Non-involutive twist-structures

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Abstract. A recent paper by Jakl, Jung and Pultr succeeded for the first time in establishing a very natural link between bilattice logic and the duality theory of d-frames and bitopological spaces. In this paper we further exploit, extend and investigate this link from an algebraic and a logical point of view. In particular, we introduce classes of algebras that extend bilattices, d-frames and N4-lattices (the algebraic counterpart of Nelson paraconsistent logic) to a setting in which the negation is not necessarily involutive, and we study corresponding logics. We provide product representation theorems for these algebras as well as completeness, algebraizability (and some non-algebraizability) results for the corresponding logics.

1 Introduction

Bilattices and bilattice-based logics are well-known formalisms of paraconsistent logic which have had a considerable impact in AI and computer science ([9], [2]). In recent years, interesting connections have been highlighted ([17], [18]), both on a formal level and from the point of view of motivation, with logics of so-called *strong negation* such as the paraconsistent version of Nelson logic ([1], [13]). On the other hand, a clear parallelism also seems to exist between bilattices and other formalisms motivated by the attempt to deal with inconsistency in computer science, notably the theory of d-frames and bitopological spaces [11]. This latter connection, however, had never been clearly stated in formal terms until the recent paper [10] introduced a mathematical framework that may be a possible way of bridging this gap. The present paper is an attempt at connecting, further exploring and developing both the above-mentioned links, introducing a uniform logical and algebraic framework which encompasses paraconsistent Nelson systems, bilattice-based logics and (the finitary aspects of) d-frame theory.

One of the main intuitions behind bilattices is that truth values may be viewed as split into two components, representing respectively positive and negative evidence concerning a given proposition. Since positive and negative evidence are not assumed to be the complement of each other as in classical logic, this allows one to deal with partial as well as inconsistent information. On an algebraic level, this intuition is reflected in the fact that every bilattice can be represented as a special product (known in the literature as *bilattice product* or *twist-structure*) of two lattices, the positive-evidence lattice and the negative-evidence lattice. In principle, the two need not be related, that is, the domains of positive and negative evidence may not have the same structure. To give an example from computability, consider the question of whether a given Turing machine will stop, i.e., the "halting problem". Positive evidence for the machine stopping is the observation that it actually has stopped. Until this has happened, we do not have any positive evidence at all, and so the lattice of positive evidence has just two elements, "unknown" and "has stopped". Negative evidence, on the other hand, should be treated quite differently, since we can not observe non-halting behaviour directly. Instead, we employ the lattice of natural numbers together with a top element, where each natural number n indicates that we have observed that the machine has been running for n steps (or units of time) and has not yet stopped. The top element means non-termination, but it is an "ideal" value that can not be observed directly but is the supremum of the infinite set of propositions "has not stopped after n steps".

If one wants to have a negation connective in the language, then the only available candidate in the literature until recently was "strong negation" (essentially the same in bilattices and Nelson lattices), which requires the two domains to be isomorphic lattices. The situation changed with the recent [10], which introduced a novel and very natural way of defining a weaker negation operator that allows for the positive and the negative domain to be truly independent of each other, and gives rise to interesting structures that are moreover supported by a clear topological interpretation. To continue the discussion of the halting problem from this perspective, negation would allow us to formalise evidence for the statement "it is not true that the machine will stop". However, this need not change our distinction between positive and negative evidence. We can continue to insist that positive evidence must be "real evidence", for example, the observation that the machine has returned to a state that it had assumed before, hence will be trapped in an infinite loop forever. Again, this is a binary observation; once we make it, we know that the program will loop, but until we have made it we know nothing. The negative lattice, on the other hand, can again be used to express doubt about the statement, and it may be useful to have an infinite scale to express shades of doubt. For example, if the program contains *nondeterministic* constructs (such as the ones that arise from parallelism) then negative evidence could be that the program always stopped on n previous runs.

The present work expands and exploits the main ideas of [10] introducing algebraic structures, called *non-involutive bilattices*, that have a pre-bilattice reduct and a negation operator that is no longer required to be involutive nor to satisfy all the De Morgan identities. This algebraic framework allows us to rigorously formulate a very natural and expected connection between bilattice-based logics on the one hand and the topological setting of d-frames and bitopological spaces on the other. We show in particular how many well-known structures can be seen as special cases of non-involutive bilattices, namely pre-bilattices, bilattices with an involutive negation, and the nd-frames of [10]. If we further introduce Nelson-type implications into the language, we can show how N4-lattices, Nelson algebras and implicative bilattices nicely fit into the picture as well. We axiomatize the logics corresponding to these algebraic structures, showing how some of them turn out to be more algebraically well-behaved than others, and we provide equational presentations as well as twist-structure representation theorems. A preliminary version of this paper (containing results which roughly correspond to the present Section 3, but with a more categorical focus) has been presented in [12].

The paper is organized as follows. In Section 2 we introduce the notation and recall some preliminary results. Section 3 introduces the class of *non-involutive bilattices*, a generalization of bilattices in which the negation operator $\neg$ is not necessarily involutive, i.e. does not satisfy the identity $\neg\neg x = x$. We provide an abstract presentation for these algebras, as well as a product representation, and we characterize the congruence lattice of a non-involutive bilattice in terms of those of its factors. In Section 4 we add two implication operators to the algebraic language of non-involutive bilattices (reflecting the fact that both factors of a d-frame carry a definable Heyting implication), we give an abstract axiomatization as well as a product representation for the corresponding algebras, that we call *non-involutive implicative bilattices.* The expressive power gained thanks to the implications allows us to define a Hilbert calculus (Section 5) whose consequence is equivalent to the equational consequence of the (equationally-definable) class of non-involutive implicative bilattices, i.e. we prove that the syntactically-defined logic is algebraizable (in the sense of [3, Definition 2.10]) and has the variety of non-involutive implicative bilattices as its equivalent algebraic semantics [3, Definition 2.8]. We also consider a weaker logic which can be defined in a natural way using non-involutive implicative bilattices as its semantics; even though this logic is not algebraizable in the above-mentioned sense, we are able to introduce a complete calculus for it. In Sections 6 and 7 we show that the algebraizability result holds true even if we consider a more restricted algebraic language, essentially disregarding the bilattice knowledge order and operations. This allows us to establish a link between our setting and that of paraconsistent Nelson logic, introducing a class of algebras that generalize both non-involutive implicative bilattices and N4-lattices (the algebraic counterpart of paraconsistent Nelson logic). We obtain in this way a generalized version of a well-known result which characterizes N4-lattices as subreducts of implicative bilattices. Finally, in Section 8 we present a negative result that explains our choice of focussing on a richer algebraic and logical language than the one considered in [10] and [4]: we show that the logic that one could naturally associate to the class of algebras introduced in [10] is not equivalential, and so not algebraizable either (i.e. it does not correspond to the equational consequence of any class of algebras).

2 Preliminaries

2.1 Logics, algebras and matrices

Given an algebraic signature, we denote by $\mathbf{Fm}$ the absolutely free algebra built over a countable set of propositional variables. A *logic* defined over $\mathbf{Fm}$, denoted $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$, is a structural consequence relation.

We will be dealing with matrix semantics for logics (see [19] for further details). A *matrix* is a pair $M = \langle \mathbf{A}, D \rangle$ where $\mathbf{A}$ is an algebra (a non-empty set A equipped with a family of finitary operations) and $D \subseteq A$ is a subset of *designated elements*. Each matrix $M = \langle \mathbf{A}, D \rangle$ determines a logic $\models_M$ by defining $\Gamma \models_M \varphi$ if and only if, for all homomorphisms $h \colon \mathbf{Fm} \to \mathbf{A}$, we have that $h(\Gamma) \subseteq D$ implies $h(\varphi) \in D$. We say that M is a *matrix for a logic* $\mathcal{L}$ when $\vdash_{\mathcal{L}} \subseteq \models_M$ (that is, M is sound for $\mathcal{L}$). A class of matrices $\mathcal{M} = \{M_i : i \in I\}$ defines a logic $\models_{\mathcal{M}}$ by setting $\Gamma \models_{\mathcal{M}} \varphi$ if and only if $\Gamma \models_{M_i} \varphi$ for all $i \in I$.

The *Leibniz congruence* of a matrix $M = \langle \mathbf{A}, D \rangle$, usually denoted $\boldsymbol{\Omega}_{\mathbf{A}}(D)$, is the largest congruence of $\mathbf{A}$ that is compatible with D, meaning that, for all elements $a, b \in A$, if $a \in D$ and $\langle a, b \rangle \in \boldsymbol{\Omega}_{\mathbf{A}}(D)$, then $b \in D$. The *reduction* of $M = \langle \mathbf{A}, D \rangle$ is the matrix $M^* = \langle \mathbf{A}/\boldsymbol{\Omega}_{\mathbf{A}}(D), D/\boldsymbol{\Omega}_{\mathbf{A}}(D) \rangle$, where $\mathbf{A}/\boldsymbol{\Omega}_{\mathbf{A}}(D)$ is the usual quotient algebra and $D/\boldsymbol{\Omega}_{\mathbf{A}}(D) = \{a/\boldsymbol{\Omega}_{\mathbf{A}}(D) : a \in D\}$. A matrix is $M = \langle \mathbf{A}, D \rangle$ *reduced* when $\boldsymbol{\Omega}_{\mathbf{A}}(D)$ is the identity, that is, when M is isomorphic to its own reduction M^*. Any matrix M defines the same logic as its reduction M^*, which makes reduced matrices particularly important in the semantical study of logics. In fact, any logic is complete with respect to the class of all reduced matrices for it. The class

$$\mathbf{Alg}^*(\mathcal{L}) := \{\mathbf{A} : \langle \mathbf{A}, D \rangle \text{ is a reduced matrix for } \mathcal{L}\}$$

consists of all algebras $\mathbf{A}$ that are the reducts of some reduced matrix for $\mathcal{L}$.

Any logic $\mathcal{L}$ is (trivially) complete with respect to the class of matrices $\mathcal{M}_{\mathcal{L}} = \{\langle \mathbf{Fm}, \mathcal{T} \rangle : \mathcal{T} \text{ is a theory of } \mathcal{L}\}$. This class can itself be reduced in the following way. $\mathcal{M}_{\mathcal{L}}$ is an example of a *generalized matrix* (g-matrix), that is, a pair $\langle \mathbf{A}, \mathcal{C} \rangle$ where $\mathbf{A}$ is an algebra and $\mathcal{C}$ is a closure system on $\mathbf{A}$ (i.e. a family $A \in \mathcal{C} \subseteq P(A)$ closed under arbitrary intersections). The *Tarski congruence* of a g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ is the largest *logical* congruence θ of $\mathbf{A}$, i.e. the largest congruence such that $\langle a, b \rangle \in \theta$ implies that the closure of a equals the closure of b. The *reduction* of $\langle \mathbf{A}, \mathcal{C} \rangle$ is the g-matrix $\langle \mathbf{A}/\theta, \mathcal{C}/\theta \rangle$, where $\mathcal{C}/\theta = \{D/\theta : D \in \mathcal{C}\}$. A g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ is just a particular class of matrices that share the same underlying algebra $\mathbf{A}$, hence all definitions about classes of matrices are extended to g-matrices. The *Lindenbaum-Tarski g-matrix* of a logic $\mathcal{L}$ is the reduction $\mathcal{M}^*_{\mathcal{L}}$ of the g-matrix $\mathcal{M}_{\mathcal{L}} = \{\langle \mathbf{Fm}, \mathcal{T} \rangle : \mathcal{T} \text{ is a theory of } \mathcal{L}\}$. The algebraic reduct of $\mathcal{M}^*_{\mathcal{L}}$, denoted $\mathbf{Fm}^*$ is the *Lindenbaum-Tarski algebra* of $\mathcal{L}$. The class of $\mathcal{L}$*-algebras*

$$\mathbf{Alg}(\mathcal{L}) := \{\mathbf{A} : \langle \mathbf{A}, \mathcal{C} \rangle \text{ is a reduced g-matrix for } \mathcal{L}\}$$

consists of all algebras $\mathbf{A}$ that are the reducts of some reduced g-matrix for $\mathcal{L}$. So, in particular, $\mathbf{Fm}^* \in \mathbf{Alg}(\mathcal{L})$. The inclusion $\mathbf{Alg}^*(\mathcal{L}) \subseteq \mathbf{Alg}(\mathcal{L})$ holds for any logic, while the converse need not hold in general (we refer the reader to [8] for further details).

We will be dealing mainly with quasiequational and equational classes of algebras, also known as *quasivarieties* and *varieties* (see [6] for further details). For our purposes, it will be enough to know that a quasivariety is a class of algebras that is definable via quasiequations, i.e. universally quantified implications whose premiss is a finite conjunction of equations and whose conclusion is a single equation. We shall also refer to the fact that quasivarieties are closed under the operation of taking isomorphic images and subalgebras, but not necessarily under homomorphic images. Varieties are quasivarieties that can be axiomatized using equations only, i.e. implications of the above-defined type with an empty set of premisses. A quasivariety is a variety if and only if it is closed under homomorphic images.

2.2 Bilattices

In this section we introduce definitions and well-known results about bilattices (see [4] for further details and proofs).

Definition 1 (Interlaced pre-bilattice). *An* interlaced pre-bilattice *is an algebra* $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup \rangle$ *such that* $\langle B, \wedge, \vee \rangle$ *and* $\langle B, \sqcap, \sqcup \rangle$ *are lattices, and each one of the four operations* $\{\vee, \wedge, \sqcup, \sqcap\}$ *is monotonic with respect to both lattice orders.*

The lattice $\langle B, \wedge, \vee \rangle$ is called the truth lattice (t-lattice), and its order is denoted by $\leq$ and is called the truth order (t-order). The lattice $\langle B, \sqcap, \sqcup \rangle$ is called the knowledge (or information) lattice (k-lattice), and its order $\sqsubseteq$ the knowledge order (k-order).

The following construction is known as *product bilattice* in the bilattice literature and as *(full) twist-structure* in the literature on Nelson logics[4]. Besides showing an easy way of constructing an interlaced pre-bilattice, its importance lies in the fact that all interlaced pre-bilattices can be obtained in this way.

Definition 2 (Product pre-bilattice). *Let* $\mathbf{L}_+ = \langle L_+, \wedge_+, \vee_+ \rangle$ *and* $\mathbf{L}_- = \langle L_-, \wedge_-, \vee_- \rangle$ *be lattices. The* product pre-bilattice $\langle L_+ \times L_-, \wedge, \vee, \sqcap, \sqcup \rangle$ *is defined as follows. For all*

[4] The word "full" refers to the fact that the universes of algebras thus built are direct products, whereas a non-full twist-structure might correspond to a subreduct of one such product, see e.g. Section 6.

$\langle a_+, a_- \rangle, \langle b_+, b_- \rangle \in L_+ \times L_-$,

$$\begin{aligned}
\langle a_+, a_- \rangle \wedge \langle b_+, b_- \rangle &= \langle a_+ \wedge_+ b_+, a_- \vee_- b_- \rangle \\
\langle a_+, a_- \rangle \vee \langle b_+, b_- \rangle &= \langle a_+ \vee_+ b_+, a_- \wedge_- b_- \rangle \\
\langle a_+, a_- \rangle \sqcap \langle b_+, b_- \rangle &= \langle a_+ \wedge_+ b_+, a_- \wedge_- b_- \rangle \\
\langle a_+, a_- \rangle \sqcup \langle b_+, b_- \rangle &= \langle a_+ \vee_+ b_+, a_- \vee_- b_- \rangle.
\end{aligned}$$

Thus, the lattice reduct $\langle L_+ \times L_-, \sqcap, \sqcup \rangle$ is just the standard direct product $\mathbf{L}_+ \times \mathbf{L}_-$, while the reduct $\langle L_+ \times L_-, \wedge, \vee \rangle$ is the direct product $\mathbf{L}_+ \times (\mathbf{L}_-)^{op}$ where $(\mathbf{L}_-)^{op}$ denotes the lattice $\langle L_-, \vee_-, \wedge_- \rangle$.

It is straightforward to check that a product pre-bilattice is always an interlaced pre-billatice in which the two orders are given, for all $\langle a_+, a_- \rangle, \langle b_+, b_- \rangle \in L_+ \times L_-$, by

$$\langle a_+, a_- \rangle \leq \langle b_+, b_- \rangle \text{ iff } a_+ \leq_+ b_+ \text{ and } b_- \leq_- a_-$$

$$\langle a_+, a_- \rangle \sqsubseteq \langle b_+, b_- \rangle \text{ iff } a_+ \leq_+ b_+ \text{ and } a_- \leq_- b_-$$

where $\leq_+$ and $\leq_-$ denote the lattice orders of $\mathbf{L}_+$ and $\mathbf{L}_-$ respectively. This reflects the intuition that an element $\langle a_+, a_- \rangle \in L_+ \times L_-$ can be thought of as encoding evidence about some assertion: evidence for it (represented by a_+), and evidence against (represented by a_-). Then an increase in information (knowledge) amounts to saying that overall evidence goes up, while an increase in truth means that *evidence for* increases and *evidence against* decreases.

Theorem 1. *Every interlaced pre-bilattice* $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup \rangle$ *is isomorphic to the product pre-bilattice of* $\mathbf{B}_+ = \langle B/{\equiv_+}, \wedge, \vee \rangle$ *and* $\mathbf{B}_- = \langle B/{\equiv_-}, \vee, \wedge \rangle$*, where*

$$\equiv_+ \; = \{\langle a, b \rangle \in B^2 : a \wedge b = a \sqcup b\} \qquad \equiv_- \; = \{\langle a, b \rangle \in B^2 : a \wedge b = a \sqcap b\}$$

through the map $\iota \colon B \to B/{\equiv_+} \times B/{\equiv_-}$ *given by* $\iota(a) = \langle [a]_+, [a]_- \rangle$ *for all* $a \in B$*, where* $[a]_+$ *and* $[a]_-$ *denote the equivalence classes of* $a \in B$ *in the quotients* $B/\equiv_+$ *and* $B/\equiv_-$*.*

The above result is proved in full generality (for unbounded pre-bilattices) in [5, Theorem 3.2]. It is useful to notice that the relations defined in Theorem 1 correspond, in a product pre-bilattice $L_+ \times L_-$, to the following:

$$\equiv_+ \; = \{\langle \langle a_+, a_- \rangle, \langle a_+, a'_- \rangle \rangle \in (L_+ \times L_-)^2\} \quad \equiv_- \; = \{\langle \langle a_+, a_- \rangle, \langle a'_+, a_- \rangle \rangle \in (L_+ \times L_-)^2\}.$$

Theorem 1 provides a very convenient way of proving properties about interlaced pre-bilattices: by checking that they hold in product pre-bilattices. The following corollary lists a few that will be used in subsequent proofs.

Corollary 1. *Let* $\mathbf{B}$ *be an interlaced pre-bilattice and let* $a, b, c \in \mathbf{B}$ *be such that* $a \sqcup b \sqsubseteq c$*. Then,*

(i) $(a \wedge b \wedge c) \sqcap ((a \wedge b) \vee c) = a \wedge b$,
(ii) $(a \sqcup (b \wedge c)) \sqcap ((a \wedge b) \vee c)\rangle = a \sqcup b$,
(iii) $(a \wedge c) \vee (b \wedge c) = (a \vee b) \wedge c$,
(iv) $(a \wedge c) \vee (b \wedge c) = (a \vee b) \wedge c$.

3 Non-involutive bilattices

According to the original definition of [9], a bilattice is a pre-bilattice which has an additional unary operator (called *negation*) that satisfies the involutive and De Morgan identities (see below). For a product bilattice, the existence of such an operator is equivalent to the requirement that the underlying lattices $\mathbf{L}_+$ and $\mathbf{L}_-$ be isomorphic. However, as shown in [10, Definition 3.1], even in the absence of an isomorphism, a weaker notion of negation can be defined, as follows.

Definition 3 (Non-involutive product bilattice). *Let* $\mathbf{L}_+ = \langle L_+, \leq_+, \wedge_+, \vee_+ \rangle$ *and* $\mathbf{L}_- = \langle L_-, \leq_-, \wedge_-, \vee_- \rangle$ *be lattices, and let* $n\colon L_+ \to L_-$ *and* $p\colon L_- \to L_+$ *be maps satisfying the following properties:*

(i) n, p *are both meet-semilattice homomorphisms;*
(ii) n, p *preserve the lattice bounds of* $\mathbf{L}_+$ *and* $\mathbf{L}_-$ *(if present);*
(iii) $n \circ p$, $p \circ n \leq Id$.

The non-involutive product bilattice *is the algebra* $\mathbf{L}_+ \bowtie \mathbf{L}_- = \langle L_+ \times L_-, \wedge, \vee, \sqcap, \sqcup, \neg \rangle$ *where* $\langle L_+ \times L_-, \wedge, \vee, \sqcap, \sqcup \rangle$ *is the product pre-bilattice of Definition 2 and the* negation *is given by*

$$\neg\langle a_+, a_- \rangle = \langle p(a_-), n(a_+) \rangle.$$

Observe that, if $\mathbf{L}_+ = \langle L_+, \wedge_+, \vee_+, 0_+, 1_+ \rangle$ and $\mathbf{L}_- = \langle L_-, \wedge_-, \vee_-, 0_-, 1_- \rangle$ are bounded lattices, then maps n, p satisfying Definition 3 can always be defined by letting $n(a_+) = 0_-$ for all $a_+ \neq 1_+$ and $n(1_+) = 1_-$, $p(a_-) = 0_+$ for all $a_- \neq 1_-$ and $p(1_-) = 1_+$. Thus, any bounded interlaced (product) pre-bilattice can be endowed in a canonical way with a negation that turns it into a non-involutive product bilattice. In case there exists an isomorphism $\iota\colon \mathbf{L}_+ \cong \mathbf{L}_-$, we can obtain the usual product bilattice [5, Definition 3.10] by letting e.g. $n = \iota$ and $p = \iota^{-1}$.

We are going to prove that non-involutive product bilattices coincide with the class of algebras defined by the following abstract presentation.

Definition 4 (Non-involutive bilattice). *A* non-involutive bilattice *is an interlaced pre-bilattice* $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup, \neg \rangle$ *endowed with a negation* $\neg$ *satisfying the following identities:*

(i) $\neg(x \sqcap y) = \neg x \sqcap \neg y$,
(ii) $\neg\bot = \bot \quad \neg\top = \top \quad \neg\mathsf{t} = \mathsf{f} \quad \neg\mathsf{f} = \mathsf{t}$ *(if bounds are present),*
(iii) $\neg\neg x \sqsubseteq x$.
(iv) $\neg(x \wedge y) \equiv_+ \neg(x \sqcup y) \quad \neg(x \wedge y) \equiv_- \neg(x \sqcap y)$.

An (involutive) bilattice can thus be defined as a non-involutive bilattice that additionally satisfies $x \sqsubseteq \neg\neg x$ and $\neg(x \sqcup y) = \neg x \sqcup \neg y$ (in which case, the usual De Morgan laws $\neg(x \wedge y) = \neg x \vee \neg y$ and $\neg(x \vee y) = \neg x \wedge \neg y$ also hold).

Lemma 1. *Condition (iv) in Definition 4 can be equivalently replaced by the following quasiequations*[5]*:* $x \equiv_+ y \Rightarrow \neg x \equiv_- \neg y$ *and* $x \equiv_- y \Rightarrow \neg x \equiv_+ \neg y$.

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[5] Following standard usage, by *quasiequation* we mean a formula of (classical) first-order logic of type $(e_1 \,\&\, \ldots \,\&\, e_n) \Rightarrow e_0$, where the e_i are equations in the algebraic language considered, & denotes first-order conjunction and $\Rightarrow$ first-order implication.

Proof. Let $\langle B, \wedge, \vee, \sqcap, \sqcup, \neg\rangle$ a non-involutive bilattice according to Definition 4, and assume $a \equiv_+ b$, i.e. $a \wedge b = a \sqcup b$. Then,

$$\begin{aligned}
\neg a &= \neg(a \sqcap (a \sqcup b)) && \text{lattice identities}\\
&= \neg(a \sqcap (a \wedge b)) && a \wedge b = a \sqcup b\\
&\equiv_- \neg(a \wedge a \wedge b) && \neg(x \wedge y) \equiv_- \neg(x \sqcap y)\\
&= \neg(b \wedge a \wedge b) && \text{lattice identities}\\
&\equiv_- \neg(b \sqcap (a \wedge b)) && \neg(x \wedge y) \equiv_- \neg(x \sqcap y)\\
&= \neg(b \sqcap (a \sqcup b)) && a \wedge b = a \sqcup b\\
&= \neg b && \text{lattice identities.}
\end{aligned}$$

We conclude that $\neg a \equiv_- \neg b$ as required. Similarly, using $\neg(x \wedge y) \equiv_+ \neg(x \sqcap y)$ we have $a \equiv_+ b$ implies $\neg a \equiv_- \neg b$. This shows that every non-involutive bilattice satisfies $x \equiv_+ y \Rightarrow \neg x \equiv_- \neg y$ and $x \equiv_- y \Rightarrow \neg x \equiv_+ \neg y$. The converse is easy, because $x \wedge y \equiv_+ x \sqcap y$ holds in any interlaced bilattice [5, Proposition 3.4], so by applying the quasiequation $x \equiv_+ y \Rightarrow \neg x \equiv_- \neg y$ we obtain $\neg(x \wedge y) \equiv_- \neg(x \sqcap y)$. The proof of $\neg(x \wedge y) \equiv_+ \neg(x \sqcup y)$ is similar.

While Definition 4 ensures that the class of non-involutive bilattices is equationally definable (a variety of algebras), Lemma 1 provides a presentation that is often the easier to work with. We are now able to prove a representation theorem for non-involutive bilattices that is analogous to the product representation of interlaced bilattices.

Proposition 1. *Every non-involutive product bilattice* $\mathbf{L}_+ \bowtie \mathbf{L}_- = \langle L_+ \times L_-, \wedge, \vee, \sqcap, \sqcup, \neg\rangle$ *is a non-involutive bilattice.*

Proof. Since the negation-free reduct of $\mathbf{L}_+ \bowtie \mathbf{L}_-$ is an interlaced pre-bilattice, we only need to show that properties (i)–(iv) of Definition 4 are satisfied, which is routine checking. Concerning (iv) notice that, for verifying e.g. $\neg(x \wedge y) \equiv_+ \neg(x \sqcup y)$, it is sufficient to check that the first component of the left-hand side is equal to the first component of right-hand side.

Theorem 2. *Every non-involutive bilattice* $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup, \neg\rangle$ *is isomorphic to the non-involutive product bilattice of* $\mathbf{B}_+ = \langle B/{\equiv_+}, \wedge, \vee\rangle$ *and* $\mathbf{B}_- = \langle B/{\equiv_-}, \vee, \wedge\rangle$*, constructed according to Definition 3, with the negation defined as* $\neg\langle [a]_+, [a]_-\rangle = \langle p([a]_-), n([a]_+)\rangle$ *for all* $a \in B$*. The isomorphism is given by the map* $\iota\colon B \to B/{\equiv_+} \times B/{\equiv_-}$ *defined as* $\iota(a) = \langle [a]_+, [a]_-\rangle$ *for all* $a \in B$*.*

Proof. We know from Theorem 1 that $\langle B/{\equiv_+}, \wedge, \vee\rangle$ and $\langle B/{\equiv_-}, \vee, \wedge\rangle$ are lattices, and that the map ι is a pre-bilattice isomorphism. Define $n\colon B/{\equiv_+} \to B/{\equiv_-}$ by $n([a]_+) = [\neg a]_-$ and $p\colon B/{\equiv_-} \to B/{\equiv_+}$ by $p([a]_-) = [\neg a]_+$. Lemma 1 guarantees that these maps are well defined, and it is straightforward to check that they satisfy Definition 3. It remains to show that $\iota(\neg a) = \neg\iota(a)$. This is immediate: $\iota(\neg a) = \langle [\neg a]_+, [\neg a]_-\rangle = \langle p([a]_-), n([a]_+)\rangle = \neg\langle [a]_+, [a]_-\rangle = \neg\iota(a)$.

As in the case of pre-bilattices, the correspondence between non-involutive bilattices and non-involutive product bilattices (that we can view as quadruples $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$) can be formulated as a covariant categorical equivalence between two naturally associated algebraic categories (see [12] for details). This connection can then be exploited to obtain further insight into the structure non-involutive bilattices.

One can prove, for example, that the congruence lattice $\mathrm{Con}(\mathbf{B}_+)$ of a non-involutive bilattice $\mathbf{B} \cong \mathbf{B}_+ \bowtie \mathbf{B}_-$ is isomorphic (as a complete lattice) to a certain sub-lattice of $\mathrm{Con}(\mathbf{B}_+) \times \mathrm{Con}(\mathbf{B}_-)$, where $\mathrm{Con}(\mathbf{B}_+)$ and $\mathrm{Con}(\mathbf{B}_-)$ denote the congruence lattices of $\mathbf{B}_+$ and $\mathbf{B}_-$ respectively; a result that can be viewed as a generalization of [5, Proposition 3.8]. Let $\langle \theta_+, \theta_- \rangle \in \mathrm{Con}(\mathbf{B}_+) \times \mathrm{Con}(\mathbf{B}_-)$ be a pair of congruences which satisfy, for all $a, b \in B$,

$$\text{if } \langle [a]_-, [b]_- \rangle \in \theta_-, \text{ then } \langle p([a]_-), p([b]_-) \rangle \in \theta_+ \tag{1}$$

$$\text{if } \langle [a]_+, [b]_+ \rangle \in \theta_+, \text{ then } \langle n([a]_+), n([b]_+) \rangle \in \theta_- \tag{2}$$

Denote by $\mathrm{Con}^*(\mathbf{B}_+, \mathbf{B}_-) \subseteq \mathrm{Con}(\mathbf{B}_+) \times \mathrm{Con}(\mathbf{B}_-)$ the set of pairs of congruences which satisfy (1) and (2), and notice that it is the universe of a complete lattice in which the meet is set-theoretic intersection.

Lemma 2. *Let $\mathbf{B}$ be an interlaced pre-bilattice, $\theta \in \mathrm{Con}(\mathbf{B})$ and $a, b \in \mathbf{B}$. The following conditions are equivalent:*

(i) $\langle a \wedge b, a \sqcup b \rangle \in \theta$*;*
(ii) $\langle a \wedge c, b \wedge c \rangle \in \theta$ *for some $c \in \mathbf{B}$ such that $a \sqcup b \sqsubseteq c$;*
(iii) $\langle a \wedge c, b \wedge c \rangle \in \theta$ *for all $c \in \mathbf{B}$ such that $a \sqcup b \sqsubseteq c$.*

Proof. Obviously (iii) implies (ii). To show that (ii) implies (i), assume $\langle a \wedge c, b \wedge c \rangle \in \theta$ for some c with $a \sqcup b \sqsubseteq c$. Then, on the one hand, we have $\langle a \wedge a \wedge c, a \wedge b \wedge c \rangle = \langle a \wedge c, a \wedge b \wedge c \rangle \in \theta$, and on the other $\langle a \sqcup (a \wedge c), a \sqcup (b \wedge c) \rangle = \langle a \wedge c, a \sqcup (b \wedge c) \rangle \in \theta$ (the equality $a \sqcup (a \wedge c) = a \wedge c$ holds because, by the interlacing conditions, $a \sqsubseteq c$ implies $a \sqsubseteq a \wedge c$). Thus, by symmetry and transitivity of θ, we have $\langle a \wedge b \wedge c, a \sqcup (b \wedge c) \rangle \in \theta$. From this we obtain $\langle (a \wedge b \wedge c) \sqcap ((a \wedge b) \vee c), (a \sqcup (b \wedge c)) \sqcap ((a \wedge b) \vee c) \rangle \in \theta$. The two equalities $(a \wedge b \wedge c) \sqcap ((a \wedge b) \vee c) = a \wedge b$ and $(a \sqcup (b \wedge c)) \sqcap ((a \wedge b) \vee c)) = a \sqcup b$, which hold by Corollary 1.(i)-(ii), imply that $\langle a \wedge b, a \sqcup b \rangle \in \theta$ as required.
To conclude the proof it remains to show that (i) implies (iii). Assume $\langle a \wedge b, a \sqcup b \rangle \in \theta$. Then $\langle a \wedge (a \wedge b), a \wedge (a \sqcup b) \rangle = \langle a \wedge b, a \wedge (a \sqcup b) \rangle \in \theta$ and $\langle b \wedge (a \wedge b), b \wedge (a \sqcup b) \rangle = \langle a \wedge b, b \wedge (a \sqcup b) \rangle \in \theta$. By symmetry and transitivity of θ we thus have $\langle a \wedge (a \sqcup b), b \wedge (a \sqcup b) \rangle \in \theta$, Now, for any $c \in \mathbf{B}$ such that $a \sqcup b \sqsubseteq c$, we have $a \wedge (a \sqcup b) \wedge c = a \wedge c$ and $b \wedge (a \sqcup b) \wedge c = b \wedge c$ (this can again be checked using the product representation of pre-bilattices). Thus we have $\langle a \wedge (a \sqcup b) \wedge c, b \wedge (a \sqcup b) \wedge c \rangle = \langle a \wedge c, b \wedge c \rangle \in \theta$ as required.

We omit the proof of the following lemma as it is entirely analogous to that of Lemma 2.

Lemma 3. *Let $\mathbf{B}$ be an interlaced pre-bilattice, $\theta \in \mathrm{Con}(\mathbf{B})$ and $a, b \in \mathbf{B}$. The following conditions are equivalent:*

(i) $\langle a \wedge b, a \sqcap b \rangle \in \theta$*;*
(ii) $\langle a \vee c, b \vee c \rangle \in \theta$ *for some $c \in \mathbf{B}$ such that $a \sqcup b \sqsubseteq c$;*
(iii) $\langle a \vee c, b \vee c \rangle \in \theta$ *for all $c \in \mathbf{B}$ such that $a \sqcup b \sqsubseteq c$.*

Proposition 2. *The lattice* $\mathrm{Con}(\mathbf{B})$ *of any non-involutive bilattice* $\mathbf{B}$ *is isomorphic to* $\mathrm{Con}^*(\mathbf{B}_+, \mathbf{B}_-)$.

Proof. The isomorphism is given by the two maps $\mathsf{L}\colon \mathrm{Con}(\mathbf{B}) \to \mathrm{Con}^*(\mathbf{B}_+, \mathbf{B}_-)$ and $\mathsf{B}\colon \mathrm{Con}^*(\mathbf{B}_+, \mathbf{B}_-) \to \mathrm{Con}(\mathbf{B})$ defined as follows. For $\theta \in \mathrm{Con}(\mathbf{B})$, let $\mathsf{L}(\theta) = \langle \theta_+, \theta_- \rangle$ where $\theta_+ = \{\langle [a]_+, [b]_+ \rangle \in \mathbf{B}_+ \times \mathbf{B}_+ : \langle a \wedge b, a \sqcup b \rangle \in \theta\}$ and $\theta_- = \{\langle [a]_-, [b]_- \rangle \in \mathbf{B}_- \times \mathbf{B}_- :$

$\langle a \wedge b, a \sqcap b \rangle \in \theta\}$. For $\langle \theta_+, \theta_- \rangle \in \mathrm{Con}^*(\mathbf{B}_+, \mathbf{B}_-)$, let $\mathsf{B}\langle \theta_+, \theta_- \rangle = \{\langle a, b \rangle \in \mathbf{B} \times \mathbf{B} : \langle [a]_+, [b]_+ \rangle \in \theta_+, \langle [a]_-, [b]_- \rangle \in \theta_-\}$. To check that the map L is well-defined, suppose $[a]_+ = [a']_+$, $[b]_+ = [b']_+$ (i.e. $a \wedge a' = a \sqcup a'$ and $b \wedge b' = b \sqcup b'$) and $\langle [a]_+, [b]_+ \rangle \in \theta_+$ (i.e. $\langle a \wedge b, a \sqcup b \rangle \in \theta$). Let $c = a \sqcup b \sqcup a' \sqcup b'$. By Lemma 2, we have $a \wedge c = a' \wedge c$, $b \wedge c = b' \wedge c$ and $\langle a \wedge c, b \wedge c \rangle \in \theta$. Thus we immediately obtain $\langle a' \wedge c, b' \wedge c \rangle = \langle a \wedge c, b \wedge c \rangle \in \theta$, which again by Lemma 2 gives us $\langle a' \wedge b', a' \sqcup b' \rangle \in \theta$ and so $\langle [a']_+, [b']_+ \rangle \in \theta_+$ as required. A similar reasoning (relying on Lemma 3) shows that $[a]_- = [a']_-$, $[b]_- = [b']_-$ and $\langle [a]_-, [b]_- \rangle \in \theta_-$ imply $\langle [a']_-, [b']_- \rangle \in \theta_-$. So the map L is well-defined. It remains to show that $\mathsf{L}(\theta) \in \mathrm{Con}^*(\mathbf{B}_+, \mathbf{B}_-)$. We shall check compatibility of θ_+ with $\vee_+$ and leave the other cases to the reader. Assume $\langle [a]_+, [b]_+ \rangle, \langle [a']_+, [b']_+ \rangle \in \theta_+$, which means that $\langle a \wedge b, a \sqcup b \rangle, \langle a' \wedge b', a' \sqcup b' \rangle \in \theta$. Letting $c = a \sqcup b \sqcup a' \sqcup b'$ and invoking Lemma 2, we have $\langle a \wedge c, b \wedge c \rangle, \langle a' \wedge c, b' \wedge c \rangle \in \theta$, from which we obtain $\langle (a \wedge c) \vee (a' \wedge c), (b \wedge c) \vee (b' \wedge c) \rangle \in \theta$. Since $(a \wedge c) \vee (a' \wedge c) = (a \vee a') \wedge c$ and $(b \wedge c) \vee (b' \wedge c) = (b \vee b') \wedge c$ by Corollary 1.(iii)-(iv), we can use Lemma 2) again to obtain $\langle [a \vee a']_+, [b \vee b']_+ \rangle = \langle [a]_+ \vee_+ [a']_+, [b]_+ \vee_+ [b']_+ \rangle \in \theta_+$. To see that the pair $\mathsf{L}(\theta)$ satisfies conditions (1) and (2) which define the sublattice $\mathrm{Con}^*(\mathbf{B}_+, \mathbf{B}_-)$, assume for instance $\langle [a]_-, [b]_- \rangle \in \theta_-$. By Lemma 2, this means that $\langle a \vee c, b \vee c \rangle \in \theta$ for some $c \in \mathbf{B}$ with $a \sqcup b \sqsubseteq c$. Then $\langle \neg(a \vee c), \neg(b \vee c) \rangle \in \theta$ as well. Now observe that $a \sqsubseteq c$ implies $\neg(a \vee c) = \neg a \wedge \neg c$ (invoking Theorem 2, we can easily check this in a product bilattice), and similarly we have $\neg(b \vee c) = \neg b \wedge \neg c$. Thus we have $\langle \neg a \wedge \neg c, \neg b \wedge \neg c \rangle \in \theta$. Since $a \sqcup b \sqsubseteq c$ implies $\neg a \sqcup \neg b \sqsubseteq \neg c$, we can use Lemma 2 once more to conclude that $\langle \neg a \wedge \neg b, \neg a \sqcup \neg b \rangle \in \theta$ and so $\langle [\neg a]_+, [\neg b]_+ \rangle = \langle p[a]_-, p[b]_- \rangle \in \theta_+$. This establishes (1); the proof of (2) is similar.
The map B is obviously well-defined, and checking that $\mathsf{B}\langle \theta_+, \theta_- \rangle \in \mathrm{Con}(\mathbf{B})$ is straightforward. It is also easy to see that the maps L and B are mutually inverse. For example, $\theta = \mathsf{B}(\mathsf{L}(\theta))$ because $\theta = \{\langle a, b \rangle \in \mathbf{B} \times \mathbf{B} : \langle a \wedge b, a \sqcup b \rangle, \langle a \wedge b, a \sqcap b \rangle \in \theta\}$. Also, L and B are monotone and order-reflecting, which implies that they are order isomorphisms between the lattice $\langle \mathrm{Con}(\mathbf{B}), \subseteq \rangle$ and the lattice $\langle \mathrm{Con}^*(\mathbf{B}_+, \mathbf{B}_-), \subseteq \rangle$.

When the maps n and p are mutually inverse isomorphisms between $\mathbf{B}_+$ and $\mathbf{B}_-$ (so $\mathbf{B}$ is an involutive bilattice), then (1) and (2) imply that, for any $\langle \theta_+, \theta_- \rangle \in \mathrm{Con}(\mathbf{B}_+) \times \mathrm{Con}(\mathbf{B}_-)$, $\langle [a]_+, [b]_+ \rangle \in \theta_+$ if and only if $\langle n([a]_+), n([b]_+) \rangle \in \theta_-$ and likewise $\langle [a]_-, [b]_- \rangle \in \theta_-$ if and only if $\langle p([a]_-), p([b]_-) \rangle \in \theta_+$. Thus we recover, as a corollary of Proposition 2, the isomorphism $\langle \mathrm{Con}(\mathbf{B}), \subseteq \rangle \cong \langle \mathrm{Con}(\mathbf{B}_+), \subseteq \rangle$ that is proved in e.g. [5, Proposition 3.8].

4 Adding implications

In this section we generalize the construction of [4] in order to define implication connective(s) on non-involutive bilattices.

Recall that an *implicative lattice* (also known in the literature as a *Brouwerian lattice*) is a lattice $\mathbf{L} = \langle L, \wedge, \vee, \to \rangle$ expanded with an extra binary operation $\to$ (called *implication*) which satisfies the following *residuation* property: $a \wedge b \leq c$ if and only if $b \leq a \to c$, for all $a, b, c \in L$. Implicative lattices are the algebraic counterpart of the negation-free fragment of intuitionistic logic, and correspond precisely to the 0-free subreducts of Heyting algebras. This implies, in particular, that any implicative lattice is distributive and has a top element, which we denote by 1. For our purposes, it will also be useful to recall that implicative lattices form an equational class[6].

[6] See e.g. [16, p. 55], but notice that implicative lattices are called *relatively pseudo-complemented lattices* in this book.

Definition 5 (Non-involutive implicative product bilattice). *Let* $\mathbf{L}_+ = \langle L_+, \wedge_+, \vee_+, \rightarrow_+, 1_+\rangle$ *and* $\mathbf{L}_- = \langle L_-, \wedge_-, \vee_-, \rightarrow_-, 1_-\rangle$ *be implicative lattices, and let* $n\colon L_+ \to L_-$ *and* $p\colon L_- \to L_+$ *be maps satisfying properties (i)–(iii) of Definition 3. The* non-involutive implicative product bilattice *is the algebra* $\mathbf{L}_+ \bowtie \mathbf{L}_- = \langle L_+ \times L_-, \wedge, \vee, \sqcap, \sqcup, \supset, \not\subset, \neg\rangle$*, whose* $\{\supset, \not\subset\}$*-free reduct is the product bilattice of Definition 3 and the two* implications *are given by*

$$\langle a_+, a_-\rangle \supset \langle b_+, b_-\rangle = \langle a_+ \rightarrow_+ b_+,\ n(a_+) \wedge_- b_-\rangle$$

$$\langle a_+, a_-\rangle \not\subset \langle b_+, b_-\rangle = \langle p(a_-) \wedge_+ b_+,\ a_- \rightarrow_- b_-\rangle.$$

Definition 5 generalizes both the construction given in [4] for the algebras there called "Brouwerian bilattices" and that of *nd*-frames of [10, Definition 3.1]. In fact, any Brouwerian bilattice can be seen as a non-involutive implicative product bilattice $\mathbf{L} \bowtie \mathbf{L}$ where the maps n, p are both the identity on $\mathbf{L}$ and the $\not\subset$ operation is given by $x \not\subset y = \neg(\neg x \supset \neg y)$. The operation $\not\subset$, though not considered in [10], is definable in any *nd*-frame (for both underlying frames of an *nd*-frame are completely distributive lattices in which the implications $\rightarrow_+$ and $\rightarrow_-$ are the residua of the lattice meets). As for involutive implicative bilattices, a *strong implication* connective can be defined by

$$x \rightarrow y = (x \supset y) \wedge \neg(y \not\subset x).$$

One can compute that

$$\langle a_+, a_-\rangle \rightarrow \langle b_+, b_-\rangle = \langle (a_+ \rightarrow_+ b_+) \wedge_+ p(b_- \rightarrow_- a_-), n(a_+) \wedge_- b_-\rangle$$

which implies in particular that $x \rightarrow y = (x \rightarrow y) \supset (x \rightarrow y)$ holds if and only if $x \leq y$.

As we have done in Section 3 for the implicationless algebras, we will provide an abstract axiomatization for the class of products introduced in Definition 5. In order to do this we introduce some further auxiliary notation. Let $a = \langle a_+, a_-\rangle$, $b = \langle b_+, b_-\rangle \in \mathbf{L}_+ \bowtie \mathbf{L}_-$. We write $\varepsilon(a)$ as an abbreviation for $a \supset a$. We further define:

$a \preccurlyeq_+ b$ if and only if $a \supset b = \varepsilon(a \supset b)$

$a \preccurlyeq_- b$ if and only if $\neg(a \not\subset b) = \varepsilon(\neg(a \not\subset b))$

Doing the calculations, one can check that

$a \preccurlyeq_+ b$ if and only if $a_+ \leq_+ b_+$

$a \preccurlyeq_- b$ if and only if $a_- \leq_- b_-$.

It follows that $\preccurlyeq_+$ and $\preccurlyeq_-$ are preorders, which induce the equivalence relations $\equiv_+$ and $\equiv_-$ that we have considered earlier. The intersection $\preccurlyeq_+ \cap \preccurlyeq_-$ is precisely the knowledge order of $\mathbf{L}_+ \bowtie \mathbf{L}_-$ and the intersection $\preccurlyeq_+ \cap (\preccurlyeq_-)^{-1}$ is the truth order. We state below a few useful facts for further reference.

Given $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ implicative lattices with maps which satisfy properties (i)–(iii) in Definition 3, let us call a lattice filter $F \subseteq L_+$ *open* when $a_+ \in F$ implies $p(n(a_+)) \in F$ for all $a_+ \in L_+$. Likewise we say that a lattice filter $G \subseteq L_-$ is open when $a_- \in G$ implies $n(p(a_-)) \in G$ for all $a_- \in L_-$.

Proposition 3. *Let* $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ *be implicative lattices with maps* n *and* p *which satisfy properties (i)–(iii) in Definition 3, and let* $a_+, b_+ \in L_+$*. Then,*

(i) $n(a_+) = 1_-$ *implies* $a_+ = 1_+$,
(ii) $n(a_+ \rightarrow_+ b_+) \leq_- n(a_+) \rightarrow_- n(b_+)$,

(iii) if $F \subseteq L_+$ is a non-empty (open) lattice filter, then so is $p^{-1}[F] \subseteq L_-$.

Proof. (i). If $n(a_+) = 1_-$, then $p(n(a_+)) = 1_+$ because p preserves the bounds. But $p(n(a_+)) \leq_+ a_+$ and so $a_+ = 1_+$.
(ii). By residuation, we have $n(a_+ \to_+ b_+) \leq_- n(a_+) \to_- n(b_+)$ iff $n(a_+ \to_+ b_+) \wedge_- n(a_+) \leq_- n(b_+)$. Also, $n(a_+ \to_+ b_+) \wedge_- n(a_+) = n((a_+ \to_+ b_+) \wedge_+ a_+)$ because n preserves meets and $(a_+ \to_+ b_+) \wedge_+ a_+ = a_+ \wedge_+ b_+$ which holds in any implicative lattice. Thus $n(a_+ \to_+ b_+) \leq_- n(a_+) \to_- n(b_+)$ is equivalent to $n(a_+ \wedge_+ b_+) = n(a_+) \wedge_- n(b_+) \leq_- n(b_+)$ which is certainly true.
(iii). If F is non-empty, then $1_+ \in F$ and so $1_- \in p^{-1}[F]$ because p preserves the bounds. Suppose $a_-, b_- \in p^{-1}[F]$, i.e. $p(a_-), p(b_-) \in F$. Then $p(a_-) \wedge_+ p(b_-) = p(a_- \wedge_- b_-) \in F$ and so $a_- \wedge_- b_- \in p^{-1}[F]$. Also, if $a_- \in p^{-1}[F]$ and $a_- \leq_- b_-$, then $p(a_-) \in F$ and $p(b_-) \in F$ as well since p is order-preserving. So $b_- \in p^{-1}[F]$. Finally, if F is open and $a_- \in p^{-1}[F]$, then $p(a_-) \in F$ and so $p(n(p(a_-))) \in F$, which implies $n(p(a_-)) \in p^{-1}[F]$.

Obviously the preceding proposition implies its dual stated below.

Corollary 2. *Let $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ be implicative lattices with maps n and p which satisfy properties (i)–(iii) in Definition 3, and let $a_-, b_- \in L_-$. Then,*

(i) $p(a_-) = 1_+$ implies $a_- = 1_-$,
(ii) $p(a_- \to_- b_-) \leq_+ p(a_-) \to_+ p(b_-)$,
(iii) if $F \subseteq L_-$ is a non-empty (open) lattice filter, then so is $n^{-1}[F] \subseteq L_+$.

From Proposition 3.(iii) and Corollary 2.(iii) it follows that the lattice of open filters of $\mathbf{L}_+$ is isomorphic to the lattice of open filters of $\mathbf{L}_-$. This in turn implies that the congruences of $\mathbf{L}_+$ that are compatible with the maps n and p (in the sense explained below) also correspond to those of $\mathbf{L}_-$. Let us say that $\theta_+ \in \mathrm{Con}(\mathbf{L}_+)$ is a *pn-congruence* if $\langle a_+, b_+\rangle \in \theta_+$ implies $\langle p(n(a_+)), p(n(b_+))\rangle \in \theta_+$ for all $a_+, b_+ \in L_+$. Likewise we define an *np-congruence* as a congruence $\theta_- \in \mathrm{Con}(\mathbf{L}_-)$ such that $\langle a_-, b_-\rangle \in \theta_-$ implies $\langle n(p(a_-)), n(p(b_-))\rangle \in \theta_-$ for all $a_-, b_- \in L_-$. Let us denote by $\mathrm{Con}_{pn}(\mathbf{L}_+)$ ($\mathrm{Con}_{np}(\mathbf{L}_-)$) the set of pn-congruences of $\mathbf{L}_+$ (np-congruences of $\mathbf{L}_-$). Both sets are closed under arbitrary intersections, and thus form complete lattices ordered by set-theoretic inclusion.

Corollary 3. *Let $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ be implicative lattices with maps n and p which satisfy properties (i)–(iii) in Definition 3. Then,*

(i) the lattice of open filters of $\mathbf{L}_+$ and of $\mathbf{L}_-$ are isomorphic (via the maps p^{-1} and n^{-1}),
(ii) $\langle \mathrm{Con}_{pn}(\mathbf{L}_+), \subseteq\rangle$ is isomorphic (as a complete lattice) to $\langle \mathrm{Con}_{np}(\mathbf{L}_-), \subseteq\rangle$.

Proof. (i). Follows from Proposition 3.(iii) and Corollary 2.(iii), as soon as one notices that $F_+ = n^{-1}[p^{-1}[F_+]]$ and $F_- = p^{-1}[n^{-1}[F_-]]$ for all open filters $F_+ \subseteq L_+$, $F_- \subseteq L_-$.
(ii). We are going to use the preceding item together with the following fact. It is well known that the lattice of congruences of any implicative lattice is isomorphic to the lattice of its filters via the following maps. To a congruence, say $\theta \in \mathrm{Con}(\mathbf{L}_+)$, one associates the filter $F_\theta = 1_+/\theta$, and to a filter $F \subseteq L_+$ one associates the congruence θ_F defined by $\langle a_+, b_+\rangle \in \theta_F$ if and only if $a_+ \to_+ b_+, b_+ \to_+ a_+ \in F$. We claim that these maps also establish an isomorphism between $\langle \mathrm{Con}_{pn}(\mathbf{L}_+), \subseteq\rangle$ and the lattice of open filters of $\mathbf{L}_+$. For $\theta \in \mathrm{Con}_{pn}(\mathbf{L}_+)$, the filter F_θ is open because $\langle a_+, 1_+\rangle \in \theta$ implies $\langle p(n(a_+)), p(n(1_+))\rangle = \langle p(n(a_+)), 1_+\rangle \in \theta$. Conversely, suppose F is open and $\langle a_+, b_+\rangle \in \theta_F$, i.e. $a_+ \to_+ b_+, b_+ \to_+ a_+ \in F$. Then

$p(n(a_+ \to_+ b_+)) \in F$, which implies $p(n(a_+)) \to_+ p(n(b_+)) \in F$ because $p(n(a_+ \to_+ b_+)) \leq_+ p(n(a_+)) \to_+ p(n(b_+))$. This last inequality holds because of (ii) in Proposition 3 and Corollary 2. By symmetry we also have $p(n(b_+)) \to_+ p(n(a_+)) \in F$ which allows us to conclude that $\langle p(n(a_+)), p(n(b_+)) \rangle \in \theta_F$ as required. The same reasoning shows that $\langle \mathrm{Con}_{np}(\mathbf{L}_-), \subseteq \rangle$ is isomorphic to the lattice of open filters of $\mathbf{L}_-$. Thus, by item (i) above, we have an isomorphism $\langle \mathrm{Con}_{pn}(\mathbf{L}_+), \subseteq \rangle \cong \langle \mathrm{Con}_{np}(\mathbf{L}_-), \subseteq \rangle$. One can then check that to a congruence $\theta_+ \in \mathrm{Con}_{pn}(\mathbf{L}_+)$ corresponds the congruence $\theta_- \in \mathrm{Con}_{np}(\mathbf{L}_-)$ defined by $\langle a_-, b_- \rangle \in \theta_-$ iff $\langle p(a_- \to_- b_-), 1_+ \rangle, \langle p(b_- \to_- a_-), 1_+ \rangle \in \theta_+$.

Recalling properties (1) and (2) from Section 3, we can notice that $\mathrm{Con}^*(\mathbf{L}_+, \mathbf{L}_-) \subseteq \mathrm{Con}_{pn}(\mathbf{L}_+) \times \mathrm{Con}_{np}(\mathbf{L}_-)$. Thus, in case $\mathbf{L}_+$ and $\mathbf{L}_-$ are implicative lattices, Corollary 3.(ii) tells us that each pair $\langle \theta_+, \theta_- \rangle \in \mathrm{Con}^*(\mathbf{L}_+, \mathbf{L}_-)$ is determined by the first (or, equivalently, the second) component. For example, θ_+ is the unique congruence in $\mathrm{Con}_{pn}(\mathbf{L}_+)$ satisfying $\langle p(a_- \to_- b_-), 1_+ \rangle, \langle p(b_- \to_- a_-), 1_+ \rangle \in \theta_+$ iff $\langle a_-, b_- \rangle \in \theta_-$. Thus we also have isomorphisms $\mathrm{Con}^*(\mathbf{L}_+, \mathbf{L}_-) \cong \mathrm{Con}_{pn}(\mathbf{L}_+) \cong \mathrm{Con}_{np}(\mathbf{L}_-)$.

Definition 6. *A* non-involutive implicative bilattice *is an algebra* $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup, \supset, \not\subset, \neg \rangle$ *satisfying the following properties:*

(i) *the relations* $\preccurlyeq_+ = \{\langle a, b \rangle \in B \times B : a \supset b = \varepsilon(a \supset b)\}$ *and* $\preccurlyeq_- = \{\langle a, b \rangle \in B \times B : \neg(a \not\subset b) = \varepsilon(\neg(a \not\subset b))\}$ *are preorders (i.e. reflexive and transitive),*
(ii) $\leq = \preccurlyeq_+ \cap (\preccurlyeq_-)^{-1}$,
(iii) *the equivalence relation* $\equiv_+$ *induced by* $\preccurlyeq_+$ *is compatible with the operations* $\wedge, \vee, \supset$,
(iv) *the equivalence relation* $\equiv_-$ *induced by* $\preccurlyeq_-$ *is compatible with the operations* $\wedge, \vee, \not\subset$,
(v) *the quotients* $\mathbf{B}_+ = \langle B/\equiv_+, \wedge, \vee, \supset \rangle$ *and* $\mathbf{B}_- = \langle B/\equiv_-, \vee, \wedge, \not\subset \rangle$ *are implicative lattices*[7],
(vi) $x \equiv_+ y \Rightarrow \neg x \equiv_- \neg y$ *and* $x \equiv_- y \Rightarrow \neg x \equiv_+ \neg y$,
(vii) $x \not\subset y \equiv_+ \neg x \wedge y$ *and* $x \supset y \equiv_- \neg x \vee y$,
(viii) $x \not\subset x \equiv_- \neg(x \supset x)$ *and* $x \supset x \equiv_+ \neg(x \not\subset x)$,
(ix) $\neg(x \vee y) \equiv_+ \neg x \wedge \neg y$ *and* $\neg(x \wedge y) \equiv_- \neg x \vee \neg y$,
(x) $\neg\neg x \preccurlyeq_+ x$ *and* $\neg\neg x \preccurlyeq_- x$.
(xi) $\neg\bot = \bot \quad \neg\top = \top \quad \neg\mathsf{t} = \mathsf{f} \quad \neg\mathsf{f} = \mathsf{t}$ *(if any of those bounds is present),*
(xii) $\langle B, \wedge, \vee, \sqcap, \sqcup \rangle$ *is an interlaced pre-bilattice where the relations* $\equiv_+, \equiv_-$ *coincide with those defined in Theorem 1.*

It is easy to show (recalling Lemma 1) that any algebra satisfying all properties in Definition 6 also satisfies all items of Definition 4. Hence, as expected, any non-involutive implicative bilattice has a non-involutive bilattice reduct. The reader might have noticed that some items in Definition 6 are redundant; our reason for having them is that they will make it easier to generalize the definition to N4-like structures which lack some of the bilattice operations (see Section 6). It is easy to see that all conditions in Definition 6 can be expressed as quasiequations; therefore, the class of non-involutive implicative bilattices (from now on denoted NIB) is a quasivariety. In fact, the product representation that we are going to prove next will allow us to verify that NIB is actually a variety.

Given our previous considerations, it is straightforward to check that any non-involutive implicative product bilattice satisfies all the conditions in Definition 6, which gives us the following.

Proposition 4. *Every non-involutive implicative product bilattice is a non-involutive implicative bilattice.*

[7] Notice that $\mathbf{B}_-$ has $\vee$ as meet (whose residuum is $\not\subset$) and $\wedge$ as join.

Conversely, given a non-involutive implicative bilattice $\mathbf{B}$, we can construct the product $\mathbf{B}_+ \bowtie \mathbf{B}_-$ and show that the two algebras are isomorphic. As before, we define the maps $p \colon \mathbf{B}_- \to \mathbf{B}_+$ and $n \colon \mathbf{B}_+ \to \mathbf{B}_-$ by $p([a]_-) = [\neg a]_+$ and $n([a]_+) = [\neg a]_-$. Item (vi) of Definition 6 guarantees that these maps are well defined, while items (viii)–(xi) ensure that they satisfy properties (i)–(iii) of Definition 3.

Theorem 3. *Let* $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup, \supset, \not\subset, \neg \rangle$ *be a non-involutive implicative bilattice. Then the map* $\iota \colon \mathbf{B} \to \mathbf{B}_+ \bowtie \mathbf{B}_-$ *given by* $\iota(a) = ([a]_+, [a]_-)$ *for all* $a \in B$ *is an isomorphism.*

Proof. We already know that ι is a non-involutive bilattice isomorphism. It remains to show that ι preserves the $\supset, \not\subset$ operations, that is $\iota(a \supset b) = \iota(a) \supset \iota(b)$ and $\iota(a \not\subset b) = \iota(a) \not\subset \iota(b)$. As to the former, using Definition 6 (vii), we have $\iota(a \supset b) = \langle [a \supset b]_+, [a \supset b]_- \rangle = \langle [a]_+ \supset [b]_+, [\neg a \vee b]_- \rangle = \langle [a]_+ \supset [b]_+, [\neg a]_- \vee [b]_- \rangle = \langle [a]_+ \supset [b]_+, n([a]_+) \vee [b]_- \rangle = \iota(a) \supset \iota(b)$. As to the latter, using Definition 6 (vii) again we get $\iota(a \not\subset b) = ([a \not\subset b]_+, [a \not\subset b]_-) = ([\neg a \wedge b]_+, [a]_- \not\subset [b]_-) = ([\neg a]_+ \wedge [b]_+, [a]_- \not\subset [b]_-) = (p([a]_-) \wedge [b]_+, [a]_- \not\subset [b]_-) = \iota(a) \not\subset \iota(b)$.

It is easy to show that the lattice of congruences of a non-involutive implicative bilattice is isomorphic (through the same maps defined earlier) to $\mathrm{Con}^*(\mathbf{B}_+, \mathbf{B}_-)$, defined as before as the set of pairs of congruences (of implicative lattices) that satisfy (1) and (2) from Section 3.

Theorem 4. *The lattice* $\mathrm{Con}(\mathbf{B})$ *of any non-involutive implicative bilattice* $\mathbf{B}$ *is isomorphic to* $\mathrm{Con}^*(\mathbf{B}_+, \mathbf{B}_-)$ *and also to each of* $\mathrm{Con}_{pn}(\mathbf{B}_+)$ *and* $\mathrm{Con}_{np}(\mathbf{B}_-)$*.*

Proof. For the first part of the statement it suffices to check that the isomorphism defined in the proof of Proposition 2 preserves the implications, which is straightforward. The second part of the statement follows from Corollary 3.(ii).

Proposition 5. *The class* NIB *of non-involutive implicative bilattices is a variety.*

Proof. We know that NIB is a quasivariety, so it remains to show that this class is closed under homomorphic images. We will check this for product bilattices, which we can do without loss of generality by Theorem 3. Let then $\mathbf{B} = \mathbf{B}_+ \bowtie \mathbf{B}_-$ be a product bilattice and let $\mathbf{C}$ be a homomorphic image of $\mathbf{B}$ via some homomorphism $h \colon \mathbf{B} \to \mathbf{C}$. Denote by θ the kernel of h, and consider the congruences $\theta_+ \subseteq \mathbf{B}_+ \times \mathbf{B}_+, \theta_- \subseteq \mathbf{B}_- \times \mathbf{B}_-$ defined according to Proposition 2. We claim that $\mathbf{C}$ can be embedded into $\mathbf{B}_+/\theta_+ \bowtie \mathbf{B}_-/\theta_-$, which implies (again by Theorem 3, plus the fact that implicative lattices form a variety) that $\mathbf{C} \in IS(\mathsf{NIB}) = \mathsf{NIB}$ (this last equality obviously holds because a quasivariety is closed under isomorphisms and subalgebras). To show this, consider the map $\iota \colon \mathbf{C} \to \mathbf{B}_+/\theta_+ \bowtie \mathbf{B}_-/\theta_-$ given by $\iota(h(a)) = \langle [a]_+/\theta_+, [a]_-/\theta_- \rangle$ for all $a \in B$, where $[a]_+, [a]_-$ denote the equivalence classes of a under $\equiv_+$ and $\equiv_-$ respectively. It is easy to check that ι is well defined. Let us show that ι preserves the meet with respect to the truth order:

$$\begin{aligned}
\iota(h(a) \wedge_{\mathbf{C}} h(b)) &= \iota(h(a \wedge_{\mathbf{B}} b)) \\
&= \langle [a \wedge_{\mathbf{B}} b]_+/\theta_+, [a \wedge_{\mathbf{B}} b]_-/\theta_- \rangle \\
&= \langle ([a]_+ \wedge_{\mathbf{B}_+} [b]_+)/\theta_+, ([a]_- \vee_{\mathbf{B}_-} [b]_-)/\theta_- \rangle \\
&= \langle [a]_+/\theta_+ \wedge_{\mathbf{B}_+/\theta_+} [b]_+/\theta_+, [a]_-/\theta_- \vee_{\mathbf{B}_-/\theta_-} [b]_-/\theta_- \rangle \\
&= \langle [a]_+/\theta_+, [a]_-/\theta_- \rangle \wedge_{\mathbf{B}_+/\theta_+ \bowtie \mathbf{B}_-/\theta_-} \langle [b]_+/\theta_+, [b]_-/\theta_- \rangle \\
&= \iota(h(a)) \wedge_{\mathbf{B}_+/\theta_+ \bowtie \mathbf{B}_-/\theta_-} \iota(h(b)).
\end{aligned}$$

Preservation of the other connectives can be easily proved in the same way. To prove injectivity of ι, assume $\iota(h(a)) = \iota(h(b))$, that is $\langle [a]_+/\theta_+, [a]_-/\theta_- \rangle = \langle [b]_+/\theta_+, [b]_-/\theta_- \rangle$. Then, by definition of θ_+ and θ_-, we have $\langle a \wedge b, a \sqcup b \rangle, \langle a \wedge b, a \sqcap b \rangle \in \theta$. Then $\langle a \sqcap b, a \sqcup b \rangle \in \theta$ which implies (in any lattice) $\langle a, b \rangle \in \theta$, i.e. $h(a) = h(b)$ as required.

In order to introduce and study a logic of non-involutive implicative bilattices, the notion of bifilter (see e.g. [5, Section 3.3]) will be useful.

Definition 7. *A* bifilter *of a (non-involutive implicative) bilattice* $\mathbf{B}$ *is a non-empty set* $F \subseteq B$ *that is upward closed (in both lattice orders) and is furthermore closed under binary meets of both orders, that is* $a, b \in F$ *imply* $a \wedge b, a \sqcap b \in F$. *An* open bifilter *is a bifilter such that* $\neg\neg a \in F$ *whenever* $a \in F$.

Bifilters are well known in the bilattice literature since the works of Arieli and Avron [2], whereas the new notion of open bifilter clearly poses a non-trivial constraint only when the negation might be non-involutive.

Proposition 6. *Any bifilter* F *of a (non-involutive implicative) bilattice* $\mathbf{B} = \mathbf{B}_+ \bowtie \mathbf{B}_-$ *is of the form* $F = F_+ \times B_-$ *where* F_+ *is a non-empty lattice filter of* $\mathbf{B}_+$. *Moreover,* F *is open if and only if* F_+ *is such that* $a_+ \in F_+$ *implies* $p(n(a_+)) \in F_+$.

Proof. The first part of the statement is well known [5, Proposition 3.18] and the second is an immediate consequence of the first.

Proposition 6 immediately implies that any $\mathbf{B} \in \mathsf{NIB}$ has a minimal bifilter $F_\varepsilon = \{1_+\} \times B_-$, which is also open, and is given by $F_\varepsilon = \{\varepsilon(a) : a \in B\}$, where $\varepsilon(a)$ abbreviates $a \supset a$.

Proposition 7. *Let* $\mathbf{B} \in \mathsf{NIB}$ *and* $F \subseteq B$. *The following are equivalent:*

(i) F *is an (open) bifilter.*
(ii) F *is non-empty and closed under (mp), i.e.* $a, a \supset b \in F$ *imply* $b \in F$ *(and* $\neg\neg a \in F$ *whenever* $a \in F$*).*

Proof. (i)⇒(ii). To take advantage of the characterization of Proposition 6, we will assume that $\mathbf{B} = \mathbf{B}_+ \bowtie \mathbf{B}_-$ and so $F = F_+ \times B_-$ for some non-empty lattice filter $F_+ \subseteq B_+$. Then the result easily follows from the fact that a lattice filter of an implicative lattice is closed under (mp) relative to the Heyting implication of $\mathbf{B}_+$.
(ii)⇒(i). Also assuming $\mathbf{B} = \mathbf{B}_+ \bowtie \mathbf{B}_-$, by (ii) we have that $[F]_+$ is non-empty and is closed under (mp) relative to the Heyting implication of $\mathbf{B}_+$. Hence $[F]_+$ is a lattice filter of $\mathbf{B}_+$ and the result follows again by Proposition 6.

5 The logic of non-involutive implicative bilattices

Following [10], we can consider the logic $\models_s$ defined by all matrices $\langle \mathbf{B}, F_\varepsilon \rangle$ where $\mathbf{B} \in \mathsf{NIB}$ and $F_\varepsilon = \{\varepsilon(a) : a \in B\}$. Alternatively, one could consider the (weaker) logic $\models_w$ defined by all matrices $\langle \mathbf{B}, F \rangle$ where $\mathbf{B} \in \mathsf{NIB}$ and F is any bifilter. The two logics coincide if the negation is involutive, but in general only the inclusion $\models_w \subseteq \models_s$ holds. It is also an immediate consequence of the definitions that the two logics share the same valid formulas, and therefore only differ when it comes to consequences of non-empty sets of formulas. In particular we have $p \models_s \neg\neg p$ which does not hold in $\models_w$, although it is true that $\emptyset \models_w p$ implies $\emptyset \models_w \neg\neg p$. Another rule that is sound in $\models_s$ but not in $\models_w$ is $p \wedge \neg q \vdash \neg(p \supset q)$, which is reminiscent of the stronger axiom $(p \wedge \neg q) \supset \neg(p \supset q)$ that holds in involutive bilattice logic.

We now introduce a Hilbert calculus $\vdash_{\mathsf{NIB}}$ that we will prove to be complete with respect to the above-defined semantic consequence $\models_s$ and algebraizable in the sense of [3] with respect to the class of non-involutive implicative bilattices.

Definition 8. *The logic $\vdash_{\mathsf{NIB}}$ is defined by the following axioms and rules.*

Axioms for the $\{\wedge, \vee, \sqcap, \sqcup, \supset\}$-fragment (corresponding to intuitionistic/implicative bilattice logic):

$(\supset 1)$	$p \supset (q \supset p)$	
$(\supset 2)$	$(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$	
$(\wedge \supset)$	$(p \wedge q) \supset p$	$(p \wedge q) \supset q$
$(\supset \wedge)$	$p \supset (q \supset (p \wedge q))$	
$(\supset \vee)$	$p \supset (p \vee q)$	$q \supset (p \vee q)$
$(\vee \supset)$	$(p \supset r) \supset ((q \supset r) \supset ((p \vee q) \supset r))$	
$(\sqcap \supset)$	$(p \sqcap q) \supset p$	$(p \sqcap q) \supset q$
$(\supset \sqcap)$	$p \supset (q \supset (p \sqcap q))$	
$(\supset \sqcup)$	$p \supset (p \sqcup q)$	$q \supset (p \sqcup q)$
$(\sqcup \supset)$	$(p \supset r) \supset ((q \supset r) \supset ((p \sqcup q) \supset r))$	

Axioms for the $\{\wedge, \vee, \sqcap, \sqcup, \not\subset, \neg\}$-fragment:

$(\not\subset 1)$	$\neg(p \not\subset (q \not\subset p))$	
$(\not\subset 2)$	$\neg((p \not\subset (q \not\subset r)) \not\subset ((p \not\subset q) \not\subset (p \not\subset r)))$	
$(\vee \not\subset)$	$\neg((p \vee q) \not\subset p)$	$\neg((p \vee q) \not\subset q)$
$(\not\subset \vee)$	$\neg(p \not\subset (q \not\subset (p \vee q)))$	
$(\not\subset \wedge)$	$\neg(p \not\subset (p \wedge q))$	$\neg(q \not\subset (p \wedge q))$
$(\wedge \not\subset)$	$\neg((p \not\subset r) \not\subset ((q \not\subset r) \not\subset ((p \wedge q) \not\subset r)))$	
$(\sqcap \not\subset)$	$\neg((p \sqcap q) \not\subset p)$	$\neg((p \sqcap q) \not\subset q)$
$(\not\subset \sqcap)$	$\neg(p \not\subset (q \not\subset (p \sqcap q)))$	
$(\not\subset \sqcup)$	$\neg(p \not\subset (p \sqcup q))$	$\neg(q \not\subset (p \sqcup q))$
$(\sqcup \not\subset)$	$\neg((p \not\subset r) \not\subset ((q \not\subset r) \not\subset ((p \sqcup q) \not\subset r)))$	

Interaction axioms:

(*A1*)	$\neg(p \not\subset q) \supset (\neg p \supset \neg q)$
(*A2*)	$\neg((\neg p \vee q) \not\subset (p \supset q))$
(*A3*)	$\neg((p \supset q) \not\subset (\neg p \vee q))$
(*A4*)	$(p \not\subset q) \supset (\neg p \wedge q)$
(*A5*)	$(\neg p \wedge q) \supset (p \not\subset q)$
(*A6*)	$\neg\neg p \supset \neg(q \not\subset (p \supset q))$
(*A7*)	$\neg\neg(p \supset q) \supset \neg(\neg p \not\subset \neg q)$
(*A8*)	$\neg((p \not\subset p) \supset \neg(p \supset p))$

Non-involutive negation axioms:

(*NI1*)	$\neg(p \vee q) \supset (\neg p \wedge \neg q)$
(*NI2*)	$(\neg p \wedge \neg q) \supset \neg(p \vee q)$
(*NI3*)	$\neg(\neg(p \wedge q) \not\subset (\neg p \vee \neg q))$
(*NI4*)	$\neg((\neg p \vee \neg q) \not\subset \neg(p \wedge q))$
(*NI5*)	$\neg\neg p \supset p$
(*NI6*)	$\neg(\neg\neg p \not\subset p)$

Axioms for the constants (if present):

$(\supset \mathsf{t})$	$p \supset \mathsf{t}$
$(\supset \mathsf{f})$	$\mathsf{f} \supset p$
$(\not\subset \mathsf{t})$	$\neg(\mathsf{t} \not\subset p)$
$(\not\subset \mathsf{f})$	$\neg(p \not\subset \mathsf{f})$
$(\supset \top)$	$p \supset \top$
$(\supset \bot)$	$\bot \supset p$
$(\not\subset \top)$	$\neg(p \not\subset \top)$
$(\not\subset \bot)$	$\neg(\bot \not\subset p)$

The rules are modus ponens *(mp) and* double negation *(dn):*

(*mp*)	$p, p \supset q \vdash q$
(*dn*)	$p \vdash \neg\neg p$

Lemma 4. *(Soundess) The class of matrices* $\langle \mathbf{B}, F_\varepsilon \rangle$ *where* $\mathbf{B} \in \mathsf{NIB}$ *and* $F_\varepsilon = \{\varepsilon(a) : a \in B\}$ *is sound for the logic* $\vdash_{\mathsf{NIB}}$.

Proof. A matter of routine checking, using the product representation of non-idempotent implicative bilattices (Theorem 3 and Proposition 6).

We state here the main result but we delay the proof until Section 7.

Theorem 5. *The logic $\vdash_{\mathsf{NIB}}$ is algebraizable with translations $\tau\colon Fm \to Eq$ given by $\tau(p) = \{p \approx \varepsilon(p)\}$ and $\rho\colon Eq \to Fm$ given by $\rho(x \approx y) = \{x \supset y, y \supset x, \neg(x \not\subset y), \neg(y \not\subset x)\}$. The equivalent algebraic semantics of $\vdash_{\mathsf{NIB}}$ is the variety NIB of non-involutive implicative bilattices.*

Proof. See Corollaries 4 and 5.

Theorem 5 implies, in particular, that the calculus $\vdash_{\mathsf{NIB}}$ is complete with respect to the intended semantics for our logic, i.e. the class of matrices $\langle \mathbf{B}, F_\varepsilon \rangle$ with $\mathbf{B} \in \mathsf{NIB}$. Moreover, we can exploit the algebraizability result to obtain a complete axiomatization for the other logic $\models_w$ that we introduced above as the logic of all matrices $\langle \mathbf{B}, F \rangle$ where $\mathbf{B} \in \mathsf{NIB}$ and F is an arbitrary bifilter.

Lemma 5. *Denote by $\vdash_w$ the calculus having all theorems of $\vdash_{\mathsf{NIB}}$ as axioms and (mp) as the only rule of inference. Then $\mathbf{Alg}^*(\vdash_w) = \mathbf{Alg}(\vdash_w) = \mathsf{NIB}$.*

Proof. The equality $\mathbf{Alg}^*(\vdash_w) = \mathbf{Alg}(\vdash_w)$ holds for all protoalgebraic logics [8, Proposition 3.2.] and $\vdash_w$ is protoalgebraic by [7, Theorem 1.1.3]. Moreover, since $\vdash_w \subseteq \vdash_{\mathsf{NIB}}$, by Theorem 5 we have $\mathsf{NIB} = \mathbf{Alg}^*(\vdash_{\mathsf{NIB}}) \subseteq \mathbf{Alg}^*(\vdash_w)$. For the other inclusion, it suffices to show that the Lindenbaum-Tarski algebra $\mathbf{Fm}^*$ of $\vdash_w$ belongs to NIB. This is so because $V(\mathbf{Alg}^*(\vdash_w)) = V(\mathbf{Fm}^*)$ by [8, Proposition 2.26], and so $\mathbf{Fm}^* \in \mathsf{NIB}$ implies $\mathbf{Alg}^*(\vdash_w) \subseteq V(\mathbf{Alg}^*(\vdash_w)) = V(\mathbf{Fm}^*) \subseteq V(\mathsf{NIB}) = \mathsf{NIB}$. We claim that Tarski congruence of $\vdash_w$ is $\varOmega = \{\langle \varphi, \psi \rangle \in Fm \times Fm : \emptyset \vdash_w \varphi \leftrightarrow \psi\} = \{\langle \varphi, \psi \rangle \in Fm \times Fm : \emptyset \vdash_g \varphi \leftrightarrow \psi\}$, which is (by Theorem 5) the Leibniz congruence of the matrix $\langle \mathbf{Fm}, \mathrm{Th}(\vdash_g) \rangle$. This last remark immediately implies that $\varOmega$ is a congruence of $\mathbf{Fm}$, therefore it remains to check that it is the greatest logical congruence (i.e. contained in the inter-derivability relation of $\vdash_w$). To this purpose, suppose θ is a logical congruence of $\langle \mathbf{Fm}, \vdash_w \rangle$ and $\langle \varphi, \psi \rangle \in \theta$. Then $\langle \varphi \supset \varphi, \varphi \supset \psi \rangle \in \theta$ and also $\langle \neg(\varphi \not\subset \varphi), \neg(\varphi \not\subset \psi) \rangle \in \theta$. This implies that $\varphi \supset \varphi \vdash_w \varphi \supset \psi$ and $\neg(\varphi \not\subset \varphi) \vdash_w \neg(\varphi \not\subset \psi)$. Both $\varphi \supset \varphi$ and $\neg(\varphi \not\subset \varphi)$ are theorems of $\vdash_{\mathsf{NIB}}$ (and thus theorems of $\vdash_w$), so we have $\emptyset \vdash_w \varphi \supset \psi$ and $\emptyset \vdash_w \neg(\varphi \not\subset \psi)$. In a similar way we obtain $\emptyset \vdash_w \psi \supset \varphi$, $\emptyset \vdash_w \neg(\psi \not\subset \varphi)$, and so $\emptyset \vdash_w \varphi \leftrightarrow \psi$. This means that $\langle \varphi, \psi \rangle \in \varOmega$ and so $\theta \subseteq \varOmega$. Thus, the Lindenbaum-Tarski algebra $\mathbf{Fm}^*$ of $\vdash_w$ is $\mathbf{Fm}/\varOmega$, which by Theorem 5 belongs to NIB as claimed.

Lemma 6. *Let $\mathbf{B} \in \mathsf{NIB}$ and $F \subseteq B$. The following are equivalent:*

(i) F is a logical filter of $\vdash_w$
(ii) F is a bifilter.

Proof. (i)⇒(ii). F is non-empty since it contains the interpretation of all theorems of $\vdash_w$. Moreover, F is closed under (mp), so we can apply Proposition 7 to conclude that F is a bifilter.
(ii)⇒(i). Proposition 4 implies that F contains the interpretation of all axioms of $\vdash_{\mathsf{NIB}}$, and closure of F under (mp) follows from Proposition 7.

Theorem 6. *The logic $\models_w$ is axiomatized by the calculus $\vdash_w$ having all theorems of $\vdash_{\mathsf{NIB}}$ as axioms and (mp) as the only rule of inference.*

Proof. Soundness is easy. On the one hand, as we have observed earlier, $\models_w$ and $\models_s$ share the same set of valid formulas. Therefore (by soundness of $\vdash_{\mathsf{NIB}}$), any theorem of $\vdash_{\mathsf{NIB}}$ is valid in $\models_w$ too. On the other hand, any bifilter is closed under (mp) by Lemma 6. Thus all matrices $\langle \mathbf{B}, F\rangle$, with $\mathbf{B} \in \mathsf{NIB}$ and F a bifilter, are models of $\vdash_w$. For completeness, assume $\Gamma \nvdash_w \varphi$. Then there is a reduced matrix $\langle \mathbf{B}, F\rangle$ and a valuation $h\colon \mathbf{Fm} \to \mathbf{B}$ such that $h(\Gamma) \subseteq F$ and $h(\varphi) \notin F$. By Lemma 5, $\mathbf{B} \in \mathsf{NIB}$ and, by Lemma 6, F is a bifilter. Hence, $\Gamma \not\models_w \varphi$ as required.

6 Starting from N4-lattices

In this section we generalize the non-involutive bilattice product construction, introducing a common framework for bilattices, *nd*-frames and N4-lattices. We are going to work with the $\{\wedge, \vee, \supset, \not\subset, \neg\}$-fragment of the bilattice language (in general, we do not assume the presence of any constant), but all the notation is consistent with the one used in the preceding sections. In particular, we write $\varepsilon(a)$ as an abbreviation for $a \supset a$, and $a \preccurlyeq_+ b$ as an abbreviation for $a \supset b = \varepsilon(a \supset b)$, and $a \preccurlyeq_- b$ as an abbreviation for $\neg(a \not\subset b) = \varepsilon(\neg(a \not\subset b))$. As before, we also define $x \to y = (x \supset y) \wedge \neg(y \not\subset x)$ and $x \leftrightarrow y = \{x \supset y, y \supset x, \neg(x \not\subset y), \neg(y \not\subset x)\}$.

Definition 9. *Let $\mathbf{L}_+ \bowtie \mathbf{L}_-$ be a non-involutive implicative product bilattice (Definition 5). A* non-involutive twist-structure over $\langle \mathbf{L}_+, \mathbf{L}_-\rangle$ *is any $\{\wedge, \vee, \supset, \not\subset, \neg\}$-subalgebra of $\mathbf{L}_+ \bowtie \mathbf{L}_-$ having the property that $\pi_1(A) = L_+$ and $\pi_2(A) = L_-$.*

Definition 10. *A* non-involutive N4-lattice *is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \supset, \not\subset, \neg\rangle$ where $\langle A, \wedge, \vee\rangle$ is a lattice (a bounded lattice, in case the bounds f and t are present) and:*

- *(i) the relations $\preccurlyeq_+$ and $\preccurlyeq_-$ are preorders (i.e. reflexive and transitive),*
- *(ii) $x \leq y$ if and only if $x \preccurlyeq_+ y$ and $y \preccurlyeq_- x$.*
- *(iii) the equivalence relation $\equiv_+$ induced by $\preccurlyeq_+$ is compatible with the operations $\wedge, \vee, \supset$,*
- *(iv) the equivalence relation $\equiv_-$ induced by $\preccurlyeq_-$ is compatible with the operations $\wedge, \vee, \not\subset$,*
- *(v) the quotients $\mathbf{A}_+ = \langle A/{\equiv_+}, \wedge, \vee, \supset\rangle$ and $\mathbf{A}_- = \langle A/{\equiv_-}, \vee, \wedge, \not\subset\rangle$ are implicative lattices,*
- *(vi) $x \equiv_+ y \Rightarrow \neg x \equiv_- \neg y$ and $x \equiv_- y \Rightarrow \neg x \equiv_+ \neg y$,*
- *(vii) $x \not\subset y \equiv_+ \neg x \wedge y$ and $x \supset y \equiv_- \neg x \vee y$,*
- *(viii) $x \not\subset x \equiv_- \neg(x \supset x)$ and $x \supset x \equiv_+ \neg(x \not\subset x)$,*
- *(ix) $\neg(x \vee y) \equiv_+ \neg x \wedge \neg y$ and $\neg(x \wedge y) \equiv_- \neg x \vee \neg y$,*
- *(x) $\neg\neg x \preccurlyeq_+ x$ and $\neg\neg x \preccurlyeq_- x$,*
- *(xi) $\neg \mathsf{t} = \mathsf{f}$ $\neg \mathsf{f} = \mathsf{t}$ (if the constants are present).*

Non-involutive N4-lattices ($\mathsf{NN4}$) are obviously a generalization of non-involutive implicative bilattices. It is also easy to check that any N4-lattice [14, Definition 8.4.1] satisfies all items of Definition 10 if we let $x \not\subset y = \neg(\neg x \supset \neg y)$. That is, non-involutive N4-lattices can also be seen as a generalization of N4-lattices. Next we show the equivalence between Definition 9 and Definition 10. The following proposition is straightforward.

Proposition 8. *Every non-involutive twist-structure is a non-involutive N4-lattice.*

Let $\mathbf{A} \in \mathsf{NN4}$ and $\mathbf{A}_+, \mathbf{A}_-$ be as in Definition 10.(v). Observe that $a \preccurlyeq_+ b$ implies $[a]_+ \supset [b]_+ = ([a]_+ \supset [b]_+) \supset ([a]_+ \supset [b]_+)$ and so $[a]_+ \leq_+ [b]_+$ in $\mathbf{A}_+$. Conversely, it is not difficult to show that $[a]_+ \leq_+ [b]_+$ implies $a \preccurlyeq_+ b$. Similarly we have $a \preccurlyeq_- b$ iff $[a]_+ \leq_- [b]_-$ in $\mathbf{A}_-$.

As before, we define maps $n\colon \mathbf{A}_+ \to \mathbf{A}_-$ by $n([a]_+) = [\neg a]_-$ and $p\colon \mathbf{A}_- \to \mathbf{A}_+$ by $p([a]_-) = [\neg a]_+$. Definition 10.(vi) guarantees that these are well defined. Moreover we have $n([a \supset a]_+) = [\neg(a \supset a)]_- = [a \not\subset a]_-$ by item (viii), $n([a \wedge b]_+) = [\neg(a \wedge b)]_- = [\neg a \vee \neg b]_- =$

$[\neg a]_- \vee [\neg b]_- = n([a]_+) \vee n([b]_+)$ by item (ix) and $p(n([a]_+)) = [\neg\neg a]_+ \leq_+ [a]_+$ by item (x). A similar argument shows that the map p also satisfies properties (i)–(iii) of Definition 3. Thus we have a non-involutive twist-structure $\mathbf{A}_+ \bowtie \mathbf{A}_-$.

Theorem 7. *Let $\mathbf{A} = \langle A, \wedge, \vee, \supset, \not\subset, \neg \rangle$ be a non-involutive N4-lattice. Then the map $\iota\colon \mathbf{A} \to \mathbf{A}_+ \bowtie \mathbf{A}_-$ given by $\iota(a) = ([a]_+, [a]_-)$ for all $a \in A$ is an embedding such that $\pi_1(\iota[A]) = A_+$ and $\pi_2(\iota[A]) = A_-$, i.e $\mathbf{A}$ is isomorphic to a non-involutive twist-structure over $\langle \mathbf{A}_+, \mathbf{A}_- \rangle$.*

Proof. It is obvious that $\pi_1(\iota[A]) = A_+$ and $\pi_2(\iota[A]) = A_-$. Also, injectivity of ι is an immediate consequence of Definition 10.(ii). It remains to check that ι preserves the algebraic operations. The case of $\neg, \wedge$ and $\vee$ is immediate. Let us check that $\iota(a \supset b) = \iota(a) \supset \iota(b)$ and $\iota(a \not\subset b) = \iota(a) \not\subset \iota(b)$. As to the former, using Definition 10.vii, we have $\iota(a \supset b) = \langle [a \supset b]_+, [a \supset b]_- \rangle = \langle [a]_+ \supset [b]_+, [\neg a \vee b]_- \rangle = \langle [a]_+ \supset [b]_+, [\neg a]_- \vee [b]_- \rangle = \langle [a]_+ \supset [b]_+, n([a]_+) \vee [b]_- \rangle = \iota(a) \supset \iota(b)$. As to the latter, again by Definition 10.(vii) we have $\iota(a \not\subset b) = ([a \not\subset b]_+, [a \not\subset b]_-) = ([\neg a \wedge b]_+, [a]_- \not\subset [b]_-) = ([\neg a]_+ \wedge [b]_+, [a]_- \not\subset [b]_-) = (p([a]_-) \wedge [b]_+, [a]_- \not\subset [b]_-) = \iota(a) \not\subset \iota(b)$.

Theorem 7 implies that we can view any non-involutive N4-lattice as a subalgebra of some product $\mathbf{L}_+ \bowtie \mathbf{L}_-$. We can use this fact to simplify our proofs and establish basic properties of non-involutive N4-lattices. For example, we can say that the lattice reduct of a non-involutive N4-lattice is distributive, because implicative lattices are distributive and therefore so must be their product $\mathbf{L}_+ \bowtie \mathbf{L}_-$. We list below other properties that can be easily checked in twist-structures.

Proposition 9. *Every non-involutive N4-lattice satisfies the following equations:*

(i) $(x \supset x) \supset y = y = (x \not\subset x) \not\subset y = \neg(x \supset x) \not\subset y$
(ii) $(x \wedge y) \supset z = x \supset (y \supset z) = y \supset (x \supset z) = (x \supset y) \supset (x \supset z)$
(iii) $(x \vee y) \not\subset z = x \not\subset (y \not\subset z) = y \not\subset (x \not\subset z) = (x \not\subset y) \not\subset (x \not\subset z)$
(iv) $(x \vee y) \supset z \leq (x \supset z) \wedge (y \supset z) \equiv_+ (x \vee y) \supset z$
(v) $(x \wedge y) \not\subset z \equiv_- (x \not\subset z) \vee (y \not\subset z) \leq (x \wedge y) \not\subset z$
(vi) $x \leq (x \to y) \supset y$
(vii) $x \supset x \leq \neg\neg(x \supset x)$
(viii) $\neg\neg(x \not\subset x) \leq x \not\subset x$

7 The logic of non-involutive N4-lattices

In this section we are going to introduce a logic that is algebraizable and has the class of non-involutive N4-lattices as its equivalent algebraic semantics. As a corollary, we will obtain the above-stated algebraizability of the logic of non-involutive implicative bilattices (Theorem 5).

The translations witnessing algebraizability are $\tau\colon Fm \to Eq$ given by $\tau(p) = \{p \approx \varepsilon(p)\}$, and $\rho\colon Eq \to Fm$ given by $\rho(x \approx y) = \{x \supset y, y \supset x, \neg(x \not\subset y), \neg(y \not\subset x)\}$, or equivalently $\rho(x \approx y) = \{x \to y, y \to x\}$.

Definition 11. *The logic $\vdash_{\mathsf{NN4}}$ is defined by all the axioms and rules from Definition 8 which do not mention the knowledge connectives*[8].

[8] We notice that axiom (A8) and all the (NI1)–(NI6) are not needed to show that the logic is algebraizable, so if we remove them we obtain a logic which is still algebraizable, but with respect to some weaker structures than $\mathsf{NN4}$.

Remark 1. All rules of the $\{\wedge, \vee, \supset\}$-fragment of intuitionistic logic are rules of $\vdash_{\mathsf{NN4}}$ as well. Therefore, all derivations of positive intuitionistic logic (or, indeed, of the $\{\wedge, \vee, \supset\}$-fragment of Brouwerian bilattice logic and N4-logic, see [4,13]) can be reproduced in $\vdash_{\mathsf{NN4}}$ as well. This fact will often be used to shorten our proofs.

The following lemma lists a few useful properties of the $\not\subset$-implication.

Lemma 7. *The following hold:*

(i) $\neg p, \neg(p \not\subset q) \vdash_{\mathsf{NN4}} \neg q$ ($\not\subset$-mp)
(ii) $\neg p \vdash_{\mathsf{NN4}} \neg(q \not\subset p)$
(iii) $\neg(p \not\subset q), \neg(q \not\subset r) \vdash_{\mathsf{NN4}} \neg(p \not\subset r)$ ($\not\subset$-transitivity)
(iv) $\neg(p \not\subset (q \not\subset r)) \vdash_{\mathsf{NN4}} \neg(q \not\subset (p \not\subset r))$
(v) $\vdash_{\mathsf{NN4}} \neg((p \not\subset q) \not\subset ((q \not\subset r) \not\subset (p \not\subset r)))$
(vi) $\neg(p \not\subset q), \neg(p \not\subset r) \vdash_{\mathsf{NN4}} \neg(p \not\subset (q \vee r))$

Proof. (i). By (A1) and (mp).
(ii). By ($\not\subset$ 1) and item (i) above ($\not\subset$-mp).
(iii). By item (ii) above we have $\neg(q \not\subset r) \vdash_{\mathsf{NN4}} \neg(p \not\subset (q \not\subset r))$. Hence by ($\not\subset$ 2) and (mp) we get $\neg(q \not\subset r) \vdash_{\mathsf{NN4}} \neg((p \not\subset q) \not\subset (p \not\subset r))$, which gives us $\neg(p \not\subset q), \neg(q \not\subset r) \vdash_{\mathsf{NN4}} \neg(p \not\subset r)$ by ($\not\subset$-mp).
(iv). From ($\not\subset$ 2) and ($\not\subset$-mp) we obtain $\neg(p \not\subset (q \not\subset r)) \vdash_{\mathsf{NN4}} \neg((p \not\subset q) \not\subset (p \not\subset r))$. Since $\neg(q \not\subset (p \not\subset q))$ is an instance of ($\not\subset$ 1), by $\not\subset$-transitivity we have $\neg((p \not\subset q) \not\subset (p \not\subset r)) \vdash_{\mathsf{NN4}} \neg(q \not\subset (p \not\subset r))$. So by transitivity of $\vdash_{\mathsf{NN4}}$ we get $\neg(p \not\subset (q \not\subset r)) \vdash_{\mathsf{NN4}} \neg(q \not\subset (p \not\subset r))$ as required.
(v). From $\neg((q \not\subset r) \not\subset (p \not\subset (q \not\subset r)))$, which is an instance of ($\not\subset$ 1), and ($\not\subset$ 2) we obtain, by $\not\subset$-transitivity, that $\neg((q \not\subset r) \not\subset ((p \not\subset q) \not\subset (p \not\subset r)))$ is a theorem. Using item (iv) above, we have then that $\neg((p \not\subset q) \not\subset ((q \not\subset r) \not\subset (p \not\subset r)))$ is a theorem as well.
(vi). $\neg(q \not\subset (r \not\subset (q \vee r)))$ is an instance of ($\not\subset$ $\vee$). Hence by item (iii) above we have $\neg(p \not\subset q) \vdash_{\mathsf{NN4}} \neg(p \not\subset (r \not\subset (q \vee r)))$. Since $\neg((p \not\subset (r \not\subset (q \vee r))) \not\subset ((p \not\subset r) \not\subset (p \not\subset (q \vee r))))$ is an instance of ($\not\subset$ 2), we have, by ($\not\subset$-mp), $\neg(p \not\subset q) \vdash_{\mathsf{NN4}} \neg((p \not\subset r) \not\subset (p \not\subset (q \vee r)))$. Then, using ($\not\subset$-mp) again, we have $\neg(p \not\subset q), \neg(p \not\subset r) \vdash_{\mathsf{NN4}} \neg(p \not\subset (q \vee r))$.

Theorem 8. *The logic $\vdash_{\mathsf{NN4}}$ is algebraizable with translations $\tau\colon Fm \to Eq$ given by $\tau(p) = \{p \approx \varepsilon(p)\}$ and $\rho\colon Eq \to Fm$ given by $\rho(x \approx y) = \{x \supset y, y \supset x, \neg(x \not\subset y), \neg(y \not\subset x)\}$.*

Proof. Recall that $p \leftrightarrow q$ is a shorthand for the set $\{p \supset q, q \supset p, \neg(p \not\subset q), \neg(q \not\subset p)\}$. We also write $\Gamma \vdash \Delta$ as a to mean that $\Gamma \vdash \delta$ for all $\delta \in \Delta$ (beware: we depart here from the widespread interpretation of $\Gamma \vdash \Delta$ as $\Gamma \vdash \delta$ for *some* $\delta \in \Delta$). By [3, Theorem 4.7], in order to prove the result it is sufficient to check that the following conditions are met:

(i) $\vdash_{\mathsf{NN4}} p \leftrightarrow p$,
(ii) $p \leftrightarrow q \vdash_{\mathsf{NN4}} q \leftrightarrow p$,
(iii) $p \leftrightarrow q, q \leftrightarrow r \vdash_{\mathsf{NN4}} p \leftrightarrow r$,
(iv) $p \leftrightarrow q \vdash_{\mathsf{NN4}} \neg p \leftrightarrow \neg q$,
(v) $p \leftrightarrow q, r \leftrightarrow s \vdash_{\mathsf{NN4}} (p * r) \leftrightarrow (q * s)$ for all $* \in \{\wedge, \vee, \supset, \not\subset\}$,
(vi) $p \dashv\vdash_{\mathsf{NN4}} p \leftrightarrow (p \supset p)$.

(i). In light of Remark 1, we only need to prove that $\neg(p \not\subset p)$ is a theorem of the logic. Notice that $\neg((p \not\subset ((q \not\subset p) \not\subset p)) \not\subset ((p \not\subset (q \not\subset p)) \not\subset (p \not\subset p)))$ is an instance of ($\not\subset$ 2) and

both $\neg(p \not\subset ((q \not\subset p) \not\subset p))$ and $\neg(p \not\subset (q \not\subset p))$ are instances of ($\not\subset$ 1). Then the result is obtained by applying ($\not\subset$-mp) twice.
(ii). Immediate.
(iii). By Remark 1 we have $\{p \leftrightarrow q, q \leftrightarrow r\} \vdash_{\mathsf{NN4}} \{p \supset r, r \supset p\}$. The remaining part follows by $\not\subset$-transitivity that we proved in Lemma 7.(iii).
(iv). By (A1) we have $p \leftrightarrow q \vdash_{\mathsf{NN4}} \{\neg p \supset \neg q, \neg q \not\subset \neg p\}$. To see that $p \leftrightarrow q \vdash_{\mathsf{NN4}} \{\neg(\neg p \not\subset \neg q), \neg(\neg q \not\subset \neg p)\}$, notice that by (dn), (A7) and (mp) we have $(p \supset q) \vdash_{\mathsf{NN4}} \neg\neg(p \supset q) \vdash_{\mathsf{NN4}} \neg(\neg p \not\subset \neg q)$.
(v). We need to consider each connective in $\{\wedge, \vee, \supset, \not\subset\}$.
($\wedge$). We have $\{p \leftrightarrow q, r \leftrightarrow s\} \vdash_{\mathsf{NN4}} \{(p \wedge r) \supset (q \wedge s), (q \wedge s) \supset (p \wedge r)\}$ by Remark 1. To complete the proof we are going to show that $\{\neg(p \not\subset q), \neg(r \not\subset s)\} \vdash_{\mathsf{NN4}} \neg((p \wedge r) \not\subset (q \wedge s))$. Both $\neg(q \not\subset (q \wedge s))$ and $\neg(s \not\subset (q \wedge s))$ are instances of ($\not\subset \wedge$). Hence, by $\not\subset$-transitivity, we obtain $\{\neg(p \not\subset q), \neg(r \not\subset s)\} \vdash_{\mathsf{NN4}} \{\neg(p \not\subset (q \wedge s)), \neg(r \not\subset (q \wedge s))\}$. Since $\neg((p \not\subset (q \wedge s)) \not\subset ((r \not\subset (q \wedge s)) \not\subset ((p \wedge r) \not\subset (q \wedge s))))$ is an instance of ($\wedge \not\subset$), we can apply ($\not\subset$-mp) twice to obtain the required result.
($\vee$). We have $\{p \leftrightarrow q, r \leftrightarrow s\} \vdash_{\mathsf{NN4}} \{(p \vee r) \supset (q \vee s), (q \vee s) \supset (p \vee r)\}$ by Remark 1. To finish the proof it is enough to show that $\{\neg(p \not\subset q), \neg(r \not\subset s)\} \vdash_{\mathsf{NN4}} \neg((p \vee r) \not\subset (q \vee s))$. Notice that $\neg((p \vee r) \not\subset p)$ and $\neg((p \vee r) \not\subset r)$ are instances of ($\vee \not\subset$). Hence, by $\not\subset$-transitivity, we have $\{\neg(p \not\subset q), \neg(r \not\subset s)\} \vdash_{\mathsf{NN4}} \{\neg((p \vee r) \not\subset q, \neg((p \vee r) \not\subset s\}$. The result then follows by Lemma 7.(vi).
($\supset$). We have $\{p \leftrightarrow q, r \leftrightarrow s\} \vdash_{\mathsf{NN4}} \{(p \supset r) \supset (q \supset s), (q \supset s) \supset (p \supset r)\}$ by Remark 1. To finish the proof it is enough to show that $\{p \supset q, \neg(r \not\subset s)\} \vdash_{\mathsf{NN4}} \neg((p \supset r) \not\subset (q \supset s))$. We have observed above, in the proof of item (iv), that $p \supset q \vdash_{\mathsf{NN4}} \neg(\neg p \not\subset \neg q)$. Hence, $\{p \supset q, \neg(r \not\subset s)\} \vdash_{\mathsf{NN4}} \{\neg(\neg p \not\subset \neg q), \neg(r \not\subset s)\}$. Also, as shown in the proof of ($\vee$), $\{\neg(\neg p \not\subset \neg q), \neg(r \not\subset s)\} \vdash_{\mathsf{NN4}} \neg((\neg p \vee r) \not\subset (\neg q \vee s))$. We have $\neg((\neg p \vee r) \not\subset (\neg q \vee s)) \vdash_{\mathsf{NN4}} \neg((\neg p \vee r) \not\subset (q \supset s))$ by (A2) and $\not\subset$-transitivity and $\neg((\neg p \vee r) \not\subset (q \supset s)) \vdash_{\mathsf{NN4}} \neg((p \supset r) \not\subset (q \supset s))$ by (A3) and $\not\subset$-transitivity. Then using transitivity of $\vdash_{\mathsf{NN4}}$ we obtain the desired result.
($\not\subset$). First we show that $\{\neg(q \not\subset p), \neg(r \not\subset s)\} \vdash_{\mathsf{NN4}} \neg((p \not\subset r) \not\subset (q \not\subset s))$. By Lemma 7.(v) we have $\vdash_{\mathsf{NN4}} \neg((q \not\subset p) \not\subset ((p \not\subset r) \not\subset (q \not\subset r)))$, so by ($\not\subset$-mp) we obtain $\{\neg(q \not\subset p), \neg(r \not\subset s)\} \vdash_{\mathsf{NN4}} \neg((p \not\subset r) \not\subset (q \not\subset r))$. On the other hand $\neg((q \not\subset (r \not\subset s)) \not\subset ((q \not\subset r) \not\subset (q \not\subset s)))$ is an instance of ($\not\subset$ 2) and we have $\neg(r \not\subset s) \vdash_{\mathsf{NN4}} \neg(q \not\subset (r \not\subset s))$ by Lemma 7.(ii). Thus by ($\not\subset$-mp) we get $\neg(r \not\subset s) \vdash_{\mathsf{NN4}} \neg((q \not\subset r) \not\subset (q \not\subset s))$. From this and $\{\neg(q \not\subset p), \neg(r \not\subset s)\} \vdash_{\mathsf{NN4}} \neg((p \not\subset r) \not\subset (q \not\subset r))$, using $\not\subset$-transitivity, we have $\{\neg(q \not\subset p), \neg(r \not\subset s)\} \vdash_{\mathsf{NN4}} \neg((p \not\subset r) \not\subset (q \not\subset s))$ as required. To complete the proof it is sufficient to show that $\{p \leftrightarrow q, r \leftrightarrow s\} \vdash_{\mathsf{NN4}} (p \not\subset r) \supset (q \not\subset s)$. By item (iv) above, we have $\{p \leftrightarrow q, r \leftrightarrow s\} \vdash_{\mathsf{NN4}} \neg p \leftrightarrow \neg q$. Then, as we have seen in the proof of ($\wedge$), we have $\{p \leftrightarrow q, r \leftrightarrow s\} \vdash_{\mathsf{NN4}} (\neg p \wedge r) \supset (\neg q \wedge s)$. We have $\vdash_{\mathsf{NN4}} (p \not\subset r) \supset (\neg p \wedge r)$ by (A4) and $\vdash_{\mathsf{NN4}} (\neg q \wedge s) \supset (q \not\subset s)$ by (A5), so by transitivity of $\supset$ (which we have by Remark 1) we conclude that $\{p \leftrightarrow q, r \leftrightarrow s\} \vdash_{\mathsf{NN4}} (p \not\subset r) \supset (q \not\subset s)$.
(vi). By Remark 1 we have $p \dashv\vdash_{\mathsf{NN4}} \{p \supset (p \supset p), (p \supset p) \supset p\}$. It remains to show that $p \vdash_{\mathsf{NN4}} \{\neg(p \not\subset (p \supset p)), \neg((p \supset p) \not\subset p)\}$ The formula $\neg((p \supset p) \not\subset p)$ is actually a theorem. To see this, notice that $\neg((p \supset p) \not\subset (\neg p \vee p))$ is an instance of (A3) and $\neg((\neg p \vee p) \not\subset p)$ is an instance of ($\vee \not\subset$). Then the result follows by $\not\subset$-transitivity. Finally, $\neg\neg p \supset \neg(p \not\subset (p \supset p))$ is an instance of (A6), so by (dn) and (mp) we obtain $p \vdash_{\mathsf{NN4}} \neg\neg p \vdash_{\mathsf{NN4}} \neg(p \not\subset (p \supset p))$.

Corollary 4. *The logic $\vdash_{\mathsf{NIB}}$ is algebraizable with translations $\tau\colon Fm \to Eq$ given by $\tau(p) = \{p \approx \varepsilon(p)\}$ and $\rho\colon Eq \to Fm$ given by $\rho(x \approx y) = \{x \supset y, y \supset x, \neg(x \not\subset y), \neg(y \not\subset x)\}$.*

Proof. The logic $\vdash_{\mathsf{NIB}}$ is by definition an expansion of $\vdash_{\mathsf{NN4}}$ that we have seen to be algebraizable. Looking at the proof of Theorem 8, one sees that the only additional condition that needs to be checked for $\vdash_{\mathsf{NIB}}$ is (v), i.e. that $p \leftrightarrow q, r \leftrightarrow s \vdash_{\mathsf{NN4}} (p * r) \leftrightarrow (q * s)$ for all

$* \in \{\sqcap, \sqcup\}$. Let us consider both cases.
($\sqcap$). The positive part, i.e. $p \leftrightarrow q, r \leftrightarrow s \vdash_{\mathsf{NN4}} \{(p \sqcap r) \supset (q \sqcap s), (q \sqcap s) \supset (p \sqcap r)\}$, holds by Remark 1. To prove, e.g., that $p \leftrightarrow q, r \leftrightarrow s \vdash_{\mathsf{NN4}} \neg((p \sqcap r) \not\subset (q \sqcap s))$, we reason as in the proof of Theorem 8, case ($\vee$). Since $\neg((p \sqcap r) \not\subset p)$ and $\neg((p \sqcap r) \not\subset r)$ are instances of ($\sqcap \not\subset$), we apply $\not\subset$-transitivity to obtain $\{\neg(p \not\subset q), \neg(r \not\subset s)\} \vdash_{\mathsf{NN4}} \{\neg((p \sqcap r) \not\subset q, \neg((p \sqcap r) \not\subset s\}$. At this point we cannot apply directly Lemma 7.(vi), but we can mimic its proof (using axiom ($\not\subset \sqcap$) instead of ($\not\subset \vee$)) to obtain $\{\neg(p \not\subset q), \neg(r \not\subset s)\} \vdash_{\mathsf{NN4}} \neg((p \sqcap r) \not\subset (q \sqcap s))$.
($\sqcup$). The positive part, i.e. $p \leftrightarrow q, r \leftrightarrow s \vdash_{\mathsf{NN4}} \{(p \sqcup r) \supset (q \sqcup s), (q \sqcup s) \supset (p \sqcup r)\}$, holds by Remark 1. To prove, e.g., that $p \leftrightarrow q, r \leftrightarrow s \vdash_{\mathsf{NN4}} \neg((p \sqcup r) \not\subset (q \sqcup s))$, we reason as in the proof of Theorem 8, case ($\wedge$). That is, we show that $\{\neg(p \not\subset q), \neg(r \not\subset s)\} \vdash_{\mathsf{NN4}} \neg((p \sqcup r) \not\subset (q \sqcup s))$. Both $\neg(q \not\subset (q \sqcup s))$ and $\neg(s \not\subset (q \sqcup s))$ are instances of ($\not\subset \sqcup$). Hence, by $\not\subset$-transitivity, we obtain $\{\neg(p \not\subset q), \neg(r \not\subset s)\} \vdash_{\mathsf{NN4}} \{\neg(p \not\subset (q \sqcup s)), \neg(r \not\subset (q \sqcup s))\}$. Since $\neg((p \not\subset (q \sqcup s)) \not\subset ((r \not\subset (q \sqcup s)) \not\subset ((p \sqcup r) \not\subset (q \sqcup s))))$ is an instance of ($\sqcup \not\subset$), we can apply ($\not\subset$-mp) twice to obtain the required result.

Theorem 9. *The equivalent algebraic semantics of* $\vdash_{\mathsf{NN4}}$ *is the class* NN4 *of non-involutive N4-lattices.*

Proof. Taking advantage of Theorem 7, it is easy to check that any algebra $\mathbf{A} \in \mathsf{NN4}$ satisfies all equations and quasiequations which correspond (via τ) to the axioms and rules of $\vdash_{\mathsf{NN4}}$ (see [3, Theorem 2.17]). Conversely, we need to check that any algebra satisfying these equations and quasiequations also satisfies all conditions of Definition 10. We omit the proof that the operations $\wedge$ and $\vee$ satisfy all lattice equations, which is straightforward, and provide a sketch of the non-trivial proofs of the other items[9].
(i). Reflexivity of $\preccurlyeq_+$ and $\preccurlyeq_-$ follows from the fact that $p \supset p$ and $\neg(p \not\subset p)$ are theorems of the logic–see the proof of Theorem 8.(i). Transitivity follows from (the translations of) axioms ($\supset 2$) and ($\not\subset 2$), see also Lemma 7.(iii).
(ii). Taking $a \leq b$ as an abbreviation of $a = a \wedge b$, the "only if" part can be proved using axioms ($\wedge \supset$) and ($\not\subset \wedge$). For the converse one needs to show that $a \supset b = \varepsilon(a \supset b)$ and $\neg(b \supset a) = \varepsilon(\neg(b \supset a))$ imply $a \leftrightarrow (a \wedge b) = \varepsilon(a \leftrightarrow (a \wedge b))$, that is $a \supset (a \wedge b) = \varepsilon(a \supset (a \wedge b))$, $(a \wedge b) \supset a = \varepsilon((a \wedge b) \supset a)$, $\neg(a \not\subset (a \wedge b)) = \varepsilon(\neg(a \not\subset (a \wedge b)))$ and $\neg((a \wedge b) \not\subset a) = \varepsilon(\neg((a \wedge b) \not\subset a))$. The first, $a \supset (a \wedge b) = \varepsilon(a \supset (a \wedge b))$, follows from $p \supset q \vdash_{\mathsf{NN4}} p \supset (p \wedge q)$ (see Remark 1). The second and third follow from ($\wedge \supset$) and ($\not\subset \wedge$). For the last we need to show that $\neg(q \supset p) \vdash_{\mathsf{NN4}} \neg((p \wedge q) \not\subset p)$ which can be obtained from ($\wedge \not\subset$) instantiated as $\neg((p \not\subset p) \not\subset ((q \not\subset p) \not\subset ((p \wedge q) \not\subset p)))$ with two applications of ($\not\subset$-mp).
(iii) and (iv) follow from the proof of Theorem 8.(iv).
(v). It is easy to show that the translations of the first twelve axioms of $\vdash_{\mathsf{NN4}}$ imply that the quotients $\mathbf{A}_+$ and $\mathbf{A}_-$ are implicative lattices.
(vi). The first quasiequation can be shown as follows. If $a \equiv_+ b$, then $a \supset b = \varepsilon(a \supset b)$. Then $\neg\neg(a \supset b) = \varepsilon(\neg\neg(a \supset b))$ as well by (dn). By axiom (A7) and (mp) we have then $\neg(\neg a \not\subset \neg b) = \varepsilon(\neg(\neg a \not\subset \neg b))$. By symmetry, from $b \supset a = \varepsilon(b \supset a)$ we obtain $\neg(\neg b \not\subset \neg a) = \varepsilon(\neg(\neg b \not\subset \neg a))$ and so $\neg a \equiv_- \neg b$ as required. Analogously, using (A1), one can show that $a \equiv_- b \Rightarrow \neg a \equiv_+ \neg b$.
(vii). The first property follows from (A4) and (A5), the second from (A2) and (A3).
(viii). The first property follows from (A8), the second from Theorem 8.(i) or item (i) above.
(ix). Both properties follow easily from axioms (NI1)–(NI4).
(x). Follow from axioms (NI5) and (NI6).

[9] We also omit the easy proof that $\mathtt{t}$ and $\mathtt{f}$ are actually the lattice bounds, in case the axioms ($\supset \mathtt{t}$)–($\not\subset \mathtt{f}$) are included in the logic.

Corollary 5. *The equivalent algebraic semantics of* $\vdash_{\mathsf{NIB}}$ *is the class* NIB *of non-involutive implicative bilattices.*

Proof. Since the translations witnessing algebraizability of $\vdash_{\mathsf{NN4}}$ and $\vdash_{\mathsf{NIB}}$ are the same, Corollary 4 implies that every algebra $\mathbf{B} \in \mathbf{Alg}(\vdash_{\mathsf{NIB}})$ has a $\{\wedge, \vee, \supset, \not\subset, \neg\}$-reduct which is a non-involutive N4-lattice. Moreover, $\mathbf{B}$ satisfies all the τ-translations of the additional axioms of $\vdash_{\mathsf{NIB}}$. Let us check that this implies $\mathbf{B} \in \mathsf{NIB}$, and thus $\mathbf{Alg}(\vdash_{\mathsf{NIB}}) \subseteq \mathsf{NIB}$. As a non-involutive N4-lattice, $\mathbf{B}$ can be viewed as a twist-structure (Theorem 7) and so we can assume that $\mathbf{B} \subseteq \mathbf{B}_+ \bowtie \mathbf{B}_-$. Let $a = \langle a_+, a_- \rangle$, $b = \langle b_+, b_- \rangle \in \mathbf{B}$ be arbitrary elements. By axiom $(\supset \sqcap)$ we have $\varepsilon(a \supset (b \supset (a \sqcap b))) = a \supset (b \supset (a \sqcap b)) = (a \wedge b) \supset (a \sqcap b)$, where the latter equality holds by Proposition 9.(ii). On the other hand, applying $\supset$-transitivity to axioms $(\supset \wedge)$ and $(\sqcap \supset)$, we have $\vdash_{\mathsf{NIB}} (p \sqcap q) \supset (q \supset (p \wedge q))$. Thus, $\varepsilon((a \sqcap b) \supset (b \supset (a \wedge b))) = (a \sqcap b) \supset (b \supset (a \wedge b)) = b \supset ((a \sqcap b) \supset (a \wedge b))$, the last equality holding by Proposition 9.(ii). Since by $(\sqcap \supset)$ we have $\varepsilon((a \sqcap b) \supset b) = (a \sqcap b) \supset b$, we obtain (again by $\supset$-transitivity) that $\varepsilon((a \sqcap b) \supset (a \wedge b)) = (a \sqcap b) \supset (a \wedge b)$. It follows that $\pi_1(a \sqcap b) = \pi_1(a \wedge b)$. In a similar way, using $(\not\subset \sqcap)$, $(\not\subset \vee)$ and $(\sqcap \not\subset)$, we can show that $\pi_2(a \sqcap b) = \pi_2(a \vee b)$. Thus we have $\langle a_+, a_- \rangle, \sqcap \langle b_+, b_- \rangle = \langle a_+ \wedge_+ b_+, a_- \wedge_- b_- \rangle$. This (by Theorem 3) ensures that the $\sqcap$ operation satisfies all identities that hold in NIB. A similar reasoning can be used to show that $\langle a_+, a_- \rangle, \sqcup \langle b_+, b_- \rangle = \langle a_+ \vee_+ b_+, a_- \vee_- b_- \rangle$. By $(\vee \supset)$ and $(\not\subset \sqcup)$, we have $\pi_1(a \vee b) \leq_+ \pi_1(a \sqcup b)$, and by $(\supset \vee)$ and $(\sqcup \supset)$ we obtain $\pi_1(a \sqcup b) \leq_+ \pi_1(a \vee b)$. Using $(\not\subset \wedge)$ and $(\sqcup \not\subset)$ we obtain $\pi_2(a \sqcup b) \leq_+ \pi_2(a \wedge b)$. The other inequality, $\pi_2(a \wedge b) \leq_+ \pi_2(a \sqcup b)$, is obtained using $(\wedge \not\subset)$ and $(\not\subset \sqcup)$. It is equally easy to check that the axioms for the constants, in case they are present, ensure that $\bot = \langle 0_+, 0_- \rangle$ and $\top = \langle 1_+, 1_- \rangle$. Thus, using Theorem 3) we conclude that $\mathbf{B}$ is a bilattice. To show that $\mathsf{NIB} \subseteq \mathbf{Alg}(\vdash_{\mathsf{NIB}})$, let $\mathbf{B} \in \mathsf{NIB}$. Then the matrix $\langle \mathbf{B}, F_\varepsilon \rangle$, where F_ε is the least open bifilter of $\mathbf{B}$, is a model of $\vdash_{\mathsf{NIB}}$ (Lemma 4). Moreover, we know by Theorem 9 that the matrix $\langle \mathbf{B}, F_\varepsilon \rangle$ is reduced, if we view $\mathbf{B}$ as a non-involutive N4-lattice. A fortiori, $\langle \mathbf{B}, F_\varepsilon \rangle$ must be reduced if we view $\mathbf{B}$ as a bilattice, which means that $\mathbf{B} \in \mathbf{Alg}(\vdash_{\mathsf{NIB}})$ as required.

In analogy with the two consequence relations ($\models_w$ and $\models_s$) associated to non-involutive bilattices, we might define a second consequence relation $\models^w_{\mathsf{NN4}}$ determined by all matrices $\langle \mathbf{A}, F \rangle$ such that $\mathbf{A} \in \mathsf{NN4}$ and F is an implicative filter of $\mathbf{A}$, i.e. (cf. Proposition 7) a non-empty set closed under ($\supset$-mp). Reproducing the proofs of Lemmas 5 and 6, it is not difficult to prove that $\models^w_{\mathsf{NN4}}$ is axiomatized by calculus having all theorems of $\vdash_{\mathsf{NN4}}$ as axioms and (mp) as the only rule of inference.

To conclude the section, we are going to obtain a characterization of the congruences of a non-involutive N4-lattice which is analogous to those of Proposition 2 and Theorem 4 (see also [5, Proposition 3.8]). We could have proven this result directly, but we can now take advantage of algebraizability of $\vdash_{\mathsf{NN4}}$ (Theorems 8 and 9) to obtain a shorter proof.

Let us begin by noticing that the logical filters of a non-involutive N4-lattice $\mathbf{A}$ are in correspondence with the open lattice filters of $\mathbf{A}_+$.

Proposition 10. *The lattice of* $\vdash_{\mathsf{NN4}}$*-filters of any non-involutive N4-lattice* $\mathbf{A}$ *is isomorphic to the lattice of open lattice filters of* $\mathbf{A}_+$*, where a filter* $F_+ \subseteq A_+$ *is open when* $[a]_+ \in F_+$ *implies* $[\neg\neg a]_+ \in F_+$*.*

Proof. Let F be a $\vdash_{\mathsf{NN4}}$-filter and let us check that $F_+ = \{[a]_+ \in A_+ : a \in F\}$ is a lattice filter of $\mathbf{A}_+$. Notice that $[a]_+ \in F_+$ implies $a \in F$, because the relation $\equiv_+$ is compatible with F, in the sense that $a \in F$ and $a \equiv_+ b$ imply $b \in F$. This holds because $a \equiv_+ b$ implies $a \supset b = \varepsilon(a \supset b)$, so $a \supset b \in F$, and F is closed under (mp). Thus $F_+ = G_+$ implies $F = G$.

Using this remark, in order to show that $[a]_+, [b]_+ \in F_+$ implies $[a]_+ \wedge_+ [b]_+ = [a \wedge b]_+ \in F_+$, it is sufficient to observe that, by ($\supset \wedge$) and (mp), we have that $a, b \in F$ imply $a \wedge b \in F$. Similarly, if $[a]_+ \in F_+$ and $[a]_+ \leq_+ [b]_+$, then $a \in F$ and, by ($\supset \vee$) and (mp), $a \vee b \in F$, which means that $[a \vee b]_+ = [a]_+ \vee_+ [b]_+ = [b]_+ \in F_+$ as required. Also, if $[a]_+ \in F_+$ (hence $a \in F$), then by (dn) we have $\neg\neg a \in F$ and so $[\neg\neg a]_+ \in F_+$. Conversely, if F_+ is an open lattice filter, then $F_+^* = \{a \in A : [a]_+ \in F_+\}$ is a $\vdash_{\mathsf{NN4}}$-filter. This is so because, for any axiom φ of $\vdash_{\mathsf{NN4}}$ and any homomorphism $h\colon \mathbf{Fm} \to \mathbf{A}$, $h(\varphi) = \varepsilon(h(\varphi))$ and so $[h(\varphi)]_+ = 1_+ \in F_+$. Moreover, F_+^* is closed under (mp) because $a, a \supset b \in F_+^*$ (that is, $[a]_+, [a \supset b]_+ \in F_+$) imply $[b]_+ \in F_+$ and so $b \in F_+^*$. Similarly, $a \in F_+^*$ means that $[a]_+ \in F_+$ and so we have $[\neg\neg a]_+ \in F_+$ and $\neg\neg a \in F_+^*$. So F_+^* is closed under (dn) as well. It is also clear that $F_+^* = G_+^*$ implies $F_+ = G_+$ and that $(F_+^*)_+ = F$ and $(F_+)^* = F$.

Theorem 10. *For any non-involutive N4-lattice $\mathbf{A}$, the lattice $\langle \mathrm{Con}_{\mathsf{NN4}}(\mathbf{A}), \subseteq \rangle$ is isomorphic to $\langle \mathrm{Con}_{pn}(\mathbf{A}_+), \subseteq \rangle$.*

Proof. By algebraizability of $\vdash_{\mathsf{NN4}}$ and [3, Theorem 5.1], we have an isomorphism $\mathrm{Con}_{\mathsf{NN4}}(\mathbf{A}) \cong \mathcal{F}i_{\vdash_{\mathsf{NN4}}}(\mathbf{A})$, where $\mathrm{Con}_{\mathsf{NN4}}(\mathbf{A})$ denotes the lattice of $\mathsf{NN4}$-congruences of $\mathbf{A}$ (i.e., all congruences θ such that $\mathbf{A}/\theta \in \mathsf{NN4}$) and $\mathcal{F}i_{\vdash_{\mathsf{NN4}}}(\mathbf{A})$ denotes the lattice of all $\vdash_{\mathsf{NN4}}$-filters on $\mathbf{A}$. By Proposition 10, we have an isomorphism $\mathcal{F}i_{\vdash_{\mathsf{NN4}}}(\mathbf{A}) \cong \mathcal{F}i_{\neg\neg}(\mathbf{A}_+)$, where $\mathcal{F}i_{\neg\neg}(\mathbf{A}_+)$ denotes the lattice of all lattice filters of $\mathbf{A}_+$ which satisfy the property mentioned in Proposition 10. Finally, the isomorphism $\mathcal{F}i_{\neg\neg}(\mathbf{A}_+) \cong \mathrm{Con}_{pn}(\mathbf{A}_+)$ follows from the proof of Corollary 3.(ii). It is then sufficient to compose these isomorphisms to obtain $\mathrm{Con}_{\mathsf{NN4}}(\mathbf{A}) \cong \mathrm{Con}_{pn}(\mathbf{A}_+)$. It may be instructive to see how, given $\theta \in \mathrm{Con}_{\mathsf{NN4}}(\mathbf{A})$, the congruence $\theta_+ \in \mathrm{Con}_{pn}(\mathbf{A}_+)$ is defined, and vice versa. One has $\langle [a]_+, [b]_+ \rangle \in \theta_+$ iff $\langle a \supset b, \varepsilon(a \supset b) \rangle, \langle b \supset a, \varepsilon(b \supset a) \rangle \in \theta$. Conversely, for $\eta \in \mathrm{Con}_{pn}(\mathbf{A}_+)$, the congruence $\eta^* \in \mathrm{Con}_{\mathsf{NN4}}(\mathbf{A})$ is defined by $\langle a, b \rangle \in \eta^*$ iff $[a \supset b]_+, [b \supset a]_+, [\neg(a \not\subset b)]_+, [\neg(b \not\subset a)]_+ \in 1_+/\eta$.

Proposition 11. *The class $\mathsf{NN4}$ of non-involutive N4-lattices is a variety.*

Proof. Taking advantage of Theorem 10, we can reason as in the proof of Proposition 5 to show that $\mathsf{NN4}$ (or rather, using Theorem 7, that the corresponding class of twist-structures) is closed under homomorphic images. Given $\mathbf{A} \in \mathsf{NN4}$ and a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$, we consider $\theta = \ker(h)$ and define congruences $\theta_+ \subseteq \mathbf{A}_+ \times \mathbf{A}_+, \theta_- \subseteq \mathbf{A}_- \times \mathbf{A}_-$ defined to Theorem 10. That is, we let $\langle [a]_+, [b]_+ \rangle \in \theta_+$ iff $\langle a \supset b, \varepsilon(a \supset b) \rangle, \langle b \supset a, \varepsilon(b \supset a) \rangle \in \theta$, and $\langle [a]_-, [b]_- \rangle \in \theta_+$ iff $\langle \neg(a \not\subset b), \varepsilon(\neg(a \not\subset b)) \rangle, \langle \neg(b \not\subset a), \varepsilon(\neg(b \not\subset a)) \rangle \in \theta$. We then define the map $\iota\colon \mathbf{B} \to \mathbf{A}_+/\theta_+ \bowtie \mathbf{A}_-/\theta_-$ given by $\iota(h(a)) = \langle [a]_+/\theta_+, [a]_-/\theta_- \rangle$, where $[a]_+, [a]_-$ denote the equivalence classes of $a \in A$ under $\equiv_+$ and $\equiv_-$ respectively. It is easy to check that ι is well defined. Also, the proof that ι is a homomorphism is the same as for Proposition 5. To prove injectivity of ι, assume $\iota(h(a)) = \iota(h(b))$, which by definition means that $\langle a \supset b, \varepsilon(a \supset b) \rangle, \langle b \supset a, \varepsilon(b \supset a) \rangle, \langle \neg(a \not\subset b), \varepsilon(\neg(a \not\subset b)) \rangle, \langle \neg(b \not\subset a), \varepsilon(\neg(b \not\subset a)) \rangle \in \theta$. Then we have $\langle (a \supset b) \wedge \neg(b \not\subset a), \varepsilon(a \supset b) \wedge \varepsilon(\neg(b \not\subset a)) \rangle = \langle a \to b, \varepsilon(a \supset b) \wedge \varepsilon(\neg(b \not\subset a)) \rangle \in \theta$ and $\langle (a \to b) \supset b, (\varepsilon(a \supset b) \wedge \varepsilon(\neg(b \not\subset a))) \supset b \rangle = \langle (a \to b) \supset b, b \rangle \in \theta$, where the last equality holds because the equation $(\varepsilon(x) \wedge \varepsilon(y)) \supset z = z$, as can be easily checked in a twist-structure, is valid in $\mathsf{NN4}$. At this point, using Proposition 9.(vi), we obtain $\langle a \wedge ((a \to b) \supset b), a \wedge b \rangle = \langle a, a \wedge b \rangle \in \theta$. A symmetrical reasoning shows that $\langle b, a \wedge b \rangle \in \theta$ and so $\langle a, b \rangle \in \theta$ as required. This shows that $\mathbf{B}$ is isomorphic to a subalgebra of some algebra in $\mathsf{NN4}$, and therefore that $\mathbf{B} \in \mathsf{NN4}$ as required.

Thanks to the preceding proposition, we can sharpen the result of Theorem 10, for in a variety congruences and relative congruences coincide.

Corollary 6. *For any non-involutive N4-lattice* $\mathbf{A}$*, the lattice* $\langle \mathrm{Con}(\mathbf{A}), \subseteq \rangle$ *is isomorphic to* $\langle \mathrm{Con}_{pn}(\mathbf{A}_+), \subseteq \rangle$*.*

8 On the logic of nd-frames

In [10, Section 4] the logic of nd-frames is compared with the Arieli-Avron implicative bilattice logic [2], and the authors observe that a large part of the Arieli-Avron logic is valid in nd-frames.

We may ask how large this shared part actually is, or in other words, what is a complete axiomatization of the logic of nd-frames.

The question may be formulated more precisely as follows. Since in [10] only the $\supset$ implication is considered, let us call *nd-bilattice* any algebra which is the $\{\not\subset\}$-free subreduct of a non-involutive implicative bilattice. Denote by $\models_{nd}$ be the logic defined by the class of all matrices $\langle \mathbf{B}, F_\varepsilon \rangle$ such that $\mathbf{B}$ is an *nd*-bilattice and $F_\varepsilon = \{\varepsilon(a) : a \in B\}$ is defined as before. All axioms mentioned in [10, Theorem 4.2] are valid in $\models_{nd}$, but there is no guarantee that these provide a complete axiomatization of the logic. In particular, some axioms of the Arieli-Avron logic which are not sound in $\models_{nd}$ can be reintroduced in a weakened form or through rules. For example, $p \supset \neg\neg p$ is not sound but it is easy to check that the rule $p \vdash \neg\neg p$ is. The same holds for the axiom $(p \wedge \neg q) \supset \neg(p \supset q)$ that is not sound in $\models_{nd}$ while the rule $p \wedge \neg q \vdash \neg(p \supset q)$ is. Notice also that the axiom $\neg\neg(p \wedge \neg q) \supset \neg(p \supset q)$ is sound. The failure of the deduction theorem relative to $\supset$ (which the Arieli-Avron logic enjoyed) is obviously crucial here.

The logic $\models_{nd}$ is certainly protoalgebraic thanks to the $\supset$ implication [7, Theorem 1.1.3], and it is easy to show that it is truth-equational as well [15]. Thus, it is weakly algebraizable [7, Ch. 4]. However, we are going to prove that $\models_{nd}$ is not equivalential [7, Ch. 3] and, a fortiori, not algebraizable. In order to prove this result, we are going to take a look at reduced models of $\models_{nd}$. We state the next lemma without proof, for although we will not need it in what follows, it gives some insight on the counterexample presented in Proposition 13.

Lemma 8. *Let* $\langle \mathbf{B}, F \rangle$ *be a model of* $\models_{nd}$ *with* $\mathbf{B}$ *an nd-bilattice. Denote by* Ω *the Leibniz congruence of* $\langle \mathbf{B}, F \rangle$*. Then, for all* $a, b \in B$*, the following are equivalent:*

(i) $\langle a, b \rangle \in \Omega$*,*
(ii) $\{a \supset b, b \supset a\} \cup \{\neg(a \wedge c) \supset \neg(b \wedge c), \neg(a \wedge c) \supset \neg(b \wedge c) : c \in B\} \subseteq F$*.*

From the preceding lemma one easily obtains the following characterization.

Proposition 12. *Let* $\mathbf{B}$ *be an nd-bilattice and* $F_\varepsilon = \{\varepsilon(a) : a \in B\}$*. The matrix* $\langle \mathbf{B}, F \rangle$ *is a reduced model of* $\models_{nd}$ *if and only if the following condition is met: for all* $a, b \in B$*,*

$$\text{if } a \equiv_+ b \text{ and } \neg(a \wedge c) \equiv_+ \neg(b \wedge c) \text{ for all } c \in B, \text{ then } a = b.$$

The preceding proposition can give a hint on how to single out matrices that are not reduced. A protoalgebraic logic is *equivalential* if and only if the class of its reduced matrix models is closed under submatrices [7, Theorem 3.2.1]. Hence, if $\models_{nd}$ is not equivalential, then it must be possible to find some reduced matrix for $\models_{nd}$ which has a submatrix that is not reduced. The following proposition presents an example of this.

Proposition 13. *The logic $\models_{nd}$ is not equivalential (hence, not algebraizable either).*

Proof. We are going to show that the class of reduced models of $\models_{nd}$ is not closed under submodels, hence the result will follow by [7, Theorem 3.2.1]. Consider the product $\mathbf{2}_+ \bowtie \mathbf{4}_-$ where $\mathbf{2}_+ = \langle \{0_+, 1_+\}, \wedge_+, \vee_+, \rightarrow_+ \rangle$ is the two-element Boolean algebra and $\mathbf{4}_- = \langle \{0_-, a, b, 1_-\}, \wedge_-, \vee_- \rangle$ is the distributive lattice which is the $\{\wedge, \vee\}$-reduct of the four-element Boolean algebra. The map $n \colon \mathbf{2}_+ \to \mathbf{4}_-$ is defined in the only possible way (both bounds have to be respected), and $p \colon \mathbf{4}_- \to \mathbf{2}_+$ is defined by $p(0_-) = p(a) = p(b) = 0_+$ and $p(1_-) = 1_+$. It is clear that $\mathbf{2}_+ \bowtie \mathbf{4}_-$ is a subreduct of a non-involutive implicative bilattice, namely of the very product $\mathbf{2}_+ \bowtie \mathbf{4}_-$ where $\mathbf{4}_-$ is viewed as a Boolean algebra. Also observe that the matrix $\langle \mathbf{2}_+ \bowtie \mathbf{4}_-, F_\varepsilon \rangle$, where $F_\varepsilon = \{1_+\} \times \{0_-, a, b, 1_-\}$, is reduced. On the other hand, the submatrix determined by the subuniverse $\{0_+, 1_+\} \times \{0_-, a, 1_-\}$, whose filter is $\{1_+\} \times \{0_-, a, 1_-\}$, is not reduced. This is because, as is easy to check, the congruence generated by $\{\langle 1_+, a \rangle, \langle 1_+, 0_+ \rangle\}$ is compatible with the filter.

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