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Novel applications of Discrete Mereotopology to Mathematical Morphology

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Abstract

This paper shows how the Discrete Mereotopology notions of adjacency and neighbourhood between regions can be exploited through Mathematical Morphology to accept or reject changes resulting from traditional morphological operations such as closing and opening. This leads to a set of six morphological operations (here referred to generically as *minimal opening* and *minimal closing*) where minimal changes fulfil specific spatial constraints. We also present an algorithm to compute the RCC5D and RCC8D relation sets across multiple regions resulting in a performance improvement of over three orders of magnitude over our previously published algorithm for Discrete Mereotopology.

Keywords: mathematical morphology, discrete mereotopology, image processing, spatial reasoning

1. Introduction

This paper focuses on the processing of spatial relationships between discrete regions using Mathematical Morphology (MM). There has been a long-standing interest in formal definitions of adjacency and containment between image regions, as those types of relations can form a basis for model building in image contents retrieval and analysis. This has applications to problems where the description of hierarchical structure is important, for instance, in biological imaging numerous problems revolve around the characterisation of relations of diverse nature, for instance molecules in organelles, organelles in cells, cells in tissue compartments and tissues in organs. The subject has

11 been approached from a variety of points of view, including digital polygon
 12 geometry [1], digital topology [2, 3], hierarchical modelling [4] and connected
 13 filtering operators [5, 6, 7].

14 Bloch [8, 9, 10, 11] has provided an extensive body of work on spatial re-
 15 lations in computer vision and identified ways to symbolically and program-
 16 matically harness and represent the inherent imprecision arising from image
 17 formation, post-processing, perception and the semantics related to certain
 18 spatial relationships sought. In [11] Bloch shows how MMI can function as
 19 unifying framework for spatial knowledge representation and provides con-
 20 nections to formal logics, in particular raising the possibility of implementing
 21 Region Connection Calculus (RCC) [12] operators (as well as providing a MM
 22 definition for the RCC's *tangential proper part* (TPP) relation). In [9], it is
 23 proposed to construct modal logics using MM from the notion of adjunction
 24 [13] to define modal operators that can be utilised for symbolic representa-
 25 tion and interpretation of spatial relationships. In [14] the notion of fuzzy
 26 adjacency between image objects was investigated and formally defined so
 27 the concept of adjacency can be extended (e.g. using fuzzy MM formulations) to
 28 accommodate *degrees of adjacency* by means of *admissible transformations*
 29 that lead to strict adjacency and thus allow consistent representations and
 30 the management of imprecision mentioned earlier.

31 Research has also focused on applying MM and spatial reasoning to discrete
 32 spaces with the purpose of applying spatial reasoning to digital images.
 33 In this context, Galton [2, 3] introduced the notion of Discrete Mereotopology
 34 (DM) where he develops various mereotopological concepts for discrete
 35 spaces. Our work in [15, 16] shows that a subset of DM functions (closure
 36 and interior) map directly to the MM dilation and erosion operators [17]
 37 respectively, commonly used in image processing. In [2] that mapping was
 38 exploited to implement the full spatial relation set given by the RCC5D and
 39 RCC8D logics [12] in terms of MM. Briefly, the relation sets RCC5D and
 40 RCC8D encode five and eight set of relations respectively that capture vari-
 41 ous notions of neighbourhood, overlap and contact. After mechanically verifying
 42 DM theorems adopted in the imaging algorithms (using the theorem prover
 43 SPASS [18]), we implemented the RCC5/8D relation sets and exploited sev-
 44 eral DM theorems as short-cuts in imaging algorithms to compute operations
 45 on pairs of regions. DM can therefore be used to perform certain types of
 46 segmentation and model-testing analyses based on MM procedures. Those
 47 analyses have applications in histological imaging, where segmented histolog-
 48 ical components regions of interest (those corresponding to, e.g., nuclei and

49 cell bodies) represent valid theoretical models of histological reality that are
 50 related in specific ways in terms of their spatial relations [15, 16]. This log-
 51 ical, model-based approach to image interpretation provides a clean formal
 52 semantic framework in which to interpret image segmentation results and,
 53 furthermore, guarantees that the imaging algorithms encoding theorems in
 54 DM are provably sound. It also enables development of algorithms that ex-
 55 plicitly encode and ‘reason’ about spatial relations and local structure (e.g.,
 56 cell and tissue organisation) as well as facilitating the encoding of other struc-
 57 tural data of interest, such as the spatial localisation of molecular markers
 58 in cells and tissues.

59 Next we report new applications of DM that enrich MM operations. The
 60 paper is organised as follows. First, we revisit the definitions of adjacency, con-
 61 nection and region neighbourhood in DM and their MM counterparts. Next
 62 we present a new, more efficient version of the RCC5D and RCC8D algo-
 63 rithm that that outlined in a previous publication [16]. Finally, we discuss a
 64 novel application of DM that extends MM with the notions of morphologi-
 65 cal minimal closing and minimal opening, where DM is used to restrict the
 66 changes of the traditional MM closing and opening operations so the original
 67 region shape is minimally modified, while still achieving a desirable result.
 68 The paper concludes with a discussion.

69 2. Methods

70 The convention adopted here is that images consist of 2D square pixel
 71 arrays with 8-adjacency, meaning every non-boundary pixel of the array is
 72 surrounded by 8 neighbours forming a 3×3 pixel matrix. Image regions
 73 are sets of pixels locally-connected under 8-neighbour adjacency, representing
 74 objects of interest in the image. We assume that these regions exist in binary
 75 images but can include multiple planes or slices representing the same spatial
 76 reality, so that regions can share the same image space without being merged.

77 2.1. Adjacency

78 The adjacency relation between pixels is captured by a reflexive and sym-
 79 metric relation $A(x, y)$, meaning that pixel x is adjacent to or equal to pixel
 80 y . $A(x, y)$ is satisfied if $d(x, y) \leq \sqrt{2}$, where $d : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ is the two-
 81 dimensional Euclidean distance function defined on pixel coordinates in \mathbb{Z}^2 .
 82 In DM terms [15], the adjacency relation between regions X and Y is referred

83 to as *external contact* and is denoted $EC(X, Y)$. It is built from two other
 84 relations, namely *contact*:¹

$$C(X, Y) \equiv_{\text{def}} \exists x, y [x \in X \ \& \ y \in Y \ \& \ A(x, y)], \quad (1)$$

85 and *overlap*:

$$O(X, Y) \equiv_{\text{def}} X \cap Y \neq \emptyset \quad (2)$$

86 that is, the intersection between overlapping regions X and Y is non-null.
 87 External contact is then defined as:

$$EC(X, Y) \equiv_{\text{def}} C(X, Y) \ \& \ \neg O(X, Y). \quad (3)$$

88 In [14] Bloch et al. showed that the adjacency relation (or external contact
 89 in DM [15]) reworked in MM is equivalent to.

$$EC(X, Y) \equiv (X \cap Y = \emptyset) \ \& \ (X \oplus B) \cap Y \neq \emptyset, \quad (4)$$

90 where \oplus represents a morphological dilation operation with a 3×3 square
 91 structuring element or kernel B [17] (assumed to be centered at the origin
 92 of space to guarantee the extensivity of the dilation). Thus region X has
 93 external contact with region Y if the two regions do not intersect and the
 94 dilation of X leads to a non-empty intersection with Y .

95 2.2. Disconnection and region neighbourhood

96 In DM, a pair of regions X and Y are said to be *disconnected* if they are
 97 not in contact, i.e., $\neg C(X, Y)$; this is denoted $DC(X, Y)$. This relation can
 98 also be defined in terms of the mereotopological *discrete closure* operation
 99 (cl_D), instead of connection, as follows:

$$DC(X, Y) \equiv cl_D(X) \cap Y = \emptyset. \quad (5)$$

100 Here the function $cl_D(X)$ is defined as the union of the set of pixels whose im-
 101 mediate neighbourhoods overlap X , where the immediate neighbourhood of
 102 a pixel x , $N(x)$, contains just those pixels which are adjacent to x , including
 103 pixel x itself.

¹The symbols \exists , $\&$, \in , \cap , \neg and \equiv are read “there exists”, “and”, “is a member of”,
 “intersection”, “not”, and “if and only if”, respectively; \emptyset denotes the empty set.

$$\text{cl}_D(X) \equiv_{\text{def}} \{x \mid \mathbf{O}(N(x), X)\}. \quad (6)$$

104 In the case of our assumed 8-connected square grid, $\text{cl}_D(X)$ is equivalent to
 105 the dilation of X using a structuring element B , which in our model consists
 106 of an arbitrary pixel and its immediate neighbourhood, so

$$\text{cl}_D(X) = X \oplus B. \quad (7)$$

107 Therefore definition (5) translates into MM as:

$$\text{DC}(X, Y) \equiv (X \oplus B) \cap Y = \emptyset. \quad (8)$$

108 We also define a special type of neighbourhood relation between pairs
 109 of regions that is not part of the RCC5, cl_D sets but is particularly useful
 110 when considering binary regions residing in the same image: region Y is a
 111 neighbour of X and separated from it by one pixel. We name this relation
 112 NC (for *neighbourhood connection*)² and define it as:

$$\text{NC}(X, Y) \equiv_{\text{def}} \neg \text{FC}(\text{cl}_D(X), Y), \quad (9)$$

113 which in MM terms corresponds to

$$\text{NC}(X, Y) \equiv \text{EC}((X \oplus B), Y). \quad (10)$$

114 These formulae allow implementation of the extended MM functions that
 115 follow in Section 4. Figure 1 shows examples of the RCC8D relation set and
 116 the special cases of NC and PC*.

117 2.3. Region Connection Calculus via Mathematical Morphology

118 In [15] we introduced equivalences between DM and MM allowing DM to
 119 be implemented and understood in terms of MM procedures. Those equiv-
 120 alences make it convenient to develop DM using standard image processing
 121 applications supporting basic MM operations (erosion, dilation, reconstruc-
 122 tion). In [16], an FM algorithm implementation was presented which made
 123 use of the overlap of binary regions in images. That algorithm computes
 124 the spatial relations between two regions (self-connected or not) residing in
 125 different images. For many applications, however, it is required to find the

²In DM the relation NC is symmetric, i.e., $\text{NC}(X, Y) \rightarrow \text{NC}(Y, X)$.

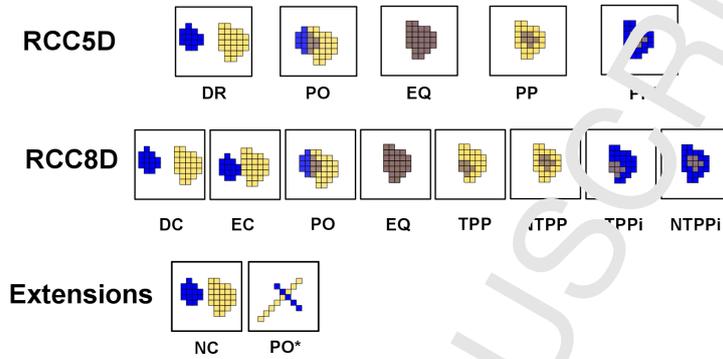


Figure 1: The five and eight spatial relations that hold between regions in the RCC5D and RCC8D sets in the discrete domain. The blue regions represent region X and the yellow regions Y , the intersection $X \cap Y$ being shown in brown. The names in RCC5D stand for disjoint (DR), partial overlap (PO), equal (EQ), proper part (PP) and proper part inverted (PPI). The RCC8D set makes additional distinctions: disconnection (DC), external connection (EC), partial overlap (PO), tangential proper part (TPP), non-tangential proper part (NTPP). TPPi and NTPPi are the inverse relations, e.g., TPPi(X, Y) means the same as TPP(Y, X). The extensions considered here are NC for ‘neighbourhood connection’ (a case of a DC relation where the regions are some dilation away from adjacency) and PO* (a case of EC occurring on ‘crossing objects’ that do not share any overlapping pixels), which while possible in the discrete domain, is counter-intuitive with real-world objects.

126 relations held between multiple regions³ contained in pairs of images (e.g., bi-
 127 ological objects across different confocal microscopy imaging planes, or stain
 128 channels). In such cases the computation can be decomposed into a se-
 129 quence of analyses between pairs of self-connected regions: first extract two
 130 given regions into two empty images (maintaining their relative positions),
 131 next compute the relation held between them using the said algorithm, and
 132 repeat this for all remaining region pairs. That implementation exploits the
 133 ‘start pixels’ of regions (the first pixel in a given region encountered in raster
 134 scan order) and uses morphological reconstruction [19] to extract each region
 135 separately and apply the RCC test to the extracted pair. Such an approach,
 136 however, quickly becomes computationally expensive; when dealing with ei-

³While a non-null region in DM is simply the union of an arbitrary set of pixels, the algorithmic manipulation of regions being assumed here is typically restricted to connected components, or simple regions.

137 ther large images (for which morphological reconstruction is slow) or images
 138 featuring many regions (the number of tests is given by the product of the
 139 number of regions across the images, complexity $O(nm)$). Some shortcuts
 140 have been identified, for instance in RCC5D, the disjoint relation DR can be
 141 assumed by default for all region pairs and other spatial relations only com-
 142 puted in cases of overlap, avoiding a considerable number of tests. Similarly,
 143 EQ can be identified in those regions pairs whose minimum pixel value is 3 in
 144 the sum of image X (labelled as 0 and 1) and image Y (labelled as 0 and 2).
 145 However, the distribution of DM relations varies with the image content and
 146 therefore such shortcuts do not necessarily lead to noticeable execution time
 147 improvements. The next section presents a more efficient algorithm which
 148 avoids the decomposition of the computation into an exhaustive sequence
 149 of region pairs. The procedure shows a considerable advantage in execution
 150 time compared to our previous algorithm 2 and it enables DM analysis to be
 151 more efficient and therefore applicable to high-throughput workflows.

152 3. Fast RCC5/8 Algorithm

153 We assume n binary regions in image X and m binary regions in image
 154 Y . The aim is to identify the spatial relations of the regions in X with the
 155 regions in Y . Those relations can be stored in an $n \times m$ matrix, here called
 156 the ‘RCC table’ (stored as an image) where the x and y coordinates are
 157 indices pointing to the x th and y th regions in X and Y respectively.

158 3.1. Computing RCC D

159 First, two images are generated using connected component-labelling, one
 160 where all regions in X have unique labels (according to their raster scan or-
 161 der) and the other similarly with the labels of the regions in Y . We call
 162 these images X_{labelled} and Y_{labelled} respectively. Two additional images are
 163 computed, one where pixels belonging to regions in X are labelled as 1 (or
 164 foreground) and 0 otherwise (background) and the other where pixels belong-
 165 ing to regions in Y are labelled as 2 (foreground) and 0 otherwise. These two
 166 images are summed to produce a third image XY , where pixels now have
 167 values 0 or 1 (the pixel is in X but not Y), 2 (it is in Y but not X), 3 (a region
 168 of X overlaps a region of Y at that location) or 0 (image background). A
 169 further binary image O is computed as the intersection (overlap) of X and
 170 Y . These overlaps arise in the case of RCC5D relations PO, EQ, PP and

171 PPI. Inspection of the values of the pixels of the overlaps in O (by redirecting
 172 ing to X_{labelled} and Y_{labelled}) reveals which two regions form a given overlap.
 173 We store the label values of the regions of O in arrays $\text{ovX}[]$ and $\text{ovY}[]$.
 174 The regions in images X and Y involved in overlapping relations are also
 175 inspected by redirection to image XY , and their minimum pixel values are
 176 stored in arrays $\text{minX}[]$ and $\text{minY}[]$. These arrays store information on
 177 whether a given region contains non-overlapping pixel values of 1 or 2 (which
 178 occur in PPI and PO cases) or whether all the pixels in a region are over-
 179 lapping (value 3, which occurs in PP and EQ cases). Adding minX and minY
 180 provides enough information to compute four of the five RCC5D relations
 181 (i.e., all those that involve region overlaps)

Relation	minX	minY	minX+minY
PO(x, y)	1	2	3
EQ(x, y)	3	3	6
PP(x, y)	3	2	5
PPI(x, y)	1	3	4

Table 1: Minimum values of pixel composition of overlapping regions X and Y . Region labels are: background=0, $X=1$, $Y=2$, $X+Y=3$. The columns minX and minY indicate the minimum value in regions X and Y respectively when a given relation holds.

182 From this scheme, it can be worked out that the relation R between
 183 regions $X_{\text{ovX}[i]}$ and $Y_{\text{ovY}[i]}$ in image X and Y (given by the overlap region O_i)
 184 is:

$$R[i] = \text{out}[\text{minX}[\text{ovX}[i]] + \text{minY}[\text{ovY}[i]]], \quad (11)$$

185 where $\text{out}[]$ is a look-up table holding labels for relations PO = 3, EQ = 6,
 186 PP = 5 and PPI = 4 (see Table 1, rightmost column). Since the only remain-
 187 ing RCC5D relation, DR, does not involve an overlap, DR can conveniently
 188 be assumed by default for all possible region pairs and during the analysis
 189 the values in the RCC table are only updated for those regions involved in
 190 overlapping relations using the procedure described. The procedure is shown
 191 in pseudocode in Algorithm 1.

192 3.2. From RCC5D to RCC8D

193 RCC5D introduces the notion of contact between regions, covering both
 194 overlap and adjacency [1] and resulting in eight spatial relations which pro-

Algorithm 1 Pseudocode for RCC5D computation across multiple regions in images X and Y .

1. Default all relations between regions in X and Y to DR.
 2. Compute labelled images X_{labelled} and Y_{labelled} where each region has a unique label.
 3. Compute image XY , coded as 1 \rightarrow pixel of a region in X but not Y , 2 \rightarrow pixel of a region in Y but not X , 3 \rightarrow pixel of a region in both X and Y .
 4. Compute binary image O , coded as 0 \rightarrow background, 1 \rightarrow $X \cap Y$.
 5. Create arrays $ovX[]$ and $ovY[]$ holding the information of which regions in X and Y form overlaps in O , by inspecting region labels in X_{labelled} and Y_{labelled} .
 6. Create arrays $minX[]$ and $minY[]$ by inspecting for each region in O the minimum pixel value for that region in image XY .
 7. For each region in O , $minX + minY$ gives the RCC5D relation: 3 \rightarrow PO, 4 \rightarrow PR, 5 \rightarrow PP, 6 \rightarrow EQ.
-

195 vide a more fine-grained spatial description than RCC5D. The RCC5D re-
 196 lation DR is split into the RCC8D relations EC (external connection) and
 197 DC (disconnection), the RCC5D relation PP is split into TPP and NTPP
 198 (tangential and non-tangential proper part respectively), the former occur-
 199 ring when the proper part abuts the background regions, the latter when it
 200 does not; the same thing happens, mutatis mutandis, with the inverse rela-
 201 tions. The RCC5D results obtained by the method described earlier can be
 202 reprocessed to capture the RCC8D relations of the same set of regions by
 203 performing single forward image scans testing for adjacency patterns (rather
 204 than processing region-pairs one at a time). The computation of RCC8D
 205 could be seen as a decomposition of the problem into a set of sub-problems
 206 (first compute RCC5D, then re-process the image without having to consider
 207 all region pairs, while exploiting the previously obtained results), similar to
 208 the type of problem reduction sought in dynamic programming [20]. We
 209 search for the presence/absence of certain patterns of adjacent pixels occu-
 210 pancy which, in conjunction with the known RCC5D relations, are indicative
 211 of specific RCC8D relations. The cases PC and EQ are the same in RCC5D
 212 and RCC8D. Of the remaining cases, suppose that we know the RCC5D re-
 213 lation between regions X and Y is DP. Then the RCC8D relation can only
 214 be either DC or EC. For it to be EC there must be at least one instance where
 215 a pixel of X is adjacent to a pixel of Y . The relation is DC is assumed by
 216 default and then we scan the image looking for the adjacency pattern; if it is
 217 found, EC is returned, if the pattern is not found, then the default DC holds
 218 good.

219 The following notation is used to describe the two-pixel patterns. Con-
 220 sider a pixel p and let n be one of its immediate neighbours. Set $p(X)$
 221 to be 1 or 0 according as p does or does not belong to region X ; and
 222 likewise with $p(Y)$, $n(X)$, and $n(Y)$. Then the two-pixel pattern exhib-
 223 ited by the pair p, n with respect to X and Y is denoted by the quadruple
 224 $(p(X), p(Y), n(X), n(Y))$.

225 From the above we can say that a DR relation between X and Y will
 226 be DC unless one of the quadruple patterns $(0,1,1,0)$ or $(1,0,0,1)$ is exhibited
 227 for some p, n pair in the image, in which case the relation is EC. Similarly,
 228 a case of $PP(X, Y)$ will be $NTPP(X, Y)$ unless patterns $(0,0,1,1)$ or $(1,1,0,0)$
 229 occur, in which case it will be $TPP(X, Y)$; and likewise with PPi , $NTPPi$,
 230 and $TPPi$. To perform these tests, the image is scanned using the ‘forward
 231 mask’ of pixel p , shown in Figure 2.

232 At each p we determine the two-pixel patterns formed by p with each

$N_{(x-1,y-1)}$	$N_{(x,y-1)}$	$N_{(x+1,y-1)}$
$N_{(x-1,y)}$	$P_{(x,y)}$	$N_{(x+1,y)}$
$N_{(x-1,y+1)}$	$N_{(x,y+1)}$	$N_{(x+1,y+1)}$

Figure 2: The forward mask of pixel p . The pixel patterns for occupancy of regions in image X and Y are tested between the central pixel p in the neighbours n in the ‘forward mask’ (shaded pixels). The pixels in the ‘backward mask’ do not need to be tested because the patterns have already been visited during the restoration.

233 of the shaded elements of the mask pattern. As the scan progresses, an
 234 accumulator records whether these patterns have arisen, and the relabelling
 235 of the region relations is done after the scan is finished. The form of the
 236 mask is dictated by the fact that the image is scanned top-to-bottom and
 237 left-to-right (no need to look at the pattern for, e.g., $p = (x, y)$, $n = (x, y - 1)$,
 238 since this will already have been detected when $(x, y - 1)$ played the role of
 239 p , with the pattern $p = (x, y - 1)$, $n = (x, y)$).

240 3.3. Extended relations NC and PO^*

241 The NC relation (definition 3) describes two regions separated by a one-
 242 pixel gap (Figure 1). This occurs when a region is detected as DC and
 243 the pixel patterns over the next-nearest neighbours (the external shell of a
 244 5×5 neighbourhood) show that pixels p (in the neighbourhood centre) and
 245 n (in the shell) are occupied by pixels of regions of X and Y , or Y and X ,
 246 respectively. The PO^* relation arises when two 8-connected regions ‘cross’
 247 each other in corner-connected regions, without overlapping or sharing any
 248 pixels (Figure 1). Such a pattern can commonly arise in the square lattice
 249 and it is interpreted as EC in $RCC8D$. In practical applications, however such
 250 results can be unintuitive (e.g. a linear object crosses another without ever
 251 “passing through” it) and it might be useful to identify these occurrences.
 252 This is done by inspecting 2×2 n and p pixel patterns for exclusive corner-
 253 connected pixel pairs in relations that have been identified as EC .

254 3.4. Complexity analysis and performance

255 The old algorithm in [16] uses morphological binary reconstruction to
 256 extract every pair of regions before calculating the relational model held
 257 between them. It has been shown in [21] that morphological reconstruction
 258 is a computationally expensive, highly non-linear procedure. Its complexity
 259 depends on the number of component/pixels to be reconstructed. Even for
 260 the efficient/best-compromise algorithms [22] it is recognised that a mean
 261 case complexity analysis would be extremely difficult to compute because
 262 of the variety of input images that may be used. In addition to utilizing
 263 reconstruction, the computational complexity of the core of the old algorithm
 264 (in the worst case scenario) is quadratic, $O(nm)$ (when $m \approx n$) because the
 265 relations are computed between all possible region pairs (n and m) one at a
 266 time, or subquadratic when $m \neq n$. While some shortcuts were identified (e.g.
 267 to avoid computing relations between those regions that are further away
 268 than two dilations, guaranteed to be DC), an important bottleneck remains
 269 with the binary reconstruction steps necessary to extract the region pairs.

270 The new Algorithm 1, first, avoids extracting individual pairs of regions
 271 into new images to compute the relations, thus avoiding morphological re-
 272 construction altogether. Secondly, it computes the RCC5D relation from a
 273 sequence of steps that reduce the complexity from quadratic to linear yield-
 274 ing to an average case complexity of $O(n+m)$. In particular, steps 1, 3 and
 275 4 in Algorithm 1 have a constant-time algorithm of order 1 ($O(1)$). Step 2
 276 (image labelling) requires a maximum time complexity of $O(n+m)$. Steps
 277 5, 6 and 7 process the overlapping subregions that occur across the two im-
 278 ages. It should be noted that relations PP, PPI and EQ are one to one, and
 279 result in one overlapping segment per region pair. A worst case scenario
 280 where all the relations held are any of the above (therefore $n=m$) would lead
 281 to a scaling of these steps to $O(n)$ which is still less than $O(n+m)$. The
 282 PO relation, however, is a special case in the sense that a region can have
 283 more than one overlapping subregion (with one or multiple other regions).
 284 For instance large and convoluted regions could potentially lead to a scaling
 285 higher than $O(n)$. While it is not possible to foresee what regions configura-
 286 tions may be found in segmented images, it is nevertheless possible to clarify
 287 the impact of this unknown, experimentally. In a series of performance tests
 288 on random binary images (detailed below) we found that on average, the
 289 number of overlapping subregions across 500 tests (average 7152, maximum
 290 19431 regions) was smaller than the number of regions $n+m$ (average 11168,
 291 maximum 38934). The running time of the proposed algorithm would there-
 292 fore, on average, increase linearly with the total number of regions $O(n+m)$,

293 with some exceptional configurations where it could be higher depending of
 294 the number and nature of the PO relations. As examined experimentally,
 295 situations where this is above the quadratic complexity of the old algorithm
 296 appear to be unlikely. The successive forward passes on the labelled images
 297 to compute the RCC8D relation set, as well as the extended relations NC
 298 and PO*, are of the order $O(1)$ and therefore do not increase the algorithm
 299 complexity.

300 Figure 3 shows the difference in performance, in seconds, of the previ-
 301 ously published [16] and the new algorithms on 512×512 pixels, random
 302 binary images with varying probabilities, p , of foreground pixels. The tests
 303 were performed on the ImageJ platform, version 1.51 [23] under the Linux
 304 operating system on an Intel Xeon CPU (E31225) at 3.1GHz. The plot shows
 305 the average of 5 runs at each p in steps of 0.01. The average difference over
 306 all p was an improvement of 491 times faster than the previous algorithm,
 307 while largest difference was found at $p = 0.42$ where the new algorithm was
 308 1684 times faster than the old one. The execution times appear to be depen-
 309 dent not only on the number of regions per image but also on the proportion
 310 of the different types of relations that occur at various p (not shown). A
 311 slight advantage was noticed for the old algorithm implementation on im-
 312 ages with the highest p , (where only very few regions exist, the images being
 313 mostly occupied by one large region), but this difference, in practical terms,
 314 becomes negligible as the execution times in those cases are all at a fraction
 315 of a second.

316 4. New morphological filters: Minimal closing and opening

317 In addition to the applications of DM in histological imaging [15, 16,
 318 24], the fast algorithm enables new MM operators with reasonable speed
 319 performance to be designed, exploiting the relations between image regions
 320 and the changes they undergo after other morphological operations.

321 In MM, the operation closing ϕ with a kernel B is defined as the dilation
 322 of a region, followed by an erosion [17]:

$$\phi_B(X) =_{\text{def}} (X \oplus B) \ominus B. \quad (12)$$

323 Closing is an extensive transformation, where voids in regions, and de-
 324 tails that cannot contain the translations of kernel B , are filled. Note that
 325 in reotopological closure, which refers to a topological operator defined on

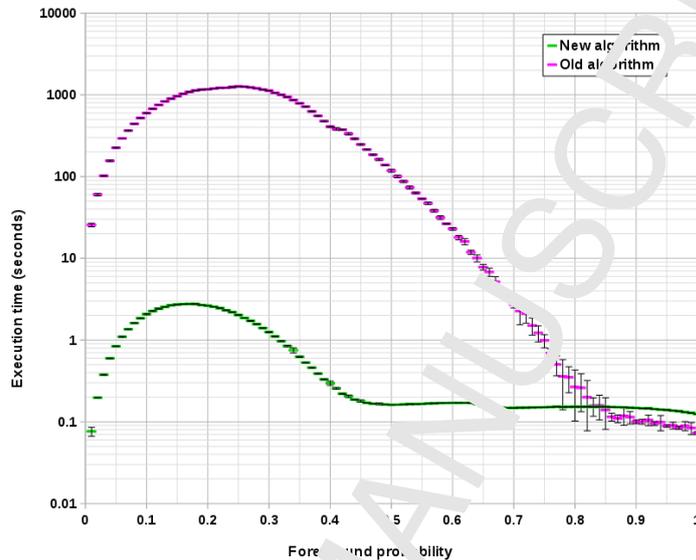


Figure 3: Differences between the average execution times of the old and new RCCSD algorithms. The tests were done, on random binary 512×512 pixel images with varying foreground pixel probabilities. Each point is the average of 5 runs and the vertical bars indicate one standard deviation from the mean.

326 a discrete space, does not correspond to closing but to dilation in MM, de-
 327 spite the similarity in their names. When closing binary regions, voids are
 328 filled with the foreground value. While the rest of this section only deals
 329 with closings, there is a dual MM operation with respect to the set comple-
 330 ment, namely opening, an anti-extensive transform, which instead of filling,
 331 removes those object pixels that cannot be fully covered by the translations
 332 of kernel B .

333 Closing is commonly used to ‘fill in’ gaps between nearby regions desired
 334 to be joined, to fill small holes in regions and to reduce the complexity of
 335 region boundaries (‘shorelines’ from now on). These actions are, however,
 336 not independent: gaps, holes and shoreline irregularities are processed con-
 337 currently as the operation does not differentiate between them. In certain
 338 circumstances, however, it might be desirable to achieve only one of those
 339 results, e.g., joining nearby regions while avoiding major modifications of the
 340 shoreline details that do not yield a connection to another region. Using

341 DM this can be modelled as follows. In MM, the set of pixels added to the
 342 original region by the closing operator is known as the *black top-hat*:

$$\text{BTH}(X) = \phi_B(X) - X. \quad (13)$$

343 The black top-hat often consists of a set of disconnected components,
 344 the elements of which we call *black top-hat segments*. In DM this is defined
 345 immediately following, where $\text{BTH}_{\text{seg}}(Y, X)$ is read as ‘Y is a connected component
 346 of the black top-hat of X’ and $\text{BTH}_{\text{segs}}(X)$ is the set of black top-hat
 347 segments of X. The relation $\text{CC}(Y, X)$ which we use here, read as ‘Y is a
 348 connected component of X’, is also defined in DM—see [15], p 572:

$$\text{BTH}_{\text{seg}}(Y, X) \equiv_{\text{def}} \text{CC}(Y, \text{BTH}(X)), \quad (14)$$

$$\text{BTH}_{\text{segs}}(Y, X) =_{\text{def}} \{Y \mid \text{CC}(Y, \text{BTH}(X))\}. \quad (15)$$

349 The set-builder notation used to define function $\text{BTH}_{\text{segs}}(X)$ in definition 15
 350 returns a set of regions, namely the Y s. In DM, however, where a region
 351 rather than a set of regions is required as the output of a function⁴, the set
 352 union operator is added, i.e.,

$$\text{BTH}_{\text{segs}}(X) = \bigcup \{Y \mid \text{CC}(Y, \text{BTH}(X))\}, \quad (16)$$

353 and the same principle applies for equations 19, 20 and 25. The relation
 354 between Y and X , given $\text{BTH}_{\text{seg}}(Y, X)$, is always EC, that is to say they
 355 are adjacent (externally connected) regions—remembering here that the seg-
 356 ments are the result of an extensive transformation. In addition, the segments
 357 could also be adjacent to other regions in their neighbourhood; if the distance
 358 ϵ separating pairs of regions is no more than half the width of kernel B , the
 359 black top-hat segments create ‘bridges’ between originally disconnected (i.e.,
 360 DC) regions. When considering region X and all other regions Y in the seg-
 361 mented image, two black top-hat segment types can arise. First we have
 362 what we call *shorelines* where $\text{BTH}_{\text{shoreline}}(Y, X)$ is read as ‘Y is a connected
 363 shoreline component of the black top-hat of X’. In this case the black top-hat
 364 segment Y adjoins exactly one connected component of X :

⁴ remembering that in DM, a region can comprise several disjoint, region-parts as well as being a simple region.

$$\text{BTH}_{\text{shoreline}}(Y, X) \equiv_{\text{def}} \text{CC}(Y, \text{BTH}(X)) \ \& \ \exists Z[\forall U[\text{CC}(U, X) \ \& \ \text{EC}(U, X)] \leftrightarrow U = Z], \quad (17)$$

365 i.e., Y is a connected shoreline component of the black top-hat of X if and
 366 only if Y is a connected component of the black top-hat of X , and there
 367 exists exactly one connected component of X that is EC to Y .

368 The second case is where we have a black top-hat segment that forms a
 369 *bridge* between two regions. $\text{BTH}_{\text{bridge}}(Y, X)$ is read as “ Y is a black top-hat
 370 bridge of X ”:

$$\text{BTH}_{\text{bridge}}(Y, X) \equiv_{\text{def}} \text{CC}(Y, \text{BTH}(X)) \ \& \ \exists Z, U[\text{CC}(Z, X) \ \& \ \text{CC}(U, X) \ \& \ Z \neq U \ \& \ \text{EC}(Z, Y) \ \& \ \text{EC}(U, Y)], \quad (18)$$

371 which is similar to definition (17) except there are now at least two connected
 372 components of X externally connected to the black top-hat segment Y of X ,
 373 not one.

374 The spatial relations that hold between the black top-hat segments and
 375 the original regions provides a means for identifying those which act as
 376 bridges (between DC region pairs), and those which do not (and consequently
 377 only modify a region shoreline). From this it follows that the black top-hat
 378 segments adjacent to only one region are shoreline modifiers (including hole
 379 filling, when the holes can be filled by the kernel), and those adjacent to more
 380 than one region are bridges. Retaining one or another type, (e.g., by means
 381 of binary reconstruction [19]), gives rise to two types of conditional minimal
 382 closing, shoreline smoothing without region merging:

$$\phi_B^{\text{shoreline}}(X) = \{Y \mid \text{BTH}_{\text{shoreline}}(Y, X)\}, \quad (19)$$

383 and region merging without boundary smoothing:

$$\phi_B^{\text{bridge}}(X) = \{Y \mid \text{BTH}_{\text{bridge}}(Y, X)\}. \quad (20)$$

384 Note that in MCC8D, the notion of shoreline or boundary of a region does
 385 not differentiate between the ‘outside’ boundary and the boundary with an
 386 internal hole. The DM treatment of region holes is dealt with later in this
 387 paper.

388 With regards to implementation, the different black top-hat variants are
 389 sorted by an exhaustive analysis of the relations between all original regions
 390 versus all black top-hat segments generated after an MM closing (i.e., $X \oplus$
 391 $B) \ominus B$). Those results, arranged in an $m \times n$ matrix or table indexed
 392 by regions and black top-hat segments in scan order (here named the RCC
 393 table), provide a convenient way to search for those special relations. The
 394 DM relation between a given region and a black top-hat segment can be one
 395 of two, out of the eight possible outcomes of the PCU8D region set: either
 396 DC or EC. To identify ‘bridge’ black top-hat segments, we use indexing of
 397 the original regions and black top-hat segments in the x and y axis of the
 398 RCC table respectively: the number of EC instances in a row indicates the
 399 number of different regions a given black top-hat segment is adjacent to.
 400 Black top-hat segments with total EC counts per row equal to 1 are therefore
 401 shoreline modifiers (i.e., they are adjacent to only one region), while those
 402 instances with counts exceeding 1 are guaranteed to be bridges. As will be
 403 seen comparing Figures 4f and 4g, black top-hat shoreline segments include
 404 those completely surrounded by a region; we call these segments *lakes*. In DM
 405 this can be defined as follows, where $\text{BTH}_{\text{lake}}(Y, X)$ is read as ‘Y is a black
 406 top-hat lake of X’; the definition uses the DM definition of a hole defined
 407 later:

$$\text{BTH}_{\text{lake}}(Y, X) \equiv_{\text{def}} \text{BTH}_{\text{seg}}(Y, X) \ \& \ \text{Hole}(Y, X). \quad (21)$$

408 The crucial distinction between a shoreline and lake black top-hat segment
 409 of a given region is that a lake also satisfies what it is to be a hole in that
 410 region which again is encoded in another RCC table indexing regions and
 411 holes. Examples of binary region merging with minimal shoreline smoothing
 412 and shoreline smoothing without region merging are given in Figure 4.

413 While black top-hat segments have the same connectivity as the origi-
 414 nal regions (e.g., 8-connected) the minimal closing can be minimised fur-
 415 ther by considering only the adjacency relations of their 4-connected sub-
 416 components. The rationale for this is that retaining a given black top-hat
 417 segment is similar to adding some background pixels to the foreground. Since
 418 the 8-connected foreground convention implies a 4-connected background, it
 419 is possible to restrict minimal closing to the 4-connected sub-components of
 420 a given black top-hat segment that satisfies the bridge or shoreline properties
 421 described earlier and not including the whole black top-hat region. Figure
 422 5 shows the effect of retaining such 4-connected components in cases of pro-

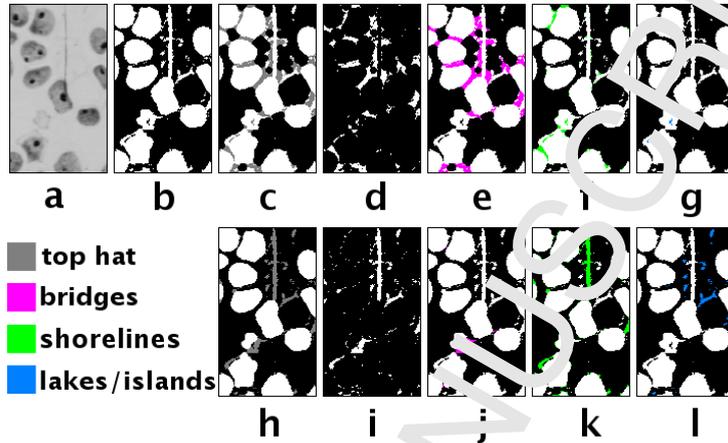


Figure 4: Closing and minimal closing of binary images with a disc of radius 3. The original greyscale image of lymphocytes stained with silver nitrate for detection of nucleolar organising regions (dark spots)(a) was segmented with the minimum error thresholding algorithm [25] (b). In (c) the traditional closing (with the added pixels in grey that make the ‘black top-hat’ (d)). Panel (e) shows in magenta the black top-hat segments that have an adjacency relation with more than one region in (b), acting as bridges. We call this operation ‘minimal closing bridges’. Panel (f) shows those black top-hat segments that have adjacency to only one other region in (b) (minimal closing shorelines), while in (g) are shown the lakes which are black top-hat segments that have no connection to the rest of the background’s subset that intersects the image boundaries. Panel (h) shows the traditional opening (with added pixels in grey that make the ‘white top-hat’ (i)). Panels (j-l) shown the minimal opening of bridges, shorelines and islands respectively.

423 ccessing regions with null interior.

424 Finally, the dual operation of the closing is opening, Υ :

$$\Upsilon_B(X) =_{\text{def}} (X \ominus B) \oplus B, \quad (22)$$

425 and the corresponding top-hat transformation for the opening is called *white*
426 *top-hat*:

$$\text{WTH}(X) =_{\text{def}} X - \Upsilon_B(X), \quad (23)$$

427 which identifies the segments that were removed from the original after the
428 opening operation. As before and mirroring definitions for the black top-hat
429 we define a white top-hat segment Y of region X , and the set of white top-hat
430 segments of X :

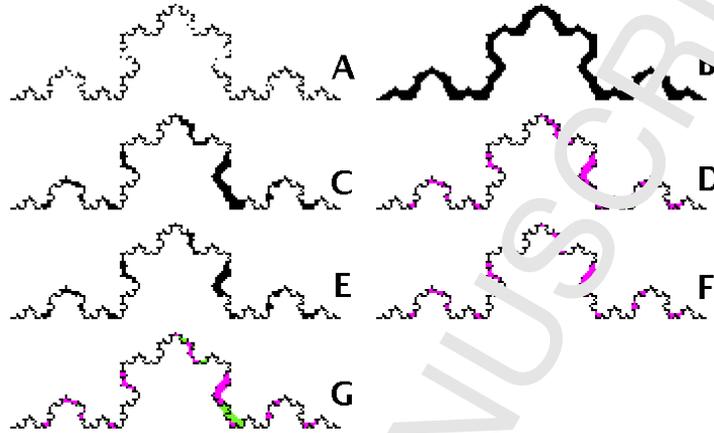


Figure 5: Closing versus minimal closing. (a) A digitised version of a 4th order von Koch curve with discontinuities, resulting in 30 fragments. The target is to merge all pieces into a single region. (b) The classical result using morphological closing with a circular kernel of radius 4 (the smallest kernel that closes all gaps). Note the loss of detail in the result. (c) A minimal closing where the gaps between any two fragments were filled independently with the smallest kernel possible until a single region was obtained. (d) The detail of the minimal closing (black is the original set, magenta (dark grey in B/W version) represents the black top-hat segments). (e) and (f) show the same example, but this time retaining only the connected subregions of each black top-hat segment (BTH_{segs} in the text) that acts as a homeotopological bridge between fragments. Note that this closing modifies the original even less than in (c). (g) shows in green (bright grey) the part sub-regions of the black top-hat segments that were not necessary to retain to achieve the minimal closing (i.e., the difference between (d) and (f)).

$$WTH_{\text{seg}}(Y, X) \equiv_{\text{def}} CC(Y, WTH(X)), \quad (24)$$

$$WTH_{\text{segs}}(X) = \{Y \mid CC(Y, WTH(X))\}. \quad (25)$$

431 It is therefore possible to implement minimal opening operations as the
 432 dual of minimal closing. Note that while opening is an anti-extensive trans-
 433 formation, the white top-hat segments are in relation EC to the regions in the
 434 opened image that is: $WTH_{\text{seg}}(Y, X) \rightarrow EC(Y, \Upsilon_B(X))$. The two new dual
 435 minimal opening operations are *open shorelines* and *open bridges*, depend-
 436 ing on which type of white top-hat segments are retained or removed. It is
 437 also possible to define an additional minimal opening operation that removes

438 those white top-hat segments that are DC to all other regions in the opened
 439 image. We call this procedure *opening islands*, and its dual, *closing lakes*. In-
 440 terestingly, these *opening islands* and *closing lakes* are equivalent to opening
 441 and closing by reconstruction, respectively [26]. This sequence of morpho-
 442 logical operations combining MM with the explicit relations of DM shows
 443 the potential for defining a variety of fine-grained morphological operators
 444 that target a particular goal. It also highlights the importance of securing
 445 computationally efficient ways to compute and store relations between pairs
 446 of regions when processing segmented images such as those assumed in the
 447 RCC table, where these relations are explicitly used in these new operators.
 448 An example of the advantage of these new operators is shown in Figure 5.
 449 Here connecting fragments in a discontinued curve can be restricted to places
 450 where the closing leads to fragment connections, without interference at lo-
 451 cations where the connection is not necessary. By so doing we preserve the
 452 original as much as possible with a less dramatic loss of global detail than
 453 traditional closing.

454 Similarly to minimal binary opening and closing, the procedures above
 455 can directly be applied to process greyscale images via threshold decomposi-
 456 tion (although the threshold decomposition it is usually an inefficient pro-
 457 cedure). Figure 6 shows examples of the greyscale versions of the minimal
 458 closing and opening respectively.

459 5. Discussion

460 Bloch [11, 9] originally suggested that RCC relations can be defined in
 461 MM, and specifically provides the translation for the $TPP(X, Y)$ relation [11],
 462 which is equivalent to ours in [15]. It should be noted that while MM is not
 463 specific about discrete or continuous space, that is not an exact translation
 464 of the RCC8 T_{TPP} relation on discrete space, because RCC presupposes an
 465 infinitely divisible one. Instead, for the case of discrete space, the connections
 466 drawn are with the RCC8D relation set of discrete mereotopology.

467 The implementation of RCC5D, RCC8D and additional DM relations as
 468 MM procedures opens a range of new opportunities to extend some opera-
 469 tions beyond their original design by means of exploiting spatial relations
 470 held between regions. This is specially useful when designing analytical pro-
 471 cedures that can benefit from mechanically reasoning about image contents.

472 The approach presented here allows the results of closing and openings
 473 to be made conditional on certain types of modifications which might not

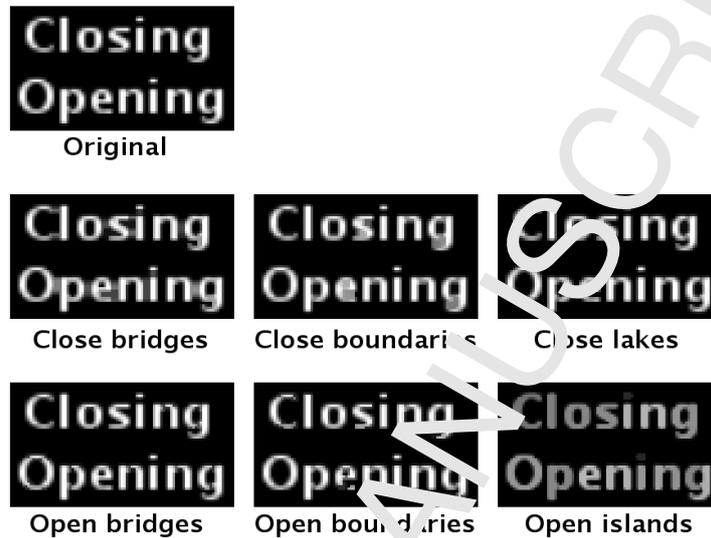


Figure 6: Greyscale minimal closing and opening. The examples were computed using a 3×3 kernel on a greyscale image of text. Note (second row) how minimal closing bridges connect nearby regions without modifying the shorelines or filling lakes and how the closing of lakes does not affect the shoreline features. The open bridges procedure leads to fragmentation of regions in the original without affecting other shoreline features (compared to open shorelines), while the opening of islands removes the white top-hat segments with no adjacency relations to any other patterns at a given grey level.

474 be straightforward to achieve otherwise or might require more complex ap-
 475 proaches such as multiscale operators and directional information [27]. While
 476 conditional filtering is not new, traditional conditional morphological opera-
 477 tors apply their constraints in a given local sub-image (given by the kernel).
 478 To replicate this type of filtering region-wise is challenging because classical
 479 methods require additional processing to account for relations between re-
 480 gions and conditions on these to be met, whereas in DM it is built into its
 481 very foundations.

482 The bridges, boundaries, island and lakes regions in relation to opening
 483 and closing (i.e. the white top-hat and black top-hat segments) have similar-
 484 ities to what Soille and Vogt call ‘binary patterns’ [28] for which they iden-
 485 tified formulae for their computation (and include some additional patterns:
 486 core, perforations, branches and loops). For minimal closing and opening,

487 however DM has the advantage of being able to relate, via the RCC table,
488 which original regions are adjacent to those segments and therefore open the
489 possibility to control algorithmically whether segments are included or re-
490 moved from particular configurations of regions. That would require
491 further computation in the approach presented in [28].

492 There has been interest in other types of conditional operations, for ex-
493 ample homotopic sequential filtering to preserve the topology of an image
494 [29, 30] or multiscale top-hat transforms to improve image contrast [31]. Here
495 we described how processing can be applied to changes of regions or across
496 regions. A number of new uses for DM via MM has been recently identified
497 in applications that require dealing with models where image regions fulfil
498 specific spatial relation between their parts [16, 24, 32]. Such models com-
499 monly arise in histological imagery, where detected regions represent regions
500 with special biological meanings (such as cells, nuclei, tissues, organs) that
501 not only can be distinctly detected, but also exist in specific spatial relations
502 and hierarchies. Such relations need to be fulfilled if the extraction of bio-
503 logically relevant information from images is to be related to a given context
504 in terms of ontological levels of organisation [33]. On a different kind of ap-
505 plication, Cointepas [34] proposed the use of MM combined with adjacency
506 relations to construct homotopically deformable cellular models and resolve
507 complex problems, such as 3D cerebral cortex segmentation, where topology
508 preservation is essential to yield not only accurate but anatomically plausible
509 results.

510 The procedures presented here stem from our work in histological imag-
511 ing using digital images of 2D tissue sections, and as such are based on
512 a 2D Cartesian grid representation. It would be desirable to further develop
513 these concepts and algorithms in n-dimensions so they can be applied to
514 e.g. temporal, volumetric and higher dimensional data sets. Furthermore,
515 alternative schemes such as simplicial complexes (used to represent multidim-
516 ensional data) [35], graphs [36] and hypergraphs [37, 38] (for non-lattice
517 implementations of MM) might be advantageous for such generalisation to
518 higher dimensions.

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AUTHOR DECLARATION

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

We confirm that the manuscript has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. We further confirm that the order of authors listed in the manuscript has been approved by all of us.

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Novel applications of Discrete Mereotopology to Mathematical Morphology.**Highlights**

- Six new mathematical morphology operators using mereotopological concepts.
- Novel “minimal closing” and “minimal opening” morphological operations
- A new discrete region connection calculus algorithm with improved execution speed.

