$ of non-empty lists. Pulling back along e_L we also get a coproduct diagram for E_L; let E_L^* be the pullback of $\mathsf{List}^*(A)$.

Next, the morphism $\mathsf{p} : N \times \mathsf{List}^*(A) \to A$ is defined so that $\mathsf{p}_n(s)$ is the nth element of s (with suitable treatment of cases where n is out of bounds), and one can then show that it restricts to a morphism $N \times E_L^* \to E$. Knowing (via e_L) the length function on E_L, we can then derive a morphism $E_L^* \to \mathsf{List}^*(E)$ and from that we obtain a morphism $E_L \cong 1 + E_L^* \to \mathsf{List}(E)$. It is then straightforward to show that it composes with $\mathsf{List}(e)$ to give e_L. $\square$

Theorem 19. *Let U be an object of an AU $\mathcal{A}$. Then $\mathcal{A}[c : 1 \to U]$ is equivalent to the slice category $\mathcal{A}/U$.*

Proof. It is known that $\mathcal{A}/U$ is an arithmetic universe (see Proposition 2.13 in [11] and also [12]). Our statement is a strengthening of ex. 6 page 71 of [10] to arithmetic universes (moreover our definition of $\mathcal{A}[c : 1 \to U]$ differs from that in [10] in that it classifies non-strict AU-functors) and hence, we here give the proof in full detail.

We can think of $\mathcal{A}/U$ as extending $\mathcal{A}$ via the AU-functor

$$\xi : \mathcal{A} \longrightarrow \mathcal{A}/U$$

sending an object V to $\pi_1 : U \times V \to U$. Moreover the diagonal

$$\Delta_U : U \to U \times U$$

is a global element of $\pi_1 : U \times U \to U$. Therefore we get a strict AU-functor

$$\widetilde{\xi} : \mathcal{A}[c : 1 \to U] \longrightarrow \mathcal{A}/U$$

extending ξ and taking c to Δ_U.

Conversely, we can define a functor

$$\gamma : \mathcal{A}/U \to \mathcal{A}[c : 1 \to U]$$

sending any object $f : V \to U$ to the equalizer

$$\gamma(f) \hookrightarrow \mathcal{I}(V) \underset{c\cdot !}{\overset{\mathcal{I}(f)}{\rightrightarrows}} \mathcal{I}(U).$$

We note immediately that $\widetilde{\xi} \cdot \gamma \cong \mathrm{Id}_{\mathcal{A}/U}$. This is because any object $f : V \to U$ of $\mathcal{A}/U$ is an equalizer of the morphisms $U \times f$, $\Delta_U \cdot \pi_1 : U \times V \to U \times U$ over U, and hence is isomorphic to $\widetilde{\xi}(\gamma(f))$.

Next, we find $\gamma \cdot \xi(V)$ is isomorphic to the equalizer

$$\mathcal{I}(V) \overset{\langle c\cdot !, \mathrm{Id}\rangle}{\hookrightarrow} \mathcal{I}(U) \times \mathcal{I}(V) \cong \mathcal{I}(U \times V) \underset{c\cdot !}{\overset{\mathcal{I}(\pi_1)}{\rightrightarrows}} \mathcal{I}(U)$$

giving $\gamma \cdot \xi \cong \mathcal{I}$. Similarly, $\gamma(\Delta_U) : \gamma \cdot \xi(1) \to \gamma \cdot \xi(U)$ corresponds under the isomorphisms to $c : 1 \to \mathcal{I}(U)$.

We shall need to show that γ is an AU-functor. Let $\delta : \mathcal{A}/U \to \mathcal{A}$ be the functor, left adjoint to ξ, that takes each object $f : V \to U$ to V; δ preserves finite colimits and pullback. Note that if $h : f \to g$ in $\mathcal{A}/U$ (so $f = gh$) then the following square is a pullback.

$$\begin{array}{ccc} \gamma(f) & \hookrightarrow & \mathcal{I}\delta(f) \\ \gamma(h)\downarrow & & \downarrow \mathcal{I}\delta(h) \\ \gamma(g) & \hookrightarrow & \mathcal{I}\delta(g). \end{array}$$

Now suppose that Γ is a finite diagram in $\mathcal{A}/U$, with colimit h. The above observation shows that to get the image under γ of the colimit cocone we first take the image under $\mathcal{I}\delta$ and then pullback along $\gamma(h) \hookrightarrow \mathcal{I}\delta(h)$. But finite colimits are preserved by $\mathcal{I}\delta$ and in an AU they stable under pullback, so it follows that γ preserves finite colimits. A similar argument shows that it preserves pullbacks. It also preserves the terminal object $\mathrm{Id}_U \cong \xi(1)$.

Using [12] the list object of $f : V \to U$ in $\mathcal{A}/U$, let us write $\mathsf{List}_U(f)$, is calculated as an equalizer of two morphisms in $\mathcal{A}$,

$$\mathsf{List}(V) \times U \xrightarrow{\pi_1} \mathsf{List}(V) \xrightarrow{\mathsf{List}(f)} \mathsf{List}(U), \text{ and}$$

$$\mathsf{List}(V) \times U \xrightarrow{\mathsf{List}(!) \times U} N \times U \xrightarrow{\mathsf{mult}} \mathsf{List}(U).$$

(Note that $\mathsf{List}(1)$ is N.) We define $\mathsf{mult}(n, u)$ to be a list of length n all of whose elements are u. More categorically, it is defined according to Definition 1 using $\mathsf{rec}(b, g) : U \times \mathsf{List}(1) \to U \times \mathsf{List}(U)$, where $b : U \to U \times \mathsf{List}(U)$ is $b(u) = (u, \mathsf{r}_0^U)$ and $g : U \times \mathsf{List}(U) \times 1 \to U \times \mathsf{List}(U)$ is $g(u, l, *) = (u, \mathsf{r}_1^U(l, u))$. We can re-express the equalizer as one in $\mathcal{A}/U$, namely of

$$\xi(\mathsf{List}(V)) \xrightarrow{\xi(\mathsf{List}(f))} \xi(\mathsf{List}(U)), \text{ and}$$

$$\xi(\mathsf{List}(V)) \xrightarrow{\langle \Delta_U \cdot !, \xi(\mathsf{List}(!))\rangle} \xi(U) \times \xi(N) \xrightarrow{\cong} \xi(N \times U) \xrightarrow{\xi(\mathsf{mult})} \xi(\mathsf{List}(U)).$$

Then, since γ preserves finite limits and $\gamma \cdot \xi \cong \mathcal{I}$, we have that $\gamma(\mathsf{List}_U(f))$ is an equalizer of

$$\mathcal{I}(\mathsf{List}(V)) \xrightarrow{\mathcal{I}(\mathsf{List}(f))} I(\mathsf{List}(U)), \text{ and}$$

$$\mathcal{I}(\mathsf{List}(V)) \xrightarrow{\langle c\cdot !, \mathcal{I}(\mathsf{List}(!))\rangle} \mathcal{I}(U) \times \mathcal{I}(N) \xrightarrow{\cong} \mathcal{I}(N \times U) \xrightarrow{\mathcal{I}(\mathsf{mult})} \mathcal{I}(\mathsf{List}(U)).$$

The second of these is equal to $\mathcal{I}(\mathsf{List}(c)) \cdot \mathcal{I}(\mathsf{List}(!))$. Since $\mathcal{I}$ preserves List, we see that we are equalizing

$$\mathsf{List}(\mathcal{I}(V)) \xrightarrow{\mathsf{List}(\mathcal{I}(f))} \mathsf{List}(I(U)), \text{ and}$$

$$\mathsf{List}(\mathcal{I}(V)) \xrightarrow{\mathsf{List}(\mathcal{I}(!))} \mathsf{List}(1) \xrightarrow{\mathsf{List}(\mathcal{I}(c))} \mathsf{List}(\mathcal{I}(U)).$$

Using Lemma 18 we deduce that the equalizer we seek for $\gamma(\mathsf{List}_U(f))$ is isomorphic to $\mathsf{List}(\gamma(f))$.

Since $(\mathcal{I}, c) \cong (\gamma \cdot \widetilde{\xi} \cdot \mathcal{I}, \gamma \cdot \widetilde{\xi} \cdot c) = (\gamma \cdot \xi, \gamma(\Delta_U))$ as models of the constant c, we can use Theorem 17 to show $\gamma \cdot \widetilde{\xi} \cong \mathrm{Id}_{\mathcal{A}[c]}$, and we conclude that $\widetilde{\xi}$ and γ form an adjoint equivalence. $\square$

3.2. Closed subspaces

We now turn to the closed subspace presented by the relation $\phi \leq \bot$, where ϕ is a subobject of 1.

In topos theory, where this is the closed complement of the open subspace for ϕ, we can analyse the situation as follows. A closed embedding is Stone over its codomain, and the corresponding Boolean algebra of clopens (internal in the topos of sheaves over the codomain) can be described as the coequalizer B_ϕ of two maps $\phi \hookrightarrow 1 \to 2$, where the second morphism is either of the two coproduct injections. Thus B_ϕ has the two elements $\bot_B$ and $\top_B$, but they are equal (giving an inconsistent theory) over ϕ. The internal frame is then the ideal completion of B_ϕ, and sheaves over it are equivalent to presheaves over B_ϕ satisfying a finitary pasting condition.

A presheaf F over B_ϕ is a restriction morphism $F(\top_B) \to F(\bot_B)$ that is an isomorphism if ϕ holds – in other words, $F(\top_B) \times \phi \to F(\bot_B) \times \phi$ is an isomorphism. The only significant effect of the sheaf pasting conditions is that $F(\bot_B)$ must be a singleton. Hence a finitary sheaf is determined by $F(\top_B)$, under the condition that it has exactly one element if ϕ holds – in other words, the projection $F(\top_B) \times \phi \to \phi$ is an isomorphism. We can rephrase the discussion in terms of a certain coequalizer.

For the rest of this subsection, we do not use presume the truth of this topos-theoretic analysis, but by mimicking its construction for AUs we obtain our Theorem 28. We take ϕ to be a subobject of 1 in an AU $\mathcal{A}$.

Definition 20. The endofunctor $\mathcal{V}_\phi$ on $\mathcal{A}$ is defined on objects U by letting $\mathcal{V}_\phi(U)$ be the following coequalizer:

$$\begin{tikzcd}
 & U \arrow[dr, "i_1"] & & \\
U \times \phi \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & & U + \phi \arrow[r, two heads, "e"] & \mathcal{V}_\phi(U) \\
 & \phi \arrow[ur, "i_2"'] & &
\end{tikzcd}$$

Equivalently, $\mathcal{V}_\phi(U)$ is the pushout of the two projections from $U \times \phi$.

This is extended to morphisms in the obvious way.

We shall use the following definition to calculate the equivalence relation corresponding to the epi e.

Definition 21. If U is any object of $\mathcal{A}$, we define the relation $\sim_\phi^U$ on U as $\Delta \vee (\phi \times U \times U)$, where we are writing Δ for the equality relation, i.e. the image of the diagonal morphism. (We shall sometimes omit the superscript U.) It is clear that $\sim_\phi^U$ is an equivalence relation.

Proposition 22. *The equivalence relation for the epi* $e : U + \phi \twoheadrightarrow \mathcal{V}_\phi(U)$ *is* $\sim_\phi^{U+\phi}$.

Proof. First, the image of $\langle i_1 \cdot \pi_1, i_2 \cdot \pi_2 \rangle$ in $(U+\phi)^2$ is less than $\phi \times (U+\phi)^2$, so $U + \phi \twoheadrightarrow (U+\phi)/\sim_\phi^{U+\phi}$ factors via e.

It remains to show that $\sim_\phi^{U+\phi}$ is less than the equivalence relation generated by the relation implicit in the definition of e, and it suffices to consider the disjunct $\phi \times (U+\phi)^2 \cong \phi \times U \times U + \phi \times U + U \times \phi + \phi$. The part $U \times \phi$ is what we have in the definition of e, and $\phi \times U$ follows by symmetry then $\phi \times U \times U$ by transitivity. $\square$

Lemma 23. *Let U be an object of $\mathcal{A}$. Then the following conditions are equivalent.*

1. *The projection* $\pi_2 : U \times \phi \xrightarrow{\pi_2} \phi$ *is an isomorphism.*
2. *The morphism* $\eta_U = e \cdot i_1 : U \to \mathcal{V}_\phi(U)$ *is an isomorphism.*

Proof. (In the internal logic, condition (1) says that if ϕ holds then U is a singleton.)

(1)⇒(2) is clear, since η_U is the pushout of π_2 along π_1.

For (2)⇒(1) first note that π_1 is monic, and it follows that π_2 is also monic. Now from [6, A1.4.8] (valid in pretoposes) it follows that the pushout square is also a pullback, and we deduce that π_2 is an isomorphism.[1] □

Proposition 24. *$\mathcal{V}_\phi$ preserves finite limits (non-strictly).*

Proof. That $\mathcal{V}_\phi$ preserves 1 follows from Lemma 23.

Next we show it preserves binary products. Given objects U and V, the morphism $e_U \times e_V : (U + \phi) \times (V + \phi) \to \mathcal{V}_\phi(U) \times \mathcal{V}_\phi(V)$ is epi. However, by definition of e the images of the parts $U \times \phi$ and $\phi \times V$ are less than the image of ϕ, and it follows that the morphism $U \times V + \phi \twoheadrightarrow \mathcal{V}_\phi(U \times V) \to \mathcal{V}_\phi(U) \times \mathcal{V}_\phi(V)$ is also epi. We calculate its kernel pair as a subobject of $(U \times V + \phi)^2$. Any part with ϕ is a subobject of $\phi \times (U \times V + \phi)^2 \leq \sim_\phi^{U\times V+\phi}$, so it remains to calculate the kernel pair restricted to $(U \times V)^2$, which is $\sim_\phi^U \times \sim_\phi^V \leq \sim_\phi^{U\times V+\phi}$. It follows that $\mathcal{V}_\phi(U \times V) \twoheadrightarrow \mathcal{V}_\phi(U) \times \mathcal{V}_\phi(V)$ is an iso.

Finally we show it preserves equalizers. Let $E \hookrightarrow U$ be the equalizer of $f, g : U \to V$, and let $E' \hookrightarrow \mathcal{V}_\phi U$ be the equalizer of $\mathcal{V}_\phi f$ and $\mathcal{V}_\phi g$. The inverse image of E' under e_U is the inverse image of $\sim_\phi^{V+\phi}$ under $\langle f + \phi, g + \phi\rangle : U + \phi \to (V + \phi)^2$, namely $(E + \phi) \vee \phi \times (U + \phi)$, and it follows that the restriction of e_U from $(E + \phi) \vee \phi \times (U + \phi)$ to E' is the pullback of an epi and hence epi. But by definition of e_U the image of $\phi \times (U + \phi)$ is contained in that of $i_1(\phi)$ and it follows that $E + \phi \twoheadrightarrow \mathcal{V}_\phi(E) \to E'$ is also epi. Its kernel pair is $\sim_\phi^{U+\phi}$ restricted to $(E + \phi)^2$, which is just $\sim_\phi^{E+\phi}$. It follows that $\mathcal{V}_\phi(E) \twoheadrightarrow E'$ is an isomorphism. □

It follows that $\mathcal{V}_\phi$ preserves monics.

Lemma 25. *Let $m : V \hookrightarrow U$ be a monic in $\mathcal{A}$.*

1. *The pullback of*

$$\begin{array}{ccccc} & & & & \mathcal{V}_\phi(V) \\ & & & & \downarrow \mathcal{V}_\phi(m) \\ U & \xrightarrow{i_1} & U+\phi & \overset{e}{\twoheadrightarrow} & \mathcal{V}_\phi(U) \end{array}$$

is the subobject $V \vee (\phi \times U)$ of U.

2. *$\mathcal{V}_\phi(m)$ is invertible iff $U \leq V \vee (\phi \times U)$.*

Proof. 1. Every monic is regular, and it follows that we can use the calculation for equalizers in the proof of Proposition 24. This shows that the pullback of $\mathcal{V}_\phi(m)$ along e is $(V + \phi) \vee \phi \times (U + \phi)$. Pulling that back along i_1 we get the result.

2. The ⇒ direction follows from part (1). For the converse, we see that $V + \phi \to (V \vee (\phi \times U)) + \phi \twoheadrightarrow \mathcal{V}_\phi(U)$ is epi, and so $\mathcal{V}_\phi(V) \twoheadrightarrow \mathcal{V}_\phi(U)$ is epi. □

Proposition 26. *The functor $\mathcal{V}_\phi : \mathcal{A} \to \mathcal{A}$ is the functor part of a monad whose multiplication is an isomorphism.*

Proof. Defining the unit η as in Lemma 23 (2), we show two properties of it.

First, $\eta_{\mathcal{V}_\phi(U)}$ is an isomorphism. By Lemma 23 we need to show that the projection $\mathcal{V}_\phi(U) \times \phi \to \phi$ is an isomorphism, in other words there is some $\delta : \phi \to \mathcal{V}_\phi(U)$ such that $\delta \cdot \pi_2 = \pi_1$. We define $\delta = e_U \cdot i_2$. Since $e_U \times \phi : (U+\phi) \times \phi \to \mathcal{V}_\phi(U) \times \phi$ is epi, it suffices to show that $\delta \cdot \pi_2$ and π_1 compose equally with it. On $\phi \times \phi$ this is immediate, while on $U \times \phi$ it follows from the definition of e_U.

Second, $\eta_{\mathcal{V}_\phi(U)} = \mathcal{V}_\phi(\eta_U)$. It suffices to check that they compose equally with the epi e_U, and

$$\mathcal{V}_\phi(\eta_U) \cdot e_U = e_{\mathcal{V}_\phi(U)} \cdot (\eta_U + \phi)$$

$$\eta_{\mathcal{V}_\phi(U)} \cdot e_U = e_{\mathcal{V}_\phi(U)} \cdot i_1 \cdot e_U.$$

These agree on both summands of $U + \phi$.

Given these, we can define the multiplication μ_U as $(\eta_{\mathcal{V}_\phi(U)})^{-1}$. From the second property of η it follows that $\mu_{\mathcal{V}_\phi(U)} = \mathcal{V}_\phi(\mu_U)$. The monad properties now follow. □

For any monad, the multiplication is an isomorphism iff for each Eilenberg–Moore algebra the structure map is an isomorphism, its inverse being the unit. The category of Eilenberg–Moore algebras is then equal to the full subcategory of the base category whose objects are those for which the unit is an isomorphism.

The discussion at the start of this subsection now suggests that we *define* the "category of finitary sheaves over B_ϕ", which we shall write $\mathrm{Sh}(B_\phi)$, to be the category of Eilenberg–Moore algebras of the monad $\mathcal{V}_\phi$. It is a reflective subcategory of $\mathcal{A}$. We write $\mathcal{J} : \mathcal{A} \to \mathrm{Sh}(B_\phi)$ for the reflection and $\mathrm{inc} : \mathrm{Sh}(B_\phi) \to \mathcal{A}$ for the inclusion.

Proposition 27. *$\mathrm{Sh}(B_\phi)$ is an AU, and $\mathcal{J} : \mathcal{A} \to \mathrm{Sh}(B_\phi)$ is an AU-functor.*

[1] We thank the anonymous referee for pointing out this argument.

Proof. In general, the AU-structure of $\mathrm{Sh}(B_\phi)$ is obtained by first taking the corresponding structure in $\mathcal{A}$ and then applying $\mathcal{J}$. We discuss the different kinds of structure in more detail.

The inclusion inc creates limits, so $\mathrm{Sh}(B_\phi)$ has all finite limits. Since $\mathcal{V}_\phi$ preserve finite limits, so does $\mathcal{J}$. For finite colimits, let D be a finite diagram in $\mathrm{Sh}(B_\phi)$ and let $\gamma : \mathsf{inc}(D) \to C$ be the colimit cocone in $\mathcal{A}$. $\mathcal{J}$, as a left adjoint, preserves all existing colimits, so $\mathcal{J}(\gamma)$ makes $\mathcal{J}(C)$ a colimit of $\mathcal{J} \cdot \mathsf{inc}(D) \cong D$ in $\mathrm{Sh}(B_\phi)$. It follows that $\mathrm{Sh}(B_\phi)$ has all finite colimits. The other conditions for colimits follow from the fact that $\mathcal{J}$ preserves finite limits.

For list objects, suppose we have $b : B \to Y$ and $g : Y \times \mathcal{J}(A) \to Y$ in $\mathrm{Sh}(B_\phi)$. We then get a composite $g' : \mathsf{inc}(Y) \times A \to \mathsf{inc}(Y)$ as

$$\mathsf{inc}(Y) \xleftarrow{\mathsf{inc}(g)} \mathsf{inc}(Y \times \mathcal{J}(A)) \cong \mathsf{inc}(Y) \times \mathcal{V}_\phi(A) \xleftarrow{\mathsf{inc}(Y)\times\eta_A} \mathsf{inc}(Y) \times A$$

and hence $\mathsf{rec}(\mathsf{inc}(b), g') : \mathsf{inc}(B) \times \mathsf{List}(A) \to \mathsf{inc}(Y)$. Applying $\mathcal{J}$ to the diagram in Definition 1, and using the isomorphism $\mathcal{J} \cdot \mathsf{inc} \cong \mathrm{Id}$ we get the corresponding diagram for $\mathcal{J}(\mathsf{List}(A))$ as list object of $\mathcal{J}(A)$ in $\mathrm{Sh}(B_\phi)$. To prove uniqueness of $\mathcal{J}(\mathsf{rec}(\mathsf{inc}(b), g'))$ in making this diagram commute we use the fact that morphisms $r : B \times \mathcal{J}(\mathsf{List}(A)) \to Y$ are equivalent under the adjunction to morphisms $\mathsf{inc}(B) \times \mathsf{List}(A) \to \mathsf{inc}(Y)$, and commutativity of the two diagrams is preserved by that equivalence.

To summarize: for any object A of $\mathcal{A}$, $\mathcal{J}(\mathsf{List}(A))$ serves as a list object of $\mathcal{J}(A)$ in $\mathrm{Sh}(B_\phi)$. From this we deduce that $\mathrm{Sh}(B_\phi)$ has list objects and that $\mathcal{J}$ preserves them. □

Theorem 28. *Let $\mathcal{A}$ be an AU, let ϕ be a subobject of* 1 *in $\mathcal{A}$ and let $\mathrm{Sh}(B_\phi)$ and $\mathcal{J}$ be defined as above. Then $\mathrm{Sh}(B_\phi)$ is equivalent to $\mathcal{A}[\phi \le \bot]$.*

Proof. Here $\mathcal{A}[\phi \le \bot]$ is constructed according to Definition 16. We write $\mathcal{I} : \mathcal{A} \to \mathcal{A}[\phi \le \bot]$ for the canonical AU-functor. By Theorem 17 we can extend the AU-functor $\mathcal{J} : \mathcal{A} \to \mathrm{Sh}(B_\phi)$ to a strict AU-functor

$$\widetilde{\mathcal{J}} : \mathcal{A}[\phi \le \bot] \to \mathrm{Sh}(B_\phi)$$

because there is a morphism from $\mathcal{J}(\phi)$ to $\mathcal{J}(\bot)$.

Conversely we can define a functor

$$\gamma = \mathcal{I} \cdot \mathsf{inc} : \mathrm{Sh}(B_\phi) \to \mathcal{A}[\phi \le \bot].$$

These functors form an adjoint equivalence.

Immediately, $\widetilde{\mathcal{J}} \cdot \gamma = \widetilde{\mathcal{J}} \cdot \mathcal{I} \cdot \mathsf{inc} = \mathcal{J} \cdot \mathsf{inc}$ is naturally isomorphic to the identity. To see that $\gamma \cdot \widetilde{\mathcal{J}}$ is naturally isomorphic to the identity we need that $\gamma \cdot \widetilde{\mathcal{J}} \cdot \mathcal{I} \cong \gamma \cdot \mathcal{J} = \mathcal{I} \cdot \mathcal{V}_\phi$ is isomorphic to $\mathcal{I}$, which is obvious by construction of $\mathcal{V}_\phi$. This also shows that γ is an AU-functor, since the AU constructions in $\mathrm{Sh}(B_\phi)$ are calculated by applying $\mathcal{V}_\phi$ to the constructions in $\mathcal{A}$. After that the result follows from Theorem 17. □

4. Subspaces

In this section we examine subspaces and show (Theorem 42) how the open subspaces and closed subspaces generate a Boolean algebra of subspaces that is free over the distributive lattice of subobjects of 1. This result is wholly constructive, but has the important consequence that we can reuse some classical arguments as though we had a Boolean algebra of subobjects of 1 in an AU – see Section 5.

Our treatment is developed from that of [22], albeit with substantial changes: the underlying idea is that subspaces of an AS are analogous to inductively generated subtopologies of a formal topology. Interestingly, however, it is dualized, with meets and joins exchanged. This is because of the differing behaviours of two approaches to formal topology. For formal topologies in general, specified by a full cover relation, arbitrary *joins* of subspaces are easily seen to exist, but meets take more work insofar as they exist at all. On the other hand, for inductively generated formal topologies, specified by an axiom set, *meets* are easy, while finitary joins exist but take a little more work. [22] deals with general formal topologies, using joins of subspaces, and then treats inductively generated topologies as a special case. In the present AU setting we do not have a good account of general formal topologies and so are intrinsically in the inductively generated case.

Regarding closed subspaces, one should note that the classical property splits into various inequivalent formulations in constructive point-free topology. We follow the notion of closed subspace as complement of open subspace, using Boolean complementation in a lattice of point-free subspaces. The other notion is that a subspace is closed if it contains all its closure points. This is the notion developed constructively by Sambin in his Basic Picture ([17,16]) and also leads to definitions such as that of "weakly closed sublocale". The two notions are compared in [22].

For locales, a subspace (sublocale) can be understood as given by a family of pairs (ϕ_i, ψ_i) $(i \in I)$, where ϕ_i, ψ_i are subobjects of 1 in the topos of sheaves. These can be understood as extra relations $\phi_i \le \psi_i$ used for presenting the frame, and in terms of points they are extra constraints: a point x of the superlocale is in the sublocale iff, for every i for which ϕ_i is a neighbourhood of x, so too is ψ_i. This point of view is systematically taken in [22].

We can take a similar approach in arithmetic spaces. (Some other well known characterizations of sublocales from topos theory, for instance as nuclei on frames, do not adapt to the AU setting.) If ϕ, ψ are subobjects of 1 in the arithmetic universe $\mathcal{A}X$ then the subspace for $\phi \le \psi$ has AU $\mathcal{A}X[\phi \to \psi]$. We may write $X[\phi \le \psi]$ or $X[\phi \to \psi]$ for the corresponding AS.

However, the question arises as to what the indexing set I might be. If it is external, then the AU is got by adjoining morphisms for all its elements, obtaining $\mathcal{A}X[\phi_i \to \psi_i \ (i \in I)]$. On the other hand it could be internal in $\mathcal{A}X$, giving two subobjects U and V of I and the subspace AU presented as $\mathcal{A}X[U \to V \text{ over } I]$ (or $\mathcal{A}X[U \leq_I V]$). It is not clear to us what is going to be the right notion to adopt (and maybe it varies). For the present work we shall take the internal view, which is in line with the philosophy that the infinities one uses should be the ones that can be characterized internally. However, in our present applications I will be a finite cardinal, given by an external natural number, and so there is no essential distinction between the two views.

We can simplify these presentations. Consider the pullback square

$$\begin{array}{ccc} U \times_I V & \xrightarrow{q} & V \\ p \downarrow & & \downarrow f \\ U & \xrightarrow[e]{} & I \end{array}$$

If e and f are both monic then so are p and q, and a morphism $U \to V$ over I is equivalent to a morphism $U \to U \times_I V$ inverting p. Hence every subspace presented as $\mathcal{A}X[U \leq_I V]$ can equivalently be presented as $\mathcal{A}X[m^{-1}]$ for some monic m. To put it another way, in considering the presentations using U, I, V, we can without loss of generality take $I = U$ and we invert $U \hookleftarrow V$.

Definition 29. Let X be an arithmetic space. If m_1, m_2 are monics in $\mathcal{A}X$ then we say $m_1 \leq m_2$ if m_2 is invertible in $\mathcal{A}X[m_1^{-1}]$ – in other words, by Theorem 17, there is an AU-functor $\mathcal{A}X[m_2^{-1}] \to \mathcal{A}X[m_1^{-1}]$ under $\mathcal{A}X$, i.e. an AS-map $X[m_1^{-1}] \to X[m_2^{-1}]$ over the whole space X. This defines a preorder on the set of monics. We call a *subspace* of X an equivalence class of monics under $\leq$, and write $\mathrm{Subsp}(X)$ for the poset of subspaces. It is a $\wedge$-semilattice, with $m_1 \wedge m_2$ defined as the coproduct monic $m_1 + m_2$.

Note that if m_1 and m_2 have the same codomain, i.e. they are $U \hookleftarrow V_i$, then the subspace meet, got by inverting $U + U \hookleftarrow V_1 + V_2$, can equivalently be got using the subobject meet by inverting $U \hookleftarrow V_1 \wedge V_2$.

So far our knowledge of the structure of $\mathrm{Subsp}(X)$ is rather limited.

Lemma 30. *If $\mathcal{A}X$ is an AU with m_1 and m_2 two monics, then*

$$\mathcal{A}X[m_1^{-1}, m_2^{-1}] \simeq \mathcal{A}X[(m_1 + m_2)^{-1}].$$

Writing $\mathcal{I}_1 : \mathcal{A}X \to \mathcal{A}X[m_1^{-1}]$ for the canonical AU-functor, we also have

$$\mathcal{A}X[m_1^{-1}, m_2^{-1}] \simeq \mathcal{A}X[m_1^{-1}][\mathcal{I}_1(m_2)^{-1}].$$

Proof. We write $\mathcal{I}_2 : \mathcal{A}X[m_1^{-1}] \to \mathcal{A}X[m_1^{-1}][\mathcal{I}_1(m_2)^{-1}]$ and $\mathcal{I}_{12} : \mathcal{A}X \to \mathcal{A}X[m_1^{-1}, m_2^{-1}]$ for the other canonical AU-functors. Then we can define strict AU-functors

$$\mathcal{A}X[m_1^{-1}, m_2^{-1}] \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{A}X[m_1^{-1}][\mathcal{I}_1(m_2)^{-1}]$$

by $F \cdot \mathcal{I}_{12} = \mathcal{I}_2 \cdot \mathcal{I}_1$ and $G \cdot \mathcal{I}_2 = G'$ where the strict AU-functor $G' : \mathcal{A}X[m_1^{-1}] \to \mathcal{A}X[m_1^{-1}, m_2^{-1}]$ has $G' \cdot \mathcal{I}_1 = \mathcal{I}_{12}$. Then $G \cdot F = \mathrm{Id}_{\mathcal{A}X[m_1^{-1}, m_2^{-1}]}$ because $G \cdot F \cdot \mathcal{I}_{12} = G \cdot \mathcal{I}_2 \cdot \mathcal{I}_1 = G' \cdot \mathcal{I}_1 = \mathcal{I}_{12}$. Also (using Theorem 17) $F \cdot G' \cong \mathcal{I}_2$ because $F \cdot G' \cdot \mathcal{I}_1 = F \cdot \mathcal{I}_{12} = \mathcal{I}_2 \cdot \mathcal{I}_1$, and then $F \cdot G \cong \mathrm{Id}_{\mathcal{A}X[m_1^{-1}][\mathcal{I}_1(m_2)^{-1}]}$ because $F \cdot G \cdot \mathcal{I}_2 = F \cdot G' \cong \mathcal{I}_2$. $\square$

Lemma 31. *Let m, m_1 and m_2 be monics in $\mathcal{A}X$, and let $Y = X[m^{-1}]$ with canonical AU-functor $\mathcal{I} : \mathcal{A}X \to \mathcal{A}Y$. Then*

$$m + m_1 \leq m_2 \text{ over } X \text{ iff } \mathcal{I}(m_1) \leq \mathcal{I}(m_2) \text{ over } Y.$$

Proof. From Lemma 30 it is clear that m_2 is inverted in $\mathcal{A}X[m^{-1}, m_1^{-1}]$ iff $\mathcal{I}(m_2)$ is inverted in $\mathcal{A}Y[\mathcal{I}(m_1)^{-1}]$. $\square$

4.1. Boolean logic conservative over coherent logic

In this section we prove a well known conservativity result, but in a way that is adapted to our subsequent development in Section 4.2.

Theorem 32. *The category of Boolean algebras is a reflexive subcategory of the category of distributive lattices and the unit component is full, i.e. a distributive lattice L order embeds in its free Boolean algebra.*

We shall be applying this theorem in the case where the distributive lattice is the lattice of subobjects of an object in an arithmetic universe.

Throughout, we shall understand "lattice" and "semilattice" in a bounded sense: $\wedge$-semilattices have top $\top$, $\vee$-semilattices have bottom $\perp$, and lattices have both.

We write $\mathcal{F}X$ for the Kuratowski finite powerset of X, equivalently (under $\cup$) the free semilattice over X.

The free Boolean algebra generated from a distributive lattice can be characterized algebraically as follows:

Proposition 33. *Let L be a distributive lattice. Then the free Boolean algebra over it,* $\mathrm{BA}\langle L\ (\text{qua DL})\rangle$*, can be presented as a distributive lattice as*

$$\mathrm{DL}\langle L\ (\text{qua DL}), \neg\phi\ (\phi \in L) \mid \neg\phi \text{ a Boolean complement of } \phi\rangle$$

and as a meet semilattice as

$$\begin{aligned}\wedge\text{-}semi\langle L^{op} \times L\ (\text{qua poset}) \mid\ & (\phi, \psi_1) \wedge (\phi, \psi_2) \leq (\phi, \psi_1 \wedge \psi_2)\\ & (\phi_1, \psi) \wedge (\phi_2, \psi) \leq (\phi_1 \vee \phi_2, \psi)\\ & \top \leq (\phi, \psi)\ (\text{if } \phi \leq \psi)\\ & (\phi, \psi) \wedge (\psi, \chi) \leq (\phi, \chi)\rangle.\end{aligned}$$

Proof. The first part is well known: the set of complementable elements is a sublattice containing L and the elements $\neg\phi$ and so is the whole of the lattice. Hence the distributive lattice so presented is already a Boolean algebra, which must be freely generated by L.

For the second part, we first enlarge the generator set $L \cup \{\neg\phi \mid \phi \in L\}$ to include joins $\neg\phi \vee \psi$, giving a $\vee$-preserving function from $L^{op} \times L$. We then find that the distributive lattice as presented in the first part is isomorphic to that generated by $L^{op} \times L$ (qua $\vee$-semilattice) subject to the same relations as given in the second presentation. The appropriate coverage theorem [23] says that the same algebra can be presented as a $\wedge$-semilattice using the same generators and relations but with "qua poset" instead of "qua $\vee$-semilattice" — provided that the relations are join stable, which they are here. This is the $\wedge$-semilattice presentation given. □

We now give a concrete representation.

Proposition 34. *Let L be a distributive lattice. Then the free Boolean algebra* $\mathrm{BA}\langle L\ (\text{qua DL})\rangle$ *is order isomorphic to $\mathcal{F}(L \times L)/\leq$, where $S \leq T$ if for every $(t_1, t_2) \in T$, and for every decomposition $S = S_1 \cup S_2$ (with S_1 and S_2 both finite) we have*

$$t_1 \wedge \bigwedge_{(s_1,s_2)\in S_2} s_2 \leq t_2 \vee \bigvee_{(s_1,s_2)\in S_1} s_1.$$

Meet is given by union. (Note that the cases where S_1 and S_2 intersect give us no information, for then the inequality always holds.)

Proof. First, $\leq$ is a preorder. For reflexivity, with $(s_1', s_2') \in S$ and $S = S_1 \cup S_2$, we consider which of S_1 or S_2 contains (s_1', s_2'). For transitivity we use induction on the length of an enumeration[2] of T to show that if $S \leq T \leq U$ then $S \leq U$. For suppose $S = S_1 \cup S_2$ and $(u_1, u_2) \in U$. Let $\sigma_1 = \bigvee_{(s_1,s_2)\in S_1} s_1$ and $\sigma_2 = \bigwedge_{(s_1,s_2)\in S_2} s_2$, so we want $u_1 \wedge \sigma_2 \leq u_2 \vee \sigma_1$. If $T = \emptyset$ then from $T \leq U$ we deduce $u_1 \leq u_2$, which suffices. Now suppose $T = T' \cup \{(t_1, t_2)\}$. From $S \leq T$ we find $S \leq T'$ and $t_1 \wedge \sigma_2 \leq t_2 \vee \sigma_1$. From $T \leq U$ we see that for every decomposition $T' = T_1' \cup T_2'$ we get two decompositions $T = (T_1' \cup \{(t_1, t_2)\}) \cup T_2'$ and $T = T_1' \cup (T_2' \cup \{(t_1, t_2)\})$ giving

$$\begin{aligned} u_1 \wedge \bigwedge_{(t_1',t_2')\in T_2'} t_2' &\leq u_2 \vee \bigvee_{(t_1',t_2')\in T_1'} t_1' \vee t_1,\\ u_1 \wedge \bigwedge_{(t_1',t_2')\in T_2'} t_2' \wedge t_2 &\leq u_2 \vee \bigvee_{(t_1',t_2')\in T_1'} t_1' \end{aligned}$$

and so $T' \leq \{(u_1, u_2 \vee t_1), (u_1 \wedge t_2, u_2)\}$. It follows by induction that $S \leq \{(u_1, u_2 \vee t_1), (u_1 \wedge t_2, u_2)\}$ and hence

$$\begin{aligned} u_1 \wedge \sigma_2 &\leq u_2 \vee t_1 \vee \sigma_1,\\ u_1 \wedge t_2 \wedge \sigma_2 &\leq u_2 \vee \sigma_1. \end{aligned}$$

Combining these with $t_1 \wedge \sigma_2 \leq t_2 \vee \sigma_1$, we obtain

$$\begin{aligned} u_1 \wedge \sigma_2 &\leq (u_1 \wedge \sigma_2) \wedge (u_2 \vee \sigma_1 \vee t_1)\\ &\leq u_2 \vee \sigma_1 \vee ((u_1 \wedge \sigma_2) \wedge t_1)\\ &\leq u_2 \vee \sigma_1 \vee (u_1 \wedge \sigma_2 \wedge (t_2 \vee \sigma_1))\\ &\leq u_2 \vee \sigma_1 \vee (u_1 \wedge \sigma_2 \wedge t_2)\\ &\leq u_2 \vee \sigma_1. \end{aligned}$$

It is immediate from the definition of $\leq$ that union provides a meet for it, so $\mathcal{F}(L \times L)/\leq$ is a meet semilattice quotient of $\mathcal{F}(L \times L)$, as is (by Proposition 33) $\mathrm{BA}\langle L\ (\text{qua DL})\rangle$. It is easy to check that the homomorphism $\mathcal{F}(L \times L) \to \mathcal{F}(L \times L)/\leq$

2 Our finite subsets are *Kuratowski* finite, so each has a finite enumeration. Note that we cannot guarantee to eliminate duplicates, because L need not have decidable equality, so a finite set need not have a well defined cardinality as a natural number.

respects the meet semilattice relations in Proposition 33 and so factors via $\mathrm{BA}\langle L\,(\text{qua DL})\rangle$. Inversely, suppose $S \leq T$ in $\mathcal{F}(L \times L)$. Calculating their images in $\mathrm{BA}\langle L\,(\text{qua DL})\rangle$ we can use distributivity there and find

$$\begin{aligned}\bigwedge_{(s_1,s_2)\in S}(\neg s_1 \vee s_2) &= \bigvee_{S=S_1\cup S_2}\ \bigwedge_{(s_1,s_2)\in S_1}\neg s_1 \wedge \bigwedge_{(s_1,s_2)\in S_2} s_2\\ &= \bigvee_{S=S_1\cup S_2}\neg\left(\bigvee_{(s_1,s_2)\in S_1} s_1\right)\wedge \bigwedge_{(s_1,s_2)\in S_2} s_2\\ &\leq \bigwedge_{(t_1,t_2)\in T}(\neg t_1 \vee t_2)\end{aligned}$$

so the homomorphism $\mathcal{F}(L \times L) \to \mathrm{BA}\langle L\,(\text{qua DL})\rangle$ factors via $\mathcal{F}(L \times L)/\leq$. $\square$

We can now prove the second part of Theorem 32.

Corollary 35. *A distributive lattice L order embeds in its free Boolean algebra.*

Proof. $\phi \in L$ maps to $\{(\top, \phi)\}$ in $\mathcal{F}(L \times L)/\leq$. Suppose $\{(\top, \phi_1)\} \leq \{(\top, \phi_2)\}$. Taking $\{(\top, \phi_1)\} = \emptyset \cup \{(\top, \phi_1)\}$ we see $\top \wedge \phi_1 \leq \phi_2 \vee \bot$. $\square$

4.2. A Boolean algebra of subspaces

Our main result now (Theorem 42) is to show that the free Boolean algebra over $\mathrm{Sub}_{\mathcal{A}X}(1)$ (the distributive lattice of subobjects of 1 in $\mathcal{A}X$) order embeds in the $\wedge$-semilattice $\mathrm{Subsp}(X)$.

This, together with the results of Section 4.1, implies that one can use Boolean reasoning in terms of subspaces, and that it is conservative over the coherent reasoning with subobjects.

Definition 36. Let X be an AS, let $L = \mathrm{Sub}_{\mathcal{A}X}(1)$ and let $S = \{(\phi_i, \psi_i) \mid 1 \leq i \leq n\} \in \mathcal{F}(L \times L)$. We write $\sigma(S)$ for the subspace in X for the monic $\coprod_{i=1}^n \phi_i \hookleftarrow \coprod_{i=1}^n \phi_i \wedge \psi_i$. (This is well defined, since different enumerations of S give equivalent monics.) By definition σ is a $\wedge$-semilattice homomorphism.

Our aim now is to show that $\sigma(S) \leq \sigma(T)$ iff $S \leq T$ as in Proposition 34. In fact we do slightly more, since we show that σ, thus factoring as an embedding $\mathrm{BA}\langle L\,(\text{qua DL})\rangle \cong \mathcal{F}(L \times L)/\leq \to \mathrm{Subsp}(X)$, preserves the finite meets and joins of $\mathrm{BA}\langle L$ (qua DL)$\rangle$.

For the rest of this section, we fix an AS X and write L for $\mathrm{Sub}_{\mathcal{A}X}(1)$.

Proposition 37. *If $S \leq T$ in $\mathcal{F}(L \times L)$ then $\sigma(S) \leq \sigma(T)$.*

Proof. Combining Propositions 33 and 34 we obtain a $\wedge$-semilattice presentation for $\mathcal{F}(L \times L)/\leq$ as quotient of $\mathcal{F}(L \times L)$, so it suffices to check that σ respects the relations in Proposition 33. These are all clear. (The first was remarked on after Definition 29.) $\square$

The difficult part is the converse, essentially because we do not have a general concrete description of $\mathcal{A}X[m^{-1}]$. Nor do we have a general way to translate the condition $m_1 \leq m_2$, which is defined in terms external to $\mathcal{A}X$, into an explicit description internal there. However, we can use the representation results of Section 3 to gain some concrete knowledge for the open and closed subspaces and their finite meets and finite joins.

Definition 38. Let X be an AS, and let ϕ, ψ be subobjects of 1 in $\mathcal{A}X$. Then –

- $X[\top \leq \phi] = \sigma(\{(\top, \phi)\})$ is the *open* subspace for ϕ, written as ϕ;
- $X[\phi \leq \bot] = \sigma(\{(\phi, \bot)\})$ is the *closed* subspace for ϕ, written as $X - \phi$;
- $(X - \phi) \wedge \psi = \sigma(\{(\top, \psi), (\phi, \bot)\})$ is a *crescent* subspace, and
- $X[\phi \leq \psi] = \sigma(\{(\phi, \psi)\})$ a *cocrescent*.

(In Proposition 43 we shall see that the cocrescent is a join $(X - \phi) \vee \psi$.)

From Theorem 19 we see that for an open subspace, $\mathcal{A}X[\top \leq \phi] \cong \mathcal{A}X[1 \to \phi] \simeq \mathcal{A}X/\phi$, with AU-functor $\mathcal{A}X \to \mathcal{A}X/\phi$ given by $A \mapsto (\pi_2 : A \times \phi \to \phi)$.

We now exploit Theorem 28 to find information about $\mathcal{A}X[\phi \leq \bot]$.

Proposition 39. *Let X be an AS, and let ϕ, ψ be subobjects of 1 in $\mathcal{A}X$. Then for the crescent $(X - \phi) \wedge \psi$ and for any subspace $Z = X[U \leq V]$ given $U \hookleftarrow V$ in $\mathcal{A}X$, we have $(X - \phi) \wedge \psi \leq Z$ iff $U \times \psi \leq V \vee U \times \phi$ in $\mathcal{A}X$.*

Proof. Using Lemma 30, $(X - \phi) \wedge \psi$ is got by taking the closed subspace for $\phi \times \psi \hookrightarrow \psi$ in $\mathcal{A}X/\psi$, and by Lemma 25 we calculate that $U \leq V$ there iff $U \times \psi \leq V \vee U \times \phi$ in $\mathcal{A}X$. $\square$

For our development of a calculus of subspaces, we shall find it convenient to define an *action* of $\mathrm{BA}\langle L\,(\text{qua DL})\rangle$ on $\mathrm{Subsp}(X)$, $(Y, a) \mapsto Y \cdot a$ (recall that L is $\mathrm{Sub}_{\mathcal{A}X}(1)$).

Definition 40. Let $Y = X[U \leq V]$ (where already we have $V \hookrightarrow U$ in $\mathcal{A}X$) be a subspace of X. We define for $S \in \mathcal{F}(L \times L)$

$$Y \cdot \bigwedge \{\neg\phi \vee \psi \mid (\phi, \psi) \in S\} \equiv \bigwedge \{X[U \times \phi \leq V \times \phi \vee U \times (\phi \wedge \psi)] \mid (\phi, \psi) \in S\}.$$

To show that this definition is good, and preserves finite meets, it suffices to check that the function $L \times L \to \mathrm{Subsp}(X)$, mapping (ϕ, ψ) to $X[U \times \phi \leq V \times \phi \vee U \times (\phi \wedge \psi)]$, respects the relations of the semilattice presentation in Proposition 33. This is straightforward.

Note that the definition uses the presentation of Y as $X[U \leq V]$. Presentation independence will follow from part 2 of Theorem 42.

We also define the bottom subspace $\perp$ as $X[1 \leq 0]$. Then we have $\sigma(S) = \perp \cdot \bigwedge \{\neg\phi \vee \psi \mid (\phi, \psi) \in S\}$. Also, the open ψ is $\perp \cdot \psi$, the closed $X - \phi$ is $\perp \cdot \neg\phi$, the crescent $(X - \phi) \wedge \psi$ is $\perp \cdot (\neg\phi \wedge \psi)$ and the cocrescent $X[\phi \leq \psi]$ is $\perp \cdot (\neg\phi \vee \psi)$.

Lemma 41. *If ϕ and ψ are subobjects of* 1*, and Y and Z are subspaces, then*

$$Y \wedge \perp \cdot (\neg\phi \wedge \psi) \leq Z \iff Y \leq Z \cdot (\phi \vee \neg\psi).$$

Proof. Proposition 39 proves this in the case $Y = \top$. The full generality can be proved by working over Y and using Lemma 31. □

Theorem 42. 1. *The action $Z \cdot a$ preserves finite meets in both arguments.*
2. *For any $a \in \mathrm{BA}\langle L\,(qua\ DL)\rangle$, the function $Z \mapsto Z \cdot a$ is right adjoint to the function $Y \mapsto Y \wedge \perp \cdot \neg a$.*
3. *$Z \cdot \perp = Z$ and $Z \cdot (a \vee b) = (Z \cdot a) \cdot b$.*
4. *The function $a \mapsto \perp \cdot a$ is an order isomorphism from $\mathrm{BA}\langle L\,(qua\ DL)\rangle$ to a sublattice of $\mathrm{Subsp}(X)$. Meet in $\mathrm{Subsp}(X)$ distributes over joins of subspaces of the form $\perp \cdot a$.*

Proof. We prove the parts out of order.

One half of (1) (that $Z \cdot (-)$ preserves finite meets) is by definition. Of course, this also implies that $Z \cdot (-)$ is monotone. The first part of (3) (that $Z \cdot \perp = Z$) is obvious.

Next, note that $\perp \cdot a \leq Z \cdot a$ for all Z $(= X[U \leq V]$, say) and a. Since a can be expressed as a meet of cocrescents, it suffices to consider $a = \neg\phi \vee \psi$, and then $\perp \cdot a = X[\phi \leq \phi \wedge \psi]$. Clearly if we impose this then we also have $U \times \phi \leq V \times \phi \vee U \times (\phi \wedge \psi)$, the defining relation for $Z \cdot a$.

We now prove (2): we must show that $Y \wedge \perp \cdot \neg a \leq Z$ iff $Y \leq Z \cdot a$. For the $\Leftarrow$ direction, we have

$$Y \wedge \perp \cdot \neg a \leq Z \cdot a \wedge Z \cdot \neg a = Z \cdot (a \wedge \neg a) = Z \cdot \perp = Z.$$

For $\Rightarrow$, take $a = \bigwedge_{i=1}^n (\phi_i \vee \neg\psi_i)$ so $\neg a = \bigvee_{i=1}^n (\neg\phi_i \wedge \psi_i)$. Hence if $Y \wedge \perp \cdot \neg a \leq Z$ then we have $Y \wedge \perp \cdot (\neg\phi_i \wedge \psi_i) \leq Z$ for all i, i.e. (by Lemma 41) $Y \leq Z \cdot (\phi_i \vee \neg\psi_i)$ for all i, i.e. $Y \leq \bigwedge_{i=1}^n Z \cdot (\phi_i \vee \neg\psi_i) = Z \cdot \bigwedge_{i=1}^n (\phi_i \vee \neg\psi_i) = Z \cdot a$.

From the adjunction we can deduce that $Z \mapsto Z \cdot a$ is monotone (which we did not know initially), and preserves all existing meets in Z. Thus this also completes the proof of (1).

(3) (second part): By applying (2) and preservation of meets

$$\begin{aligned} Y \leq Z \cdot (a \vee b) &\iff Y \wedge \perp \cdot \neg b \wedge \perp \cdot \neg a = Y \wedge \perp \cdot \neg(a \vee b) \leq Z \\ &\iff Y \wedge \perp \cdot \neg b \leq Z \cdot a \iff Y \leq (Z \cdot a) \cdot b. \end{aligned}$$

(4): Taking $a = \bigvee_{i=1}^n (\neg\phi_i \wedge \psi_i)$, our earlier discussion showed that $Y \wedge \perp \cdot a \leq Z$ iff $Y \wedge \perp \cdot (\neg\phi_i \wedge \psi_i) \leq Z$ for all i. In the case $Y = \top$ this shows that $a \mapsto \perp \cdot a$ preserves finite joins; and for general Y it shows that meet distributes over those joins. (Note that we do not know whether the whole of $\mathrm{Subsp}(X)$ is a lattice.)

We now show that the monotone function $a \mapsto \perp \cdot a$ is an order embedding. Suppose that $\perp \cdot a \leq \perp \cdot b$, i.e. $\top \leq \perp \cdot (\neg a \vee b)$ using parts (2) and (3). We show that $\neg a \vee b = \top$, for then $a \leq b$. Let $c = \neg a \vee b = \bigwedge_{i=1}^n (\neg\phi_i \vee \psi_i)$, so $\top \leq \perp \cdot (\neg\phi_i \vee \psi_i) = X[\phi_i \leq \psi_i]$ for all i. It follows that $\phi_i \leq \psi_i$ for all i, so $\neg\phi_i \vee \psi_i = \top$ and $c = \top$. □

Corollary 43. *Let X be an AS, and let ϕ, ψ be subobjects of* 1 *in $\mathcal{A}X$. Then $X[\phi \leq \psi]$ is a least upper bound $(X - \phi) \vee \psi$.*

Proof.

$$X[\phi \leq \psi] = \perp \cdot (\neg\phi \vee \psi) = \perp \cdot \neg\phi \vee \perp \cdot \psi = (X - \phi) \vee \psi \quad \square$$

Corollary 44. *Let X be an AS, and let ϕ be a subobject of* 1 *in $\mathcal{A}X$. Then amongst subspaces of X, $X - \phi$ is a complement of ϕ.*

Proof. From Corollary 43 we see that $(X - \phi) \vee \phi$ exists and is $X[\phi \leq \phi]$, which is the whole of X. In $(X - \phi) \wedge \phi$ we have $\top \leq \phi \leq \perp$, which gives the empty space. □

Of our two structure theorems in Section 3, the first, Theorem 19, works not only for AUs but for a wide range of categorical structures, including those involving stable exponentials like a locally cartesian closed category. By contrast the second, Theorem 28, is more restricted. Our use of sheaves for $\mathcal{A}[\phi \leq \perp]$ for an AU matches that known in topos theory, and the functor $\mathcal{A} \to \mathcal{A}[\phi \leq \perp]$ corresponds to the inverse image functor of the geometric morphism that is the topos inclusion. By (e.g. [7, C3.1.5]) we know that that inverse image functor preserves exponentials only if the inclusion is open, which of course is not in general true of our closed inclusions.

This should not come as a surprise. Suppose, for example, we had a similar result for Heyting pretoposes. Then the preservation of exponentials would imply a conservativity theorem of the classical logic of subspaces over the Heyting pretopos one, which would imply that any Heyting pretopos is a Boolean one.

5. An induction principle

We now give an example that, in fact, was the original motivation for the work in this paper. Suppose in an AU $\mathcal{A}X$ we have a subobject ϕ of the natural numbers object N, in other words a predicate $\phi(n)$ where $n : N$. There is an obvious induction principle arising from the fact that N is an initial induction algebra. (An *induction algebra* is a set – or, more generally, an object of a category – equipped with a constant and a unary operator.) If we have both the base case $\phi(0)$ and an induction step $(\forall n)(\phi(n) \to \phi(n+1))$, then ϕ as subobject of N is a sub-(induction algebra): it contains 0 and is closed under the successor operation s. It follows by the initiality property of N that there is a unique induction algebra homomorphism $f : N \to \phi$ and with a little more reasoning one sees that it is inverse to the inclusion $\phi \hookrightarrow N$, which is therefore an isomorphism. In other words, we have $(\forall n)\phi(n)$.

Now suppose we have *two* predicates $\phi(n)$ and $\psi(n)$ and we wish to use induction to show $(\forall n)(\phi(n) \to \psi(n))$. If $\mathcal{A}X$ were locally cartesian closed, then we could form an implication formula $\phi(n) \to \psi(n)$ as subobject of N and use the same argument as above for ϕ. However, in general an AU is *not* locally cartesian closed. Surprisingly, we get some clues from classical logic. The formula $\phi(n) \to \psi(n)$ is classically equivalent to $\neg\phi(n) \vee \psi(n)$, so classically our induction step is

$$(\forall n)((\neg\phi(n) \vee \psi(n)) \to (\neg\phi(n+1) \vee \psi(n+1))),$$

which reduces to

$$(\forall n)(\phi(n+1) \to \phi(n) \vee \psi(n+1)) \text{ and}$$
$$(\forall n)(\phi(n+1) \wedge \psi(n) \to \psi(n+1)).$$

These are two sequents that can be interpreted in an AU. Of course, the classical reasoning cannot apply directly to subobjects in the AU. However, we shall show how to exploit the fact that for sub*spaces* we have a Boolean algebra. There we can apply the classical reasoning, and it turns out that the sequents just described are a satisfactory description of the induction step.

Let us examine in more detail what induction principle we might hope for. First, we want a base case $\phi(0) \to \psi(0)$. Categorically, it appears like this. $\phi(0)$ is the subobject $0^*\phi$ of 1 got by pulling $\phi \hookrightarrow N$ back along the constant $0 : 1 \to N$:

$$\begin{array}{ccc} \phi(0) & \longrightarrow & \phi \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{0} & N \end{array}$$

$\psi(0)$ is similar, and then the base case is the condition that there is a morphism from $\phi(0) \to \psi(0)$.

Next, we want an induction step $(\forall n)((\phi(n) \to \psi(n)) \Rightarrow (\phi(n+1) \to \psi(n+1)))$. We have to take care to explain this correctly. Note that the induction hypothesis $\phi(n) \to \psi(n)$ is not a formula in our arithmetic logic – because AUs are not locally cartesian closed. But neither is it a sequent or judgement $n : N, \phi(n) \vdash \psi(n)$, for that would be implicitly universally quantified as $(\forall n)(\phi(n) \to \psi(n))$, the very thing we are trying to prove. The induction hypothesis amounts to a context in which n has been fixed (generically), and $\phi(n) \to \psi(n)$ has been hypothesized. In other words, it is a context corresponding to an AU $\mathcal{A}X[n : N][\phi(n) \to \psi(n)]$. (This is a slight abuse of notation – "ϕ" and "ψ" here denote the images of ϕ and ψ in $\mathcal{A}X[n : N]$.) The induction step is then a construction that shows how in this AU we also have $\phi(n+1) \to \psi(n+1)$, and the induction principle (which we shall prove) says that if we have both the base case and the induction step then, back in $\mathcal{A}X$, we have already $(\forall n)(\phi(n) \to \psi(n))$ – in other words, a morphism $\phi \to \psi$ over N.

The induction hypothesis is the subspace $X[n : N][\phi(n) \leq \psi(n)]$ of $X[n : N]$, and from this point of view the induction step is to show that it is less than the subspace $X[n : N][\phi(n+1) \leq \psi(n+1)]$: in other words, by Corollary 43

$$(X[n : N] - \phi(n)) \vee \psi(n) \leq (X[n : N] - \phi(n+1)) \vee \psi(n+1).$$

That is equivalent to two conditions on $X[n : N]$,

$$X[n : N] - \phi(n) \leq (X[n : N] - \phi(n+1)) \vee \psi(n+1)$$
$$\psi(n) \leq (X[n : N] - \phi(n+1)) \vee \psi(n+1)$$

and those are equivalent, by Theorem 42 and Corollary 35, to

$$\phi(n+1) \leq \phi(n) \vee \psi(n+1) \tag{IS1}$$

$$\phi(n+1) \wedge \psi(n) \leq \psi(n+1). \tag{IS2}$$

These two conditions, in which $n : 1 \to N$ is the generic natural number in $\mathcal{A}X[n : N]$, are the induction step rephrased as internal properties of $\mathcal{A}X[n : N]$. However (Theorem 19), we have concrete knowledge of $\mathcal{A}X[n : N]$ as equivalent to the slice category $\mathcal{A}X/N$, and this enables us to rephrase the conditions again as internal properties of $\mathcal{A}X$. In $\mathcal{A}X/N$ we have that 1 is the morphism $\mathrm{Id} : N \to N$ and N is the projection $\pi_2 : N \times N \to N$. The generic n is the diagonal morphism $\Delta : N \to N \times N$. The predicate ϕ becomes the projection $\pi_2 : \phi \times N \to N$. To calculate the truth value (i.e. subobject of 1) $\phi(n) = n^*\phi$, we calculate this pullback:

$$\begin{array}{ccc} \phi(n) & \longrightarrow & \phi \times N \\ \downarrow & & \downarrow \\ N & \xrightarrow{\Delta} & N \times N \end{array}$$

It can be calculated using generalized elements as comprising the triples (m, m', m'') such that $\phi(m')$ and $(m, m) = (m', m'')$; and this is just ϕ. Hence $\phi(n)$ as object of the slice is given by the morphism $\phi \to N$. Next, $\phi(n+1)$ is got as the pullback

$$\begin{array}{ccccc} \phi(n+1) & & \longrightarrow & & \phi \times N \\ \downarrow & & & & \downarrow \\ N & \xrightarrow{\Delta} & N \times N & \xrightarrow{s \times N} & N \times N \end{array}$$

and by similar reasoning we see that this is the pullback $s^*\phi$,

$$\begin{array}{ccc} \phi(n+1) & \longrightarrow & \phi \\ \downarrow & & \downarrow \\ N & \xrightarrow{s} & N \end{array}$$

The definitions of $\psi(n)$ and $\psi(n+1)$ are, of course, similar. Thus, when $\phi(n)$ etc. are defined this way in $\mathcal{A}X$, we see that the induction step is equivalent to Conditions (IS1) and (IS2) in $\mathcal{A}X$.

We have now reduced the induction principle to a result about the internal structure of $\mathcal{A}X$, with no reference to AUs presented over it.

Lemma 45. *Let X be an AS and let ϕ and ψ be two subobjects of N in $\mathcal{A}X$. As above, we shall write $\phi(n)$ and $\psi(n)$ for ϕ and ψ, and $\phi(n+1)$ and $\psi(n+1)$ for their pullbacks along $s : N \to N$. If we have $\phi(0) \leq \psi(0)$ and Conditions (IS1) and (IS2), then we also have $\phi \leq \psi$.*

Proof. Define $A(k)$ $(k : N)$ as the subobject of N comprising those j for which $j \leq k$ and $\phi(j), \ldots, \phi(k)$.

In the internal language of an arithmetic universe, the subobject $A(k)$ can be represented as the embedding in N whose domain is

$$\{ j \in N \mid \exists_{l \in List(\{x \in N \mid \phi(x)\})} \overline{\pi_1}(l) =_{List(N)} [j, \ldots, k] \,\&\, j \leq k \}$$

where $\overline{\pi_1}$ is the lifting of the first projection on lists and $[j, \ldots, k]$ is the list of numbers from j to k.

We define recursively a function $f_k : A(k) \to \{x \in N \mid x = k \,\&\, \psi(k)\}$ as follows, with $j + k$ as recursion variant. Of course, the value of $f_k(j)$ will always be k with a proof that $\psi(k)$ holds.

If $j = k = 0$, then we have $\phi(0)$ and from the base case we deduce $\psi(0)$ and can take $f_0(0) = 0$.

If $j = k > 0$, we have $\phi(j)$. From condition (IS1) we deduce $\phi(j-1) \vee \psi(k)$. In the latter case we define $f_k(j) = k$, and in the former we can recursively define $f_k(j) = f_k(j-1)$.

If $j < k$, we have $\phi(k)$ and recursively calculating $f_{k-1}(j)$ gives us $\psi(k-1)$. Now condition (IS2) gives us $\psi(k)$. $\square$

We can summarize the above discussion in our induction principle.

Theorem 46 (*Principle of Sequent Induction*)**.** *Let X be an AS, and let ϕ and ψ be subobjects of N in $\mathcal{A}X$. Suppose we have the following two conditions.*

1. *(Base case) Over X, we have $\phi(0) \leq \psi(0)$.*
2. *(Induction step) Over $X[n : N][\phi(n) \leq \psi(n)]$ (this context is the* induction hypothesis*) we also have $\phi(n+1) \leq \psi(n+1)$.*

Then $\phi \leq \psi$ holds over X.

Remark 47. With the same technique we can prove an induction principle for list-objects analogous to that of natural numbers.

We can in fact prove the induction principle (over N) for arbitrary formulae corresponding to finite conjunctions of implications $\bigwedge_i(\phi_i(n) \to \psi_i(n))$. We might try to prove this by separate inductions, one for each $\phi_i(n) \to \psi_i(n)$, but the next Theorem tells us that we can assume *all* the conditions $\phi_i(n) \to \psi_i(n)$ as induction hypotheses when trying to prove $\phi_i(n+1) \to \psi_i(n+1)$.

Theorem 48. *Let X be an AS, and for $1 \leq i \leq r$ let ϕ_i and ψ_i be subobjects of N in $\mathcal{A}X$. Suppose we have the following two conditions.*

1. *(Base case) Over X, we have $\phi_i(0) \leq \psi_i(0)$ for every i.*
2. *(Induction step) Over $X[n : N][\phi_i(n) \leq \psi_i(n)$ (all i)$]$ we have $\phi_i(n+1) \leq \psi_i(n+1)$ for every i.*

Then $\phi_i \leq \psi_i$ holds over X for every i.

Proof. We sketch the proof, which is similar to that of Lemma 45 but more complicated. By the calculus of subspaces, we can redistribute the induction hypothesis over $X[n : N]$ as

$$\bigwedge_{i=1}^{r}(\neg\phi_i(n) \vee \psi_i(n)) = \bigvee_{\{1,\ldots,r\}=I+J} \left(\neg \bigvee_{j \in I} \phi_j(n) \wedge \bigwedge_{j \in J} \psi_j(n) \right)$$

where $+$ denotes disjoint union. Hence the induction step says that for every (I, J) and every i we have

$$\neg \bigvee_{j\in I} \phi_j(n) \wedge \bigwedge_{j\in J} \psi_j(n) \le \neg\phi_i(n+1) \vee \psi_i(n+1),$$

i.e.

$$\phi_i(n+1) \wedge \bigwedge_{j\in J} \psi_j(n) \le \psi_i(n+1) \vee \bigvee_{j\in I} \phi_j(n).$$

By conservativity, the corresponding condition in $\mathcal{A}X$ holds. We must show that, together with the base case, it implies the conclusion. If $1 \le i \le r$ and $k \in N$, define $A_i(k)$ as a set of finite subsets of $\{1, \ldots, r\}$, that is $A_i(k) \subseteq \mathcal{F}\{1, \ldots, r\}$, by

$$A_i(0) = \{\emptyset \mid \phi_i(0)\}$$
$$A_i(k+1) = \{F \mid \phi_i(k+1) \wedge \bigwedge_{j\in F} \phi_j(k)\}.$$

We recursively define functions $f_i^k : A_i(k) \to \{x \in N \mid x = k \,\&\, \psi_i(k)\}$. For $f_i^0(\emptyset)$, the definition is immediate from our base case $\phi_i(0) \le \psi_i(0)$. For $k+1$ we define $f_i^{k+1}(F)$ as follows. From $F \in A_i(k+1)$ we have $\emptyset \in A_j(k)$ for all $j \in F$. By recursion on k, from $f_j^k(\emptyset)$ we deduce $\psi_j(k)$ for all $j \in F$ and hence $\phi_i(k+1) \wedge \bigwedge_{j\in F} \psi_j(k)$, and then our induction step (with $I = F, J = \{1, \ldots, r\} - F$) gives us either $\psi_i(k+1)$, as required, or $\phi_j(k)$ for some $j \notin F$. Recursing on $|\{1, \ldots, r\} - F|$, we can use a recursive call to $f_i^{k+1}(F \cup \{j\})$.

From this we can deduce $\phi_i(k) \le \psi_i(k)$ for all i and k: for if we have $\phi_i(k)$ then we can use $f_i^k(\emptyset)$. □

5.1. Application: locatedness of Dedekind sections

Corresponding to the localic form of the real line (see, e.g., [5]) there is a propositional geometric theory whose models are real numbers. However, it is even more transparent to express it as a predicate theory of Dedekind sections. It uses the rationals as a sort, but since the set of rationals can be constructed geometrically out of nothing the theory is essentially propositional. This is discussed in [21]. In this form, with no infinitary disjunctions, the theory – including the construction of $\mathbb{Q}$ – can be modelled in AUs. Thus the finitary algebra of AUs deals with countably infinitary disjunctions in the logic. The signature has two unary predicates L and R on the rationals, so a model comprises two subsets L and R of $\mathbb{Q}$. They are disjoint, and both inhabited; and L is rounded lower and R rounded upper. Those conditions can be expressed as follows.

$$\begin{aligned} \top &\longrightarrow (\exists q : \mathbb{Q})L(q) \\ (\forall q : \mathbb{Q})(L(q) &\longleftrightarrow (\exists q' : \mathbb{Q})q < q' \wedge L(q')) \\ \top &\longrightarrow (\exists r : \mathbb{Q})R(r) \\ (\forall r : \mathbb{Q})(R(r) &\longleftrightarrow (\exists r' : \mathbb{Q})r > r' \wedge R(r')) \\ (\forall q : \mathbb{Q})(L(q) \wedge R(q) &\longrightarrow \bot) \end{aligned}$$

(Note that from these we can deduce that if $L(q)$ and $R(r)$ then $q < r$.)

There is a further "locatedness" condition. As expressed in [5], it says that L and R come arbitrarily close:

$$(\forall \varepsilon : \mathbb{Q})(\varepsilon > 0 \to (\exists q, r : \mathbb{Q})L(q) \wedge R(r) \wedge r - q < \varepsilon) \tag{1}$$

or, alternatively,

$$(\forall q, r : \mathbb{Q})(q < r \to L(q) \vee R(r)). \tag{2}$$

All these are compatible with the type theory for AUs, and so syntactic categories $\mathcal{A}\mathbb{R}$ can be constructed for them. However, the question arises as to whether the two conditions (1) and (2) are still equivalent when one works with AUs, for the proof that the second implies the first is non-trivial. One uses induction on n to prove a lemma that, given q, r and ε with $L(q)$, $R(r)$ and $\varepsilon > 0$, then

$$r - q < 2^n\varepsilon \to (\exists q', r' : \mathbb{Q})L(q') \wedge R(r') \wedge r' - q' < \varepsilon.$$

The base case, $r - q < \varepsilon$, is immediate. Now suppose it is true for n, and $r - q < 2^{n+1}\varepsilon$. Define $s_i = q + i(r - q)/4$ $(0 \le i \le 4)$, so $s_0 = q$ and $s_4 = r$, so we already have $L(s_0)$ and $R(s_4)$. Applying condition (2) twice, we have both $L(s_1) \vee R(s_2)$ and $L(s_2) \vee R(s_3)$, which implies $R(s_2) \vee L(s_2) \vee (L(s_1) \wedge R(s_3))$. For the three disjuncts respectively we can replace (q, r) by (s_0, s_2), (s_2, s_4) or (s_1, s_3), halving the difference $r - q$, and apply induction.

To use this to show that (2) implies (1), suppose we are given $\varepsilon > 0$. We can find some q and r with $L(q)$ and $R(r)$, and then some n with $r - q < 2^n\varepsilon$. Then the lemma gives us the conclusion we want.

In toposes, with their function types, the inductively proved implication in the lemma is not a problem. For AUs we must use Theorem 46. Let us take $\mathbb{R}$ now to mean the AS defined for the theory of reals with (2), and let $\mathbb{R}[\varepsilon > 0]$ be got by adjoining a positive rational ε. Take $\phi(n)$ to be the formula $(\exists q, r : \mathbb{Q})(L(q) \wedge R(r) \wedge r - q < 2^n\varepsilon)$, and $\psi(n)$ the formula $(\exists q', r' : \mathbb{Q})(L(q') \wedge R(r') \wedge r' - q' < \varepsilon)$ (which in fact does not use n). The induction step described above is just what is needed to show, over $\mathbb{R}[\varepsilon > 0][n : \mathbb{N}][\phi(n) \to \psi(n)]$, that we have $\phi(n+1) \to \psi(n+1)$, and it follows (given also the base case) that over $\mathbb{R}[\varepsilon > 0]$ we have $(\forall n : \mathbb{N})(\phi(n) \to \psi(n))$. This allows us to prove over $\mathbb{R}[\varepsilon > 0]$ that we have $(\exists q, r : \mathbb{Q})(L(q) \wedge R(r) \wedge r - q < \varepsilon)$, and hence that condition (1) is valid over $\mathbb{R}$.

5.2. *A conjecture on induction algebras in* **AS**

Recall that an induction algebra is an object A equipped with a constant a_0 and an endomap t. In an AU, there is an initial induction algebra, namely $(N, 0, s)$. Now suppose X is an AS. Any object A in $\mathcal{A}X$ gives a discrete space $X[a : A]$ over X, an object of $\mathbf{AS}/X$. (This slice of **AS** is not to be confused with the slice $\mathcal{A}X/A \simeq \mathcal{A}X[a : A]$, the AU of the discrete space for A over X.) If (A, a_0, t) is an induction algebra in $\mathcal{A}X$, then $X[a : A]$ is also an induction algebra in $\mathbf{AS}/X$. The constant $X \to X[a : A]$ is given by a homomorphism $\mathcal{A}X[a : A] \to \mathcal{A}X, a \mapsto a_0$, and the unary operation $X[a : A] \to X[a : A]$ by the homomorphism $\mathcal{A}X[a : A] \to \mathcal{A}X[a : A], a \longmapsto t(a)$.

Conjecture 49. *$X[n : N]$ is initial amongst the induction algebras in $\mathbf{AS}/X$.*

We are far from proving this in general, or even formulating it accurately. (The uniqueness part of the universal characterization of initiality will require care in the handling of strictness, and of uniqueness up to isomorphism.) Nonetheless, our induction principle Theorem 46 is already an example of it.

Suppose, as in Theorem 46, we have ϕ and ψ subobjects of N in $\mathcal{A}X$. The base case $\phi(0) \le \psi(0)$ then states that the map $X \to X[n : N]$ given by $n \longmapsto 0$ extends to a map $X \to X[n : N][\phi(n) \le \psi(n)]$. Next, the induction step states that the endomap of $X[n : N]$ given by $n \longmapsto n + 1$ restricts to an endomap of $X[n : N][\phi(n) \le \psi(n)]$ over X. In other words, the premisses state that $X[n : N][\phi(n) \le \psi(n)]$ is an induction subalgebra of $X[n : N]$ in $\mathbf{AS}/X$. The conjecture would then tell us that there is a map $X[n : N] \to X[n : N][\phi(n) \le \psi(n)]$ that is a homomorphism of induction algebras. This tells us that the condition $\phi(n) \le \psi(n)$ already holds in $X[n : N]$, and hence (using our concrete structural knowledge that $\mathcal{A}X[n : N] \simeq \mathcal{A}X/N$) that $\phi \le \psi$ in X.

We propose our conjecture as a general principle of induction or recursion over the natural numbers in our AU setting.

6. Conclusions

Our investigation arose out of a phenomenon seen in the Vickers geometrization programme and discussed explicitly in [20]. Although the geometric reasoning was expressed in terms of topos theory, it was explicitly intended also to be applicable in arithmetic universes. However, in certain places (such as the question of locatedness of Dedekind sections, described in Section 5.1) it clearly was not, and an argument was used that it was acceptable – in topos theory – as part of the geometric reasoning to use a non-geometric but topos-valid proof so long as the result could be stated geometrically.

The results in this paper are a first step towards filling that gap between topos-valid geometric reasoning and AU-valid ("arithmetic") reasoning. We have proved the problematic induction principle and others, and also established a significant part of the localic technology of complementable subspaces as well as proving some particular cases of the structure theorems that are taken for granted in classifying toposes.

Our methods are constructive throughout. In fact, we conjecture that, because of the way they use universal algebra, they are themselves valid in the sense of arithmetic reasoning.

Clearly the results here are only a start in the programme of creating an AU analogue of toposes as generalized spaces. We have various conjectures on how the work might proceed.

- Theorem 28 was based on an analysis of a closed embedding as a Stone map, corresponding to an internal Boolean algebra, and the algebraic notion of "finitary sheaf" as set out in [15]. We conjecture that a similar approach would work for general Stone locales or even spectral locales: that if L is a distributive lattice internal in an AU $\mathcal{A}$, then the category of finitary sheaves over L is an AU equivalent to that got by freely adjoining to $\mathcal{A}$ a prime filter of L.
- On the analogy with sublocales (see in particular [22]) we would conjecture that the meet semilattice of subspaces (Definition 29) is a distributive lattice, with binary join $(U_1 \hookleftarrow\!\!\!\!\hookrightarrow V_1) \vee (U_2 \hookleftarrow\!\!\!\!\hookrightarrow V_2)$ given by the monic $(U_1 \times U_2 \hookleftarrow\!\!\!\!\hookrightarrow V_1 \times U_2 \vee U_1 \times V_2)$.
- The Boolean algebra of subspaces that we have identified provides a technical tool for studying how one might embed an AU in a Boolean pretopos, or even in a Boolean AU, by adapting techniques from [14].

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