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## Stochastic dynamics of a parametrically base excited rotating pendulum

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### Abstract

This paper studies the rotational motion of a parametrically excited pendulum, dynamics of which is governed by a stochastic nonlinear Mathieu equation. The interest to this problem is based on the fact that this motion may be used to harness wave energy, capturing the heaving motion of waves. Thus a narrow band excitation is used, which is modeled as a harmonic process with random phase modulations. It has been established earlier that a relatively large values of noise intensity deteriorate stability of the rotational motion, leading to vibrations. To obtain robust rotational motion a single-degree-of-freedom filter is used.

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*Keywords:* Mathieu equation; instability; rotational motion; narrow band excitation;

### 1. Introduction

Excitation of a pendulum suspension point in vertical or horizontal direction leads to a phenomenon of parametric oscillations [1]. Although there are other examples of a parametric excitation in a variety of physical systems, this paper will be focused on the pendulum, since it has been suggested recently [2, 3] that such a system may be used as a power take-off system for wave energy converters. To provide steady electricity generation it is required to maintain rotational motion of the pendulum, which is excited parametrically through heaving motion of waves (Fig 1(a)). In view of this it is important to explore the effect of the system's parameters onto its behaviour.

The motion of a parametrically excited in vertical direction only, deterministic pendulum is described by the following equation:

$$Ml^2\ddot{\theta} + b\dot{\theta} + Mgl \sin \theta = ML\ddot{f} \sin \theta \quad (1)$$

where  $\theta$  is the angle of inclination,  $f(t)$  - excitation force,  $l$  - the length of the pendulum or the distance from the suspension point to the lumped mass  $M$ . If the excitation force  $f(t)$  is perfectly harmonic, the damping is negligible

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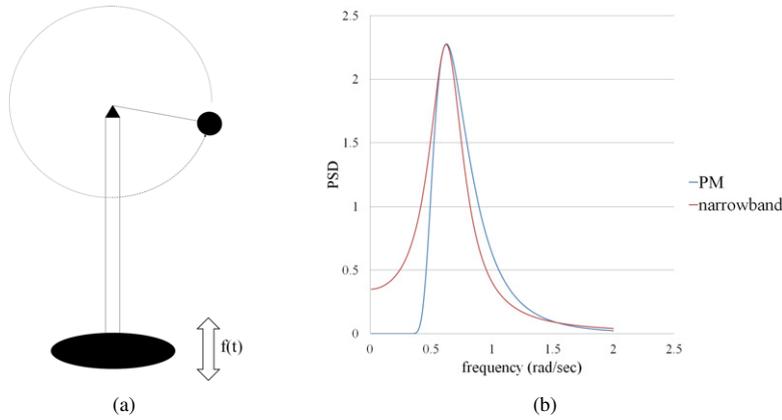


Fig. 1. (a)Pendulum excited parametrically by heaving motion of waves; (b)Comparison of PM and narrow-band spectra for D=0.3.

and the inclination angle is small ( $\sin \theta \approx \theta$ ) Eq 1 may be rewritten:

$$\ddot{\theta} + \left[ \frac{g}{l} + \frac{A}{l} \omega^2 \cos(\omega t) \right] \theta = 0 \tag{2}$$

which is the well known Mathieu equation. It can be further simplified by introducing a non-dimensional time  $\tau$  [4]:

$$\begin{aligned} \theta'' + [1 + \lambda \cos(v\tau)] \theta &= 0 \\ \tau &= \Omega t, \quad \Omega^2 = g/l, \quad v = \omega/\Omega, \quad \lambda = \frac{A}{l} v^2 = \frac{A}{g} \omega^2 \end{aligned} \tag{3}$$

Eq 3 has been studied by a number of authors for stability regions. However, it does not possess all the properties of the original nonlinear system:

$$\theta'' + [1 + \lambda \cos(v\tau)] \sin \theta = 0 \tag{4}$$

Different types of motions, including chaotic motion, rotational and periodic motion with different periods, etc., were reported by different authors, including [5, 6, 7, 8] and references therein. The important result of these investigations was the identification of areas where the rotational motion was observed. These domains were plotted in a parametric  $\lambda - v$  space, which helps to select proper system parameters.

It should be stressed that waves are not perfectly periodic and therefore it is hard to rely on the previous deterministic study since the random nature of waves may change the domains of rotational motion. The Pierson-Moskowitz (PM) spectra, is an empirical relationship that defines the distribution of energy with frequency within the ocean [9]:

$$\begin{aligned} S_{PM}(\omega) &= \frac{\alpha g^2}{\omega^5} \exp \left[ -\beta \left( \frac{\omega_p}{\omega} \right)^4 \right] \\ \text{with } \alpha &= 0.0081, \quad \omega_p = 0.877g/(\pi U_{19.5}) = 0.2\pi, \quad \beta = 1.25 \end{aligned} \tag{5}$$

where  $U_{19.5}$ -the wind speed at 19.5 meters above the sea level.

To model the narrow band wave excitation a harmonic function with random phase modulations is used [10, 4]:

$$\begin{aligned} z(t) &= \cos q(t), \quad \dot{q}(t) = v + \sigma \zeta(t), \\ E[\zeta(t)] &= 0, \quad E[\zeta(t)\zeta(t+t_0)] = \delta(t_0) \end{aligned} \tag{6}$$

where  $\zeta(t)$  - Gaussian white noise with unit intensity. Thus, the motion of the system excited in the vertical direction only by ocean waves may be modeled as

$$\theta'' + \gamma \theta' + [1 + \lambda \cos q(t)] \sin \theta = 0 \tag{7}$$

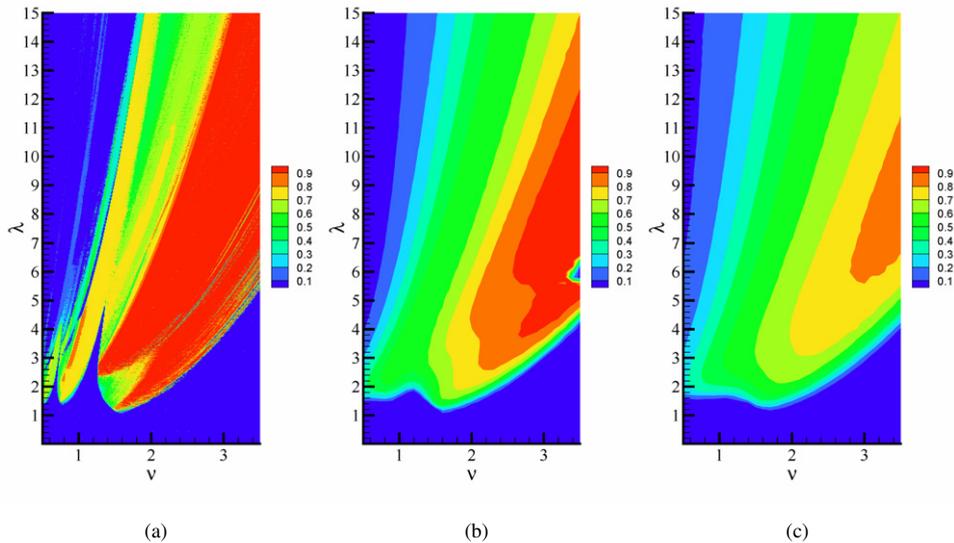


Fig. 2. (a) deterministic map of Eq 7 for  $\gamma = 0.3$ ; (b)  $D = 0.1$ ; (c)  $D = 0.3$ .

Analysis of the system in Eq 7 for small values of  $\theta$  has been conducted by different approaches: perturbation technique, multiple scale method [11], stochastic averaging approach [12, 13, 14]. Since this paper is focused on the rotational motion, assumption of small  $\theta$  would not be correct. Nonlinear Eq 7 is difficult to be analyzed analytically, since the stochastic averaging technique does not take into account the stiffness nonlinearity, whereas the quasi-conservative averaging method does not permit deriving an analytical solution in a closed form. Thus, Eq 7 has to be investigated numerically. The numerical method applied is described and explained thoroughly in [4].

## 2. Narrow band stochastic parametric excitation of the pendulum

To study the effect of noise intensity onto the rotational motion of the pendulum a set of simulations has been conducted. The value of noise intensity was selected by comparing the PM spectra with the spectra of narrow band excitation:

$$S(\omega) = \frac{\lambda^2 D}{4\pi} \frac{v^2 + \omega^2 + D^2/4}{(v^2 - \omega^2 + D^2/4)^2 + \omega^2 D^2} \quad (8)$$

Direct comparison of the spectra in Fig 1(b) described by Eq 5 and Eq 8 has revealed that the value of  $D = 0.3$  will provide reasonable prediction of PM spectra near the resonance frequency and identical bandwidth.

The results in Fig 2 demonstrate the ratio of rotational motion of the system ( $T_{rot}/T_{tot}$ ), so that the red area has over 90% of rotational motion, whereas the blue area has over 90% of oscillatory or asymptotically stable motion. Fig 2(a) represents a parametric plot for deterministic ( $D = 0$ ) system Eq 7 with  $\gamma = 0.3$  which coincides with Eq 4. There are two regions of dominant rotational motion, which join each other at values of  $v > 2$ . Fig 2(b) and Fig 2(c) demonstrate results for values of noise intensity  $D = 0.1$  and  $D = 0.3$  respectively. In [4], the effect of noise intensity on the pendulum's rotational potential was examined. It can be seen that the increase of noise intensity significantly influences the size of the domain with dominant (over 90%) rotational motion. For large values of noise intensity, presented in Fig 2(c) this subdomain has disappeared leaving a small region of only 80% of rotational motion. It should be stressed that no distinction is made here between clockwise and counterclockwise rotations. Although the direction of rotation is random in general, in the regions of dominant rotational motion, rotations once started continue in the same direction.

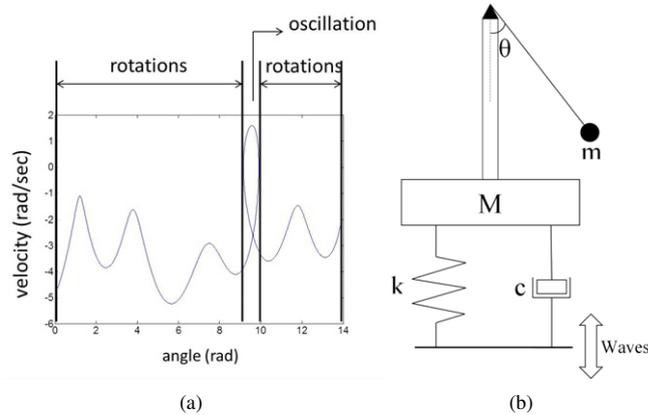


Fig. 3. (a) State space plot demonstrating the process of counting rotations; (b) Sketch of the system parametrically excited through its base.

A few words must be said about counting the rotations. It is not a trivial task to understand what can be treated as a rotational motion. Indeed, some trajectories can easily be identified as a rotational, when their motion is unmistakably simply rotational. However, there are complex or irregular trajectories (Fig 3(a)), when the system changes the direction of rotation several times within a period. In this paper we consider a rotation as the motion of the pendulum that concluded  $2\pi$  angular displacement under constant sign of the velocity, thereby we disregard here all complex trajectories.

It should be noted that it is not only important to have the dominant rotational region, but also to move down its low boundary as much as possible. This is related to the fact that  $\lambda$  is proportional to the excitation or wave amplitude, which cannot be arbitrary and exceed certain realistic values.

### 3. Nonlinear energy dissipation

In the simulations discussed so far, the assumption was made that the damping function is linear. However, in most practical applications this does not hold. In general, the mechanism dictating the energy dissipation is nonlinear and rather unclear. A better approximation of the actual behaviour of the system could be achieved by introducing a nonlinear function taking the following form:

$$d(\dot{\theta}) = \gamma\dot{\theta} + R\text{sign}(\dot{\theta}) \tag{9}$$

where  $d$  is the damping function,  $\gamma$  - the viscous damping coefficient and  $R$  - a constant representing the Coulomb friction. A typical value for the linear part is  $\gamma = 0.016$  and for the Coulomb friction  $R = 1.225 \times 10^{-4}$ . In the remaining of this section, the effect of a realistic damping function on the rotational regions of the parameter space will be discussed, focusing mainly on the Coulomb friction since the viscous damping effect has been more thoroughly studied. Note that the viscous damping coefficient is one order of magnitude smaller than the values considered previously.

In Fig 4, the parameter space plots constructed for Eq 7 with nonlinear damping function of Eq 9 are presented for the deterministic and two stochastic cases. In comparison with Fig 2, it is seen that the low viscous coefficient causes the rotational region to degenerate when the noise is induced. Even though the regions maintain their pattern, the amount of rotational motion is significantly less in Fig 4(a)-Fig 4(c)- Fig 4(e). Of course, the whole plot is shifted to lower values of  $\lambda$  as expected, indicating the lower energy dissipation in the system. When the Coulomb friction is induced to the system, the motion of the pendulum has to overcome an added constant torque as well. Fig 4(b)- Fig 4(d)-Fig 4(f) show the parameter space plots for an additional damping term  $R = 2.45 \times 10^{-4}$  which corresponds to a rather realistic scenario. In general, and as far as rotational motion is concerned there is no significant influence to the system's response. Even though the plots in the left column in Fig 4 are not identical with those in the right

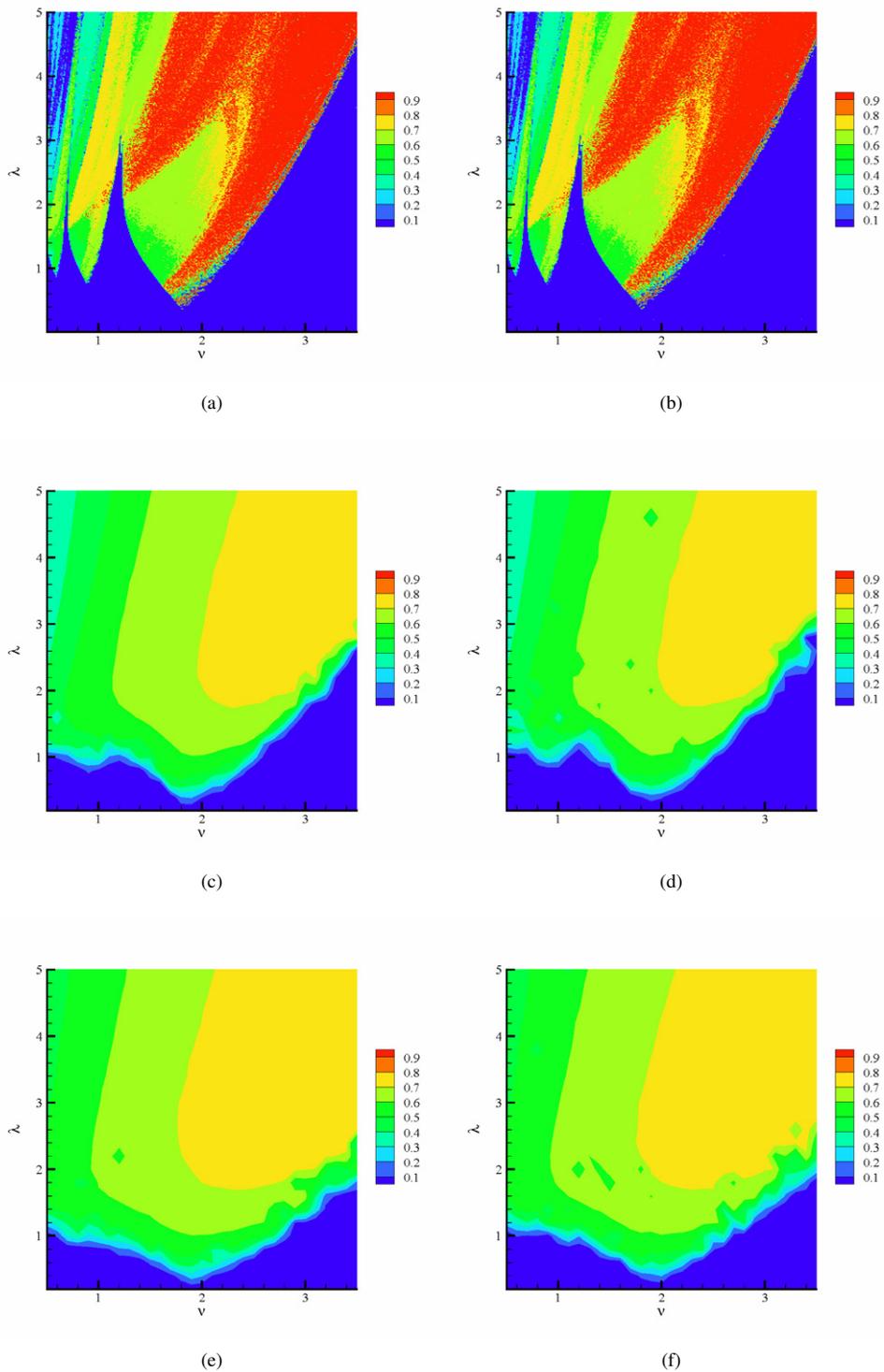


Fig. 4. Parameter space plots of Eq 7 and nonlinear damping as in Eq 9 for  $\gamma = 0.016$ : (a)  $D = 0.0, R = 0.0$ ; (b)  $D = 0.0, R = 2.45 \times 10^{-4}$ ; (c)  $D = 0.1, R = 0.0$ ; (d)  $D = 0.1, R = 2.45 \times 10^{-4}$ ; (e)  $D = 0.3, R = 0.0$ ; (f)  $D = 0.3, R = 2.45 \times 10^{-4}$ .

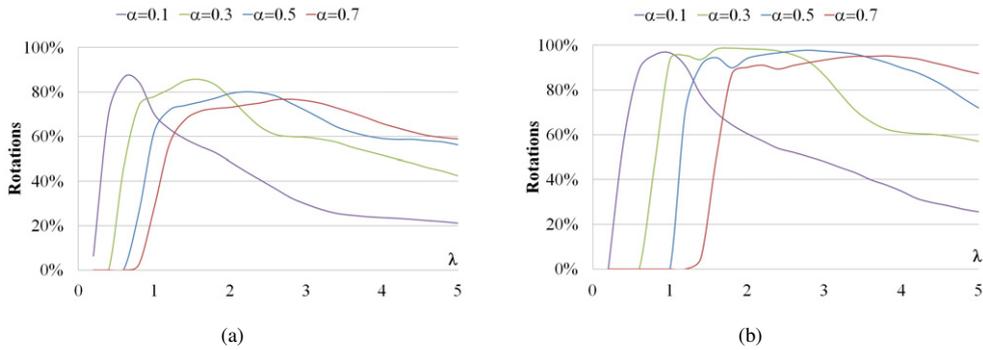


Fig. 5. Numerical simulations of Eq 10 for: (a)  $v = 2, \eta = 2, D = 0.3, \gamma = 0.3$ ; (b)  $v = 3, \eta = 3, D = 0.3, \gamma = 0.3$ .

column, one could conclude that the nonlinear term of the damping function Eq 9 could be ignored without any major change in the simulations. Thus, for the purpose of this paper the system will be treated as described in Eq 7.

#### 4. Single-degree-of-freedom base excitation

To overcome the destructive effect of large noise intensity it is proposed to filter the excitation by using a SDOF system (Fig 3(b)) to amplify the amplitude and recover the region of dominant rotational motion.

The dynamics of such a system may be described by the system of following differential equations:

$$\begin{aligned} \ddot{\theta} + c\dot{\theta} + (\Omega^2 - \ddot{x}/l) \sin \theta &= 0, \\ \ddot{x} + 2\alpha\dot{x} + \eta^2 x &= 2\alpha\omega\lambda \sin q(t) - \eta^2\lambda \cos q(t) + g + \frac{m}{M+m}l\ddot{\theta} \sin \theta + \frac{m}{M+m}l\dot{\theta}^2 \cos \theta, \\ \dot{q}(t) &= \omega + \sigma\zeta(t) \end{aligned} \tag{10}$$

where  $\omega$  - the mean excitation frequency,  $\eta$  - natural frequency of the SDOF system. Obviously in the limiting case of the infinitely large natural frequency  $\eta$  the response of the rotational system will be similar to one studied in the previous section. It should be stressed that higher values of the natural frequency  $\eta$  will result in higher response amplitude since in the base excitation the frequency explicitly amplifies the amplitude of oscillations.

In addition, considering  $m \ll M$  and transforming appropriately the second equation in Eq 10 leaves us with only the first two terms in the RHS. To compare the results of numerical simulation with previous results  $\lambda - v$  notation is kept as previously. Fig 5(a) demonstrates the results of numerical simulation for primary resonance  $v = 2, \eta = 2, D = 0.3, \gamma = 0.3$  and different values of viscous damping coefficient  $\alpha$ . It can be seen from Fig 5(a) that although the rotational motion starts at smaller values of  $\lambda$  none of the curves reaches 90% threshold. Moreover, smaller values of  $\alpha$  provide a relative short domain for high percentage rotational motion. On the contrary, relatively large values of viscous damping provide wide region of rotational motion. Obviously, the larger the rotational region the easier to keep the system in this region.

Fig 5(b) demonstrates numerical results for the system when  $v = 3, \eta = 3, D = 0.3, \gamma = 0.3$  with different values of the damping coefficient. It can be observed that all the curves cross the 90% line, however smaller values of the damping coefficient provide a narrower region of rotational motion. On the other hand, the low boundary of the dominant rotational domain (90%) starts from smaller values of  $\lambda$  when the damping coefficient is small, so that for  $\alpha = 0.1$  the dominant domain is reached at  $\lambda = 0.5$ , whereas for  $\alpha = 0.5$  it is reached when  $\lambda = 1.2$ . This result is well expected since smaller values of damping provide large response amplification.

Having seen the influence of the SDOF system introduction between the pendulum and the wave excitation for certain frequencies, a more wide investigation on the parameter space ( $v, \lambda$ ) is reasonable to follow. Fig 6 depicts results of the numerical simulations of Eq 10 over the parameter space for  $m/M = 0$ . It should be noted that the examined frequency range could not exceed certain values since large detuning would eliminate the narrow-band characteristics of the SDOF response limiting to a white noise. Thus, the previously examined frequency range (0.5

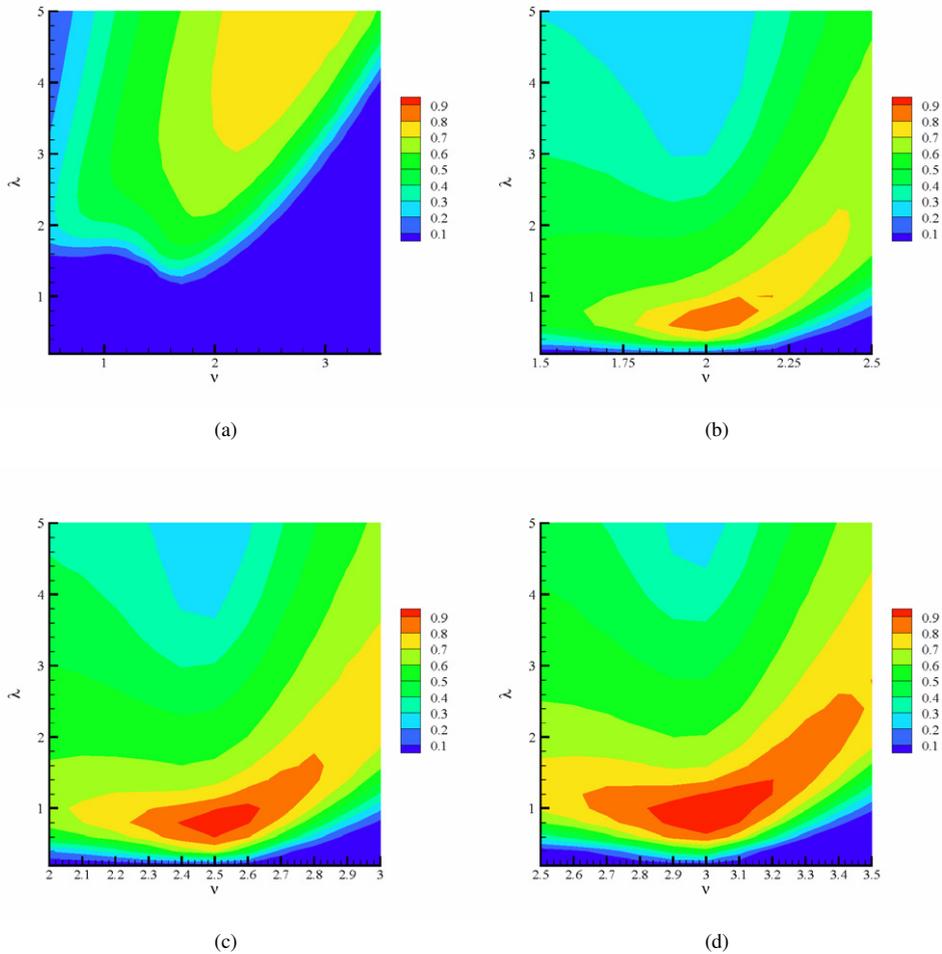


Fig. 6. Parameter space plots of Eq 10 for  $D = 0.3$ ,  $\gamma = 0.3$  and  $\alpha = 0.1$ : (a) without SDOF; (b)  $\eta = 2$ ; (c)  $\eta = 2.5$ ; (d)  $\eta = 3$ .

3.5) was divided to 3 regions (0.5 - 1.5), (1.5 - 2.5) and (2.5 - 3.5) with each one's central value also chosen as the natural frequency,  $\eta$ . Several simulations showed that the last frequency range is the most favourable, providing higher rotational motion ratio. This is why only simulations for this range and the adjacently lower one are being shown in Fig 6 for 3 different  $\eta$  values.

First, the comparison of results in the parameter space plot with and without the SDOF system for the same noise intensity (Fig 2(c)), is made for different values of the SDOF's system natural frequency  $\eta$  and  $\alpha = 0.1$ . It can be seen from Fig 6, which has already been reported from Fig 5, that namely the larger values of the natural frequency provide larger instability area with dominant rotational motion. This result is expected taking into account the deterministic map, which shows large range of  $\lambda$  for larger values of  $\nu$ . However, to reach these values, larger natural frequency of the SDOF system is needed. It is also important, that the values of  $\lambda$  for these regions are relatively small, which indicates that the rotational motion can be observed for small values of wave amplitude.

Fig 7 demonstrates the numerical results for the same value of natural frequency  $\eta = 3$  and three different values of the viscous damping coefficient  $\alpha$ . It can be seen that, while the highest rotational ratio has been between 70-80% for a rather narrow region, here a rather wide area of dominant rotational region ( $> 90\%$ ) has appeared (Fig 7(c) or Fig 7(d)). Moreover, this region instead of initiating at high amplitude values ( $\lambda > 6$ ), it starts from as low as  $\lambda = 0.6$

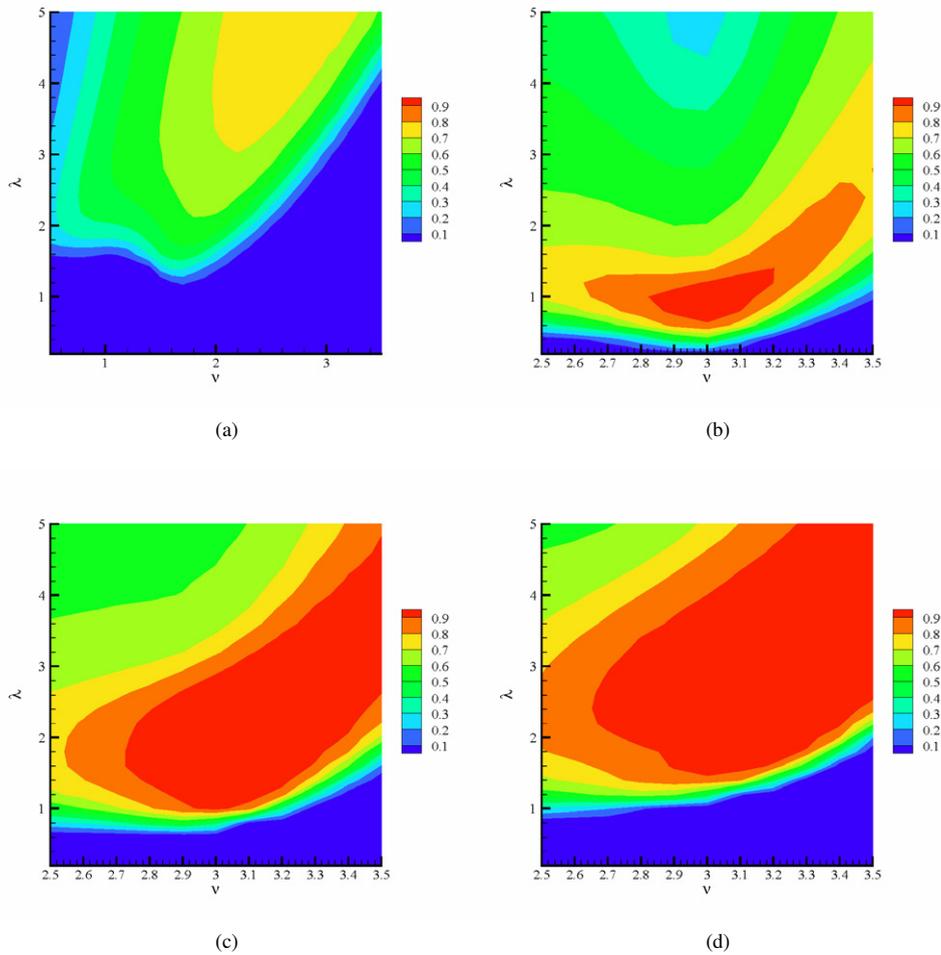


Fig. 7. Parameter space plots of Eq 10 for  $D = 0.3$ ,  $\gamma = 0.3$  and  $\eta = 3$ : (a) without SDOF system; (b)  $\alpha = 0.1$ ; (c)  $\alpha = 0.3$ ; (d)  $\alpha = 0.5$ .

even for the narrow region case of low  $\alpha$  values. Larger values of damping shift the boundary of the rotational region upwards, while simultaneously widening the area. This is not only the result of the amplitude amplification but also of the filtering out the fast fluctuations of the noise due to the property of the SDOF system. In general, the parameter space plots in Fig 7 confirm the observations made for Fig 5. Low  $\alpha$  values initiate rotational region at lower  $\lambda$  values, since the former provides higher values of mean response amplitude of the SDOF system. On the other hand, low values of  $\alpha$  provide large values of mean square amplitude, leading to large amplitude fluctuations, which results in narrowing down the instability domain. Thus, the range of  $\lambda$ 's for which rotations persist is narrower for lower  $\alpha$  values.

## 5. Conclusions

The potential of rotational motion for a lumped mass pendulum excited vertically by ocean waves was discussed. First approach on the stochastic system showed that rotations are achievable, however not as robust as one would require. Larger values of excitation amplitude may be required to force the system to the rotational mode. Furthermore, relatively large values of noise intensity, for instance  $D = 0.3$ , was chosen from the comparison against the realistic

Pierson-Moskowitz empirical spectrum, tend to tune off the system from the rotations, deteriorating the system's performance. The introduction of an SDOF linear system intermediate to the actual wave excitation and the pendulum was proven highly favourable since the resulting filtered excitation allows widening the regions of dominant rotations ( $> 90\%$ ) for smaller excitation amplitude values. In addition the influence of the system's damping coefficient  $\alpha$  and natural frequency  $\eta$  was examined. It has been established that higher values of damping result in rotational regions being more robust, intense and the rotating system is better protected against frequency variations. Furthermore, several control options arise apart from adjusting the pendulum's natural frequency, through appropriate configuration of the damping coefficient  $\alpha$ .

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