Quotients of d-Frames

Jakl, Tomas; Jung, Achim; Pultr, Ales

DOI:
10.1007/s10485-018-09553-7

Citation for published version (Harvard):
Jakl, T, Jung, A & Pultr, A 2019, 'Quotients of d-Frames', *Applied Categorical Structures*, vol. 27, no. 3, pp. 261-275. https://doi.org/10.1007/s10485-018-09553-7

Quotients of d-frames

Tomáš Jakl   Achim Jung   Aleš Pultr

December 4, 2017

Abstract

It is shown that every d-frame admits a complete lattice of quotients. Quotienting may be triggered by a binary relation on one of the two constituent frames, or by changes to the consistency or totality structure, but as these are linked by the reasonableness conditions of d-frames, the result in general will be that both frames are factored and both consistency and totality are increased.

Introduction

Motivated by the question of how to structure a knowledge base for an expert system, Belnap [Bel76, Bel76] suggested that it would be inevitable and useful to deal independently with positive and negative evidence for a given proposition. Thus he was led to a four-valued logic which, in addition to the traditional truth values, makes explicit the situation that evidence may be missing or contradictory. His proposal was taken up by several researchers and we point the interested reader to the work of Arieli and Avron, [AA96], and Rivieccio, [Riv10]. For our purposes it suffices to note that both positive and negative evidence can naturally be thought of as having the structure of a distributive lattice. Since evidence may accrue indefinitely over time, one is led furthermore to assume that they form a directed-complete partial order (dcpo) in the sense of Scott's domain theory, [Sco82, AJ94]. Combining those two structures one obtains two frames L_+ and L_-, the carrier sets of *d-frames*.

Historically, frame theory arose as a point-free approach to topology, [Joh82, PP11], but the interconnection between topology and logic is well established through the work of Smyth, Abramsky, and Vickers, [Vic89, Abr91, Smy92]. Since we are dealing with two frames, the duality between frames and topological spaces becomes a duality between d-frames and *bitopological* spaces (X, τ_+, τ_-). On the latter it is natural to consider when open sets $O \in \tau_+$ and $U \in \tau_-$ are disjoint, likewise, when they together cover the space X. On the algebraic side one therefore adds a *consistency* and a *totality* relation to capture these two fundamental situations. Thus one arrives at the idea of a *d-frame*, comprising two frames L_+ and L_- and two relations $\mathsf{con}, \mathsf{tot} \subseteq L_+ \times L_-$. The work [JM06] works out the ramifications of these ideas from the point of view of Stone duality.

In the logical interpretation one views a pair $\alpha = (\alpha_+, \alpha_-) \in L_+ \times L_-$ as evidence for the truth, respectively, falsity, of a logical proposition ϕ. It is then natural to view another such pair β as *more informative* if $\alpha_+ \leq \beta_+$ and $\alpha_- \leq \beta_-$ hold. This is the *information order* $\sqsubseteq$ on $L_+ \times L_-$. On the other hand, if $\alpha_+ \leq \beta_+$ but $\alpha_- \geq \beta_-$ then one may say that α provides more evidence than β for the *truth* of ϕ. This yields the *logical order* $\leq$ on $L_+ \times L_-$[1]. It is clear that $L_+ \times L_-$ is a frame in the information order and a distributive lattice in the logical order.

In defining d-frames, we should take note of how consistency and totality interact with both the information and logical structure of d-frames. This leads to a set of axioms dubbed *reasonableness conditions* and studied in some detail in [JM06]. It turns out that they are essential for a satisfactory theory of d-frames and their duality with bitopological spaces. However, they have hitherto posed a formidable obstacle to a treatment of *quotients* for d-frames. It is the purpose of this paper to show how to overcome this difficulty.

As we will see, the reasonableness conditions on con and tot fall naturally into two classes, which (for the purposes of this introduction only) may be called "algebraic" and "structural". While the former are inherited by frame quotients, the latter are typically not. The task, then, is to modify the well-known factorization of frames so as to maintain or regain validity of the structural axioms. The modification will take the form of a reflection from a category of "proto d-frames" where only the algebraic axioms are assumed, to the category of reasonable d-frames. This solves the problem of quotients, but also gives more information on the categories in question, notably on limits and colimits, and on an (extremal epi – mono) factorization system.

[1]We hope that the double use of $\leq$ for both the internal order on each of the two frames and the logical order on their product will not lead to too much difficulty for the reader.

The paper is organized as follows. In Section 1 we introduce the necessary notation and facts from previous work. Section 2 is then devoted to a first construction of the desired reflection. It is fairly natural, but uses an intersection of a perhaps not quite transparent system of "reasonable approximations" of the d-frame structure. Therefore we also present in Section 4 an iterative procedure which gives a more detailed picture about what is going on. Before that, in Section 3, we introduce and analyze some expedient auxiliary techniques.

1 Preliminaries

1.1. For subsets A of a poset $(X, \leq)$ we write as usual ${\downarrow}A = \{x \mid \exists a \in A,\ x \leq a\}$ and ${\uparrow}A = \{x \mid \exists a \in A,\ x \geq a\}$, and say that A is a *down-set* resp. *up-set* if $A = {\downarrow}A$ resp. $A = {\uparrow}A$, and abbreviate ${\downarrow}\{a\}$ resp. ${\uparrow}\{a\}$ to ${\downarrow}a$ resp. ${\uparrow}a$.

The suprema in lattices will be denoted by $\bigvee A$, $a \vee b$ when we are dealing with individual frames or with the logical order, and by $\bigsqcup A$, $a \sqcup b$ when we refer to the information order. We make the analogous distinction for infima.

1.2. Recall that monotone maps $f\colon X \to Y$ and $g\colon Y \to X$ are *adjoint, f to the left and g to the right*, written $f \dashv g$, if

$$f(x) \leq y \iff x \leq g(y).$$

If $f \dashv g$ then f preserves all existing suprema and g preserves all existing infima. Furthermore, if X and Y are complete lattices then a monotone map $f\colon X \to Y$ (resp. $g\colon Y \to X$) preserves all suprema (resp. infima) iff it is a left (resp. right) adjoint.

1.3. Frames. A *frame* is a complete lattice L satisfying the distributivity law

$$(\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\} \tag{frm}$$

for all $A \subseteq L$ and $b \in L$. A *frame homomorphism* preserves all joins and all finite meets.

1.3.1. The equation (frm) states that the maps $(x \mapsto x \wedge b)\colon L \to L$ preserve all joins. Hence, by 1.2, every frame is a Heyting algebra with the Heyting operation $x \mapsto (b \to x)$ satisfying

$$a \wedge b \leq c \iff a \leq b \to c.$$

1.4. Working with relations. For a relation R we write $R^{-1} = \{(x, y) \mid (y, x) \in R\}$. If $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ we write

$$R\,; S = \{(x, z) \mid \exists y, (x, y) \in R, (y, z) \in S\}$$

1.5. Quotients of frames. Taking quotients of frames is very simple. Let $R \subseteq L \times L$ be an arbitrary relation. An element $s \in L$ is said to be *R-saturated* if

$$\forall a, b \in L. \quad aRb \;\Rightarrow\; a \to s = b \to s$$

or, via the left adjoint, if

$$\forall a, b, c \in L. \quad aRb \;\Rightarrow\; (a \wedge c \leq s \text{ iff } b \wedge c \leq s).$$

The system L/R of all saturated elements is closed under meets and if we define $q_R = q\colon L \to L/R$ by setting $q_R(a) = \bigwedge\{s \in L/R \mid a \leq s\}$ we obtain a frame homomorphism such that $q_R(a) = q_R(b)$ for all $(a, b) \in R$. Furthermore, if $h\colon L \to M$ is such that for all $(a, b) \in R$, $h(a) = h(b)$, then there is precisely one $\widetilde{h}\colon L/R \to M$ with $\widetilde{h} \circ q_R = h$, and for $s \in L/R$ one has $\widetilde{h}(s) = h(s)$. (See [PP11], III.11.3.1.)

Consequently, in particular, the kernel of q_R,

$$E(q_R) = \{(x, y) \mid q_R(x) = q_R(y)\}\ ,$$

is the smallest (frame) congruence on L containing R, and we have $L/E \cong L/R$.

The symbol

$$E(h) = \{(x, y) \mid h(x) = h(y)\} \tag{E}$$

will also be used for arbitrary frame homomorphisms $h\colon L \to M$.

1.6. Conventions about pairs of frames. In the sequel we will work with pairs of frames, the first indexed with $+$, the second with $-$. As explicated in the Introduction, for a pair (L_+, L_-) we will consider two orders on the product $L_+ \times L_-$: the *information order* $\sqsubseteq$ defined by $(x_+, x_-) \sqsubseteq (y_+, y_-)$ if $x_+ \leq y_+$ and $x_- \leq y_-$, and the *logical order* $\leq$ defined by $(x_+, x_-) \leq (y_+, y_-)$ if $x_+ \leq y_+$ and $x_- \geq y_-$. We will use the symbols ff resp. tt for the smallest resp. largest element in $\leq$, that is, for the pairs $(0, 1)$ resp. $(1, 0)$. Following this convention one thinks about the product $L_+ \times L_-$ as carrying two (distributive) lattice structures

$$x \sqcup y = (x_+ \vee y_+, x_- \vee y_-), \qquad x \sqcap y = (x_+ \wedge y_+, x_- \wedge y_-),$$
$$x \vee y = (x_+ \vee y_+, x_- \wedge y_-), \qquad x \wedge y = (x_+ \wedge y_+, x_- \vee y_-).$$

An element $x \in L_+ \times L_-$ has coordinates (x_+, x_-), and a pair of maps $(h_+ \colon L_+ \to M_+,\ h_- \colon L_- \to M_-)$ is often written as $h \colon (L_+, L_-) \to (M_+, M_-)$. For such an h we consider the map $h_\times \colon L_+ \times L_- \to M_+ \times M_-$ sending (x_+, x_-) to $(h_+(x_+), h_-(x_-))$; if there is no danger of confusion, though, the subscript $\times$ will be omitted.

Using the symbol $x_\pm$ for "x_+ resp. x_-" is obvious.

1.7. Scott topology. Recall that in the Scott topology (see e.g. [AJ94, GHK$^+$03]) on a poset a subset A is closed iff it is closed under joins of directed subsets $D \subseteq A$ (which we denote by $\bigsqcup^{\uparrow} D$). If we speak of a closure in the context of a specified poset, or if we use the symbol $\overline{A}$, we have in mind the closure in the associated Scott topology.

Note that frame homomorphisms are obviously continuous with respect to the associated Scott topologies.

1.8. d-Frames. A *proto-d-frame* is a quadruple $(L_+, L_-, \mathsf{con}, \mathsf{tot})$ where L_+, L_- are frames together with the relations of *consistency* $\mathsf{con} \subseteq L_+ \times L_-$ and *totality* $\mathsf{tot} \subseteq L_+ \times L_-$ such that

(con–$\downarrow$)	con is a down-set wrt. $\sqsubseteq$,
(tot–$\uparrow$)	tot is an up-set wrt. $\sqsubseteq$,
(tt, ff)	$\{\mathit{tt}, \mathit{ff}\} \subseteq \mathsf{con} \cap \mathsf{tot}$,
($\wedge, \vee$)	both con and tot are sublattices wrt. the logical order $\leq$.

A proto-d-frame is con-*saturated* if

(con–$\bigsqcup^{\uparrow}$) con is (Scott-)closed in $(L_+ \times L_-, \sqsubseteq)$, and *balanced* if

$$(\mathsf{con}\text{–}\mathsf{tot}) \qquad x \in \mathsf{con}, y \in \mathsf{tot} \text{ and } (x_+ = y_+ \text{ or } x_- = y_-) \quad \Rightarrow \quad x \sqsubseteq y.$$

A con-saturated and balanced d-frame is referred to as a *reasonable d-frame*, or simply as a *d-frame*.

A *d-frame homomorphism* $h\colon (L_+, L_-, \mathsf{con}_L, \mathsf{tot}_L) \to (M_+, M_-, \mathsf{con}_M, \mathsf{tot}_M)$ is a pair $h\colon (L_+, L_-) \to (M_+, M_-)$ of frame homomorphisms such that $h[\mathsf{con}_L] \subseteq \mathsf{con}_M$ and $h[\mathsf{tot}_L] \subseteq \mathsf{tot}_M$. The resulting category will be denoted by

$$\mathbf{pdFrm},$$

and the full subcategory of (reasonable) d-frames by

$$\mathbf{dFrm}.$$

1.8.1. Sometimes we will consider the subcategory of the proto-d-frames that are just assumed to be con-saturated resp. balanced. Then we use the symbols

$$\mathbf{pdFrm}_c \quad \text{resp.} \quad \mathbf{pdFrm}_b.$$

1.9. We use only basic concepts of category theory. The reader may consult, e.g., [Mac71] or [AHS90].

2 A reflection of pdFrm onto dFrm

2.1. Taking quotients in the category **pdFrm** is as easy as the procedure in the category of frames, described in 1.5.

Given a proto-d-frame $\mathcal{L} = (L_+, L_-, \mathsf{con}, \mathsf{tot})$ and congruences (or, for that matter just relations) $R_\pm$ on $L_\pm$ consider the quotient maps $q_R = (q_+, q_-)$ with $q_\pm\colon L_\pm \to L_\pm/R_\pm$. Set

$$\mathcal{L}/R = (L_+/R_+, L_-/R_-, q[\mathsf{con}], q[\mathsf{tot}]).$$

2.1.1. Proposition. *$\mathcal{L}/R$ is a quotient of $\mathcal{L}$ in* **pdFrm**. *More precisely, $\mathcal{L}/R$ is a proto-d-frame, and for every d-frame homomorphism $h\colon \mathcal{L} \to \mathcal{M}$ for which $(a,b) \in R_\pm$ implies $h_\pm(a) = h_\pm(b)$, there is precisely one d-frame homomorphism $\widetilde{h}\colon \mathcal{L}/R \to \mathcal{M}$ such that $\widetilde{h} \circ q = h$.*

Proof. Checking the axioms (tt, ff) and ($\wedge,\vee$) is straightforward. Now let $x\in\mathsf{con}$ and $y\sqsubseteq q(x)$. Since q is onto we have x' such that $y=h(x')$. Then $x\sqcap x'\in\mathsf{con}$ by (con–$\downarrow$), and $h(x\sqcap x')=y$ which proves (con–$\downarrow$); similarly we see that (tot–$\uparrow$) holds and we obtain that $(L_+/R_+,L_-/R_-,q[\mathsf{con}],q[\mathsf{tot}])$ is a proto-d-frame.

The second statement immediately follows from 1.5. $\square$

2.2. With the axioms of con-saturatedness and balance it is another matter. They are not generally preserved by (d-)frame homomorphisms. In this section we will construct a reflection of **pdFrm** onto **dFrm** which addresses this problem.

2.2.1. Reasonable congruence structures. For dealing with the remaining axioms we introduce the following technical definition.

A quadruple $(\mathsf{con},\mathsf{tot},R_+,R_-)$, where $R_\pm$ are (frame) congruences on $L_\pm$, will be called a *reasonable congruence structure* on (L_+,L_-) if

(R1) con and tot are sublattices of $(L_+\times L_-,\leq)$,

(R2) $\leq_+;\mathsf{con};\geq_-\ \subseteq\ \mathsf{con}$ and $\geq_+;\mathsf{tot};\leq_-\ \subseteq\ \mathsf{tot}$,

(R3) $R_+;\mathsf{con};R_-\ \subseteq\ \mathsf{con}$ and $R_+;\mathsf{tot};R_-\ \subseteq\ \mathsf{tot}$,

(R4) $\mathsf{con};\mathsf{tot}^{-1}\ \subseteq\ R_+;\leq_+;R_+$ and $\mathsf{con}^{-1};\mathsf{tot}\ \subseteq\ R_-;\leq_-;R_-$,

(R5) $\overline{\mathsf{con}}=\mathsf{con}$ in $(L_+\times L_-,\sqsubseteq)$.

2.2.2. Observation. *For any (lattice) congruence R, $R;\leq\ =\ \leq;R$.*

Indeed, if $xRy\leq z$ then $x\leq x\vee zRy\vee z=z$ and similarly for the reverse inclusion.

2.2.3. Corollary. *Condition* (R4) *is equivalent with*

$$\mathsf{con};\mathsf{tot}^{-1}\ \subseteq\ \leq_+;R_+ \quad\text{and}\quad \mathsf{con}^{-1};\mathsf{tot}\ \subseteq\ \leq_-;R_-.$$

2.3. Proposition. *Let $(\mathsf{con},\mathsf{tot},R_+,R_-)$ be a reasonable congruence structure on (L_+,L_-) and let $q_\pm$ be the quotient homomorphisms $L_\pm\to L_\pm/R_\pm$. Then*

$$\mathcal{L}/R=(L_+/R_+,L_-/R_-,q[\mathsf{con}],q[\mathsf{tot}])$$

is a reasonable d-frame.

Proof. We already know that $\mathcal{L}/R$ is a proto-d-frame, so it remains to show that it is con-saturated and balanced. To establish the former, we need to show that $q[\mathsf{con}]$ is Scott-closed, i.e., downward closed and closed under the formation of directed suprema. Downward closure is part of being a proto-d-frame so this was shown in 2.1.1 already. So let D be a directed subset of $q[\mathsf{con}]$. We claim that $q^{-1}[D]$ is directed in $\mathcal{L}$: Let $x_1, \ldots, x_n$ be a finite set of elements in $q^{-1}[D]$. Since D is directed, there is $y \in D$ which is above all $q(x_i)$. Let x be an element of $\mathcal{L}$ that is mapped to y. We get that $q(x \sqcup \bigsqcup_{i=1}^n x_i) = y$ and hence the element $x \sqcup \bigsqcup_{i=1}^n x_i$ belongs to $q^{-1}[D]$; it is clearly an upper bound for the x_i.

Now note that (R3) implies that $q^{-1}[q[\mathsf{con}]] \subseteq \mathsf{con}$ and hence the directed set $q^{-1}[D]$ is contained in con. Condition (R5) implies that the supremum of $q^{-1}[D]$ also belongs to con; clearly, it is mapped to the supremum of D by the frame homomorphism q.

Finally we show that $\mathcal{L}/R$ satisfies (con–tot). Let $x \in \mathsf{con}$, $y \in \mathsf{tot}$ be such that $q_-(x_-) = q_-(y_-)$. This means that $x_+ \ \mathsf{con}\ x_- \ R_- \ y_- \ \mathsf{tot}^{-1}\ y_+$. By (R3) we can simplify this to $x_+ \ \mathsf{con}\ y_- \ \mathsf{tot}^{-1}\ y_+$ and (R4) now tells us that $(x_+, y_+) \in R_+ \,;\leq_+ ;R_+$, hence $q_+(x_+) \leq q_+(y_+)$. $\square$

2.3.1. Note that this proof depends on the property $E(q_\pm) = R_\pm$, guaranteed by (R3); this was not needed for proving Proposition 2.1.1.

2.4. Proposition. *The set of all reasonable congruence structures on a pair (L_+, L_-) is closed under (coordinatewise) intersections.*

Proof. Let $\{(\mathsf{con}^i, \mathsf{tot}^i, R_+^i, R_-^i) \mid i \in J\}$ be a collection of reasonable congruence structures. If J is void we have the trivial intersection (more precisely, void meet) $(L_+{\times}L_-,\ L_+{\times}L_-,\ L_+{\times}L_+,\ L_-{\times}L_-)$ which is reasonable. Hence we can consider the meets $(\bigcap_J \mathsf{con}^i, \bigcap_J \mathsf{tot}^i, \bigcap_J R_+^i, \bigcap_J R_-^i)$ with non-void J. Obviously this system satisfies (R1),(R2) and (R5).

(R3): We have $(\bigcap_J R_+^i) ; (\bigcap_J \mathsf{con}^i) ; (\bigcap_J R_-^i) \subseteq R_+^j ; \mathsf{con}^j ; R_-^j \subseteq \mathsf{con}^j$ for each individual j, and hence $(\bigcap_J R_+^i) ; (\bigcap_J \mathsf{con}^i) ; (\bigcap_J R_-^i) \subseteq (\bigcap_J \mathsf{con}^j)$.

(R4): As in (R3), we obtain that $(\bigcap_J \mathsf{con}^i) ; (\bigcap_J \mathsf{tot}^i)^{-1} \subseteq \bigcap_J (\leq_+ ; R_+^i)$. Thus, we have to show that $\bigcap_J (\leq_+ ; R_+^i) \subseteq \leq_+ ; (\bigcap_J R_+^i)$. Let $z \leq_+ x_i R_+^i w$ for all $i \in J$. Since R_+^i is a congruence, $(x_i \vee w) R_+^i w$. Since $w \leq \bigwedge_j (x_j \vee w) \leq x_i \vee w$ and since $R_\pm^i$-equivalence classes are convex we see, further, that

$\bigwedge_j (x_j \vee w) R^i_+ w$ for all $i \in J$, so that

$$\bigwedge_j (x_j \vee w)\ (\bigcap_i R^i_+)\ w.$$

Since $z \leq_+ \bigwedge_j x_j \leq_+ \bigwedge_j (x_j \vee w)$ we conclude that $(z, w) \in (\leq_+ ; \bigcap_J R^i_+)$. $\square$

2.5. Proposition. *Let $\mathcal{M} = (M_+, M_-, \mathsf{con}, \mathsf{tot})$ be a reasonable d-frame and let $h_\pm \colon L_\pm \to M_\pm$ be a pair of frame homomorphisms. Then*

$$(h^{-1}[\mathsf{con}], h^{-1}[\mathsf{tot}], E(h_+), E(h_-))$$

is a reasonable congruence structure on (L_+, L_-).

Proof. (R1) immediately follows from the definitions of ff, tt and the lattice structure of $(L_+ \times L_-, \leq)$, and the fact that $h_\pm$ are frame, hence lattice homomorphisms.

(R2): Let $x \sqsubseteq y \in h^{-1}[\mathsf{con}]$. Then $h(x) = h(x \sqcap y) = h(x) \sqcap h(y)$ and since con is a down-set, $x \in h^{-1}[\mathsf{con}]$.

(R3): If $(a, b) \in E(h_+) ; h^{-1}[\mathsf{con}] ; E(h_-)$ then we have some a', b' such that $h_+(a) = h_+(a')\, \mathsf{con}\, h(b') = h(b)$ and hence $(a, b) \in h^{-1}[\mathsf{con}]$. Similarly for tot.

(R4): Let $(a, c) \in h^{-1}[\mathsf{con}] ; (h^{-1}[\mathsf{tot}])^{-1}$. Hence, there is a b such that $(h_+(a), h_-(b)) \in \mathsf{con}$ and $(h_+(c), h_-(b)) \in \mathsf{tot}$. Since $\mathcal{M}$ is balanced, $h_+(a) \leq h_+(c)$, and hence $h_+(a \vee c) = h_+(a) \vee h_+(c) = h_+(c)$. Thus, $(a \vee c) E(h_+) c$, and $a \leq_+ (a \vee c) E(h_+) c$, making $(a, c) \in (\leq_+ ; E(h_+))$.

(R5): Since $h \colon L_+ \times L_- \to M_+ \times M_-$ is a frame homomorphism in the order $\sqsubseteq$, it is obviously Scott continuous and hence $h^{-1}[\mathsf{con}]$ is closed since con is. $\square$

2.6. For a proto-d-frame $\mathcal{L} = (L_+, L_-, \mathsf{con}, \mathsf{tot})$ consider by 2.4 the intersection of all reasonable congruence structures (that is, the smallest reasonable congruence structure) $(\mathsf{con}', \mathsf{tot}', R_+, R_-)$ on (L_+, L_-), such that $\mathsf{con} \subseteq \mathsf{con}'$ and $\mathsf{tot} \subseteq \mathsf{tot}'$ and set

$$\mathfrak{r}(\mathcal{L}) = (L_+/R_+, L_-/R_-, \kappa[\mathsf{con}'], \kappa[\mathsf{tot}'])$$

where $\kappa = \kappa_{\mathcal{L}} = (q_+, q_-)$ is the pair of the natural quotient maps $q_\pm \colon L_\pm \to L_\pm/R_\pm$.

2.6.1. Theorem. **dFrm** *is a reflective subcategory of* **pdFrm**, *with the reflection given by the homomorphisms* $\kappa_{\mathcal{L}} \colon \mathcal{L} \to \mathfrak{r}(\mathcal{L})$.

Proof. Take a proto-d-frame $\mathcal{L} = (L_+, L_-, \mathsf{con}_L, \mathsf{tot}_L)$ and a d-frame $\mathcal{M} = (M_+, M_-, \mathsf{con}_M, \mathsf{tot}_M)$. We have to prove that for each homomorphism $h \colon \mathcal{L} \to \mathcal{M}$ there is a homomorphism $\widetilde{h} \colon \mathfrak{r}(\mathcal{L}) \to \mathcal{M}$ such that $\widetilde{h} \circ \kappa_{\mathcal{L}} = h$. By 2.5, $(h^{-1}[\mathsf{con}_M], h^{-1}[\mathsf{tot}_M], E(h_+), E(h_-))$ is a reasonable congruence structure on (L_+, L_-) and hence

$$\mathsf{con}' \subseteq h^{-1}[\mathsf{con}],\ \mathsf{tot}' \subseteq h^{-1}[\mathsf{tot}],\ R_+ \subseteq E(h_+) \text{ and } R_- \subseteq E(h_-).$$

By the third and fourth inclusion and by 1.5, there are frame homomorphisms $\widetilde{h}_\pm$ such that $h_\pm = \widetilde{h}_\pm \circ \kappa_\pm$. The first and second inclusions yield $h[\mathsf{con}'] \subseteq \mathsf{con}_M$ and $h[\mathsf{tot}'] \subseteq \mathsf{tot}_M$, and since (recall 1.5 again) $\widetilde{h}_\pm$ are restrictions of $h_\pm$ we conclude that $\widetilde{h}[\mathsf{con}'] \subseteq \mathsf{con}_M$ and $\widetilde{h}[\mathsf{tot}'] \subseteq \mathsf{tot}_M$.

Unicity is obvious since the $\kappa_\pm$ are onto. □

2.6.2. Proposition. *The reflection homomorphisms $\kappa_{\mathcal{L}}$ are extremal epimorphisms in* **dFrm**.

Proof. We have to prove that if $\kappa = m \circ f$ and m is a monomorphism them m is an isomorphism. By 2.6.1 we have a homomorphism $\widetilde{f}$ such that $\widetilde{f} \circ \kappa = f$. Then $m \circ \widetilde{f} \circ \kappa = m \circ f = \kappa$ and since κ is onto, $m \circ \widetilde{f} = \mathrm{id}$. Consequently $m \circ \widetilde{f} \circ m = m$ and since m is a monomorphism we see that also $\widetilde{f} \circ m = \mathrm{id}$. □

2.6.3. Notes. 1. It is easy to see that **pdFrm** is a complete and cocomplete category. From 2.6.1 we now see that **dFrm** is complete and cocomplete, and how the limits and colimits look like.

2. Recall 1.8.1. The reflection procedure above can be easily modified to obtain reflections of **pdFrm** onto $\mathbf{pdFrm}_c$ resp. $\mathbf{pdFrm}_b$.

3. Extremal epimorphisms in the category **dFrm** were recently characterised by Imanol Carollo and M. Andrew Moshier, [CM17].

2.7. Quotients in dFrm. First, recall the standard extension of the reflector $\mathfrak{r}$ to a functor: for a morphism $h \colon \mathcal{L} \to \mathcal{M}$ in **pdFrm** there is precisely one $\mathfrak{r}(h)$ in **dFrm** such that $\mathfrak{r}(h) \circ \kappa_{\mathcal{L}} = \kappa_{\mathcal{M}} \circ h$. Hence we have commutative diagrams

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\kappa_{\mathcal{L}}} & \mathfrak{r}(\mathcal{L}) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathfrak{r}(h)} \\ \mathcal{M} & \xrightarrow{\kappa_{\mathcal{M}}} & \mathfrak{r}(\mathcal{M}) \end{array}$$

in other words, the system $(\kappa_{\mathcal{L}})_{\mathcal{L}}$ is a natural transformation. Note that for $\mathcal{L}$ in **dFrm** we have $\mathfrak{r}(\mathcal{L}) = \mathcal{L}$ and $\kappa_{\mathcal{L}} = \mathrm{id}$.

2.7.1. Let $\mathcal{L}$ be a (reasonable) d-frame and let $R = (R_+, R_-)$ be a pair of relations, $R_\pm$ on $L_\pm$. Recall the proto-d-frame $\mathcal{L}/R$ and the quotient map $q\colon \mathcal{L} \to \mathcal{L}/R$ from 2.1. Applying the reflection we obtain the d-frame $\mathfrak{r}(\mathcal{L}/R)$ and the morphism $\kappa\colon \mathcal{L}/R \to \mathfrak{r}(\mathcal{L}/R)$. We set

$$q_{\mathfrak{r}} = \kappa \circ q\colon \mathcal{L} \to \mathfrak{r}(\mathcal{L}/R)$$

and get a morphism in **dFrm**.

Proposition. 1. *$q_{\mathfrak{r}}$ is a quotient of $\mathcal{L}$ by the relation R in* **dFrm**.

2. *$q_{\mathfrak{r}}$ is an extremal epimorphism in* **dFrm**.

Proof. 1. Let $h\colon \mathcal{L} \to \mathcal{M}$ be a morphism in **dFrm** for which $(a, b) \in R_\pm$ implies $h_\pm(a) = h_\pm(b)$. By 2.1.1 there is $h'\colon \mathcal{L}/R \to M$ in **pdFrm** such that $h' \circ q = h$. Since $\mathcal{M}$ is in **dFrm** there is $\widetilde{h}$ in **dFrm** such that $\widetilde{h} \circ \kappa = h'$. Thus, $\widetilde{h} \circ q_{\mathfrak{r}} = \widetilde{h} \circ \kappa \circ q = h' \circ q = h$. Unicity is obvious since $q_{\mathfrak{r}}$ is onto.

2. Now let $q_{\mathfrak{r}} = m \circ f$ for some $f\colon \mathcal{L} \to \mathcal{M}$ and $m\colon \mathcal{M} \to \mathfrak{r}(\mathcal{L}/R)$ a monomorphism. For $(a, b) \in R_\pm$ we have $m_\pm(f_\pm(a)) = m_\pm(f_\pm(b))$ and hence $f_\pm(a) = f_\pm(b)$. Thus, there is an $f'\colon \mathcal{L}/R \to \mathcal{M}$ such that $f' \circ q = f$, and since $\mathcal{M}$ is in **dFrm** there is $\widetilde{f}\colon \mathfrak{r}(\mathcal{L}/E) \to \mathcal{M}$ such that $\widetilde{f} \circ \kappa = f'$. Consequently, $m \circ \widetilde{f} \circ q_{\mathfrak{r}} = m \circ \widetilde{f} \circ \kappa \circ q = m \circ f' \circ q = m \circ f = q_{\mathfrak{r}}$ and since $q_{\mathfrak{r}}$ is onto, $m \circ \widetilde{f} = \mathrm{id}$. Finally, $m \circ \widetilde{f} \circ m = m$, and since m is a monomorphism we see that also $\widetilde{f} \circ m = \mathrm{id}$. $\square$

2.8. Substructures in pdFrm. It is well-known that the monomorphisms in **Frm** are precisely the injective frame homomorphisms, [PP11, III, Lemma 1.1.1]. An analogous result holds for **pdFrm** and **dFrm**.

Proposition. *Let $\mathcal{L}, M$ be proto-d-frames (resp., d-frames). A d-frame homomorphism $h\colon \mathcal{L} \to \mathcal{M}$ is a monomorphism in* **pdFrm** *(resp.,* **dFrm***) iff both h_+ and h_- are injective frame homomorphisms.*

Proof. Let $h\colon \mathcal{L} \to \mathcal{M}$ be a morphism such that $h(x) = h(y)$ for different $x, y \in L_+ \times L_-$. W.l.o.g. assume $x_+ \neq y_+$. Then let S be the free frame of one generator $*$ and $\mathsf{con}_{\min}$ be the minimal consistency relation on $S \times L_-$, given by $a \in \mathsf{con}_{\min} \Leftrightarrow (a_+ = 0 \text{ or } a_- = 0)$. Likewise let $\mathsf{tot}_{\min}$ be the minimal totality relation on $S \times L_-$, given by $a \in \mathsf{tot}_{\min} \Leftrightarrow (a_+ = 1 \text{ or } a_- = 1)$. It was shown in [JM06, Proposition 5.7] that $\mathcal{I} = (S, L_-, \mathsf{con}_{\min}, \mathsf{tot}_{\min})$ is reasonable.

Since consistency and totality are chosen minimally, we have morphisms $f, f'\colon \mathcal{I} \to \mathcal{L}$, where $f_+(*) = x_+$ and $f'_+(*) = y_+$. For the other component we may choose the identity on L_- in both cases. It now holds that $h \circ f = h \circ f'$ which shows that h is not a monomorphism. $\square$

2.8.1. Proposition. *If $h\colon \mathcal{L} \to \mathcal{M}$ is a monomorphism in* **pdFrm** *and $\mathcal{M}$ is balanced, then so is $\mathcal{L}$.*

Proof. If $x \in \mathsf{con}_{\mathcal{L}}$, $y \in \mathsf{tot}_{\mathcal{L}}$ are elements such that $x_+ = y_+$ then $h(x) \in \mathsf{con}_{\mathcal{M}}$, $h(y) \in \mathsf{tot}_{\mathcal{M}}$, and $h_+(x_+) = h_+(y_+)$. Since $\mathcal{M}$ is balanced, we have $h(x) \sqsubseteq h(y)$ or equivalently, $h(x) \sqcap h(y) = h(x)$. Since h is an injective homomorphism, it follows that $x \sqcap y = x$ or $x \sqsubseteq y$. $\square$

2.8.2. A similar result for con-saturatedness holds under an additional assumption only. We say that a d-frame homomorphism $h\colon \mathcal{L} \to \mathcal{M}$ is *full* if it reflects the consistency relation, i.e., $h(x) \in \mathsf{con}_{\mathcal{M}}$ implies $x \in \mathsf{con}_{\mathcal{L}}$.

Proposition. *Let $\mathcal{L}, M$ be proto-d-frames and $h\colon \mathcal{L} \to \mathcal{M}$ a full homomorphism. Then $\mathcal{L}$ is con-saturated if $\mathcal{M}$ is.*

Proof. Fullness means $\mathsf{con}_{\mathcal{L}} = h^{-1}[\mathsf{con}_{\mathcal{M}}]$. The statement now follows from the fact that d-frame homomorphisms are Scott-continuous functions with respect to the information order and so the inverse image of a Scott-closed subset is again Scott-closed. $\square$

2.8.3. Corollary. *If $h\colon \mathcal{L} \to \mathcal{M}$ is a full monomorphism from a proto-d-frame $\mathcal{L}$ to a d-frame $\mathcal{M}$ then $\mathcal{L}$ is reasonable, that is, also a d-frame.*

2.9. Creating substructures in dFrm. The easy observations of the preceding items have a rather surprising consequence for the interplay between substructures and the reflection $\mathfrak{r}$.

Proposition. *Let $h\colon \mathcal{L} \to \mathcal{M}$ be a monomorphism from a proto-d-frame $\mathcal{L} = (L_+, L_-, \mathsf{con}, \mathsf{tot})$ to a d-frame $\mathcal{M}$. Then the reflection of $\mathcal{L}$ into* **dFrm** *is given by the reasonable congruence structure $(\overline{\mathsf{con}}, \mathsf{tot}, \Delta_+, \Delta_-)$ on (L_+, L_-) where $\Delta_\pm$ is equality on $L_\pm$. In other words, $\mathfrak{r}(\mathcal{L})$ is carried by the original frames, the original totality relation, and the Scott-closure of the original consistency relation. The underlying homomorphism of $\kappa_{\mathcal{L}}$ is the identity.*

Proof. Condition (R1) is satisfied because the logical operations are Scott-continuous with respect to the information order, so $\wedge[\overline{\mathsf{con}}\times\overline{\mathsf{con}}] = \wedge[\overline{\mathsf{con}\times\mathsf{con}}] \subseteq \overline{\wedge[\mathsf{con}\times\mathsf{con}]} \subseteq \overline{\mathsf{con}}$, and likewise for $\vee$. The new consistency relation is a lower set because all Scott-closed sets are downwards closed and this establishes (R2). For (R3) there is nothing to show because the congruences are trivial. (R4) encodes balancedness and this follows from 2.8.1 if we can show that h preserves $\overline{\mathsf{con}}$. For this observe that h is Scott-continuous, so $h[\overline{\mathsf{con}}] \subseteq \overline{h[\mathsf{con}]} \subseteq \overline{\mathsf{con}_{\mathcal{M}}} = \mathsf{con}_{\mathcal{M}}$. Finally, (R5) holds by construction. $\square$

2.10. Proposition. *The category* **dFrm** *carries the factorization system $(\mathbb{E}, \mathbb{M})$ with $\mathbb{E}$ consisting of all extremal epimorphisms and $\mathbb{M}$ consisting of all monomorphisms.*

Proof. Let $h\colon \mathcal{L} \to \mathcal{M}$ be a morphism in **dFrm** and consider the kernel E of h. By 2.1.1 we may factor $\mathcal{L}$ by E to obtain a proto-d-frame $\mathcal{L}/E$ together with a decomposition of h into morphisms $q\colon \mathcal{L} \to \mathcal{L}/E$ and $j\colon \mathcal{L}/E \to \mathcal{M}$, where the latter is injective. We apply the reflection and obtain the d-frame $\mathfrak{r}(\mathcal{L}/E)$ together with the decomposition of j into $\kappa\colon \mathcal{L}/E \to \mathfrak{r}(\mathcal{L}/E)$ and $\widetilde{j}\colon \mathfrak{r}(\mathcal{L}/E) \to \mathcal{M}$. As a commutative diagram (in **pdFrm**):

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "h"] \arrow[d, "q"'] & \mathcal{M} \\
\mathcal{L}/E \arrow[r, "\kappa"] \arrow[ur, "j"] & \mathfrak{r}(\mathcal{L}/E) \arrow[u, "\widetilde{j}"']
\end{tikzcd}$$

We know from 2.7.1 that $\kappa \circ q$ is an extremal epimorphism in **dFrm**, and from 2.9 that the underlying functions for j and $\widetilde{j}$ are the same; since they are injective, $\widetilde{j}$ is a monomorphism.

The unicity of the factorization (extremal epi, mono) is a standard categorical fact. $\square$

2.10.1. Note. By 2.9 we know a little bit more about the image factorization constructed above: The totality relation on $\mathfrak{r}(\mathcal{L}/E)$ is simply the image of $\mathsf{tot}_{\mathcal{L}}$ under q and the consistency relation is the Scott-closure of $q[\mathsf{con}_{\mathcal{L}}]$. If the morphism h was full to start with, then $\mathcal{L}/E$ is already a d-frame and the reflection has no effect on it.

3 A reformulation using quasi-congruences

3.1. The conditions (R1)–(R5) give a convenient criterion for reasonable quotients but they do not allow us easily to generate a reasonable congruence from given data. As we will now explain, the situation is much better if we incorporate the lattice orders into the congruences, that is, reformulate the conditions via *quasi-congruences*. For this, recall that a quasi-congruence on a frame L is a reflexive and transitive relation R respecting all joins and finite meets, and containing the order $\leq$.

3.1.1. Lemma. *The maps*

$$R \mapsto (\leq ; R) \quad \textit{and} \quad S \mapsto S \cap S^{-1}$$

constitute a bijection between congruences and quasi-congruences on a frame.

Proof. Checking that for R a congruence, $(\leq ; R)$ is a quasi-congruence is straighforward (for transitivity recall 2.2.2). Similarly, it is obvious that if S is a quasi-congruence then $S \cap S^{-1}$ is a congruence.

In remains to show that the translations are inverses of each other. Let R be a congruence. Then obviously $R \subseteq (\leq ; R) \cap (\leq ; R)^{-1}$. On the other hand if $x \leq uRy$ and $xRv \geq y$ for some u, v then $x = (u \wedge x)R(y \wedge v) = y$.

Conversely, let S be a quasi-congruence. We want to show that $S = \leq ; (S \cap S^{-1})$. Immediately we see that $\leq ; (S \cap S^{-1}) \subseteq S ; S \subseteq S$. On the other hand let xSy. Then $(x \vee y)S(y \vee y) = y$ and conversely $(x \vee y)S^{-1}y$ since $\geq \subseteq S^{-1}$. Thus we see that $(x \vee y, y) \in S \cap S^{-1}$ and can conclude that $x \leq (x \vee y)(S \cap S^{-1})y$. $\square$

3.2. Let us now adapt definition 2.2.1 to quasi-congruences. A quadruple $(\mathsf{con}, \mathsf{tot}, S_+, S_-)$, where $S_\pm$ are quasi-congruences on $L_\pm$, will be called a *reasonable quasi-congruence structure* or, for brevity, *reasonable qc-structure* on (L_+, L_-) if

$$\begin{array}{ll} \text{(S1)} & \mathsf{con} \text{ and } \mathsf{tot} \text{ are sublattices of } (L_+ \times L_-, \leq), \\ \text{(S3)} & S_+ \,;\mathsf{con}\,; S_-^{-1} \;\subseteq\; \mathsf{con} \;\text{ and }\; S_+^{-1}\,;\mathsf{tot}\,;S_- \;\subseteq\; \mathsf{tot}, \\ \text{(S4)} & \mathsf{con}\,;\mathsf{tot}^{-1} \;\subseteq\; S_+ \;\text{ and }\; \mathsf{con}^{-1}\,;\mathsf{tot} \;\subseteq\; S_-, \\ \text{(S5)} & \overline{\mathsf{con}} \subseteq \mathsf{con} \text{ in } (L_+ \times L_-, \sqsubseteq). \end{array}$$

(Note the absence of the counterpart to (R2) and a slightly simpler (S4); to keep the parallel we do not use "(S2)".)

3.2.1. Lemma. *In the notation from the correspondence $R \leftrightarrow S$ in* 3.1.1*, $(\mathsf{con}, \mathsf{tot}, R_+, R_-)$ is a reasonable congruence structure on (L_+, L_-) iff $(\mathsf{con}, \mathsf{tot}, S_+, S_-)$ is a reasonable quasi-congruence structure on (L_+, L_-).*

Proof. We have $S_\pm = \leq ; R_\pm$ and $R_\pm = S_\pm \cap S_\pm^{-1}$.

For $\Rightarrow$, the only requirement to check is (S3) and we have by (R3) and (R2) that

$$S_+ \,;\mathsf{con}\,; S_-^{-1} \;=\; \leq_+ ; R_+ \,;\mathsf{con}\,; R_- \,; \geq_- \;\subseteq\; \leq_+ ;\mathsf{con}\,; \geq_- \;\subseteq\; \mathsf{con}.$$

For $\Leftarrow$ we only need to prove (R2) and (R3). Since quasi-congruences contain the order we have by (S3)

$$\begin{array}{ll} (R2) & \leq_+ ;\mathsf{con}\,; \geq_- \;\subseteq\; S_+ \,;\mathsf{con}\,; S_-^{-1} \;\subseteq\; \mathsf{con}, \quad \text{and} \\ (R3) & (S_+ \cap S_+^{-1})\,;\mathsf{con}\,;(S_- \cap S_-^{-1}) \;\subseteq\; S_+ \,;\mathsf{con}\,; S_-^{-1} \;\subseteq\; \mathsf{con}. \quad \square \end{array}$$

3.2.2. Factoring by quasi-congruences The previous item assures us that given a reasonable qc-structure $(\mathsf{con}, \mathsf{tot}, S_+, S_-)$ on (L_+, L_-), we can factor $L_\pm$ by $S_\pm \cap S_\pm^{-1}$ and obtain a reasonable d-frame as described in 2.3. Alternatively, we can consider the classes $[x]_{S_\pm} := \{y \in L_\pm \mid y \, S_\pm \, x\}$ ordered by inclusion to obtain the d-frame $\mathcal{L}/(S \cap S^{-1})$ directly.

4 An iterative construction of the reflection

4.1. The conditions (S1)–(S5) tell us precisely what to add to con, tot, S_+, and S_- in order to achieve reasonableness, but since the four conditions are interdependent, it does not suffice to update the four relations just once. Instead, an iterative process of updates is required which, since $\bigsqcup$ is an infinitary operation, may even be transfinite.

However, things can be arranged in such a way that we obtain "nice" structures in each round by which we mean that $(\mathsf{con}, \mathsf{tot}, S_+, S_-)$ is such that $(L_+, L_-, \mathsf{con}, \mathsf{tot})$ is a proto-d-frame and furthermore con and tot satisfy (S3) w.r.t. $S_\pm$. We call such six-tuples *general quasi-congruence structures* or *qc-structures* for short.

4.2. The update operation. Given a qc-structure $\mathcal{Q} = (\mathsf{con}, \mathsf{tot}, S_+, S_-)$ on (L_+, L_-) we define its *update* $u(\mathcal{Q})$ as $(\mathsf{con}', \mathsf{tot}', S'_+, S'_-)$ where

$$\begin{array}{lcl} S'_+ & := & \text{the smallest quasi-congruence on } L_+ \text{ containing } S_+ \cup \mathsf{con} \,;\, \mathsf{tot}^{-1} \,; \\ S'_- & := & \text{the smallest quasi-congruence on } L_- \text{ containing } S_- \cup \mathsf{con}^{-1} \,;\, \mathsf{tot} \,; \\ \mathsf{con}' & := & S'_+ \,;\, \overline{\mathsf{con}} \,;\, {S'_-}^{-1} \,; \\ \mathsf{tot}' & := & {S'_+}^{-1} \,;\, \mathsf{tot} \,;\, S'_- \,. \end{array}$$

Proposition. *If $\mathcal{Q} = (\mathsf{con}, \mathsf{tot}, S_+, S_-)$ is a qc-structure on (L_+, L_-) then so is $u(\mathcal{Q})$.*

Proof. We showed in 2.8.3 that the Scott-closure of a logical sublattice is again a logical sublattice. This property is retained when we pre- and post-compose with quasi-congruences, for example, if $x \; S'_+ \; y \; \overline{\mathsf{con}} \; z \; {S'_-}^{-1} w$ and $x' \; S'_+ \; y' \; \overline{\mathsf{con}} \; z' \; {S'_-}^{-1} w'$ then $x \wedge x' \; S'_+ \; y \wedge y' \; \overline{\mathsf{con}} \; z \vee z' \; {S'_-}^{-1} w \vee w'$. A Scott-closed subset is always a lower set and this holds for $S'_+ \,;\, \overline{\mathsf{con}} \,;\, {S'_-}^{-1}$ as well because quasi-congruences contain the frame order and are transitive:

$$\begin{array}{rcl} x \leq x' \; S'_+ \,;\, \overline{\mathsf{con}} \,;\, {S'_-}^{-1} \; y' \geq y & \Longrightarrow & x \; S'_+ \,;\, S'_+ \,;\, \overline{\mathsf{con}} \,;\, {S'_-}^{-1} \,;\, {S'_-}^{-1} \; y \\ & \Longrightarrow & x \; S'_+ \,;\, \overline{\mathsf{con}} \,;\, {S'_-}^{-1} \; y \,. \end{array}$$

Property (S3) also follows from the transitivity of $S'_\pm$. $\square$

4.2.1. Notes. 1. The update operation increases all four relations that make up a qc-structure. On the other hand, the underlying frames remain the same.

2. A qc-structure $\mathcal{Q}$ is reasonable if and only if $u(\mathcal{Q}) = \mathcal{Q}$.

3. If we use the quasi-congruences of a qc-structure $\mathcal{Q} = (\mathsf{con}, \mathsf{tot}, S_+, S_-)$ to factor the proto-d-frame $(L_+, L_-, \mathsf{con}, \mathsf{tot})$ then we obtain another proto-d-frame as discussed in 3.2.2. The fact that con and tot satisfy (S3) has the consequence that $\mathsf{con} = q^{-1}[q[\mathsf{con}]]$ and $\mathsf{tot} = q^{-1}[q[\mathsf{tot}]]$ where $q \colon \mathcal{Q} \to \mathcal{Q}/S$ is the natural quotient morphism.

4.3. The iterative procedure. We extend the update operation to all ordinals in the obvious way; given a qc-structure on (L_+, L_-) we set

$$\begin{array}{rcl} u^0(\mathcal{Q}) & := & \mathcal{Q} \\ u^{\alpha+1}(\mathcal{Q}) & := & u(u^\alpha(\mathcal{Q})) \\ \gamma \text{ a limit ordinal: } u^\gamma(\mathcal{Q}) & := & (\bigcup_{\alpha<\gamma} \mathsf{con}^\alpha, \bigcup_{\alpha<\gamma} \mathsf{tot}^\alpha, \hat{S}_+, \hat{S}_-) \end{array}$$

where $\hat{S}_\pm$ is the smallest quasi-congruence on $L_\pm$ which contains all $(S^\alpha_\pm)$ for $\alpha < \gamma$.

Since the frame components stay constant throughout the update process, there exists a smallest ordinal λ such that $u^{\lambda+1}(\mathcal{Q}) = u^\lambda(\mathcal{Q})$ which, in light of 4.2.1(2), means that $u^\lambda(\mathcal{Q})$ is reasonable.

4.3.1. For every proto-d-frame $\mathcal{L} = (L_+, L_-, \mathsf{con}, \mathsf{tot})$ we have the qc-structure $\mathcal{Q}_\mathcal{L} = (\mathsf{con}, \mathsf{tot}, \leq_+, \leq_-)$. The above considerations, together with 2.4, now immediately yield the following.

Proposition. *Let $\mathcal{L} = (L_+, L_-, \mathsf{con}, \mathsf{tot})$ be a proto-d-frame, and let λ be the smallest ordinal such that $u^{\lambda+1}(\mathcal{Q}_\mathcal{L}) = u^\lambda(\mathcal{Q}_\mathcal{L})$. Then $u^\lambda(\mathcal{Q}_\mathcal{L})$ is the smallest reasonable qc-structure on $L_\pm$ extending $\mathcal{Q}_\mathcal{L}$. Quotienting by the resulting quasi-congruences yields the reasonable d-frame $\mathfrak{r}(\mathcal{L})$.*

4.4. A categorical perspective. We can shed a little bit more light on the above construction and its relationship to the reflection $\mathfrak{r}$ by setting qc-structures into a categorical context. To this end we consider as objects of the category **qcStruct** tuples $(L_\pm, \mathcal{Q})$ consisting of a pair $L_\pm$ of frames together with a qc-structure $\mathcal{Q}$ on (L_+, L_-). A morphism $h\colon (L_\pm, \mathcal{Q}) \to (M_\pm, \mathcal{T})$ consists of two frame homomorphisms $h_\pm\colon L_\pm \to M_\pm$ such that the four relations in $\mathcal{Q}$ are preserved, to wit:

$$h_+ \times h_-[\mathsf{con}_\mathcal{Q}] \subseteq \mathsf{con}_\mathcal{T} \quad h_+ \times h_-[\mathsf{tot}_\mathcal{Q}] \subseteq \mathsf{tot}_\mathcal{T}$$
$$h_+ \times h_+[S_+] \subseteq T_+ \qquad h_- \times h_-[S_-] \subseteq T_-$$

The full subcategory of **qcStruct** whose objects consist of reasonable qc-structures we denote by **rqcStruct**.

4.4.1. Infinite update as a functor Consider the assignment $\mathfrak{u}^\infty$ that maps objects $(L_\pm, \mathcal{Q})$ of **qcStruct** to $(L_\pm, u^\lambda(\mathcal{Q}))$ where, as before, λ is the smallest ordinal such that $u^{\lambda+1}(\mathcal{Q}) = u^\lambda(\mathcal{Q})$. On morphisms $h\colon (L_\pm, \mathcal{Q}) \to (M_\pm, \mathcal{T})$ we define $\mathfrak{u}^\infty(h)$ as h, that is, we keep the morphisms unchanged.

Proposition. $\mathfrak{u}^\infty$ *is a functor from* **qcStruct** *to* **rqcStruct***, left adjoint to the inclusion of the latter into the former. The unit of this adjunction in* **qcStruct** *is the morphism* $(\mathrm{id}_+, \mathrm{id}_-)\colon (L_\pm, \mathcal{Q}) \to (L_\pm, u^\infty(\mathcal{Q}))$.

Proof. Given $h\colon (L_\pm, \mathcal{Q}) \to (M_\pm, \mathcal{T})$ we aim to show by transfinite induction that h maps $u^\alpha(\mathcal{Q})$ into $u^\alpha(\mathcal{T})$ for all ordinals α. For $\alpha = 0$ the statement holds because h is a morphism in **qcStruct**. For $\alpha = 1$ we need to show that h also maps $u(\mathcal{Q})$ into $u(\mathcal{T})$. Indeed, if $x \;\mathsf{con}_\mathcal{Q}\; y \;\mathsf{tot}_\mathcal{Q}^{-1}\; z$ then $h_+(x) \;\mathsf{con}_\mathcal{T}\; h_-(y) \;\mathsf{tot}_\mathcal{T}^{-1}\; h_+(z)$ by assumption, and so

$$\begin{array}{rcl} h_+{\times}h_+[S_+ \cup \mathsf{con}_\mathcal{Q} \,;\mathsf{tot}_\mathcal{Q}^{-1}] & \subseteq & T'_+ \\ \text{or, equivalently:}\quad S_+ \cup \mathsf{con}_\mathcal{Q} \,;\mathsf{tot}_\mathcal{Q}^{-1} & \subseteq & (h_+{\times}h_+)^{-1}[T'_+] \end{array}$$

Since h_+ is a frame homomorphism, the last expression is a quasi-congruence on L_+ and therefore contains S'_+ which we defined as the smallest quasi-congruence containing $S_+ \cup \mathsf{con}_\mathcal{Q} \,;\mathsf{tot}_\mathcal{Q}^{-1}$.

Frame homomorphisms are Scott-continuous, and so we know that

$$h_+{\times}h_-[\overline{\mathsf{con}_\mathcal{Q}}] \;\subseteq\; \overline{h_+{\times}h_-[\mathsf{con}_\mathcal{Q}]} \;\subseteq\; \overline{\mathsf{con}_\mathcal{T}}\ .$$

Using the monotonicity of relational composition we can now conclude that $h_+{\times}h_-[\mathsf{con}'_\mathcal{Q}] \subseteq \mathsf{con}'_\mathcal{T}$. For the same reason we have $h_+{\times}h_-[\mathsf{tot}'_\mathcal{Q}] \subseteq \mathsf{tot}'_\mathcal{T}$.

Applying this argument repeatedly proves that h maps $u^n(\mathcal{Q})$ into $u^n(\mathcal{T})$, for all $n \in \mathbb{N}$, or generally, it allows us to move from an ordinal α to its successor $\alpha + 1$.

If γ is a limit ordinal and the statement holds for all $\alpha < \gamma$, then it is clear that $h_+{\times}h_-$ maps $\bigcup_{\alpha<\gamma} \mathsf{con}_\mathcal{Q}^\alpha$ into $\bigcup_{\alpha<\gamma} \mathsf{con}_\mathcal{T}^\alpha$, and likewise for tot. In the same way we see that $h_+{\times}h_+$ maps $S = \bigcup_{\alpha<\gamma} S_+^\alpha$ into $T = \bigcup_{\alpha<\gamma} T_+^\alpha$ and since h_+ is a frame homomorphism, it also maps the least quasi-congruence generated by S into the least quasi-congruence generated by T.

Since we have shown the preservation property for all ordinals, it holds in particular when both $u^\alpha(\mathcal{Q})$ and $u^\alpha(\mathcal{T})$ have stabilised.

For adjointness, assume that $h\colon (L_\pm, \mathcal{Q}) \to (M_\pm, \mathcal{T})$ and $(M_\pm, \mathcal{T})$ is reasonable. It follows that updating $(M_\pm, \mathcal{T})$ has no effect and hence h is also a morphism from $(L_\pm, \mathfrak{u}^\infty(\mathcal{Q}))$ to $(M_\pm, \mathcal{T})$. $\square$

4.5. Every proto-d-frame $\mathcal{L}$ may be extended with the orders on L_+ and L_-, respectively, to obtain the qc-structure $(L_\pm, \mathsf{con}, \mathsf{tot}, \leq_+, \leq_-)$. Since frame homomorphisms preserve the order, we thus obtain the embedding functor $\mathbf{I}\colon \mathbf{pdFrm} \to \mathbf{qcStruct}$.

On the other hand, for an $h\colon (\mathcal{L}, R_\pm) \to (\mathcal{M}, Q_\pm)$ we have by 2.1.1 precisely one $\Phi(h)$ making the diagram

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "q_R"] \arrow[d, "h"] & \mathcal{L}/R \arrow[d, "\Phi(h)"] \\
\mathcal{M} \arrow[r, "q_Q"] & \mathcal{M}/Q
\end{tikzcd}$$

commute, which gives us a functor Φ in the other direction; it is easily seen to be left adjoint to $\mathbf{I}$.

From 3.2.1 and 2.3 we infer that this adjunction restricts to **dFrm** and **rqcStruct**, and thus we have the following picture of the overall situation.

$$\begin{tikzcd}
\mathbf{dFrm} \arrow[r, hook, shift left, "\mathbf{I}"] \arrow[d, hook, shift left] & \mathbf{rqcStruct} \arrow[l, shift left, "\Phi"] \arrow[d, hook, shift left] \\
\mathbf{pdFrm} \arrow[u, shift left, "\mathfrak{r}"] \arrow[r, hook, shift left, "\mathbf{I}"] & \mathbf{qcStruct} \arrow[l, shift left, "\Phi"] \arrow[u, shift left, "\mathfrak{u}^\infty"]
\end{tikzcd}$$

It is clear by construction that the subdiagram of embeddings commutes. In Proposition 4.3.1 we showed that $\mathfrak{r} = \Phi \circ \mathfrak{u}^\infty \circ \mathbf{I}$. Finally, because of adjointness we have that $\mathfrak{r} \circ \Phi$ and $\Phi \circ \mathfrak{u}^\infty$ are naturally isomorphic.

Acknowledgements.

The results reported in this paper were obtained in Spring 2017, during visits by the third author to Birmingham and the first two authors to Prague. We gratefully acknowledge the hospitality and financial support extended to us by our two universities.

References

[AA96] O. Arieli and A. Avron. Reasoning with logical bilattices. *Journal of Logic, Language and Information*, 5:25–63, 1996.

[Abr91] S. Abramsky. Semantics of programming languages. Course Notes, 1991.

[AHS90] J. Adámek, H. Herrlich, and G. Strecker. *Abstract and Concrete Categories*. John Wiley & Sons, Inc., 1990.

[AJ94] S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Semantic Structures*, volume 3 of *Handbook of Logic in Computer Science*, pages 1–168. Clarendon Press, 1994.

[Bel76] N. D. Belnap. How a computer should think. In G. Ryle, editor, *Contemporary Aspects of Philosophy*, pages 30–56. Oriel Press, 1976.

[CM17] I. M. Carollo and M. A. Moshier. Extremal epimorphisms in dFrm and an Isbell-type density theorem. Chapman University, 2017.

[GHK+03] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *Continuous Lattices and Domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2003.

[JM06] A. Jung and M. A. Moshier. On the bitopological nature of Stone duality. Technical Report CSR-06-13, School of Computer Science, The University of Birmingham, 2006. 110 pages.

[Joh82] P. T. Johnstone. *Stone Spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1982.

[Mac71] S. Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer Verlag, 1971.

[PP11] J. Picado and A. Pultr. *Frames and Locales: Topology without points*. Springer-Birkhäuser Basel, 2011.

[Riv10] U. Rivieccio. *An Algebraic Study of Bilattice-based Logics*. PhD thesis, University of Barcelona, 2010.

[Sco82] D. S. Scott. Domains for denotational semantics. In M. Nielson and E. M. Schmidt, editors, *International Colloquium on Automata, Languages and Programs*, volume 140 of *Lecture Notes in Computer Science*, pages 577–613. Springer Verlag, 1982.

[Smy92] M. B. Smyth. Topology. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science, vol. 1*, pages 641–761. Clarendon Press, 1992.

[Vic89] S. J. Vickers. *Topology Via Logic*, volume 5 of *Cambridge Tracts in Theoretical Computer Science.* Cambridge University Press, 1989.