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On two-sided Max-Linear equations

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ABSTRACT

This paper is a contribution to the study of finite, two-sided max-linear systems $Ax = Bx$ over the max-plus algebra. These systems are known to be equivalent to mean-payoff games and the problem of deciding whether or not a non-trivial solution exists is in $NP \cap co - NP$. Yet, no polynomial solution method seems to be known to date.

We study two special types of these systems with square matrices A and B. The first type, called *minimally active*, is defined by the requirement that for every non-trivial solution x, the maximum on each side of every equation is attained exactly once. For the second type, called essential systems, we require that every component of any non-trivial solution is active on at least one side of at least one equation. Minimally active systems are shown to be a special case of essential systems and it is shown that all essential systems can be reduced to minimally active ones. Essential systems are equivalently defined as those square finite systems for which all non-trivial solutions are finite.

We prove that in every solvable two-sided max-linear system of minimally active or essential type, all positions in $C := A \oplus B$ active in any optimal permutation for the assignment problem for C, are also active for some non-trivial solution (any non-trivial solution in the minimally active case) of the two-sided system. This enables us to deduce conditions on a solution x for which it is possible in some cases to find x in polynomial time. It is also proved that any essential system can be transformed to a minimally active system in polynomial time.

1. Introduction

We start with two motivational examples. These are variants of a model called multi-processor interactive system.

Example 1. Products $P_1, \ldots, P_m$ are prepared using n processors, every processor potentially contributing to the completion of each product. It is assumed that every processor can work for all products simultaneously and that all these actions on a processor start as soon as the processor starts to work. Let a_{ij} be the duration of the work of the jth processor needed to complete the partial product for P_i ($i = 1, \ldots, m; j = 1, \ldots, n$). Let us denote by x_j the starting time of the jth processor ($j = 1, \ldots, n$). Then all partial products for P_i ($i = 1, \ldots, m$) will be ready at time

$$\max(x_1 + a_{i1}, \ldots, x_n + a_{in}).$$

Hence if $b_1, \ldots, b_m$ are given completion times of the products that have to be met exactly then the starting times have to satisfy the system of equations:

$$\max(x_1 + a_{i1}, \ldots, x_n + a_{in}) = b_i, i = 1, \ldots, m.$$

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If we denote $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ and the pair of operations $(\oplus, \otimes)$ is extended to matrices and vectors in the same way as in linear algebra, then this can be written as a compact equation:

$$A \otimes x = b. \tag{1}$$

The matrix $A = \left(a_{ij}\right)$ is called the *production matrix*.

Example 2. Now suppose that in addition to the assumptions of Example 1, k other machines prepare independently partial products for products $Q_1, \ldots, Q_m$ and the duration and starting times are b_{ij} and y_j, respectively. Then a *synchronisation problem* is to find starting times of all $n + k$ machines so that each pair (P_i, Q_i) $(i = 1, \ldots, m)$ is completed at the same time. This task is equivalent to solving the system of equations

$$\max(x_1 + a_{i1}, \ldots, x_n + a_{in}) = \max(y_1 + b_{i1}, \ldots, y_k + b_{ik}),\ i = 1, \ldots, m. \tag{2}$$

Again, using max-algebra, we can write this system as a system of max-linear equations:

$$\bigoplus_{j=1,\ldots,n} a_{ij} \otimes x_j = \bigoplus_{j=1,\ldots,k} b_{ij} \otimes y_j,\ i = 1, \ldots, m. \tag{3}$$

In the matrix–vector notation it has the form

$$A \otimes x = B \otimes y,$$

or, more generally (allowing matrix entries to be $-\infty$),

$$A \otimes x = B \otimes x. \tag{4}$$

Systems (4) have been studied since 1978 [5–7,10] and [8]. It has been proved that the solution set is finitely generated [10]. These systems have been shown to be equivalent to mean payoff games [2]. A number of solution methods exist [3,13,15] and [19]. Although none of them are polynomial, this problem is known to be in $NP \cap co - NP$ and it is therefore expected that eventually a polynomial solution method will be found.

Note that system (4) can be considered over $\mathbb{R} \cup \{-\infty\}$. The aim of the present paper is to study special cases of (4) over finite matrices (that is, matrices over $\mathbb{R}$). (For other work on special cases of systems in max-plus algebra, see [11]). More precisely:

(a) We prove that if (A, B) is a "minimally active system", then if there is a non-trivial solution, there is a finite solution x for which the optimal permutations of the assignment problem for $C := A \oplus B$ identify active elements for x.
(b) We prove that some systems can be converted to minimally active systems. This is true in particular for "essential systems" — an important special case. We prove that similar results hold for systems that can be converted in this way.
(c) Parts (a) and (b) may or may not yield a solution to $A \otimes x = B \otimes x$. Generally however, parts (a) and (b) will provide important information about a solution.

2. Prerequisites

In this section we give the definitions and some basic results which will be used in the formulations and proofs of the results of this paper. For the proofs and more information about max-algebra, the reader is referred to [1,4,9] and [16].

We assume everywhere that $m, n \geq 1$ are natural numbers and define $M = \{1, \ldots, m\}$ and $N = \{1, \ldots, n\}$. The symbol $\overline{\mathbb{R}}$ stands for $\mathbb{R} \cup \{-\infty\}$. We use the convention $\max \emptyset = -\infty$.

If $a, b \in \overline{\mathbb{R}}$ then we set

$$a \oplus b = \max(a, b)$$

and

$$a \otimes b = a + b.$$

For clarity, we use the notation $\bigoplus$ when taking the maximum over a set (max-sum) and the notation $\sum$ when taking the conventional linear sum. Throughout the paper we denote $-\infty$ by ε (the neutral element with respect to $\oplus$) and for convenience we also denote by the same symbol any vector, whose all components are $-\infty$, or a matrix whose all entries are $-\infty$. A similar convention is used for 0 vectors or matrices. If $a \in \mathbb{R}$, then the symbol a^{-1} stands for $-a$. The symbol a^k ($k \geq 1$ integer) stands for the iterated product $a \otimes a \otimes \cdots$ in which the symbol a appears k times (that is ka in conventional notation). By *max-algebra* (also called "tropical linear algebra") we understand the analogue of linear algebra developed for the pair of operations $(\oplus, \otimes)$, extended to matrices and vectors as in conventional linear algebra. That is, if $A = (a_{ij}),\ B = (b_{ij})$

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and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\overline{\mathbb{R}}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all $i \in M, j \in N$ and $C = A \otimes B$ if

$$c_{ij} = \bigoplus_k a_{ik} \otimes b_{kj} = \max_k (a_{ik} + b_{kj})$$

for all $i \in M$ and $j \in N$. If $\alpha \in \overline{\mathbb{R}}$ then $\alpha \otimes A = \left(\alpha \otimes a_{ij}\right)$. Although the use of the symbols $\otimes$ and $\oplus$ is common in max-algebra, we will apply the usual convention of not writing the symbol $\otimes$. Thus in what follows the symbol $\otimes$ will not be used and unless explicitly stated otherwise, all multiplications indicated are in max-algebra.

It will also be necessary to define the "minimum" operation. If $a, b \in \overline{\mathbb{R}}$, then we set

$$a \oplus' b = \min(a, b).$$

Note it will not be necessary to define $\min \emptyset$.

A vector or matrix is called *finite* if all its entries are real numbers. A square matrix is called *diagonal* if all its diagonal entries are real numbers and off-diagonal entries are ε. More precisely, if $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ then $diag\,(x_1, \ldots, x_n)$ or just $diag\,(x)$ is the $n \times n$ diagonal matrix

$$\begin{pmatrix} x_1 & \varepsilon & \ldots & \varepsilon \\ \varepsilon & x_2 & \ldots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \ldots & x_n \end{pmatrix}.$$

The matrix $diag\,(0)$ is called the *unit matrix* and denoted I. Obviously, $AI = IA = A$ whenever A and I are of compatible sizes. A matrix obtained from a diagonal matrix [unit matrix] by permuting the rows and/or columns is called a *generalised permutation matrix* [*permutation matrix*]. It is known that in max-algebra generalised permutation matrices are the only invertible matrices [9,12]. Clearly,

$$(diag\,(x_1, \ldots, x_n))^{-1} = diag\,\left(x_1^{-1}, \ldots, x_n^{-1}\right).$$

We have the following Lemma ([9], Lemma 7.4.1) which will be used in examples throughout this paper.

Lemma 1 (*Cancellation Rule*). *Let $v, w, a, b \in \mathbb{R}$, $a < b$. Then for any real x, we have*

$$v \oplus ax = w \oplus bx$$

if and only if

$$v = w \oplus bx.$$

Let $S \subseteq \overline{\mathbb{R}}^n$. The set S is called a *max* $-$ *algebraic subspace* if

$$\alpha u \oplus \beta v \in S$$

for every $u, v \in S$ and $\alpha, \beta \in \overline{\mathbb{R}}$. The adjective "max-algebraic" will usually be omitted.

Let $D = \left(d_{ij}\right) \in \overline{\mathbb{R}}^{m \times n}$ and $x \in \overline{\mathbb{R}}^n$. Then the set of active entries in D, scaled by x (the set of row maxima of the matrix D, where column j is scaled by x_j), is denoted by

$$\mathcal{A}\,(Dx) := \left\{ (i, j) \in M \times N : d_{ij} x_j = \bigoplus_{t \in N} d_{it} x_t \right\},$$

where the $\mathcal{A}$ stands for "active". Let $i \in M$ and define

$$\mathcal{AV}_i\,(Dx) := \{ j \in N : (i, j) \in \mathcal{A}\,(x, D) \},$$

where the $\mathcal{AV}$ stands for "active variables". Finally, let $j \in N$ and define

$$\mathcal{AE}_j\,(Dx) := \{ i \in M : (i, j) \in \mathcal{A}\,(x, D) \},$$

where the $\mathcal{AE}$ stands for "active equations".

Suppose that $A = \left(a_{ij}\right) \in \overline{\mathbb{R}}^{m \times n}$ and $B = \left(b_{ij}\right) \in \overline{\mathbb{R}}^{m \times n}$ are given. The problem of finding a non-trivial solution to the two-sided max-linear system (A, B) is the task of finding $x \in \overline{\mathbb{R}}^n$, $x \neq \epsilon$ (a non-trivial solution) such that

$$Ax = Bx. \tag{5}$$

In the rest of this paper we will assume that A and B are finite matrices. It is easy to see in this case that a non-trivial solution exists if and only if a finite solution exists. As such, we restrict our attention to finding finite solutions to (5). We define

$$V\,(A, B) = \left\{ x \in \mathbb{R}^n ; Ax = Bx \right\}. \tag{6}$$

Let $i \in M, j \in N$ and $x \in V(A, B)$. If $a_{ij}x_j = \bigoplus_{t\in N} a_{it}x_t$ ($b_{ij}x_j = \bigoplus_{t\in N} b_{it}x_t$), then we say that (i,j) is $x-active$ in A (B) and write $(i,j) \in \mathcal{A}(Ax)$ ($(i,j) \in \mathcal{A}(Bx)$). We see that $\mathcal{A}(Ax)$ ($\mathcal{A}(Bx)$) is the set of positions which are $x-active$ *in* A (B).

Let $E = (e_{ij}) \in \mathbb{R}^{n\times n}$ and denote by P_n the set of permutations on N. The $max-algebraic\ permanent$ of E is

$$maper\ (E) \bigoplus_{\sigma\in P_n} \bigotimes_{i\in N} e_{i,\sigma(i)} = \max_{\sigma\in P_n} \sum_{i\in N} e_{i,\sigma(i)}.$$

The set of optimal solutions to the assignment problem (AP) is given by

$$ap\ (E) = \left\{ \sigma \in P_n : \bigotimes_{i\in N} e_{i,\sigma(i)} = maper\ (E) \right\}.$$

We will assume everywhere in what follows that $m = n$ and $C := A \oplus B$. It is known [9] that $ap\ (C) = ap\ (C \otimes diag\ (v))$ for all $v \in \mathbb{R}^n$.

Note that

$$(\forall x \in V(A, B))\ (\forall i) \bigoplus_{t\in N} a_{it}x_t = \bigoplus_{t\in N} b_{it}x_t = \bigoplus_{t\in N} c_{it}x_t.$$

Hence, $\mathcal{A}(Cx) = \mathcal{A}(Ax) \cup \mathcal{A}(Bx)$ and for all i we have $\mathcal{AV}_i(Cx) = \mathcal{AV}_i(Ax) \cup \mathcal{AV}_i(Bx)$.

For $x \in V(A, B)$ and $i \in N$, there exists j_1, j_2 such that $(i, j_1) \in \mathcal{A}(Ax)$ and $(i, j_2) \in \mathcal{A}(Bx)$. Note that j_1 and j_2 are not necessarily distinct.

Obviously, for all $x \in V(A, B)$ and for all i, we have $|\mathcal{AV}_i(Ax)|, |\mathcal{AV}_i(Bx)| \geq 1$.

Remark 1.

$$(\forall x \in V(A, B))\ |\mathcal{A}(Ax)|, |\mathcal{A}(Bx)| \geq m = n$$

($m = n$ since we consider only square systems).

It is easily shown that if $V(A, B) \neq \emptyset$, then there exists $x \in V(A, B)$ such that for all j, $\mathcal{AE}_j(Cx) \neq \emptyset$. As such, we define

$$\tilde{V}(A, B) := \left\{ x \in V(A, B) : (\forall j)\ \mathcal{AE}_j(Cx) \neq \emptyset \right\}. \tag{7}$$

In the rest of this paper, we are interested only in finding solutions $x \in \tilde{V}(A, B)$.

Definition 1. Let $x \in \tilde{V}(A, B)$ and let $\sigma \in ap\ (C)$, σ is called "$x-optimal$" if for all $i \in N$, $(i, \sigma(i)) \in \mathcal{A}(Cx)$.

Example 3.

$$A = \begin{pmatrix} 0 & -2 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Then $\sigma = (1)(2) \in ap\ (C)$ is *x-optimal* for $x^T = (0, 0) \in \tilde{V}(A, B)$ since $(1, 1), (2, 2) \in \mathcal{A}(x, C)$.

Remark 2. The following is an important lemma, since it says that if there is an optimal permutation corresponding to a solution, then *all* optimal permutations correspond to the same solution. It means that there is no need to identify any optimal permutation as "preferable" in some way — we can be satisfied by finding all active entries of all solutions to the assignment problem for the matrix C. Note that it is not necessary to explicitly find all optimal permutations, only all entries in the matrix which correspond to optimal permutations — a subtle and technical distinction but the latter can be done in polynomial time by, say, the Hungarian algorithm [18].

Lemma 2. *If* $\left(\exists x \in \tilde{V}(A, B)\right) (\exists \sigma \in ap\ (C))\ \sigma$ *is x-optimal, then*
$\left(\forall \sigma' \in ap\ (C)\right) \sigma'$ *is x-optimal.*

Proof. Let $\sigma \in ap\ (C)$ be *x-optimal*, so that $(\forall i)\ c_{i,\sigma(i)}x_{\sigma(i)} = \bigoplus_{t\in N} c_{it}x_t$. Now consider $\sigma' \in ap\ (C), \sigma' \neq \sigma$.

Define $I := \left\{ i \in N : \sigma'(i) = \sigma(i) \right\}, \bar{I} := N \setminus I$.

Clearly, $(\forall i \in I)\ \left(i, \sigma'(i)\right) \in \mathcal{A}(Cx)$, since $\left(i, \sigma'(i)\right) = (i, \sigma(i))$. If for all $i \in \bar{I}$, $c_{i,\sigma'(i)}x_{\sigma'(i)} = c_{i,\sigma(i)}x_{\sigma(i)}$, then $\left(\forall i \in \bar{I}\right) \left(i, \sigma'(i)\right) \in \mathcal{A}(Cx)$. It follows in this case that σ' is *x-optimal*. So suppose there exists $s \in \bar{I}$ such that

$$c_{s,\sigma'(s)}x_{\sigma'(s)} \neq c_{s,\sigma(s)}x_{\sigma(s)}. \tag{8}$$

Now note that $\sigma, \sigma' \in ap\ (C)$, which implies $\sigma, \sigma' \in ap\ (C \otimes diag\ (x))$. Therefore

$$\sum_{i\in N} c_{i,\sigma'(i)}x_{\sigma'(i)} = \sum_{i\in N} c_{i,\sigma(i)}x_{\sigma(i)}, \tag{9}$$

which is equivalent to

$$\sum_{i\in I} c_{i,\sigma'(i)}x_{\sigma'(i)} + \sum_{i\in \bar{I}} c_{i,\sigma'(i)}x_{\sigma'(i)} = \sum_{i\in I} c_{i,\sigma(i)}x_{\sigma(i)} + \sum_{i\in \bar{I}} c_{i,\sigma(i)}x_{\sigma(i)}. \tag{10}$$

Hence

$$\sum_{i\in \bar{I}} c_{i,\sigma'(i)}x_{\sigma'(i)} = \sum_{i\in \bar{I}} c_{i,\sigma(i)}x_{\sigma(i)}. \tag{11}$$

It follows from (8) and (11) that $\left(\exists u \in \bar{I}\right) c_{u,\sigma'(u)}x_{\sigma'(u)} > c_{u,\sigma(u)}x_{\sigma(u)}$, but this contradicts the assumption that $(u, \sigma(u)) \in \mathcal{A}(Cx)$. We conclude that σ' is *x-optimal.* □

3. Minimally active systems

Definition 2. The system (A, B) is called minimally active if $\left(\forall x \in \tilde{V}(A, B)\right)(\forall i \in N)\, |\mathcal{AV}_i(Ax)| = |\mathcal{AV}_i(Bx)| = 1$. Equivalently, we can define minimally active systems as those for which $\left(\forall x \in \tilde{V}(A, B)\right) |\mathcal{A}(Ax)| = |\mathcal{A}(Bx)| = n$, (those attaining the lower bound in Remark 1).

Interestingly, we have the following property for minimally active systems.

Lemma 3. *Let (A, B) be a minimally active system. Then $\tilde{V}(A, B) = V(A, B)$.*

Proof. We only need to show that for the minimally active system (A, B), we have $V(A, B) \subseteq \tilde{V}(A, B)$.

Suppose for a contradiction that $\left(\exists x \in V(A, B) \setminus \tilde{V}(A, B)\right)$. Let $j \in N$ such that $\mathcal{AE}_j(Cx) = \emptyset$. We increase x_j until x_j becomes active in some equation i (this will happen due to the finiteness of A and B), producing a new solution x'. But since $|\mathcal{AV}_i(Ax)| = |\mathcal{AV}_i(Bx)| = 1$, it follows that, say, $|\mathcal{AV}_i(Ax')| \geq 2$, contradicting the assumption of minimal activity. □

It is known ([9], Lemma 7.1.1) that V is a subspace. Lemma 3 confirms that $\tilde{V}$ is a subspace also when (A, B) is minimally active. In fact, for the remainder of the paper, we will have $V = \tilde{V}$ (unless stated otherwise). An interesting property following from $V = \tilde{V}$ is that all non-trivial solutions are finite.

We state now the main result of this paper.

Theorem 1. *Let (A, B) be a minimally active system. Then $V(A, B) \neq \emptyset$ if and only if $(\exists x \in V(A, B))(\forall \sigma \in ap(C))\, \sigma$ is x-optimal.*

The importance of this Theorem should not be underestimated. Such a result would allow us to deduce important information about a solution, without any a priori knowledge of what such a solution might be. Further, this information is obtained by finding all active entries in $ap(C)$, something which is easily done (in polynomial time) with the help of, say, the Hungarian algorithm [18].

Remark 3. The 'if' statement of Theorem 1 is trivial, we need the proof of the 'only if' part only. Also, due to Lemma 2, we only need to show there exists $x \in V(A, B)$ such that σ is *x-optimal* for some $\sigma \in ap(C)$.

Before the proof, we provide examples to show the importance of such a result.

Remark 4. Note that we have not given any way to check that a system is minimally active in general. Example 4, however, is easily shown to be minimally active (since it is of small dimension). To see this, first apply the Cancellation Rule , yielding the system

$$\begin{cases} 8x_2 = 5x_1 \oplus 5x_3 \\ 7x_1 = 4x_2 \oplus 5x_3 \\ 5x_1 = 5x_2 \oplus 3x_3. \end{cases}$$

Equivalently,

$$\begin{cases} x_2 = (-3)\,x_1 \oplus (-3)\,x_3 \\ x_1 = (-3)\,x_2 \oplus (-2)\,x_3 \\ x_1 = x_2 \oplus (-2)\,x_3. \end{cases}$$

Without loss of generality, $x_1 = 0$. We see also that $x_1 \geq x_2 > (-3)\,x_2$. Therefore, $x_1 = (-2)\,x_3$ and so $x_3 = 2$. It follows that $x_2 = (-3) \oplus (-1) = -1$. We have then, after scaling, the unique solution is $x^T = (0, -1, 2)$. Note that for x as defined above we have $(\forall i = 1, 2, 3)\, |\mathcal{AV}_i(Ax)| = |\mathcal{AV}_i(Bx)| = 1$.

Example 5 is not minimally active but the application of Theorem 1 still yields a solution — we will see why in Section 4. (Example 5 is actually an example of an "essential system").

Example 4.

$$A=\begin{pmatrix}3 & \textcircled{8} & 2\\ \textcircled{7} & 1 & 4\\ 0 & 5 & \textcircled{3}\end{pmatrix}, B=\begin{pmatrix}5 & 5 & 5\\ 3 & 4 & \textcircled{5}\\ \textcircled{5} & 3 & 2\end{pmatrix}, C=\begin{pmatrix}5 & \textcircled{8} & 5\\ \textcircled{7} & 4 & \textcircled{5}\\ \textcircled{5} & 5 & \textcircled{3}\end{pmatrix}.$$

We see that $maper\,(C) = 18$ and $ap\,(C) = \{(1,2)\,(3)\,,(1,2,3)\}$ (active entries of optimal permutations in C are circled above, along with the corresponding entries in the matrices A and B). So if $V\,(A,B) \neq \emptyset$, then there exists $x \in V\,(A,B)$ such that $(1,2)\,,(2,1)\,,(3,3)\,,(2,3)\,,(3,1) \in \mathcal{A}\,(Cx)$, in which case we have

$$\begin{cases}8x_2 = 5x_1 \oplus 5x_3\\ 7x_1 = 5x_3\\ 5x_1 = 3x_3\end{cases}$$

by Theorem 1. Set, without loss of generality, $x_1 = 0$ and deduce $x_3 = 2 \Rightarrow x_2 = -1$, hence $x^T = (0,-1,2)$. Indeed, $x^T = (0,-1,2)$ is a solution.

Example 5.

$$A=\begin{pmatrix}-4 & 3 & 2\\ 5 & -1 & 3\\ 7 & 3 & 4\end{pmatrix}, B=\begin{pmatrix}0 & 2 & 2\\ 3 & 5 & 7\\ 2 & 12 & 3\end{pmatrix}, C=\begin{pmatrix}0 & 3 & 2\\ 5 & 5 & 7\\ 7 & 12 & 4\end{pmatrix}.$$

We apply the Hungarian method for finding $ap\,(C)$, as follows:

$$C=\begin{pmatrix}0 & 3 & 2\\ 5 & 5 & 7\\ 7 & 12 & 4\end{pmatrix} \to \begin{pmatrix}-3 & 0 & -1\\ -2 & -2 & 0\\ -5 & 0 & -8\end{pmatrix}$$

$$\to \begin{pmatrix}-1 & 0 & -1\\ 0 & -2 & 0\\ -3 & 0 & -8\end{pmatrix} \to \begin{pmatrix}\textcircled{0} & 0 & \textcircled{0}\\ \textcircled{0} & -3 & \textcircled{0}\\ -2 & \textcircled{0} & -7\end{pmatrix}.$$

We have highlighted the active entries of all optimal permutations. In the original matrices, these correspond to

$$A=\begin{pmatrix}-4 & 3 & \textcircled{2}\\ \textcircled{5} & -1 & 3\\ 7 & 3 & 4\end{pmatrix}, B=\begin{pmatrix}\textcircled{0} & 2 & \textcircled{2}\\ 3 & 5 & \textcircled{7}\\ 2 & \textcircled{12} & 3\end{pmatrix}, C=\begin{pmatrix}\textcircled{0} & 3 & \textcircled{2}\\ \textcircled{5} & 5 & \textcircled{7}\\ 7 & \textcircled{12} & 4\end{pmatrix}.$$

Now we apply Theorem 1. Note the implied use of the Cancellation Rule. Without loss of generality, $x_1 = 0$. From the first row, $0 = 2x_3$ and so $x_3 = -2$. From the third row, $12x_2 = 7 \oplus 4x_3 = 7 \oplus 2 = 7$, hence $x_2 = -5$. Finally then, we have $x^T = (0,-5,-2)$, which we can confirm is a solution.

It is not at all obvious that we will always have enough information about x to find it exactly, as we did in Examples 4 and 5 (since finding $ap\,(C)$ does not necessarily highlight all elements of $\mathcal{A}\,(Cx)$). In fact, we have the following Corollary of Theorem 1.

Corollary 1. *Let (A,B) be a minimally active system. If there is a non-trivial solution, then*

$$\{(i,\sigma\,(i)) : \sigma \in ap\,(C)\,, i \in N\} \subseteq \mathcal{A}\,(Cx)$$

for some $x \in V\,(A,B)$.

There are some interesting questions arising, some of which we pose in Section 5.3. For now though, we focus on the task of proving Theorem 1.

Lemma 4. *If for some $x \in V\,(A,B)$, there exists a permutation $\sigma \in P_n$ such that for all $i \in N$, $(i,\sigma\,(i)) \in \mathcal{A}\,(Cx)$, then $\sigma \in ap\,(C)$. Further, σ is x-optimal.*

Proof. We prove this by showing $\sigma \in ap\,(C \otimes diag\,(x))$, by contradiction.

Suppose not, then $(\exists i \in N)\; c_{i,\sigma(i)}x_{\sigma(i)} < \bigoplus_{t\in N} c_{it}x_t$, which contradicts $(i,\sigma\,(i)) \in \mathcal{A}\,(Cx)$.

Further, σ is *x-optimal* by definition. □

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Our task is clear. As a consequence of Lemma 4, it suffices to show that if $V(A,B) \neq \emptyset$, then there exists $x \in V(A,B)$ such that the elements of $\mathcal{A}(Cx)$ admit a permutation σ, in which case $\sigma \in ap(C)$ by Lemma 4. Theorem 1 then follows as a result of Lemma 2. See also Remark 3.

The problem of finding a permutation is equivalent to the 1-factor problem in bipartite graphs, which in turn is equivalent to the problem of finding a perfect matching in a corresponding bipartite graph.

For clarity of exposition, we will use different symbols for row/column indices.

Definition 3. Let $A, B \in \mathbb{R}^{n\times n}$ such that $V(A,B) \neq \emptyset$. Let $x \in V(A,B)$ and define the bipartite graph $G_x(A,B)$ (or simply G_x when it is clear to which system of matrices we are referring) with vertex sets S_x and T_x as follows. $S_x = \{s_1, \ldots, s_n\}$, $T_x = \{t_1, \ldots, t_n\}$, $(\forall s_i \in S)\left(\forall t_j \in T\right) s_i t_j \in E(G_x)$ if and only if $(i,j) \in \mathcal{A}(Cx)$, (effectively, $S_x = T_x = N$). We also define a $3-$*colouring* of the edges of G_x as follows:

$$c\left(s_i t_j\right) = \begin{cases} c_1, \text{ if } (i,j) \in \mathcal{A}(Ax) \setminus \mathcal{A}(Bx) \\ c_2, \text{ if } (i,j) \in \mathcal{A}(Bx) \setminus \mathcal{A}(Ax) \\ c_3, \text{ if } (i,j) \in \mathcal{A}(Ax) \cap \mathcal{A}(Bx). \end{cases}$$

We call c the activity colouring. We denote by $c(G_x)$ the graph G_x edge-coloured by c.

Definition 4.

- For $s \in S_x$, define $N(s) := \{t \in T_x : st \in E(G_x)\}$.
- For $s \in S_x$, for $r \in \{1,2,3\}$, define $N_{c_r}(s) := \{t \in T_x : t \in N(s) \text{ and } c(st) = c_r\}$.
- For $S' \subseteq S_x$, define $N\left(S'\right) := \cup_{s\in S'} N(s)$.
- Define similarly for T

Remark 5. We may refer to vertex sets S (T) when there is no ambiguity for which vector x we are considering.

Essentially,

$$N_{c_1}(s) = \mathcal{AV}_s(Ax) \setminus \mathcal{AV}_s(Bx)$$
$$N_{c_2}(s) = \mathcal{AV}_s(Bx) \setminus \mathcal{AV}_s(Ax)$$
$$N_{c_3}(s) = \mathcal{AV}_s(Ax) \cap \mathcal{AV}_s(Bx).$$

For $x \in V(A,B)$, vertex $s_i \in S_x$ corresponds to equation i in the system $Ax = Bx$ and so we may talk about $i \in S_x$ without any confusion. Similarly, $t_j \in T_x$ corresponds to x_j, so we may talk about $j \in T_x$, or even $x_j \in T_x$.

From now we assume $V(A,B) \neq \emptyset$. Our goal then, is to show there is an $x \in V(A,B)$ such that G_x has a perfect matching. Equivalently, we show there is an $x \in V(A,B)$ such that the size of the minimum vertex cover in G_x is n, due to the following lemma which follows from König–Egervary Theorem [17].

Lemma 5. *Let G_x be a bipartite graph with vertex sets S, T such that $|S| = |T| = n$. For any x, a perfect matching in G_x exists if and only if the size of a minimum vertex cover is n.*

We are ready now for the proof of the main result, Theorem 1. We complete the proof via the following equivalent Lemma.

Lemma 6. *If (A,B) is minimally active, then $(\exists x \in V(A,B))$ such that the size of the minimum vertex cover in G_x is n.*

A summary of the proof is as follows. In part **I**, we define the bipartite graph G_x for some $x \in V(A,B)$ with vertex sets S and T. We let W be a minimum vertex cover in G_x and define $W_S := S \cap W$, $W_T := T \cap W$ and $\overline{W_T} := T \setminus W_T$. We make the assumption $|\overline{W_T}| > |W_S|$. In part **II**, we describe a pairing strategy between elements of W_S and $\overline{W_T}$ according to some rules and eventually reach a contradiction when we run out of elements in W_S, concluding that $|\overline{W_T}| \leq |W_S|$, which implies $|W| = n$.

Proof. I

Let (A,B) be minimally active and $x \in V(A,B)$. Consider the bipartite graph G_x and its activity colouring $c(G_x)$. Let W be a minimum vertex cover. Note that S_x is a vertex cover of G_x (since there are active variables for each equation) and so it follows that $|W| \leq n$. If $|W| = n$, then we are done, so suppose $|W| \leq n-1$.

Also note that $\left(\forall x' \in V(A,B)\right)(\forall s \in S_{x'})\ 1 \leq |N(s)| \leq 2$, due to $x' \in V(A,B)$ (≥ 1) and the minimal activity property (≤ 2). Note $|N(s)| = 2$ corresponds to the case when $|N_{c_1}(s)| = |N_{c_2}(s)| = 1$ and $|N_{c_3}(s)| = 0$. Also, $|N(s)| = 1$ corresponds to the case when $|N_{c_1}(s)| = |N_{c_2}(s)| = 0$ and $|N_{c_3}(s)| = 1$. That is, each $s \in S$ is incident with exactly one edge (of colour c_3) or exactly two edges (of colours c_1 and c_2 respectively).

Define $W_S := W \cap S$ and $\overline{W_S} := S \setminus W_S$. Similarly, define $W_T := W \cap T$ and $\overline{W_T} := T \setminus W_T$. If we have $|\overline{W_T}| \leq |W_S|$, then it follows that $|W| \geq n$, a contradiction. So $|\overline{W_T}| > |W_S|$.

Since $(\forall s \in S_x)\ |N(s)| \geq 1$ $(x \in V(A,B))$ and $(\forall t \in T)\ |N(t)| \geq 1$ (definition of $\tilde{V}(A,B)$ and $V = \tilde{V}$ due to (A,B) being minimally active), it follows that $|W_S|, |W_T| \geq 1$. In fact, from the definition of W, we have

$$\left(\forall j \in \overline{W_T}\right)(\exists s \in W_S)\ (s,j) \in E(G_x).$$

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Every $s \in W_S$ has a neighbour in $\overline{W_T}$ (else $N(s) \subseteq W_T$ and by removing s from W we obtain a smaller vertex cover). Also, for every $t \in \overline{W_T}$, t has no neighbours in $\overline{W_S}$ (by definition of W). It follows that $N\left(\overline{W_T}\right) \subseteq W_S$. (In fact, it follows that $N\left(\overline{W_T}\right) = W_S$, though we only need that $N\left(\overline{W_T}\right) \subseteq W_S$). Similarly, $N\left(\overline{W_S}\right) \subseteq W_T$.

II

Now, we describe a pairing strategy between elements of W_S and $\overline{W_T}$. If $\left(\forall s \in N\left(\overline{W_T}\right)\right) |N_{c_1}(s) \cap \overline{W_T}| = |N_{c_2}(s) \cap \overline{W_T}| = 1$ (recall then that $\left(\forall s \in N\left(\overline{W_T}\right)\right) N(s) \cap W_T = \emptyset$), then we can define the solution x' by:

$$x'_k := \begin{cases} \alpha x_k, & \text{if } x_k \in \overline{W_T} \\ x_k, & \text{o.w.,} \end{cases}$$

where

$$\alpha := \bigoplus_{i \in \overline{W_S}}' \left\{ \bigoplus_{j \in \overline{W_T}}' \left[\left(\bigoplus_{t \in N} c_{it} x_t \right) \left(c_{ij} x_j \right)^{-1} \right] \right\}.$$

The vector x' is a solution to equations corresponding to $\overline{W_S}$ since $\left(\forall j \in \overline{W_T}\right) \left(\forall i \in \overline{W_S}\right) (i, j) \notin \mathcal{A}(Cx)$. The constant α is defined so that the variables of $\overline{W_T}$ are increased to exactly the first point where (u, t) becomes $x - active$ for some $u \in \overline{W_S}$ and some $t \in \overline{W_T}$.

It follows that $x' \in V(A, B)$ and we then have that $\left(\exists u \in \overline{W_S}\right)$ at least one of the following holds:

- $|\mathcal{AV}_u\left(Ax'\right)| \geq 2$; or
- $|\mathcal{AV}_u\left(Bx'\right)| \geq 2$.

In any case, we contradict the assumption that (A, B) is minimally active.

It follows then (for the original solution x) there exists $s_1 \in N\left(\overline{W_T}\right) = W_S$ with exactly one neighbour in $\overline{W_T}$ (say t_1, with $c(s_1 t_1) = c_{r_1}$, $r_1 \in \{1, 2, 3\}$), and at most one neighbour in W_T (no such neighbour in the case $r_1 = 3$, exactly one such neighbour otherwise).

Consider now $\overline{W_T} \setminus \{t_1\}$. Note that

$$\emptyset \neq N\left(\overline{W_T} \setminus \{t_1\}\right) \subseteq W_S \setminus \{s_1\}.$$

As before, if $\left(\forall s \in N\left(\overline{W_T} \setminus \{t_1\}\right)\right) |N_{c_1}(s) \cap \left(\overline{W_T} \setminus \{t_1\}\right)| = |N_{c_2}(s) \cap \left(\overline{W_T} \setminus \{t_1\}\right)| = 1$, then we can define a solution x' which contradicts the assumption of minimal activity of (A, B).

Again then, we conclude $\left(\exists s_2 \in N\left(\overline{W_T} \setminus \{t_1\}\right) \subseteq W_S \setminus \{s_1\}\right)$ with exactly one neighbour in $\overline{W_T} \setminus \{t_1\}$ (say t_2 with $c(s_2 t_2) = c_{r_2}$, $r_2 \in \{1, 2, 3\}$), and at most one neighbour in $T \setminus \left(\overline{W_T} \setminus \{t_1\}\right)$.

We continue in this way, pairing off elements of W_S and $\overline{W_T}$. Eventually, since $|\overline{W_T}| > |W_S|$, we run out of vertices in W_S. We have defined $t_1, \ldots, t_{|W_S|}$ and $s_1, \ldots, s_{|W_S|}$. Let $T' := \overline{W_T} \setminus \left\{t_1, \ldots, t_{|W_S|}\right\} \neq \emptyset$. It follows that $N\left(T'\right) \subseteq W_S \setminus \left\{s_1, \ldots, s_{|W_S|}\right\} = \emptyset$. This contradicts the assumption that $(\forall t \in T)\, |N(t)| \geq 1$.

We conclude that it was our initial assumption, namely that $|\overline{W_T}| > |W_S|$, which was wrong. It follows that for our original x, the size of the minimum vertex cover in G_x is n. In fact, since $x \in V(A, B)$ was arbitrary, the result holds true for all $x \in V(A, B)$. □

We have proved a stronger result than Theorem 1. To be exact, we showed that the conditions of Theorem 1 hold for all solutions, not just one.

Theorem 2. *Let (A, B) be a minimally active system. Then $V(A, B) \neq \emptyset$ if and only if $(\forall x \in V(A, B))\, (\forall \sigma \in ap(C))\, \sigma$ is x-optimal.*

We also have a stronger version of Corollary 1.

Corollary 2. *Let (A, B) be a minimally active system. If there is a non-trivial solution, then*

$$\{(i, \sigma(i)) : \sigma \in ap(C), i \in N\} \subseteq \mathcal{A}(x, C)$$

for all $x \in V(A, B)$.

4. Essential systems

In this section we show that we can generalise the results of Section 3 to a wider class of systems, which we call *essential systems*.

Definition 5. Let $A, B \in \mathbb{R}^{n \times n}$. We say that (A, B) is essential if $V(A, B) = \tilde{V}(A, B) \neq \emptyset$.

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We use the term essential since it is easily shown that an equivalent definition is that all non-trivial solutions are finite — all components of the solution vector x are essential. It should be noted that the class of systems for which all non-trivial solutions are finite is a much wider class of systems than the class of minimally active ones. Also, minimally active systems are a special case of essential systems, see Lemma 3. We stated at the beginning of the paper that we are only interested in finite solutions. The remarks made here demonstrate that we are, in fact, interested all non-trivial solutions.

It can be shown that (A, B) in Example 6 is an essential system which is not minimally active. To see that (A, B) is not minimally active, consider the unique solution $x^T = (1, 0, 2)$. To see that the system is essential, note that it is equivalent to show for $j = 1, 2, 3$ that there is no non-trivial solution for the system $\left(A', B'\right)$, where A' and B' are obtained from the matrices A and B (respectively) by deleting column j.

For the remainder of this section, (A, B) is an essential system. As in Section 3, we will only use the notation $V(A, B)$ (or simply V where no confusion can arise) but it should be remembered that $V = \tilde{V}$.

We generalise the results of Section 3 by showing that essential systems are related to minimally active ones. The following lemma is key to showing this is true.

Lemma 7. *Let $A, B \in \mathbb{R}^{n\times n}$ such that (A, B) is essential and not minimally active. Let $z \in V(A, B)$ and $r \in N$ such that, say, $|\mathcal{AV}_r(Az)| \geq 2$ (the case for $|\mathcal{AV}_r(Bz)| \geq 2$ is similar). Let $s \in \mathcal{AV}_r(Az)$. Then there exists $\delta^* > 0$ sufficiently small such that for all $0 < \delta \leq \delta^*$ the matrices $A^{(\delta)}, B^{(\delta)}$ defined by*

$$a_{ij}^{(\delta)} = \begin{cases} a_{rs}\delta^{-1}, & \text{if } i = r, j = s \\ a_{ij}, & \text{o.w.} \end{cases}$$

$$B^{(\delta)} = B$$

satisfy the following:

1. $z \in V\left(A^{(\delta)}, B^{(\delta)}\right)$,
2. $\mathcal{AV}_r\left(A^{(\delta)}z\right) = \mathcal{AV}_r(Az) \setminus \{s\}$,
3. $\emptyset \neq V\left(A^{(\delta)}, B^{(\delta)}\right) \subseteq V(A, B)$,
4. $\left(\forall x \in V\left(A^{(\delta)}, B^{(\delta)}\right)\right)(\forall i \in N)\, \mathcal{AV}_i\left(A^{(\delta)}x\right) \subseteq \mathcal{AV}_i(Ax)$ *and* $\mathcal{AV}_i\left(B^{(\delta)}x\right) \subseteq \mathcal{AV}_i(Bx)$,
5. *For all* $x \in V\left(A^{(\delta)}, B^{(\delta)}\right)$, $(r, s) \notin \mathcal{A}\left(A^{(\delta)}x\right)$,
6. $\left(A^{(\delta)}, B^{(\delta)}\right)$ *is essential.*

Lemma 7 is basically saying that by reducing the size of an element in the matrix A, say, by a sufficiently small amount, we obtain a system which is "closer" to a minimally active one while still sharing important properties with the essential system with which we started.

Before we prove Lemma 7, we have some comments.

Remark 6. For $x \in V(A, B)$, $i \in N$ and $s_i \in S_x$ consider the activity colouring $c(G_x)$. Then for s_i, at least one of the following holds:

- s_i is incident with an edge of colour c_3;
- s_i is incident with an edge of colour c_1 and an edge of colour c_2.

Note that if s_i is incident with only one edge, then that edge is coloured c_3 (the converse is not true in general).

Definition 6. If for all $j_1, j_2 \in T$, there is a path from j_1 to j_2 in G_x, then we say G_x is *variable connected*.

Remark 7. Since for all $s \in S$, $|N(s)| \geq 1$, it follows that G_x is variable connected if and only if G_x is connected. From now, we say only connected.

Clearly, if $x = \alpha x'$ for some $x, x' \in V(A, B)$, $\alpha \in \mathbb{R}$, then $G_x = G_{x'}$. In the next Lemma we show that the converse is also true (that this cannot happen otherwise).

Lemma 8. *Let $A, B \in \mathbb{R}^{n\times n}$. If G is a connected bipartite graph, then for all $x, x' \in V(A, B)$ such that $G_x = G_{x'} = G$, there exists $\alpha \in \mathbb{R}$ such that $x' = \alpha x$. That is, G corresponds to exactly one solution (up to scaling).*

Proof. Let $x \in V(A, B)$ such that $G_x = G$. Let $t_1, t_2 \in T$, $t_1 \neq t_2$ and let P be a path from t_1 to t_2 in G_x. Using the definition of $E(G_x)$, we see that $x_{t_1}x_{t_2}^{-1}$ is a fixed constant. That is

$$(\forall x \in V(A, B))(\forall t_1, t_2)\, x_{t_1}x_{t_2}^{-1} = \Delta_{t_1,t_2} \text{ (constant)}.$$

Since t_1, t_2 were arbitrary, the result follows.

Note that it does not matter which path we choose if many are available. If path P_1 yields $x_{t_1}x_{t_2}^{-1} = \alpha_1$, and path P_2 yields $x_{t_1}x_{t_2}^{-1} = \alpha_2$, $\alpha_1 \neq \alpha_2$, then $x \notin V(A, B)$, a contradiction. □

Definition 7. $x \in V(A, B)$ is called a connected solution (or just connected) if G_x is connected.

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Definition 8. Let $x \in V(A, B)$. Denote by $com(x)$ the number of components of G_x.

The following Lemma is given without proof but it should be noted that the ideas of the proof are similar to those used in the proof of Lemma 6.

Lemma 9. *Let $x \in V(A, B)$, x not connected and consider a component of G_x with the set of nodes X. Define $S' := X \cap S$ and $T' := X \cap T$. Then for all $s \in S'$ we have at least one of the following (by Remark 6):*

- $|N_{c_3}(s) \cap T'| \geq 1$,
- $|N_{c_1}(s) \cap T'|, |N_{c_2}(s) \cap T'| \geq 1$.

Define a new vector x' using

$$\alpha := \bigoplus_{i \notin S'}' \left\{ \bigoplus_{j \in T'}' \left[\left(\bigoplus_{t \in N} c_{it} x_t \right) \left(c_{ij} x_j \right)^{-1} \right] \right\},$$

and

$$x'_k := \begin{cases} \alpha x_k, & \text{if } x_k \in T' \\ x_k, & \text{if } x_k \notin T'. \end{cases}$$

Then $x' \in V(A, B)$ and $com(x') < com(x)$.

Let $x \in V(A, B)$, x not connected. By applying Lemma 9 repeatedly, we can transform x to a vector $\bar{x}$ such that $\bar{x}$ is connected and $\bar{x} \in V(A, B)$.

Note $\bar{x}$ may not be unique (the connected solution $\bar{x}$ depends on which component we use in Lemma 9). We denote by $connect(x)$ the set of connected $\bar{x} \in V(A, B)$ that can be obtained from x in this way.

We are now ready for the proof of Lemma 7.

Proof of Lemma 7. We are essentially reducing exactly one element in the system (A, B). Immediately, we can see that for all $\delta > 0, z \in V\left(A^{(\delta)}, B^{(\delta)}\right)$ (property 1), which in turn means $V\left(A^{(\delta)}, B^{(\delta)}\right) \neq \emptyset$ (first part of property 3). It is also clear for all $\delta > 0$ that $\mathcal{AV}_r\left(A^{(\delta)} z\right) = \mathcal{AV}_r(Az) \setminus \{s\}$ (property 2).

- We show next that for all $\delta > 0$ sufficiently small, $V\left(A^{(\delta)}, B^{(\delta)}\right) \subseteq V(A, B)$ (second part of property 3). First note that if $a_{rs} = b_{rs} = c_{rs}$, then this follows immediately. To see this, let $\delta > 0$ and $x \in V\left(A^{(\delta)}, B^{(\delta)}\right) \setminus V(A, B)$. It follows that $a_{rs} x_s > \bigoplus_{t \in N} b_{rt} x_t \geq b_{rs} x_s = a_{rs} x_s$, a contradiction. So assume $a_{rs} \neq b_{rs}$. Let us start with a fixed $\delta_0 > 0$. If $V\left(A^{(\delta_0)}, B^{(\delta_0)}\right) \subseteq V(A, B)$, then we are done, so suppose not. Let $\Gamma(\delta)$ be the set of connected solutions in $V\left(A^{(\delta)}, B^{(\delta)}\right) \setminus V(A, B)$ for any $\delta > 0$. Since the number of connected bipartite graphs with the set of nodes S and T is finite and each corresponds to only one solution (up to multiples), it follows that $\Gamma(\delta_0)$ is finite (up to multiples). Now, let $w \in \Gamma(\delta_0)$. We have $(\forall i)\, i \neq r$, $a_i w = b_i w$, and so $a_r w \neq b_r w$ (where a_l denotes row l of A and b_l denotes row l of B). Also, since $(\forall j \neq s)\, a_{rj}^{(\delta_0)} = a_{rj}$, it follows that $a_{rs} w_s > b_r w$ and $\bigoplus_{t \in N} a_{rt} w_t = a_{rs} w_s$. We then have

$$\begin{cases} a_{rs}^{(\delta_0)} w_s \leq b_r w \Leftrightarrow a_{rs} \delta_0^{-1} w_s \leq b_r w \\ a_{rs} w_s > b_r w. \end{cases}$$

  Therefore, $(\exists \delta'_0)\; 0 < \delta'_0 \leq \delta_0$ such that $a_{rs} \left(\delta'_0\right)^{-1} w_s = b_r w$, thus for any δ_1, $0 < \delta_1 < \delta'_0$, we have

$$a_{rs}^{(\delta_1)} w_s > b_r w$$

  and so $w \notin V\left(A^{(\delta_1)}, B^{(\delta_1)}\right) \setminus V(A, B)$. Note then that we also have for all multiples of w, namely $\alpha w, \alpha \in \mathbb{R}$, that $\alpha w \notin V\left(A^{(\delta_1)}, B^{(\delta_1)}\right) \setminus V(A, B)$. Let $\delta_1 = \frac{1}{2}\delta'_0$ (for instance). Define $\delta(w) := \delta_1$, and define $\delta\left(w'\right)$ in the same way for all $w' \in \Gamma(\delta_0)$. Since $\Gamma(\delta_0)$ is finite (up to multiples) and for all $w' \in \Gamma(\delta_0)$, for all $\alpha \in \mathbb{R}$ we have $\delta\left(w'\right) = \delta\left(\alpha w'\right)$, we can define

$$\delta^* := \min_{w \in \Gamma(\delta_0)} \delta(w) > 0.$$

  We see that $\Gamma(\delta^*) = \emptyset$. We now show that δ^* is sufficiently small so that $V\left(A^{(\delta^*)}, B^{(\delta^*)}\right) \subseteq V(A, B)$, as desired. Let δ^* be as defined and suppose for a contradiction that there exists $w \in V\left(A^{(\delta^*)}, B^{(\delta^*)}\right) \setminus V(A, B)$. Consider the graph $G_w\left(A^{(\delta^*)}, B^{(\delta^*)}\right)$. Vector w is not connected because $\Gamma(\delta^*) = \emptyset$ and since $w \notin V(A, B)$, we have $a_{rs} w_s > b_r w$. Let X_1 be the set of nodes of the component of G_w that contains w_s and define $S' := X_1 \cap S$ and $T' = X_1 \cap T$. Note that $N\left(T'\right) = S'$. Since w is not connected it follows that at least one of the following hold for all $i \in N\left(T'\right) = S'$:

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– $|N_{c_1}(i) \cap T'|, |N_{c_2}(i) \cap T'| \geq 1$,
– $|N_{c_3}(i) \cap T'| \geq 1$.

As such, we may increase w_k for all $w_k \in T'$ (similarly as in the proof of Lemma 6) by γ_1, say, until there is a new edge between T' and $S \setminus S'$. Call this new solution w'. We have $a_{rs}w'_s = a_{rs}w_s\gamma_1 > b_r w \gamma_1 \geq b_r w'$ and so $w' \in V\left(A^{(\delta^*)}, B^{(\delta^*)}\right) \setminus V(A, B)$ also. Repeat the procedure with the component of $G_{w'}$ containing w'_s, until we obtain a connected solution $\overline{w} \in V\left(A^{(\delta^*)}, B^{(\delta^*)}\right) \setminus V(A, B)$, contradicting $\Gamma(\delta^*) = \emptyset$.

- Note that property 6 follows immediately from property 3.
- Next, we show property 4 holds. Let $x \in V\left(A^{(\delta^*)}, B^{(\delta^*)}\right) \subseteq V(A, B)$ and let $i \in N$, $i \neq r$. Then $\mathcal{AV}_i\left(A^{(\delta^*)}x\right) = \mathcal{AV}_i(Ax)$ and $\mathcal{AV}_i\left(B^{(\delta^*)}x\right) = \mathcal{AV}_i(Bx)$, since $a_i^{(\delta^*)} = a_i$ and $b_i^{(\delta^*)} = b_i$. Also, since $(\forall t)\ a_{rt}x_t \leq b_r x$ and $b_r^{(\delta^*)} = b_r$, it follows that $\mathcal{AV}_r\left(A^{(\delta^*)}x\right) \subseteq \mathcal{AV}_r(Ax)$ and $\mathcal{AV}_r\left(B^{(\delta^*)}x\right) = \mathcal{AV}_r(Bx)$.
- Finally, property 5. We show that for all $x \in V\left(A^{(\delta^*)}, B^{(\delta^*)}\right)$ we have $s \notin \mathcal{AV}_r\left(A^{(\delta^*)}x\right)$. Suppose for a contradiction $\left(\exists x \in V\left(A^{(\delta^*)}, B^{(\delta^*)}\right)\right) s \in \mathcal{AV}_r\left(A^{(\delta^*)}x\right)$, that is

$$a_{rs}^{(\delta^*)}x_s = b_r^{(\delta^*)}x \Leftrightarrow a_{rs}(\delta^*)^{-1}x_s = b_r x.$$

But then $a_{rs}x_s > b_r x$ and so $x \notin V(A, B)$, a contradiction since $V\left(A^{(\delta^*)}, B^{(\delta^*)}\right) \subseteq V(A, B)$ (property 3). (The (r, s) entry has essentially become a "dead entry"). □

Remark 8. Property 6 of Lemma 7 should serve to clarify that we are safe to refer only to V in the statement of Lemma 7 and that for both systems in the statement of Lemma 7, we still have $V = \tilde{V}$.

The following Theorem is the final step to convert an essential system to a minimally active one.

Theorem 3. *Let $A, B \in \mathbb{R}^{n\times n}$, (A, B) essential and not minimally active. Then there is a sequence of systems (A, B), $\left(A^{(1)}, B^{(1)}\right)$, $\ldots$, $\left(A^{(k)}, B^{(k)}\right)$, such that*

$$\emptyset \neq V\left(A^{(k)}, B^{(k)}\right) \subseteq V\left(A^{(k-1)}, B^{(k-1)}\right) \subseteq \cdots \subseteq V\left(A^{(1)}, B^{(1)}\right) \subseteq V(A, B),$$

and $\left(A^{(k)}, B^{(k)}\right)$ is minimally active, for some $k \in \mathbb{N}$.

Proof. We construct a sequence of systems by repeated use of Lemma 7. We call this process "reduction". It is not clear immediately that reduction terminates in finite time but if it does terminate in a finite number of steps with system $\left(A^{(k)}, B^{(k)}\right)$, then, since reduction has terminated, we have

$$\left(\forall x \in V\left(A^{(k)}, B^{(k)}\right)\right)(\forall i \in N)\ |\mathcal{AV}_i\left(A^{(k)}x\right)| = |\mathcal{AV}_i\left(B^{(k)}x\right)| = 1$$

and so $\left(A^{(k)}, B^{(k)}\right)$ is minimally active by definition. Then, from repeated use of Lemma 7, property 3, we have

$$\emptyset \neq V\left(A^{(k)}, B^{(k)}\right) \subseteq V\left(A^{(k-1)}, B^{(k-1)}\right) \subseteq \cdots \subseteq V\left(A^{(1)}, B^{(1)}\right) \subseteq V(A, B).$$

It remains to show that reduction does indeed terminate in a finite number of steps. In fact, we will show that reduction terminates in no more than $2n^2$ iterations.

Suppose not for a contradiction and so we define systems $\left(A^{(1)}, B^{(1)}\right), \ldots, \left(A^{\left(2n^2\right)}, B^{\left(2n^2\right)}\right)$ using Lemma 7. Define $\left(A^{(0)}, B^{(0)}\right) := (A, B)$. Now, $(\forall r)\ 1 \leq r \leq 2n^2$, the transition from $\left(A^{(r-1)}, B^{(r-1)}\right)$ to $\left(A^{(r)}, B^{(r)}\right)$ is based on the reduction of exactly one entry of A or B. That is, either a_{ij} or b_{ij} for some i, j. We define $(i(r), j(r)) := (i, j)$.

Consider $(i(s), j(s))$, some $1 \leq s \leq 2n^2 - 1$. By Lemma 7, property 5, we have $\left(\forall x \in V\left(A^{(s)}, B^{(s)}\right)\right)(i(s), j(s)) \notin \mathcal{A}\left(A^{(s)}x\right)$. Now, let $s + 1 \leq r \leq 2n^2$ and $x' \in V\left(A^{(r)}, B^{(r)}\right)$. We claim that $(i(s), j(s)) \notin \mathcal{A}\left(x', A^{(r)}\right)$. To see this, note that (by repeated use of Lemma 7, property 4)

$$\mathcal{AV}_{i(s)}\left(A^{(r)}x'\right) \subseteq \mathcal{AV}_{i(s)}\left(A^{(r-1)}x'\right) \subseteq \cdots \subseteq \mathcal{AV}_{i(s)}\left(A^{(s)}\right).$$

We have essentially shown that once we reduce an element in the matrix A (in the matrix B) in the reduction process, then that element is now a "dead element" for all subsequent systems in the reduction process. Since there are n^2 elements in matrix A (in matrix B), there are a total of $2n^2$ elements in total which may be eliminated. (In fact, we can do better than $2n^2$ but this serves as a sufficient upper bound). This, with the fact that every system in the reduction process has non-empty solution set (Lemma 7, property 3), leads us to conclude that reduction must terminate in no more than $2n^2$ iterations. □

Remark 9. While the process of reduction is tedious and relies on an a priori knowledge of $V(A, B)$, we will see later (in the proof of Theorem 4) that it is not necessary to perform this process in practice — only to know that it can be done theoretically.

We give here an example of the process of reduction for a system of small dimension to help illustrate the process, it may be skipped.

Example 6. Let $A = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 5 & 3 \\ 1 & 0 & 1 \\ 0 & 4 & 2 \end{pmatrix}$ and consider the system $Ax = Bx$.

It is easily checked that the unique solution (after scaling and making the smallest component equal to zero) is $x^T = (1, 0, 2)$ and so $V(A, B) = \left\{\alpha(1, 0, 2)^T : \alpha \in \mathbb{R}\right\}$. Note also that x is a connected solution.

Now, $\mathcal{AV}_1(Ax) = \{1, 3\}$ and so we reduce, say, component a_{13}. Reducing a_{13} by 1 is sufficient since then $A' = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{pmatrix}$, $B' = \begin{pmatrix} 0 & 5 & 3 \\ 1 & 0 & 1 \\ 0 & 4 & 2 \end{pmatrix}$ and we have:

1. $x \in V(A', B') \Rightarrow V(A', B') \neq \emptyset$,
2. $\mathcal{AV}_1(A'x) = \{1\} = \{1, 3\} \setminus \{3\} = \mathcal{AV}_1(Ax) \setminus \{3\}$,
3. It is easily checked that $x^T = (1, 0, 2)$ is the unique solution for the system $A'x = B'x$ and so indeed we have $V(A', B') = \{\alpha(1, 0, 2) : \alpha \in \mathbb{R}\} \subseteq V(A, B)$,
4. It is clear that $(\forall u \in N)\ \mathcal{AV}_u(A'x) \subseteq \mathcal{AV}_u(Ax)$ and $\mathcal{AV}_u(B'x) \subseteq \mathcal{AV}_u(Bx)$,
5. The entry $(1, 3) \notin \mathcal{A}(A'x)$,
6. (A', B') is essential.

Now note that $\mathcal{AV}_1(B'x) = \{2, 3\}$ and so we reduce, say, component b_{12}. Reducing b_{12} by 1 is sufficient, since then $A'' = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{pmatrix}$, $B'' = \begin{pmatrix} 0 & 4 & 3 \\ 1 & 0 & 1 \\ 0 & 4 & 2 \end{pmatrix}$ and we have

1. $x \in V(A'', B'') \Rightarrow V(A'', B'') \neq \emptyset$,
2. $\mathcal{AV}_1(B''x) = \{3\} = \{2, 3\} \setminus \{2\} = \mathcal{AV}_1(B'x) \setminus \{2\}$,
3. It is easily checked that $x^T = (1, 0, 2)$ is the unique solution for the system $A''x = B''x$ and so indeed we have $V(A'', B'') = \{\alpha(1, 0, 2) : \alpha \in \mathbb{R}\} \subseteq V(A', B')$,
4. It is clear that $(\forall u \in N)\ \mathcal{AV}_u(A''x) \subseteq \mathcal{AV}_u(A'x)$ and $\mathcal{AV}_u(B''x) \subseteq \mathcal{AV}_u(B'x)$,
5. The entry $(1, 2) \notin \mathcal{A}(B''x)$,
6. (A'', B'') is essential.

Now note that $\mathcal{AV}_3(B''x) = \{2, 3\}$ and so we reduce, say, component b_{33}. Reducing b_{33} by 1 is sufficient, since then $A''' = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{pmatrix}$, $B''' = \begin{pmatrix} 0 & 4 & 3 \\ 1 & 0 & 1 \\ 0 & 4 & 1 \end{pmatrix}$ and we can show properties 1–6 hold. Crucially, we see that (A''', B''') is minimally active.

Now that we have shown that every essential system can be reduced to a minimally active system, we are able to show that Theorem 1 from Section 3 for minimally active systems, holds also for essential systems. Note that the stronger result, namely Theorem 2 from Section 3, does not hold).

Theorem 4. *Let $A, B \in \mathbb{R}^{n\times n}$ such that (A, B) is an essential system. Then $V(A, B) \neq \emptyset$ if and only if $(\exists x \in V(A, B))\ (\forall \sigma \in ap(C))$ σ is x-optimal.*

Proof. If (A, B) is minimally active then the result follows immediately from Theorem 1. So suppose (A, B) is not minimally active. We have seen in Lemma 7 and Theorem 3 that there is a sequence of systems

$(A, B), \left(A^{(1)}, B^{(1)}\right), \ldots, \left(A^{(k)}, B^{(k)}\right)$, such that

$$\emptyset \neq V\left(A^{(k)}, B^{(k)}\right) \subseteq V\left(A^{(k-1)}, B^{(k-1)}\right) \subseteq \cdots \subseteq V\left(A^{(1)}, B^{(1)}\right) \subseteq V(A, B),$$

and $\left(A^{(k)}, B^{(k)}\right)$ is minimally active for some $k \in \mathbb{N}$. Define $C^{(r)} := A^{(r)} \oplus B^{(r)}$ for $0 \le r \le k$, where $\left(A^{(0)}, B^{(0)}\right) := (A, B)$. By Lemma 6, there exists $x \in V\left(A^{(k)}, B^{(k)}\right)$ such that $\left(\forall \sigma \in ap\left(C^{(k)}\right)\right) \sigma$ is *x-optimal* for the system $\left(A^{(k)}, B^{(k)}\right)$. That is to say, there is a perfect matching M in $G_x\left(A^{(k)}, B^{(k)}\right)$. In moving from $\left(A^{(k)}, B^{(k)}\right)$ to $\left(A^{(k-1)}, B^{(k-1)}\right)$ we are not losing any edges of $G_x\left(A^{(k)}, B^{(k)}\right)$. That is to say

$$E\left(G_x\left(A^{(k)}, B^{(k)}\right)\right) \subseteq E\left(G_x\left(A^{(k-1)}, B^{(k-1)}\right)\right).$$

(To see this, recall by construction of $\left(A^{(k)}, B^{(k)}\right)$ that we reduced exactly one element in the system $\left(A^{(k-1)}, B^{(k-1)}\right)$). It follows that M is a perfect matching in $G_x\left(A^{(k-1)}, B^{(k-1)}\right)$ also. Continuing, we see that M is a perfect matching in $G_x\left(A, B\right)$, as desired and the result follows. □

A similar result to Corollary 1 holds for essential systems.

Corollary 3. *Let* (A, B) *be an essential system. If there is a non-trivial solution, then*

$$\{(i, \sigma(i)) : \sigma \in ap\,(C)\,, i \in N\} \subseteq \mathcal{A}\,(Cx)$$

for some $x \in V\,(A, B)$.

Note the stronger version, namely Corollary 2, does not necessarily hold for essential systems.

5. Next steps and open questions

The ideas in this paper allow us to take square finite systems (A, B) for which all non-trivial solutions are finite (so-called "essential systems") and find for each equation an active entry (in A, without loss of generality, see Example 7) for some finite solution x. In general, this is not enough to find x. Further, it is not clear how one should even identify a system as being essential, or even minimally active.

First, we present an example which illustrates that we may assume without loss of generality that $maper\,(A) = maper\,(C) = 0$ and $id \in ap\,(A) \cap ap\,(C)$. This example leads to some preliminary ideas relating to the problem of finding a solution to the two-sided system in the minimally active case. Next, we present a polynomially verifiable class of essential systems. Finally, we pose some open questions.

Example 7. Consider the matrices

$$A = \begin{pmatrix} 0 & -100 & -100 \\ -100 & 0 & -100 \\ -100 & 1 & -100 \end{pmatrix}, B = \begin{pmatrix} 0 & -100 & -100 \\ -100 & -100 & 1 \\ 2 & -100 & -100 \end{pmatrix}.$$

It is easily shown that (A, B) is a minimally active system with unique solution (up to scaling) $x^T = (0, 1, 0)$. The solution to the assignment problem for the matrix $C = A \oplus B$ is unique and the active entries of the optimal permutation are circled below.

$$C = \begin{pmatrix} \textcircled{0} & -100 & -100 \\ -100 & 0 & \textcircled{1} \\ 2 & \textcircled{1} & -100 \end{pmatrix}.$$

Applying the same permutation of rows to the matrices A, B and C corresponds to changing the order in which we read equations in the system (A, B) and so does not change the problem whilst applying the same permutation of columns to the matrices corresponds to a re-labelling of the variables and, again, does not change the problem. Here, we may permute rows 2 and 3, as follows:

$$A' = \begin{pmatrix} \textcircled{0} & -100 & -100 \\ -100 & \textcircled{1} & -100 \\ -100 & 0 & -100 \end{pmatrix}, B' = \begin{pmatrix} \textcircled{0} & -100 & -100 \\ 2 & -100 & -100 \\ -100 & -100 & \textcircled{1} \end{pmatrix} \text{ and}$$

$$C' = \begin{pmatrix} \textcircled{0} & -100 & -100 \\ 2 & \textcircled{1} & -100 \\ -100 & 0 & \textcircled{1} \end{pmatrix}$$

and clearly now the identity permutation is optimal in C'. The corresponding entries in the matrices A' and B' have been circled too.

Next, swapping a row in the matrix A with the same row in the matrix B simply corresponds to swapping the left and right-hand sides of that equation and does not change the problem. Here, we may swap row 3 of the matrices A and B, yielding

$$A'' = \begin{pmatrix} \textcircled{0} & -100 & -100 \\ -100 & \textcircled{1} & -100 \\ -100 & -100 & \textcircled{1} \end{pmatrix}, B'' = \begin{pmatrix} \textcircled{0} & -100 & -100 \\ 2 & -100 & -100 \\ -100 & 0 & -100 \end{pmatrix} \text{ and}$$

$$C'' = \begin{pmatrix} \textcircled{0} & -100 & -100 \\ 2 & \textcircled{1} & -100 \\ -100 & 0 & \textcircled{1} \end{pmatrix},$$

whereupon $maper\left(A''\right) = maper\left(C''\right) = 2$ and $id \in ap\left(A''\right)$. Finally, by scaling rows as appropriate, we can make the permanent equal to zero. In this example, we may scale the second and third rows by -1 to give

$$A''' = \begin{pmatrix} \textcircled{0} & -100 & -100 \\ -101 & \textcircled{0} & -101 \\ -101 & -101 & \textcircled{0} \end{pmatrix}, B''' = \begin{pmatrix} \textcircled{0} & -100 & -100 \\ 1 & -101 & -101 \\ -101 & -1 & -101 \end{pmatrix} \text{ and}$$

$$C''' = \begin{pmatrix} \textcircled{0} & -100 & -100 \\ 1 & \textcircled{0} & -101 \\ -101 & -1 & \textcircled{0} \end{pmatrix}.$$

Note that the processes used in this example work for all essential systems, not just minimally active ones.

5.1. Next steps for minimally active systems

Let (A, B) be minimally active. Without loss of generality, the diagonal elements of A are equal to zero and active for some finite solution x (see Example 7). Further, there are no other active entries in A (the minimally active property). It is easy to show that all cycles in B are non-positive and all zero cycles/loops in B are non-interesting.

We denote by K the set of indices which do not lie on a zero cycle in B. Then for $k \in K$, we have the substitution

$$x_k = \bigoplus_{t \in N, t \neq k} c_{kt} x_t.$$

In fact, we can do better than this, as follows.

Remark 10 (*Claim*)**.** For $k \in K$, $x_k = \bigoplus_{t \in N \setminus K} c_{kt} x_t$.

Proof. Proof omitted. □

With these substitutions, we can eliminate all variables in K, writing in terms of variables not in K and reduce to a necessary system of size

$(n - |K|) \times (n - |K|)$ (the $|K|$ equations corresponding to the substitutions can be removed and re-introduced later for backtracking), which is essentially a system of dual inequalities (since for each new equation we can identify also an active entry in B using the existence of zero cycles/loops). In fact, the set of solutions to the necessary system of dual inequalities is of the form $G \otimes u$, where $G \in \mathbb{R}^{r \times r}$ and r is the number of zero cycles in B. By backtracking our substitutions, we may convert our original system to an r-dimensional two-sided system, with $|K|$ equations — that is, a TSLS of dimension $|K| \times r$. Whilst this new system is smaller than the original, it is not necessarily square and it is not clear that it should be essential or even minimally active.

5.2. A class of essential systems

The theory of "symmetrised semirings" provides a useful tool for identifying some essential systems in polynomial time. See [4,9,14] for definitions relating to symmetrised semirings and "balancing". Important for us is the fact that we can check in polynomial time whether or not a square matrix has a "max-balanced" determinant. The adjective "max" is usually omitted.

We have the following necessary condition for solvability of a system ([9], Corollary 7.5.5).

Lemma 10. *Let $A, B \in \mathbb{R}^{n \times n}$. A necessary condition that the system $Ax = Bx$ has a non-trivial solution is that $C := A \oplus B$ has a balanced determinant.*

For all $i \in M$, $j \in N$ denote by $C^{[i,j]}$ the square matrix obtained from C by deleting row i and column j.

Lemma 11. *Let $A, B \in \mathbb{R}^{n \times n}$. If for all $j \in N$, there exists $i \in M$ such that $C^{[i,j]}$ does not have balanced determinant, then (A, B) is an essential system.*

Proof. For a contradiction, suppose (A, B) is not essential. Then there exists some $x \in \overline{\mathbb{R}}^n$ and some $j \in N$ such that $x_j = \epsilon$, $x \neq \epsilon$ and $Ax = Bx$. Let $A', B' \in \mathbb{R}^{n\times(n-1)}$ be the matrices obtained from A and B respectively by removing column j and let $x' \in \overline{\mathbb{R}}^{n-1}$ be the vector obtained from x by removing component j. It follows that $A'x' = B'x'$.

By hypothesis, there exists $i \in M$ such that $C^{[i,j]}$ has non-balanced determinant. Let $A'', B'' \in \mathbb{R}^{(n-1)\times(n-1)}$ be the square matrices obtained from A' and B', respectively, by removing row i. It follows that $A''x' = B''x'$. By Lemma 10, we have that $A'' \oplus B''$ has balanced determinant but $A'' \oplus B'' = C^{[i,j]}$, a contradiction. □

The author is not aware of any polynomial method for checking whether or not a system is essential in general.

Remark 11. The ideas of this paper work also when instead of finite systems (A, B), we instead take A, B over $\overline{\mathbb{R}}$ with the condition that *maper* (C) is finite.

5.3. Open questions

Here are some open questions, motivated by the results of this paper.

1. Can we identify essential systems in polynomial time? And, if so, can we identify the inessential variable (and so reduce the dimension of the problem)?
2. If we can identify a system as not essential, then can we identify an inessential variable and remove it?
3. Can we identify minimally active systems in polynomial time?
4. Is it possible for a minimally active system to have more than one solution (up to scaling)?
5. Can the ideas of this paper be adapted for the case of non-square systems?
6. What can we say about two-sided systems for which there is no finite permutation in C?

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