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Abstract

Given a minor-closed class $\mathcal{A}$ of graphs, let $\beta_{\mathcal{A}}$ denote the supremum over all graphs in $\mathcal{A}$ of the ratio of edges to vertices. We investigate the set B of all such values $\beta_{\mathcal{A}}$, taking further the project begun by Eppstein. Amongst other results, we determine the small values in B (those up to 2); we show that B is 'asymptotically dense'; and we answer some questions posed by Eppstein.

1 Introduction

For a given graph G, let $v(G), e(G)$ denote its number of vertices and edges, respectively. The *density* $\rho(G)$ of a graph G is $e(G)/v(G)$, the number of edges per vertex. Thus the average degree of G is $2\rho(G)$. Given a class $\mathcal{A}$ of graphs (closed under isomorphism), we let $\mathcal{A}_n$ denote the set of graphs in $\mathcal{A}$ on n vertices; let

$$e^*_{\mathcal{A}}(n) = \max_{G \in \mathcal{A}_n} e(G)$$

(where $e^*_{\mathcal{A}}(n) = 0$ if $\mathcal{A}_n$ is empty); and let

$$\beta_{\mathcal{A}} = \sup_{G \in \mathcal{A}} \rho(G) = \sup_{n \geqslant 1} e^*_{\mathcal{A}}(n)/n \quad \text{and} \quad \lambda_{\mathcal{A}} = \limsup_{n \to \infty} e^*_{\mathcal{A}}(n)/n.$$

A class of graphs is *proper* if it is non-empty and does not contain all graphs. A graph G contains a graph H as a *minor* if we can obtain a graph isomorphic to H from a subgraph of G by using edge contractions (discarding any loops and multiple edges, we are interested in simple graphs). A class $\mathcal{A}$ of graphs is *minor-closed* if whenever $G \in \mathcal{A}$ and H is a minor of G then H is in $\mathcal{A}$.

Let $\mathcal{A}$ be a proper minor-closed class of graphs. Then $\beta_{\mathcal{A}}$ is finite, as shown by Mader [11], and is called the *critical density* for $\mathcal{A}$. By Lemma 17 of [5], we always have $e^*_{\mathcal{A}}(n)/n \to \lambda_{\mathcal{A}}$ as $n \to \infty$: see also Norin [16], where $\mathrm{limd}(\mathcal{A})$ is the same as $2\lambda_{\mathcal{A}}$. Thus $\lambda_{\mathcal{A}}$ is called the *limiting density* of $\mathcal{A}$. By definition $\lambda_{\mathcal{A}} \leqslant \beta_{\mathcal{A}}$. Suppose for example that $\mathcal{A}$ is the minor-closed family of graphs G such that at most one component has a cycle, and any such component has at most 5 vertices: then $\lambda_{\mathcal{A}} = 1$ (because for $n \geqslant 5$ the densest graphs in $\mathcal{A}_n$ consist of K_5 and a tree on $n-5$ vertices), and $\beta_{\mathcal{A}} = 2$ (because K_5 is the densest graph in $\mathcal{A}$). If $\mathcal{A}$ is the class of series parallel graphs (those with no minor K_4) then for each $n \geqslant 2$, each edge-maximal graph in $\mathcal{A}$ has $2n-3$ edges (see [14] for more on this), so $\lambda_{\mathcal{A}} = \beta_{\mathcal{A}} = 2$.

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We are interested in the critical and limiting densities of proper minor-closed classes of graphs. The main object of study in this paper is the set of critical densities

$$B = \{\beta_{\mathcal{A}} : \mathcal{A} \text{ is a proper minor-closed class of graphs}\}.$$

We shall see shortly that for the corresponding set

$$L = \{\lambda_{\mathcal{A}} : \mathcal{A} \text{ is a proper minor-closed class of graphs}\}$$

of limiting densities we have $L = B$.

Given a proper minor-closed class $\mathcal{A}$ of graphs, a graph G is an *excluded minor* for $\mathcal{A}$ if G is not in $\mathcal{A}$ but each proper minor of G is in $\mathcal{A}$. If $\mathcal{H}$ is the set of excluded minors for $\mathcal{A}$, it is easy to see that $\mathcal{A}$ is the class of all graphs with no minor in $\mathcal{H}$: we write $\mathcal{A} = \text{Ex}(\mathcal{H})$. By the Robertson-Seymour theorem [17], the set $\mathcal{H}$ is finite.

A class of graphs is called *decomposable* when a graph G is in the class if and only if each component of G is. It is easy to see that a minor-closed class of graphs is decomposable if and only if each excluded minor is connected. By Lemma 5 in [14], if $\mathcal{A}$ is a decomposable minor-closed class of graphs, then $e^*_{\mathcal{A}}(n)/n \to \beta_{\mathcal{A}}$ as $n \to \infty$, and so $\lambda_{\mathcal{A}} = \beta_{\mathcal{A}}$. Let

$$B_1 = \{\beta_{\mathcal{A}} : \mathcal{A} \text{ is a proper decomposable minor-closed class of graphs}\}$$

be the set of critical densities of decomposable minor-closed classes (so $B_1 \subseteq B$ trivially).

We call a graph G *minor-balanced* if each minor of G has density at most that of G, and *strictly minor-balanced* (or *density-minimal* [5]) if each proper minor has density strictly less than that of G. If G is minor-balanced with density $\rho(G) = \beta > 0$, and we let $\mathcal{A}$ be the decomposable class of graphs such that each component is a minor of G, then clearly $\beta_{\mathcal{A}} = \lambda_{\mathcal{A}} = \beta$. In this case, we say that the density β is *achievable*, and that β is the *maximum density* for $\mathcal{A}$. For example, K_5 is (strictly) minor-balanced with density 2, so 2 is an achievable density. Let

$$A = \{e(G)/v(G) : G \text{ is a minor-balanced graph}\}$$

be the set of densities of minor-balanced graphs.

The first theorem presented here is a general statement concerning the structure of the set B and describing the relationships between the sets A, B, B_1 and L. It is largely taken from Eppstein [5] and contains his Theorems 19 and 20. Given a set $S \subseteq \mathbb{R}$, we let $\bar{S}$ denote its closure and let S' denote the set of accumulation (or limit) points.

Theorem 1. *(a) The set B of critical densities is countable, closed and well-ordered by $<$; and (b) $\bar{A} = B = B_1 = L$ and $A' = B'$.*

Indeed it seems that more may be known, and each critical (or limiting) density is rational. This is given as Theorem 8.3 in Norin's survey [16], where a 'glimpse' of its lengthy proof is given, based on unpublished work of Kapadia and Norin: see also the recent paper of Kapadia [9] on minor-closed classes of matroids. For further results and conjectures see [16].

We shall see shortly that the set B is unbounded (indeed, by Theorem 5, if $\beta \in B$ then $1 + \beta \in B$). It follows from the first part of Theorem 1 that B is nowhere dense. For, given $x \geqslant 0$, the set $\{\beta \in B : \beta > x\}$ has a least element x^+, and the non-empty open interval (x, x^+) is disjoint from B. In contrast, B is 'asymptotically dense', in the following sense. For each $x \geqslant 0$, let the 'gap above x' in B be $\delta_B(x) = x^+ - x$. Then B is asymptotically dense, in that $\delta_B(x) = o(1)$ as $x \to \infty$; and indeed we have the following result.

Theorem 2. *The gap* $\delta_B(x)$ *above* x *satisfies* $\delta_B(x) = O(x^{-2})$ *as* $x \to \infty$.

We would like to have been able to describe the whole set B of critical densities more fully, but at least we can give a full description of the values at most 2, in Theorems 3 and 4 (and see also Theorem 5). We identify all critical densities in the interval $[0, 2)$, and see that all are achieved. (We noted already that the density 2 is achieved.) Due to the significantly more complicated statement for the sub-interval $[1, 2)$, we divide these results into two parts. The first part concerns the interval $[0, 1)$, and is essentially due to Eppstein [5], but our approach allows us to offer a shorter proof.

Theorem 3. *We have*

$$A \cap [0, 1) = B \cap [0, 1) = \left\{ \frac{t-1}{t} : t \geqslant 1 \right\}. \tag{1}$$

The second part covers the interval $[1, 2)$, which we partition into the subintervals $[2 - \frac{1}{k-1}, 2 - \frac{1}{k})$ for $k = 2, 3, \ldots$

Theorem 4. *Let* $\mathcal{A}$ *be a proper minor-closed class of graphs. Let* $k \geqslant 2$*, and suppose that*

$$2 - \frac{1}{k-1} \leqslant \beta_{\mathcal{A}} < 2 - \frac{1}{k}.$$

Then (a) either $\beta_{\mathcal{A}}$ *is* $2 - \frac{1}{k-1}$*, or for some* $n = mk + 1 + t$*, where* $m \geqslant 1$ *and* $0 \leqslant t \leqslant k - 1$*, and*

$$n > (2k - 1 - t)(k - 1), \tag{2}$$

we have

$$\beta_{\mathcal{A}} = 2 - \frac{1}{k} - \frac{2k - t - 1}{kn}; \tag{3}$$

and (b) each such value is in A *(that is, is achievable).*

Consider the special case $k = 2$ in the above theorem, concerning values $\beta \in [1, \frac{3}{2})$. We see that these are exactly the values $\frac{3}{2} - \frac{3}{2n}$ for odd integers $n \geqslant 5$ and $\frac{3}{2} - \frac{1}{n}$ for even integers $n \geqslant 2$. This result, together with Theorem 3, constitute Theorem 22 of Eppstein [5]. We now know exactly the critical densities $\beta \leqslant 2$; and we know that for each such β there is a minor-balanced graph G with $\rho(G) = \beta$. Can we insist that G is regular or nearly regular? We shall see in Subsection 3.2 that we can always insist that the gap between maximum and minimum degrees is at most 2, but not necessarily at most 1.

An important property of a class $\mathcal{A}$ of graphs is being 'addable', see McDiarmid, Steger and Welsh [15]. We say that $\mathcal{A}$ is *bridge-addable* if for any $G \in \mathcal{A}$ and any vertices u, v belonging to different components of G, the graph obtained by adding to G the edge $\{u, v\}$ is also in $\mathcal{A}$; and $\mathcal{A}$ is *addable* if it is both decomposable and bridge-addable. As noted in [13], it is straightforward to check that a minor-closed class $\mathcal{A}$ is addable if and only if each excluded minor is 2-connected. Examples of addable minor-closed classes of graphs include forests ($\mathrm{Ex}(K_3)$), series-parallel graphs ($\mathrm{Ex}(K_4)$), and planar graphs ($\mathrm{Ex}(\{K_5, K_{3,3}\})$).

For $t = 1, 2, \ldots,$ let

$$B_t = \{\beta_{\mathcal{A}} : \mathcal{A} \text{ is a proper minor-closed class such that each excluded minor is } t\text{-connected}\}. \tag{4}$$

Observe that for B_1 this agrees with the earlier definition, and that B_2 is the set of critical densities of addable minor-closed classes. Part of the following theorem describes $B_2 \cap [0, 2]$, the set of critical densities in $[0, 2]$ of addable classes of graphs. We use the notation $1 + B$ for $\{x : x = 1 + \beta \text{ for some } \beta \in B\}$.

Theorem 5. *We have $1 + B \subsetneq B_2 \subseteq B'$, where B' is the set of accumulation points of B; $B_2 \cap [0, 1) = B' \cap [0, 1) = \emptyset$ and*

$$(1 + B) \cap [1, 2] = B_2 \cap [1, 2] = B' \cap [1, 2] = \left\{2 - \frac{1}{k} : k \geqslant 1\right\} \cup \{2\}.$$

The first part of this theorem (that $1 + B \subsetneq B_2 \subseteq B'$) extends Theorem 22 of Eppstein [5]. Both the statement that $1 + B \neq B'$ and the last part of the theorem (describing $B' \cap [1, 2]$) answer open questions (7 and 4 respectively) from the same paper (with both answers in the negative). Note that Theorem 2 and the first part of Theorem 5 together imply the following fact.

Corollary 6. *We have $\delta_{B_2}(x) = O(x^{-2})$ and $\delta_{B'}(x) = O(x^{-2})$ as $x \to \infty$.*

It is not clear to what extent the sets B_t are interesting objects of study for large values of t. However, the study of B_1 and B_2 is well-motivated by the fact that they consist of the critical densities of the decomposable and the addable minor-closed classes of graphs respectively. Moreover, since K_5 and $K_{3,3}$ giving rise to the minor-closed class of planar graphs are both 3-connected, we would also want to learn more about the structure of B_3. After proving Theorem 5, we obtain a first result in that direction, Proposition 21, which says that $1 + B_2 \subseteq B_3$. See also Propositions 18 and 23 concerning other expressions for $\beta_{\mathcal{A}}$ when all excluded minors are t-connected, for $t = 2$ and 3. In particular, we see that, if each excluded minor is t-connected and $K_{t+1} \in \mathcal{A}$, then $\beta_{\mathcal{A}}$ is the supremum of the '$(t-1)$-density' $\rho_{t-1}(G)$ (see the definition of $\rho_t(G)$ in (8)) over the t-connected graphs G in $\mathcal{A}$. (This is also true for $t = 1$ but not for $t = 4$.)

Plan of the paper

In the next section, we first prove Theorem 1 on the general structure of the set B of critical densities, and then prove Theorem 2, showing that B is 'asymptotically dense'. In Section 3 we prove Theorems 3 and 4 which give a full description of $B \cap [0, 2]$: Theorem 4 is the main work involved in the paper. In Section 4 we discuss the critical densities of minor-closed classes with all excluded minors t-connected, in particular the case $t = 2$ corresponding to addable classes; and we prove Theorem 5 (except for proving that $1 + B \neq B'$). In Section 5 we begin the study of the structure of the set B above the value 2, which allows us to show that $1 + B \neq B'$ and to answer a further open question from [5]. Finally, in Section 6 we present some open problems.

2 General structure of B

In this section we consider the general structure of the set B, and in particular we prove Theorems 1 and 2.

Proof of Theorem 1. Since there are only countably many finite sets of finite graphs, and by the Robertson-Seymour theorem [17] every minor-closed family of graphs can be characterised by such a family, it follows that B is countable. We shall see shortly that B is the closure $\bar{A}$ of A, and it will follow of course that B is closed.

Now let us show that B is well-ordered by $<$. (This follows also from Theorem 19 of Eppstein [5], but for completeness we give a short proof here.) Assume that $\{\beta_n\}_{n\geqslant 1}$ is an infinite strictly decreasing sequence of points in B. Then, for all $n \geqslant 1$ there is some minor-balanced graph G_n satisfying $\beta_{n+1} < \rho(G_n) \leqslant \beta_n$. Consequently, for all $n \geqslant 1$ we have $\rho(G_n) > \rho(G_{n+1})$. Now, let us consider the sequence $\{G_n\}_{n\geqslant 1}$. By the Robertson-Seymour theorem, there is some $1 \leqslant i < j$ such that G_i is a minor of G_j. However, G_j is minor-balanced, so every minor of G_j has density at most equal to that of G_j, and we have a contradiction. Thus no infinite decreasing sequences of points exist in B, and so B is well-ordered by $<$.

We have now proved part (a) of the theorem, and we move on to prove the first statement in part (b). We shall show in four steps that $B = B_1$, $B_1 \subseteq L$, $L \subseteq \bar{A}$ and $\bar{A} \subseteq B$; and that will of course show that $B = B_1 = L = \bar{A}$, as required.

1. $B = B_1$. Let $\beta \in B$, and let $\mathcal{A}_0$ be a proper minor-closed class of graphs with $\beta_{\mathcal{A}_0} = \beta$. Let $\mathcal{A}_1$ be the set of graphs such that each component is in $\mathcal{A}_0$. Then $\mathcal{A}_1$ is minor-closed and decomposable. Clearly $\mathcal{A}_1 \supseteq \mathcal{A}_0$, so $\beta_{\mathcal{A}_1} \geqslant \beta$. But let $G \in \mathcal{A}_1$, with components $H_1, \ldots, H_k$. Then each $H_i \in \mathcal{A}_0$, so $\frac{e(H_i)}{v(H_i)} \leqslant \beta$. Hence

$$\rho(G) = \frac{e(G)}{v(G)} = \frac{\sum_i e(H_i)}{\sum_i v(H_i)} \leqslant \beta,$$

and so $\beta_{\mathcal{A}_1} = \beta$. Thus $B \subseteq B_1$ and so $B = B_1$, as required.

2. $B_1 \subseteq L$. Let $\beta \in B_1$. Let $\mathcal{A}$ be a decomposable minor-closed class of graphs with $\beta_{\mathcal{A}} = \beta$. As we noted earlier, by Lemma 5 in [14] we have $\lambda_{\mathcal{A}} = \beta_{\mathcal{A}}$, so $\beta \in L$.

3. $L \subseteq \bar{A}$. Theorem 20 of Eppstein [5] says that $L = \bar{A}$.

4. $\bar{A} \subseteq B$. Let $\beta \in \bar{A}$: we must show that $\beta \in B$. Since $A \subseteq B$, we may assume wlog that $\beta \notin A$. Since B is well-ordered, there exists $\varepsilon > 0$ such that $A \cap [\beta, \beta + \varepsilon) = \emptyset$. Thus there is an infinite sequence of minor-balanced graphs $\{G_k\}_{k\geqslant 1}$ such that $\rho(G_k)$ strictly increases to β. Let $\tilde{\mathcal{A}}$ be the (decomposable) class of graphs such that each component is a minor of some graph G_k. Then each component of each graph $G \in \tilde{\mathcal{A}}$ has density at most $\rho(G_k)$ for some k, so arguing as in the first step above we have $\rho(G) < \beta$; and it follows that $\beta_{\tilde{\mathcal{A}}} \leqslant \beta$. But also $\beta_{\tilde{\mathcal{A}}} \geqslant \rho(G_k)$ for each k, so $\beta_{\tilde{\mathcal{A}}} \geqslant \beta$. Thus $\beta_{\tilde{\mathcal{A}}} = \beta$, and $\beta \in B$, as required.

It remains to show that $A' = B'$. Of course $A' \subseteq B'$. Let $\beta \in B'$: we must show that $\beta \in A'$. There is an infinite sequence $\beta_k \in B$ strictly increasing to β. If $\beta_k \in A$ infinitely often then of course $\beta \in A'$: thus we may assume wlog that for each k we have $\beta_k \notin A$, and so since $B = \bar{A}$ we have $\beta_k \in \bar{A} \setminus A$. Thus for each k there is a minor-balanced graph G_k such that $\beta_k - \frac{1}{k} < \rho(G_k) < \beta_k$. Then $\rho(G_k) \in A$ and $\rho(G_k) \to \beta$ as $k \to \infty$, so $\beta \in A'$, as required. □

Proof of Theorem 2. We prove the theorem by showing that all graphs in two particular families are minor-balanced, and that as their densities grow, the gaps between the densities shrink appropriately. Let us start by introducing the first family we shall work with.

Let $k \geqslant 2$, let $n = k(k+1)+2$ (observe that n is even), and let $0 \leqslant m \leqslant n$ be such that if $m > n/2$ then $(m - n/2)$ is a multiple of 3. We construct the graph $G_k(m)$ as follows. Start with a clique on a set X of k vertices. Then, put a complete bipartite graph between X and a set Y of n vertices. Finally, we construct the graph on Y as follows:

1. If $m \leqslant n/2$, let Y induce an m-edge matching.
2. Otherwise, if $m > n/2$ and $(m - n/2)$ is a multiple of 3, then let Y induce $2(m-n/2)/3$ triangles and a perfect matching on the remainder of Y.

For example, we present the graph $G_2(7)$ in Figure 1.

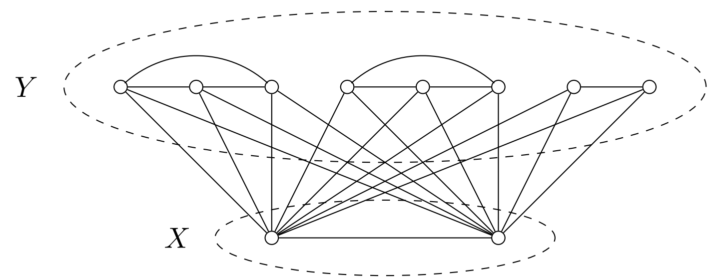

Figure 1: Graph $G_2(7)$, with the set X inducing a clique on $k = 2$ vertices, and the set Y containing $n = 2(2+1)+2 = 8$ vertices, inducing $2(7-8/2)/3 = 2$ triangles and a perfect matching on the remaining 2 vertices.

Let us start by giving bounds on the density of $G_k(m)$. When $m = 0$ we have

$$\begin{aligned}\rho(G_k(0)) &= \frac{\binom{k}{2} + nk}{n+k} = \frac{k^2 - k + 2k^3 + 2k^2 + 4k}{2(k + k^2 + k + 2)} \\ &= \frac{2k^3 + 3k^2 + 3k}{2(k^2+2k+2)} = k - \frac{1}{2} + \frac{k+2}{2(k^2+2k+2)} := a_k.\end{aligned}$$

Thus for $0 \leqslant m \leqslant n/2$ we have

$$\rho(G_k(m)) = \rho(G_k(0)) + \frac{m}{n+k} = k - \frac{1}{2} + \frac{k+2+2m}{2(k^2+2k+2)}.$$

Hence

$$\rho(G_k(m+1)) - \rho(G_k(m)) = \frac{1}{k^2+2k+2} \quad \text{for } 0 \leqslant m < n/2.$$

Also, as an aside, note that $\rho(G_k(n/2-1)) = k$. Now consider m with $n/2 < m \leqslant n$ and $m - n/2$ divisible by 3 (note that the smallest such m is $n/2+3$). Note first that

$$\rho(G_k(m+3)) - \rho(G_k(m)) = \frac{3}{k^2+2k+2}.$$

Let n_1 be the largest $m \leqslant n$ such that $m - n/2$ is divisible by 3, so $n_1 = n - \delta_1$ where δ_1 is 1 or 2 (note that n is not divisible by 3). Note that the number of vertices in Y in triangles is

$3 \cdot 2(n-\delta_1-n/2)/3 = n-2\delta_1$. Thus

$$\begin{aligned}\rho(G_k(n_1)) &= \frac{\binom{k}{2}+(n-2\delta_1)(k+1)+2\delta_1(k+1/2)}{k+n}\\ &= \frac{\binom{k}{2}+n(k+1)-\delta_1}{k+n} = \frac{(2k+2)n+k^2-k-2\delta_1}{2(k+n)}\\ &= \frac{(2k+1)(k+n)-(2k+1)k+k^2-k+n-2\delta_1}{2(k+n)}\\ &= k+\frac{1}{2}-\frac{k-2+2\delta_1}{2(k+n)} \quad := b_k.\end{aligned}$$

Therefore, for each $k \geqslant 2$, the densities of the graphs $G_k(m)$ give a cover of the interval $[a_k, b_k]$ with 'mesh' (gaps) $< 3/k^2$.

To prove Theorem 2, we introduce a second family of graphs $F_k(m)$ to close the gaps between consecutive intervals $[a_k, b_k]$, and then show that the graphs we use (the graphs $G_k(m)$ and $F_k(m)$) are minor-balanced. Then the classes of graphs with all components being minors of our graphs give us the desired values in B.

First, let us work on closing the gap between the intervals $[a_k, b_k]$ and $[a_{k+1}, b_{k+1}]$ for some $k \geqslant 2$. Given $n = k(k+1)+2$, let n_2 be the largest multiple of 3 at most n, so that $n_2 = n-\delta_2$ where δ_2 is 1 or 2, and $n_2 \geqslant k(k+1)$. (In fact $\delta_2 = 3-\delta_1$.) Given $m \geqslant n_2 \geqslant k(k+1)$ and divisible by 3, let $F_k(m)$ be the union of $m/3$ cliques on $k+3$ vertices each, all sharing a common set of k vertices. Then $v(F_k(m)) = k+m$ and $e(F_k(m)) = \binom{k}{2}+m(k+1)$. For example, we present the graph $F_2(9)$ in Figure 2.

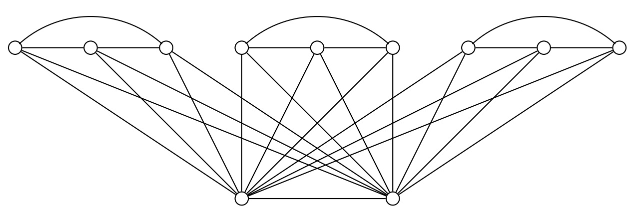

Figure 2: Graph $F_2(9)$, consisting of $9/3 = 3$ cliques each on $2+3 = 5$ vertices, sharing a common set of 2 vertices.

Thus

$$\begin{aligned}\rho(F_k(m)) &= \frac{\binom{k}{2}+m(k+1)}{k+m} = \frac{k^2-k+2m(k+1)}{2(k+m)}\\ &= \frac{(2k+1)(m+k)+m-k^2-2k}{2(k+m)}\\ &= k+\frac{1}{2}+\frac{m-k^2-2k}{2(k+m)}.\end{aligned}$$

Hence

$$\rho(F_k(m+3))-\rho(F_k(m)) = \frac{3k(3+k)}{2(k+m)(k+m+3)} \leqslant \frac{3k(3+k)}{2(k^2+2k)(k^2+2k+3)},$$

and so, since $k(k+3) < (k+2)^2$,

$$0 \leqslant \rho(F_k(m+3)) - \rho(F_k(m)) \leqslant \frac{3k(3+k)}{2k^2(k+2)^2} < \frac{3}{2k^2}.$$

Also

$$\begin{aligned}\rho(F_k(n_2)) &= k + \frac{1}{2} + \frac{k^2+k+2-\delta_2-k^2-2k}{2(k+n_2)} \\ &= k + \frac{1}{2} - \frac{k-2+\delta_2}{2(k+n-\delta_2)}.\end{aligned}$$

It is now easy to check that

$$|\rho(F_k(n_2)) - b_k| = |\rho(F_k(n_2)) - \rho(G_k(n_1))| = O(1/k^2).$$

Also, since $k^2+k \leqslant n_2 \leqslant k^2+k+1$, we have

$$\begin{aligned}\rho(F_k(n_2+4k)) &= \frac{\binom{k}{2} + n_2k + 4k^2 + n_2 + 4k}{n_2+4k+k} \\ &\geqslant \frac{\binom{k}{2} + k^3 + k^2 + 4k^2 + k^2 + k + 4k}{k^2+k+5k+1} \\ &= \frac{2k^3+13k^2+9k}{2(k^2+6k+1)} \\ &= \frac{2k(k^2+6k+1)+(k^2+6k+1)+k-1}{2(k^2+6k+1)} \\ &= k + \frac{1}{2} + \frac{k-1}{2(k^2+5k+1)} \quad := c_k.\end{aligned}$$

We have $|c_k - a_{k+1}| = O(1/k^2)$. Hence the densities $\rho(F_k(m))$ provide us with the required mesh of the gaps (b_k, a_{k+1}) between intervals covered by the densities of the graphs $G_k(m)$.

Hence what remains is to show that the graphs $G_k(m)$ and $F_k(m)$ are minor-balanced. For the graphs $G_k(m)$, since we want to find their minors with the highest density, we may assume that the minors were obtained only through vertex deletions and edge-contractions. Additionally, due to the structure of $G_k(m)$ which can be seen as unions of cliques all sharing a common set of k vertices, any vertex deletion is equivalent to some edge-contraction (with the exception of removing the last of the k common vertices, but minors with the whole common set removed have density at most 1 so we may ignore them).

Hence let H be a minor of $G_k(m)$ that was obtained only through edge-contractions. As the order of contractions is irrelevant, assume we first perform all contractions of edges with at least one vertex in Y, i.e., edges not fully contained in the common set X. If $m < n/2$ then the density of $G_k(m)$ is less than k. The contractions we perform all remove one vertex and at least k edges, hence the density can only go down. Next, we contract the edges in the common set X, but this can be seen as removing universal vertices, hence it also pulls the density down, proving that H is sparser than $G_k(m)$ in this case.

If $n/2 \leqslant m \leqslant n$ then the density of $G_k(m)$ is at most $k + 1/2$. Recall that in this case, the graph induced by Y is a union of disjoint triangles and a perfect matching on the remainder of Y. We again contract the edges with at least one end in Y first. As there are no isolated vertices in the graph induced by Y, in order to have a contraction resulting with the deletion of only k edges, we first have to remove $k+1$ edges once, or make two contractions which result in removing $k+2$ and $k+1$ edges respectively. Thus, on average, we remove at least $k+1/2$ edges per vertex. Contractions limited to X can again be seen as removing universal vertices, thus again any minor H obtained this way has to be at most as dense that $G_k(m)$. Hence the graphs $G_k(m)$ are minor-balanced.

For the graphs $F_k(m)$ we can argue similarly. This completes the proof of Theorem 2. □

Let G be a strictly minor-balanced graph with density $\rho(G) = \beta$. We know that the class of graphs with every component being a minor of G is a minor-closed class with maximum density β.

Proposition 7. *For any $\beta \in B$ there is a finite number of strictly minor-balanced graphs with $\rho(G) = \beta$.*

Proof. Assume not and let $\{G_n\}_{n\geqslant 1}$ be a sequence of distinct strictly minor-balanced graphs with density β. Then, by the Robertson-Seymour theorem, there is some $1 \leqslant i < j$ such that G_i is a minor of G_j. However, by the fact that G_j is strictly balanced we then have

$$\beta = \rho(G_i) < \rho(G_j) = \beta,$$

a contradiction. □

3 Complete characterisation of critical densities at most 2

In this section we first prove Theorem 3 (and a little more, see Proposition 8); then we prove Theorem 4, which requires most of the section; and finally, in the last subsection, we investigate to what extent, given $\beta \leqslant 2$, we can choose a minor-balanced graph G achieving this density (that is, with $\rho(G) = \beta$) which is regular or near-regular. The equation (1) is contained in Theorem 22 of Eppstein [5], but we prove it for completeness and since our proof is short.

Proof of Theorem 3. Let $\beta_{\mathcal{A}} < 1$. Observe first that some star is not in $\mathcal{A}$ and some path is not in $\mathcal{A}$, for otherwise $\beta_{\mathcal{A}} = \sup_{G\in\mathcal{A}} \rho(G) \geqslant 1$. Hence there is only a finite number of possible connected (unlabelled) graphs in $\mathcal{A}$. Clearly $\beta_{\mathcal{A}}$ is the maximum value of $\rho(G)$ for connected graphs in $\mathcal{A}$, achieved for some graph G_0 with $v(G_0) = t$ say. Then, as G_0 is connected we have $e(G_0) \geqslant t - 1$, and as $\rho(G_0) < 1$, also $e(G_0) \leqslant t - 1$. Hence we have $\beta_{\mathcal{A}} = (t - 1)/t$.

Now, fix $t \geqslant 1$ and let $\mathcal{H}_t = \{C_3, K_{1,3}, P_{t+1}\}$, so $\mathcal{A}_t = \mathrm{Ex}(\mathcal{H}_t)$ consists of the forests of paths each with at most t vertices. Then $\beta_{\mathcal{A}_t} = \frac{t-1}{t} \in A$ as intended. □

The equation (1) describes the restriction to $[0, 1)$ of (the set A and) the set B of critical densities. Recall from Theorem 1 that B is also the set of limiting densities. In this sparse region, we can describe the limiting behaviour of $e^*_{\mathcal{A}}(n)$ rather precisely.

Proposition 8. *Let $\mathcal{A}$ be a minor-closed class of graphs, and suppose that* $\liminf e^*_{\mathcal{A}}(n)/n < 1$. *Then there is a positive integer t such that*

$$e^*_{\mathcal{A}}(n) = \tfrac{t-1}{t} n + O(1). \tag{5}$$

Additionally, for each positive integer t there is a minor-closed class $\mathcal{A}$ such that (5) *holds.*

Proof. Suppose that $\liminf e^*_{\mathcal{A}}(n)/n < 1$. Then, arguing as before, there is a finite set of connected (unlabelled) graphs in $\mathcal{A}$. Let $\mathcal{C}$ be the set of connected graphs C on at least two vertices such that the disjoint union of k copies of C is in $\mathcal{A}$ for each positive integer k. If $\mathcal{C}$ is empty then $e^*_{\mathcal{A}}(n) = O(1)$, so suppose not, and let $q = \max\{\rho(C) : C \in \mathcal{C}\}$, which is achieved for some C^* with $v(C^*) = t$. Then $e^*_{\mathcal{A}}(n) = qn + O(1)$. Thus we again see that $e(C^*) < t$, which implies that $e(C^*) = t - 1$. Hence $e^*_{\mathcal{A}}(n) = \frac{t-1}{t}n + O(1)$.

The family $\mathcal{H}_t = \{C_3, K_{1,3}, P_{t+1}\}$ again shows that for any $t \geqslant 1$, there is a family $\mathcal{A}_t = \mathrm{Ex}(\mathcal{H}_t)$ for which (5) holds. □

3.1 Proof of Theorem 4

To prove Theorem 4 we first introduce a family of graphs we call 2-plants. The name of the family should indicate that our graphs are related to 2-trees but are more general.

We say that a graph G with $v(G) = k \geqslant 2$ is a 2-*plant* if there exists an ordering of its vertices $v_1, v_2, \ldots, v_k$ for which the following two conditions are satisfied:

1. $\{v_1, v_2\} \in E(G)$,
2. for all $3 \leqslant i \leqslant k$ we have $|N(v_i) \cap \{v_1, v_2, \ldots, v_{i-1}\}| \geqslant 2$.

If G is a 2-plant as above, we say that G is a 2^+-plant if for at least one value of i in the second condition above the inequality is strict, and otherwise we say that G is a $2^=$-plant. Observe that a 2^+-plant must contain at least 4 vertices, with K_4 being the unique 4-vertex 2^+-plant.

Of course, every 2^+-plant or $2^=$-plant is also a 2-plant. Clearly every 2-tree is a $2^=$-plant, but it is also clear that $2^=$-plants are a strictly broader class of graphs than 2-trees, as we do not require the two neighbours of v_i in $\{v_1, v_2, \ldots, v_{i-1}\}$ to be adjacent (for example, the 5-vertex graph obtained by subdividing one edge of K_4 once is a $2^=$-plant but not a 2-tree.) Consequently, even though each 2-tree is strictly minor-balanced, a $2^=$-plant need not be minor-balanced (as shown again by the example of the graph obtained from K_4 by subdividing one edge once), and similarly a 2^+-plant need not be minor-balanced.

The following lemma will be a crucial tool in the proof of Theorem 4.

Lemma 9. *Let $m \geqslant 0$, $k \geqslant 2$ and $0 \leqslant t \leqslant k-1$. Let G be a connected graph on $n = mk + 1 + t$ vertices, in which every edge is contained in a $(k+1)$-vertex $2^=$-plant or in some 2^+-plant. Then,*

$$e(G) \;\geqslant\; 2n - 2 - m \;=\; \left(2 - \frac{1}{k}\right) n + \frac{t+1}{k} - 2. \tag{6}$$

This bound is sharp, and indeed for each integer ℓ with

$$2n - 2 - m \leqslant \ell \leqslant \binom{n}{2}$$

there is a connected graph G with n vertices and ℓ edges such that each edge is in a $(k+1)$-vertex $2^=$-plant or in some 2^+-plant. Further, in the case $\ell = 2n - 2 - m$, we may choose G so that also it is strictly minor-balanced.

Proof. Let $W_0 = \{v\}$ for some arbitrary vertex v of G, and let G_0 be the trivial graph on W_0. For $j = 1, 2, \ldots$ we shall construct a set W_j of vertices, with corresponding induced subgraph G_j of G. If $|W_{j-1}| < n$, we obtain W_j from W_{j-1} using the following procedure. Take an arbitrary edge e of G with exactly one vertex in W_{j-1} (such an edge can always be found since G is connected). Let U be the vertex set of an arbitrary copy of a $(k+1)$-vertex $2^=$-plant or some 2^+-plant to which e belongs, and set $W_j = W_{j-1} \cup U$.

It is clear that if $|U \cap W_{j-1}| = 1$, U spans a $(k+1)$-vertex $2^=$-plant, and there are no 'extra' edges between $U \setminus W_{j-1}$ and $W_{j-1} \setminus U$, then as we form G_j from G_{j-1} we increase the number of vertices by k, and the number of edges by exactly $2k-1$. We *claim* that this is the only scenario which allows us to possibly extend to G_j in such a "sparse" way, with all other scenarios leading us to adding at least two edges per vertex.

Suppose first that $|U \cap W_{j-1}| = 1$. Then the claim is immediate if there are 'extra' edges between $U \setminus W_{j-1}$ and $W_{j-1} \setminus U$; and similarly the claim is immediate if U spans a 2^+-plant, since by the definition of 2^+-plants, for some $i \geqslant 4$ we are adding $i-1$ vertices and at least $2i-2$ edges to the graph G_j.

Hence assume that $|U \cap W_{j-1}| \geqslant 2$ and let $U = \{v_1, v_2, \ldots, v_i\}$ (where possibly $i = k+1$), as in the definition of i-vertex 2-plants. The argument below holds both when the plant on U is a $2^=$-plant and when it is a 2^+-plant. We think of adding the vertices of U to W_{j-1} one by one, in the order $v_1, v_2, \ldots$; and we say that the number of new edges v_a *contributes* to G_t is the number of edges $\{v_b, v_a\}$ with $b < a$ which are not already in the graph constructed (that is, such that at most one of v_a, v_b is in $W_{j-1} \cup \{v_1, \ldots, v_{a-1}\}$). We shall show that the total number of edges contributed is at least $2\,|U \setminus W_{j-1}|$, which will establish the claim.

Let $s_1 < s_2$ be the two smallest indices of the vertices of U contained in W_{j-1}. If $s_1 = 1$, $s_2 = 2$ then all vertices in $U \setminus W_{j-1}$ contribute at least two new edges by the definition of a 2-plant. If $s_1 = 1$, $s_2 > 2$ then $v_2 \in U \setminus W_{j-1}$ contributes only one new edge $\{v_1, v_2\}$. However, in this case $v_{s_2} \in W_{j-1}$ also contributes at least one new edge, as the only edge $\{v_b, v_{s_2}\}$ with $b < s_2$ that could already be in the graph is the possible edge $\{v_1, v_{s_2}\}$. Hence we again contribute at least two edges per vertex.

If $s_1 = 2$, then $v_1 \in U \setminus W_{j-1}$ contributes no new edges. However, $v_2 \in W_{j-1}$ contributes one new edge $\{v_1, v_2\}$, and the same applies to $v_{s_2} \in W_{j-1}$ (since the only edge $\{v_b, v_{s_2}\}$ with $b < s_2$ that could already be in the graph is the possible edge $\{v_2, v_{s_2}\}$), so the claim holds in this case.

Finally, if $s_1 > 2$ then $v_1, v_2 \in U \setminus W_{j-1}$ together contribute only one new edge to G_t. However, v_{s_1} must contribute at least 2 new edges, and v_{s_2} must contribute at least one: hence, the total count again gives at least twice as many edges as vertices, completing the proof of the claim.

Using the above procedure we recover the whole of G. Thus the bound in (6) follows immediately by observing that we start with one vertex, and add at least 2 edges per vertex, except in a special move when we add exactly k vertices and exactly one less edge. But we can do this at most m times, so

$$e(G) \geqslant 2(n-1) - m,$$

as in (6).

To show that the bounds are tight, we take a union of m disjoint $(k+1)$-vertex $2^=$-plants, and merge them into one connected graph by choosing one vertex in every plant and identifying the chosen vertices into one vertex. We then take the remaining t vertices, choose one of the $2^=$-plants and join all t vertices to the two neighbours of v_{k+1} in that plant. In the resulting graph every edge is in a $(k+1)$-vertex $2^=$-plant and it is clearly sharp for inequality (6).

We can now add edges one by one to the star of plants described above, completing the individual $(k+1)$-vertex $2^=$-plants into cliques (in the case of the plant with the vertex v_{k+1} copied t times, we first complete the original $2^=$-plant, then join all t additional vertices to the original $k+1$ vertices, and then complete the clique on $k+1+t$ vertices). Throughout this procedure the graph clearly keeps the property that all edges are contained in a $(k+1)$-vertex $2^=$-plant or in some 2^+-plant. Once we have completed a star of cliques, we add the remaining edges in an arbitrary order. Whenever we add an edge e, we can treat one of its end-vertices as v_{k+1}, the centre of the star as v_k, and the rest of the plant is then contained in the clique containing the other end-vertex of e.

It remains only to consider further the case when ℓ is as small as possible, that is $\ell = 2n-2-m$. Recall first that, for $a \geqslant 1$, $K^+_{2,a}$ is the graph obtained from the complete bipartite graph $K_{2,a}$ by connecting the vertices in the part of size 2 by an edge: it is a 2-tree, and so it is strictly balanced. All we need to do is to take the m initial $(k+1)$-vertex $2^=$-plants to be copies of $K^+_{2,k-1}$ (where we list the adjacent vertices in the part of size 2 as v_1, v_2). If $t \geqslant 1$ then the copy of $K^+_{2,k-1}$ to which we add the t extra vertices becomes a copy of $K^+_{2,k-1+t}$. To see that the graph G constructed is strictly balanced, let us think of it as a $K^+_{2,k-1+t}$ with copies of $K^+_{2,k-1}$ attached to it by identifying vertices. Now, observe that the density of $K^+_{2,k-1+t}$ is at most

$$\frac{2(2k-2)+1}{2k} = 2 - \frac{3}{2k},$$

while contracting $K^+_{2,k-1}$ to a vertex removes $2k-1$ edges and k vertices. Since only contracting a whole $K^+_{2,k-1}$ allows us to remove less than twice as many edges as vertices, and also

$$\frac{2k-1}{k} = 2 - \frac{2}{2k} > 2 - \frac{3}{2k},$$

we see that any proper minor of G is strictly sparser than G. This completes the proof of the lemma. □

We are now ready to prove Theorem 4.

Proof of Theorem 4. First, we observe that for each $k \geqslant 2$, we obtain $\beta_{\mathcal{A}} = 2 - \frac{1}{k-1}$ by taking $\mathcal{A}$ to be the family of graphs where every component is a minor of $K^+_{2,3k-5}$ since it is a (strictly) minor-balanced graph on $3k-3$ vertices and $6k-9$ edges. Hence we can focus on densities in the open intervals $(2-\frac{1}{k-1}, 2-\frac{1}{k})$ for $k \geqslant 2$.

Thus suppose that $2 - \frac{1}{k-1} < \beta_{\mathcal{A}} < 2 - \frac{1}{k}$. Let $0 < \varepsilon < \beta_{\mathcal{A}} - (2 - \frac{1}{k-1})$. There is a connected minor-balanced graph $G \in \mathcal{A}$ with $\rho(G) > \beta_{\mathcal{A}} - \varepsilon > 2 - \frac{1}{k-1}$. Assume that there is an edge e in G not in any $(k+1)$-vertex $2^=$-plant nor in any 2^+-plant. Let j^* be the maximum j such that e is in a j-vertex $2^=$-plant, so $2 \leqslant j^* \leqslant k$; and let T be the unique j^*-vertex $2^=$-plant containing e. Then, since no vertex outside T has more than one neighbour in T, the minor G' of G obtained by contracting T to a vertex satisfies $e(G') = e(G) - (2j^*-3)$ and $v(G') = v(G) - (j^*-1)$. But we *claim* that

$$\rho(G') = \frac{e(G)-(2j^*-3)}{v(G)-(j^*-1)} \geqslant \frac{e(G)-(2k-3)}{v(G)-(k-1)} > \frac{e(G)}{v(G)} = \rho(G), \qquad (7)$$

and it follows that we may assume that in G each edge is in a $(k+1)$-vertex $2^=$-plant or in a 2^+-plant.

Let us prove the claim (7). We know that $2-\frac{1}{k-1} < \frac{e(G)}{v(G)} < 2-\frac{1}{k}$. If $v(G) \leqslant k-1$ then $e(G) > 2v(G) - \frac{v(G)}{k-1} \geqslant 2v(G)-1$, so $e(G) \geqslant 2v(G)$. But this contradicts $\frac{e(G)}{v(G)} < 2-\frac{1}{k}$ so $v(G) \geqslant k$; and hence $e(G) < 2v(G) - \frac{v(G)}{k} \leqslant 2v(G)-1$, that is $e(G)-2v(G)+1<0$. Therefore for $f(t) = \frac{e(G)-2t+3}{v(G)-t+1}$ we have $f'(t) < 0$ whenever

$$-2(v(G)-t+1)+e(G)-2t+3 = e(G)-2v(G)+1 < 0,$$

which we know must hold. Thus the first inequality in (7) is proved.

For the second one, observe that as $\frac{e(G)}{v(G)} = \alpha > 2-\frac{1}{k-1} = \frac{2k-3}{k-1}$, we have

$$\frac{e(G)-(2k-3)}{v(G)-(k-1)} > \frac{e(G)-\alpha(k-1)}{v(G)-(k-1)} = \frac{\alpha(v(G)-(k-1))}{v(G)-(k-1)} = \alpha = \frac{e(G)}{v(G)},$$

completing the proof of (7).

By Lemma 9 we know that we must have

$$\frac{e(G)}{n} = 2 - \frac{1}{k} + \frac{t+1}{kn} - \frac{2}{n} + \frac{s}{n}$$

for some integer $s \geqslant 0$. However, we have $e(G)/n \leqslant 2-\frac{1}{k}$ if and only if $s \in \{0,1\}$. Since $2-\frac{1}{k}$ is the only accumulation point of the set of such values of $e(G)/n$, we know that these are the only possible values of $\beta_{\mathcal{A}} \in (2-\frac{1}{k-1}, 2-\frac{1}{k})$.

By the last part of Lemma 9, each density with $s=0$ is achievable as $e(G)/v(G)$ for a (strictly) balanced graph G. Thus the class $\mathcal{A}$ of graphs in which each component is a minor of G satisfies $\beta_{\mathcal{A}} = e(G)/v(G)$ as required.

Next, observe that the lower bound on n in (2) is exactly the requirement that

$$\frac{2}{n} - \frac{t+1}{kn} < \frac{1}{k-1} - \frac{1}{k} = \frac{1}{k(k-1)}.$$

Thus, if $s=0$ and $e(G)/n > 2-\frac{1}{k-1}$, then n must satisfy (2).

It remains only to show that the densities obtained with $s=1$ can also be obtained with $s=0$ (for some other value of n). First, observe that when $t=k-1$ and $s=1$ then we obtain $e(G)/n = 2-1/k$. Hence we can assume that $0 \leqslant t \leqslant k-2$.

Let $n_1 = m_1k+1+t_1$, for $s=1$ giving us

$$\frac{t_1+1}{kn_1} - \frac{1}{n_1} = \frac{t_1+1-k}{k(m_1k+1+t_1)}.$$

We want to show that there exists $n_2 = m_2k+1+t_2$, with $0 \leqslant t_2 \leqslant k-1$, such that

$$\frac{t_1+1-k}{k(m_1k+1+t_1)} = \frac{t_2+1}{kn_2} - \frac{2}{n_2} = \frac{t_2+1-2k}{k(m_2k+1+t_2)}.$$

Solving the above equality for t_2 we obtain

$$\begin{aligned} t_2 &= \frac{m_1(2k-1)+2t_1+1+m_2(1+t_1-k)}{m_1+1} \\ &= 2k-1+\frac{2+2t_1-2k+m_2(1+t_1-k)}{m_1+1} \\ &= 2k-1-\frac{(m_2+2)(k-1-t_1)}{m_1+1}. \end{aligned}$$

Let us take $m_2 = \ell(m_1+1)-2$, so that we have $t_2 = 2k-1-\ell(k-1-t_1)$, for some integer ℓ to be determined. Since we assume $0 \leqslant t_1 \leqslant k-2$, we have $1 \leqslant k-1-t_1 \leqslant k-1$. Thus there is some $\ell' \geqslant 2$ such that

$$0 \leqslant t_2' = 2k-1-\ell'(k-1-t_1) \leqslant k-1.$$

Thus G_2, the 'star of $2^{=}$-plants' on $n_2 = (\ell'(m_1+1)-2)k+1+t_2'$ vertices, gives us the desired density $e(G_2)/n_2$ with $s=0$. (The lower bound (2) must hold for n_2 since we have not changed the density.) This shows that we need not consider $s=1$ further, and thus completes the proof of the theorem. □

Remark 10. Given $k \geqslant 2$, when n does not satisfy (2) then we obtain a value of $\beta_{\mathcal{A}}$ in one of the preceding intervals $(2-\frac{1}{i-1}, 2-\frac{1}{i})$ for some $i<k$. This value is obviously achievable, but is already covered by the case $k=i$ of the theorem.

Remark 11. Theorem 4 implies that the least possible values of $\beta_{\mathcal{A}}$ larger than 1 are $\frac{6}{5} < \frac{5}{4} < \frac{9}{7} < \frac{4}{3} < \dots$. Indeed, taking $k=2$, by (3) we want to maximize the value of $(3-t)/2n$ for n satisfying (2), which for $k=2$ is equivalent to the condition that $n \geqslant 4$. Then, for n odd, which implies $t=0$, the lowest values of $\beta_{\mathcal{A}}$ we obtain are $\frac{3}{2}-\frac{3}{10}=\frac{6}{5}$ for $n=5$, $\frac{3}{2}-\frac{3}{14}=\frac{9}{7}$ for $n=7$, and $\frac{3}{2}-\frac{3}{18}=\frac{4}{3}$ for $n=9$. For n even, which gives $t=1$, we obtain the values $\frac{3}{2}-\frac{2}{8}=\frac{5}{4}$ for $n=4$, and $\frac{3}{2}-\frac{2}{12}=\frac{4}{3}$ for $n=6$.

3.2 Near regularity for minor-balanced graphs

Let us continue to consider critical densities $\beta \leqslant 2$. For each such β we know from Theorems 3 and 4 that there is a minor-balanced graph G with $\rho(G)=\beta$, and so with average degree 2β. Can we insist that G is regular or nearly regular?

Note that for any graph G

$$\delta(G) \leqslant \lfloor 2\rho(G) \rfloor \leqslant \lceil 2\rho(G) \rceil \leqslant \Delta(G).$$

If 2β is an integer t then we may hope to find a regular minor-balanced graph G with density β; and indeed this is easy – just take G to be K_{t+1}. What can we say in the other cases, when G cannot be regular? We shall see that we can always insist that $\Delta - \delta \leqslant 2$ but not necessarily that $\Delta - \delta = 1$. Recall that a *block* in a graph is a maximal 2-connected subgraph, or a cut-edge (bridge) together with its end vertices, or an isolated vertex.

- For $\frac{1}{2} < \beta < 1$ we must have $\delta = 1$, and we may always take G as a path, with $\delta = 1$ and $\Delta = 2$.

- For $1 < \beta < \frac{3}{2}$ we must have $\delta = 2$ and $\Delta \geqslant 3$. For $\beta = \frac{5}{4}$ we may take G as the diamond D with $\Delta = 3$: for other values β we must have $\Delta \geqslant 4$ (since the only graphs with $\Delta \leqslant 3$ in which each edge is in a triangle are C_3, D and K_4) and we can always take G with $\Delta = 4$ (indeed we can insist that each block of G is a 2-tree, by the proof of Lemma 9).
- For $\frac{3}{2} < \beta < 2$, we must have $\delta = 2$ or 3 and $\Delta \geqslant 4$. We can always take G with $\Delta = 4$, as above. Sometimes we can achieve $\delta = 3$ (for example if $\beta = \frac{9}{5}$ and G is K_5 less an edge), but not always: Proposition 12 below shows that with $\beta = \frac{20}{13}$ we must have $\delta = 2$.

Proposition 12. *Each connected minor-balanced graph G with $\rho(G) = \frac{20}{13}$ consists of 4 blocks each of which is a diamond D, and in particular $\delta(G)$=2.*

Proof. Let G be a connected n-vertex minor-balanced graph with $\rho(G) = e(G)/n = 20/13$. Write n as $3m+1+t$ for some $m \geqslant 0$ and $t \in \{0,1,2\}$. Since $2-1/(k-1) < 20/13 < 2-1/k$ with $k = 3$, by the discussion in the second paragraph of the proof of Theorem 4, each edge is in a 4-vertex $2^{=}$-plant (that is, a diamond) or a 2^{+}-plant. Hence, by Lemma 9,

$$\frac{e(G)}{n} = 2 - \frac{1}{3} - \left(\frac{2}{n} - \frac{s}{n} - \frac{t+1}{3n}\right)$$

for some $s \geqslant 0$. Thus

$$\frac{2}{n} - \frac{s}{n} - \frac{t+1}{3n} = \frac{5}{3} - \frac{20}{13} = \frac{5}{39},$$

that is $(5-3s-t)/n = 5/13$. Hence $13|n$ and so $n \geqslant 13$; and it follows that $s = t = 0$ and $n = 13$, and e(G)=20.

We have seen that $n = 13 = 3m+1$, so $m = 4$. By the proof of Lemma 9, $e(G) \geqslant 2n-2-m = 20$ with equality only if the exploration process (after setting $W_0 = \{v\}$) four times finds a 4-set U of vertices containing exactly one old vertex and spanning a $2^{=}$-plant - which is, as we noted, a diamond. But this uses up all 13 vertices, so G must have exactly 4 blocks, each of which is a diamond. ☐

By the above, for each critical density $\beta \leqslant 2$, there is a (decomposable) minor-closed class $\mathcal{A}$ such that $\beta_{\mathcal{A}} = \beta$ and there is a nearly regular minor-balanced graph $G \in \mathcal{A}$ with $\rho(G) = \beta$ – we may take $\mathcal{A}$ as the class of graphs such that each component is a minor of G. If we consider the graph classes $\mathcal{A}$ rather than just critical densities β then we can say little, even when we restrict attention to classes such that $\beta_{\mathcal{A}}$ is attained by some graph in $\mathcal{A}$. In particular, we cannot bound the maximum degree Δ. For example, if $\mathcal{A}$ is the class of graphs in which each component is a star with at most k leaves, then the unique connected minor-balanced graph $G \in \mathcal{A}$ with $\rho(G) = \beta_{\mathcal{A}}$ is the k-leaf star, with $\Delta = k$.

4 Excluding t-connected minors

In this section we discuss the possible critical densities of minor-closed families of graphs in the case when all excluded minors are (at least) t-connected for some given t. We start by introducing some general tools which should be useful in this study, though some are used here only for small values of t; and then we focus on addable minor-closed classes of graphs (where all excluded minors are 2-connected), and the corresponding set B_2. In particular we prove Theorem 5, apart from one detail. We then discuss the case when all excluded minors are 3-connected, and conclude the section by proving some initial results about the structure of the sets B_t for large values of t.

4.1 Some general tools

In this subsection we introduce the key definition of the t-density of a graph. We consider when these densities are increasing in t; we see how they behave when we add a universal vertex; we give a lower bound on $\beta_{\mathcal{A}}$ in terms of t-densities when $\mathcal{A}$ is closed under t-sums; and finally we relate a class being closed under k-sums and it having t-connected excluded minors.

Given a non-negative integer t, and a graph G with $e(G) > \binom{t}{2}$ (and thus $v(G) \geqslant t+1$), the *t-density* of G is

$$\rho_t(G) = \frac{e(G) - \binom{t}{2}}{v(G) - t}. \tag{8}$$

It is convenient to define $\rho_t(G)$ to be 0 if $e(G) \leqslant \binom{t}{2}$. Thus $\rho_t(G) > 0$ if and only if $e(G) > \binom{t}{2}$. Observe that $\rho_0(G)$ is the usual density $\rho(G)$. (See Lemma 16 for motivation for the formula (8).)

Clearly we have $\rho_1(G) > \rho_0(G)$ for each graph G with an edge. Indeed, more generally, the t-densities of a graph are non-decreasing in t when the density is sufficiently large.

Proposition 13. *Let $t \geqslant 2$ be an integer, and let the graph G satisfy $v(G) \geqslant t+1$ and $e(G) \geqslant (t-1)(v(G) - t/2)$. Then*

$$\rho_t(G) \geqslant \rho_{t-1}(G) > \cdots > \rho_0(G).$$

Proof. Note first that $e(G) \geqslant (t-1)(t+1-t/2) > \binom{t}{2}$, so $\rho_s(G) > 0$ for each $s = 0, 1, \ldots, t$. Let $1 \leqslant s \leqslant t$: we want to show that $\rho_s(G) \geqslant \rho_{s-1}(G)$, with strict inequality if $s \neq t$. But

$$\begin{aligned}
(\rho_s(G) - \rho_{s-1}(G))(v-s)(v-(s-1)) &= \left(e - \binom{s}{2}\right)(v-(s-1)) - \left(e - \binom{s-1}{2}\right)(v-s) \\
&= e - v\left(\binom{s}{2} - \binom{s-1}{2}\right) + (s-1)\binom{s}{2} - s\binom{s-1}{2} \\
&= e - (s-1)(v - s/2) \geqslant 0,
\end{aligned}$$

and indeed the last inequality is strict if $s \leqslant t-1$. □

In particular, if G is a connected graph with $v(G) \geqslant 3$, then $e(G) \geqslant v(G) - 1$ and so by the case $t = 2$ of Proposition 13 we have

$$\rho_2(G) \geqslant \rho_1(G) > \rho_0(G). \tag{9}$$

(This is in fact the only result from Proposition 13 which we shall use.)

Given a graph G, let G^+ be obtained from G by adding a universal vertex; that is, a new vertex that is connected to all vertices of G. Let us call G^+ the *complete one-vertex extension* of G. The following lemma will be used in the proofs of Theorem 5 and Lemma 15.

Lemma 14. *Let $t \geqslant 0$ be an integer and G a graph. Then $\rho_{t+1}(G^+) \leqslant \rho_t(G) + 1$, with equality if $\rho_t(G) > 0$.*

Proof. Suppose first that $\rho_t(G) > 0$. Then $v(G) > t$, and so

$$e(G^+) = e(G) + v(G) > \binom{t}{2} + t = \binom{t+1}{2}.$$

Thus

$$\rho_{t+1}(G^+) = \frac{e(G) + v(G) - \binom{t+1}{2}}{v(G) + 1 - (t+1)} = \frac{e(G) - \binom{t}{2} + v(G) - t}{v(G) - t} = \rho_t(G) + 1.$$

Now let us prove that the inequality holds in general. If $\rho_{t+1}(G^+) > 0$ then as above

$$\rho_{t+1}(G^+) = \frac{e(G) - \binom{t}{2} + v(G) - t}{v(G) - t} \leqslant \rho_t(G) + 1,$$

which completes the proof. □

We say that a graph G is *t-minor-balanced* if $\rho_t(G) > 0$ and $\rho_t(G) \geqslant \rho_t(H)$ for each minor H of G. Also, we call G *strictly t-minor-balanced* if $\rho_t(G) > 0$ and $\rho_t(G) > \rho_t(H)$ for each proper minor H of G. Thus G is (strictly) 0-minor-balanced if and only if it is (strictly) minor-balanced and $\rho(G) > 0$. The case $t = 1$ of the following lemma will be used in the proofs of Theorem 5 and Proposition 21.

Lemma 15. *If a graph G is t-minor-balanced then G^+ is $(t+1)$-minor-balanced.*

Proof. Let G be t-minor-balanced. Then by Lemma 14, $\rho_{t+1}(G^+) = \rho_t(G) + 1 > 0$. Observe that any edge-maximal minor of G^+ is either an edge-maximal minor H of G, or is the complete one-vertex extension H^+ of such a graph H. Let H be any minor of G. By Lemma 14,

$$\rho_{t+1}(H^+) \leqslant \rho_t(H) + 1 \leqslant \rho_t(G) + 1 = \rho_{t+1}(G^+);$$

and if $v(H) \geqslant 2$ and $v \in V(H)$, then

$$\rho_{t+1}(H) \leqslant \rho_t(H - v) + 1 \leqslant \rho_t(G) + 1 = \rho_{t+1}(G^+).$$

It follows that G^+ is $(t+1)$-minor-balanced. □

Let G and G' be graphs such that for $W = V(G) \cap V(G')$ we have $|W| = k$ and both graphs induce a clique on W. If H is $G \cup G'$, or is obtained from $G \cup G'$ by deleting some edges within W, then H is a *k-sum* of G and G'. Let $\omega(G)$ denote the clique number of G, i.e., the number of vertices in a largest complete subgraph of G.

Lemma 16. *If $t \geqslant 0$, the minor-closed class $\mathcal{A}$ is closed under t-sums, and $G \in \mathcal{A}$ with $\omega(G) \geqslant t$, then $\beta_{\mathcal{A}} \geqslant \rho_t(G)$.*

Proof. We may assume that $\rho_t(G) > 0$, so $v(G) \geqslant t + 1$. Let G_k be formed from k copies of G all overlapping on a given t-vertex clique of G. Then

$$\rho(G_k) = \frac{\binom{t}{2} + k(e(G) - \binom{t}{2})}{t + k(v(G) - t)} \to \rho_t(G) \text{ as } k \to \infty,$$

so $\beta_{\mathcal{A}} \geqslant \rho_t(G)$. □

We close this subsection by relating a graph class $\mathcal{A}$ having t-connected excluded minors to $\mathcal{A}$ being closed under k-sums. If a minor-closed class $\mathcal{A}$ is $\mathrm{Ex}(K_t)$ for some $t \geqslant 2$, then clearly $\mathcal{A}$ is closed under k-sums for each $k \geqslant 0$ (trivially so for $k \geqslant t$). Similarly, if $\mathcal{A}$ is the class of graphs with treewidth at most t for some given $t \geqslant 1$, then $\mathcal{A}$ is closed under k-sums for each $k \geqslant 0$ (since in a tree-decomposition of a graph each clique is contained within some bag). Observe that a minor-closed class is closed under 0-sums if and only if it is decomposable, i.e., if and only if each excluded minor is connected. The following result is used in the proofs of Lemma 20 and Proposition 23.

Proposition 17. *Let $\mathcal{A}$ be a proper minor-closed class of graphs, not of the form* $\mathrm{Ex}(K_t)$ *for some t. For each integer $t \geqslant 0$, if each excluded minor for $\mathcal{A}$ is $(t+1)$-connected then $\mathcal{A}$ is closed under k-sums for each $k \leqslant t$. For each $t \in \{0, 1, 2\}$ the converse also holds; that is, $\mathcal{A}$ is closed under k-sums for each $k \leqslant t$ if and only if each excluded minor is $(t+1)$-connected.*

Let $\mathcal{A}$ be the class of graphs with treewidth at most 3. Then $\mathcal{A}$ is closed under k-sums for each $k \geqslant 0$ and so in particular for each $k \leqslant 3$. However, of the four excluded minors for $\mathcal{A}$, two are cubic (as shown by Arnborg, Proskurowski, and Corneil [1], and by Satyanarayana and Tung [18]), and so clearly are not 4-connected. Thus the converse above does not hold for $t = 3$.

Proof. Suppose first that each excluded minor for $\mathcal{A}$ is $(t+1)$-connected. Let G and H be graphs in $\mathcal{A}$ such that for $W = V(G) \cap V(H)$ we have $|W| \leqslant t$ and both graphs induce a clique on W: we must show that $G \cup H$ is in $\mathcal{A}$. Assume for a contradiction that $G \cup H$ is not in $\mathcal{A}$. Then $G \cup H$ has as a minor some excluded minor M for $\mathcal{A}$. Thus there are disjoint non-empty subsets W_v of $V(G) \cup V(H)$ for each $v \in V(M)$, such that for each $v \in V(M)$ the induced subgraph of $G \cup H$ on W_v is connected, and for each edge uv of M there is an edge between W_u and W_v in $G \cup H$. Since W induces a clique, there must exist $v_1 \in V(M)$ such that $W_{v_1} \subseteq V(H) \setminus V(G)$ (or else M is a minor of G), and similarly there exists $v_2 \in V(M)$ such that $W_{v_2} \subseteq V(G) \setminus V(H)$. Let S be the set of vertices $x \in V(M)$ such that $W_x \cap W \neq \emptyset$. Then S separates M and $|S| \leqslant |W| \leqslant t$, so M is not $(t+1)$-connected, a contradiction.

Now let $t \in \{0, 1, 2\}$, and suppose that $\mathcal{A}$ is closed under k-sums for each $k \leqslant t$. If $t = 0$ then $\mathcal{A}$ is decomposable (as we noted earlier), so each excluded minor is connected. If $t = 1$ then a graph is in $\mathcal{A}$ if and only if each block is, and K_2 is not an excluded minor, so each excluded minor is 2-connected. Suppose that $t = 2$. By the last case, each excluded minor M is 2-connected. Suppose that M is not 3-connected: then, since M is not K_3 and so $v(M) \geqslant 4$, there exist vertex sets U_1 and U_2 such that $U_1 \cup U_2 = V(M)$, $U_1 \cap U_2$ consists of two vertices u and v which form a cut, and both $U_1 \setminus U_2$ and $U_2 \setminus U_1$ are non-empty. As M is 2-connected, there must be a $u - v$ path in U_2, so if we contract $U_2 - v$ onto u we obtain a graph H_{U_1} in $\mathcal{G}$ in which uv is an edge (H_{U_1} is in $\mathcal{G}$ because it is a strict minor of a forbidden minor M). Similarly, if we contract $U_1 - v$ onto u we obtain a graph H_{U_2} in $\mathcal{G}$ in which uv is an edge. But the 2-sum of H_{U_1} and H_{U_2} on $\{u, v\}$ is either M or M with the edge uv added, so M is in $\mathcal{A}$, a contradiction. Hence M is 3-connected, as required. □

4.2 Critical densities for addable families

Let us now focus on the structure of B_2; that is, on the critical densities of minor-closed families with all excluded minors being 2-connected. This requires us to understand the 1-density $\rho_1(G)$.

Recall that $\rho_1(G) = e(G)/(v(G)-1)$ for graphs G with $v(G) \geqslant 2$, and $\rho_1(K_1) = 0$. Note first that, for a connected graph G other than K_1, the 1-density is a convex combination of the 1-densities of its blocks: indeed, since $v(G) - 1 = \sum_B (v(B) - 1)$ where the sum is over the blocks B of G, we have

$$\rho_1(G) = \frac{\sum_B e(B)}{\sum_B (v(B)-1)} = \sum_B \alpha_B \, \rho_1(B),$$

where the positive weights $\alpha_B = \frac{v(B)-1}{v(G)-1}$ sum to 1. In particular, if each block of a connected graph G is a copy of B then $\rho_1(G) = \rho_1(B)$. Also, for any graph G with at least one edge, whether or not it is connected,

$$\rho(G) < \rho_1(G) \leqslant \max_B \rho_1(B) \tag{10}$$

where the maximum is over the blocks B of G.

Next we show that for an addable class $\mathcal{A}$ of graphs, $\beta_{\mathcal{A}}$ is the supremum of $\rho_1(B)$ over the possible blocks in $\mathcal{A}$. Indeed, we have the following more detailed result, which singles out the class $\text{Ex}(K_3)$ of forests. (Observe that no forests are 2-connected.)

Proposition 18. *Let $\mathcal{A}$ be an addable minor-closed class of graphs. Then*

(a) $\beta_{\mathcal{A}} = \sup\{\rho_1(G) : G \in \mathcal{A}\}$*;*

(b) if $K_3 \notin \mathcal{A}$, then $\mathcal{A}$ is the class $\text{Ex}(K_3)$ *of forests, $\beta_{\mathcal{A}} = 1$, and $\rho_1(T) = 1$ for each connected graph (tree) $T \in \mathcal{A}$ with at least one edge; and*

(c) if $K_3 \in \mathcal{A}$, then $\beta_{\mathcal{A}} = \sup\{\rho_1(G) : G \in \mathcal{A},\ G$ is 2-connected$\}$.

Proof. Part (b) is straightforward, and implies that (a) holds when $\mathcal{A}$ is $\text{Ex}(K_3)$. Now consider part (c). By (10) we have

$$\beta_{\mathcal{A}} \leqslant \sup\{\rho_1(G) : G \in \mathcal{A},\ G \text{ is 2-connected}\} \leqslant \sup\{\rho_1(G) : G \in \mathcal{A}\}. \tag{11}$$

(Note that we need not consider $G = K_2$ in the suprema here, since $K_3 \in \mathcal{A}$.) Now let $G \in \mathcal{A}$. Then by (10) again, there is a 2-connected graph $B \in \mathcal{A}$ with $\rho_1(G) \leqslant \rho_1(B)$. By Proposition 17, $\mathcal{A}$ is closed under 1-sums; and so, by Lemma 16,

$$\beta_{\mathcal{A}} \geqslant \rho_1(B) \geqslant \rho_1(G).$$

Hence $\beta_{\mathcal{A}}$ is at least the supremum of the values $\rho_1(G)$, and so equality holds throughout (11). This completes the proof of part (c), and of part (a). □

The next result concerns 1-minor-balanced graphs. Observe that every tree with at least one edge is 1-minor-balanced. Part (c) of Lemma 19 will be used in the proof of Theorem 5.

Lemma 19. *Let G be a 1-minor-balanced graph. Then*

(a) G is connected;

(b) If G is strictly 1-minor-balanced, then either G is K_2 or G is 2-connected; and

(c) G is strictly minor-balanced.

Proof. (a) To show that G is connected, we may assume that $v(G) \geqslant 3$, since $e(G) \geqslant 1$. There cannot be an isolated vertex v, since then $\rho_1(G-v) > \rho_1(G)$. Suppose that $V(G)$ can be partitioned into non-empty parts U_1 and U_2 with $e(U_1, U_2) = \emptyset$. Then $|U_1|, |U_2| \geqslant 2$. Write G_{U_1} for the induced subgraph $G[U_1]$, and similarly for G_{U_2}. Then

$$\rho_1(G) = \frac{e(G_{U_1}) + e(G_{U_2})}{|U_1| + |U_2| - 1} < \frac{e(G_{U_1}) + e(G_{U_2})}{(|U_1| - 1) + (|U_2| - 1)} \leqslant \max\{\rho_1(G_{U_1}), \rho_1(G_{U_2})\},$$

which contradicts G being 1-minor-balanced.

(b) As before, we may suppose that $v(G) \geqslant 3$. Suppose that $V(G) = U_1 \cup U_2$ where $|U_1 \cap U_2| = 1$, $U_1 \setminus U_2$ and $U_2 \setminus U_1$ are non-empty, and $e(U_1 \setminus U_2, U_2 \setminus U_1) = \emptyset$. Then $|U_1|, |U_2| \geqslant 2$, and with G_{U_1} and G_{U_2} as before,

$$\rho_1(G) = \frac{e(G_{U_1}) + e(G_{U_2})}{(|U_1| - 1) + (|U_2| - 1)} \leqslant \max\{\rho_1(G_{U_1}), \rho_1(G_{U_2})\},$$

which contradicts the assumption that G is strictly 1-minor-balanced.

(c) Let G be 1-minor-balanced, and let H be a proper minor of G with at least one edge. Then

$$\frac{e(H)}{v(H) - 1} \leqslant \frac{e(G)}{v(G) - 1},$$

so

$$e(H)v(G) - e(G)v(H) \leqslant e(H) - e(G) < 0,$$

where the last inequality follows from the fact that a 1-minor-balanced graph must be connected, and therefore any strict minor must contain strictly fewer edges. Hence $\rho(H) < \rho(G)$, and thus G is strictly minor-balanced. □

Proof of Theorem 5. Let us start by proving that $1 + B \subseteq B_2$ and then that $B_2 \subseteq B'$. For the sake of the flow of the argument, we defer the proof of the fact that $1 + B \neq B_2$ to Section 5, as it will require us to introduce a particular family of graphs, which we shall analyze in more detail there. For now, let us say only that in Section 5 we show that $25/11 \in B_2 \setminus (1 + B)$.

Let $\beta_0 \in B$. Let $\mathcal{A}_0$ be a decomposable class with $\beta_{\mathcal{A}_0} = \beta_0$. We want to show that $1 + \beta_0 \in B_2$. Since $1 \in B_2$ (consider the class of forests) we may assume that $\beta_0 > 0$. There are two cases.

Suppose first that β_0 is achieved in $\mathcal{A}_0$; that is, there is a graph $G \in \mathcal{A}_0$ with $\rho(G) = \beta_0$. Then G is minor-balanced, and we may assume that G is connected. Hence the complete one-vertex extension G^+ is 2-connected, and G^+ is 1-minor-balanced by Lemma 15.

Let $\mathcal{G}$ be the class of graphs such that each block is a minor of G^+. Then $\mathcal{G}$ is minor-closed and addable. Since G^+ is 1-minor-balanced, the supremum of the 1-densities of the blocks in $\mathcal{G}$ is $\rho_1(G^+)$, which by Lemma 14 equals $\rho(G) + 1 = \beta_0 + 1$, and thus $\beta_{\mathcal{G}} = \beta_0 + 1$ by Proposition 18, so $1 + \beta_0 \in B_2$ as required.

The second case is when β_0 is not achieved in $\mathcal{A}_0$; that is $\rho(G) < \beta_0$ for each $G \in \mathcal{A}_0$. There is a sequence of connected minor-balanced graphs $G_1, G_2, \ldots$ in $\mathcal{A}_0$ such that $\rho(G_j)$ strictly increases to β_0. By Lemma 14, $\rho_1(G_j^+)$ strictly increases to $1 + \beta_0$. Let $\mathcal{G}$ be the class of graphs such that each block is a minor of G_j^+ for some j. Then $\mathcal{G}$ is minor-closed and addable. By Lemma 15, each graph G_j^+ is 1-minor-balanced, so the supremum of the 1-densities of the 2-connected graphs in $\mathcal{G}$ is $1+\beta_0$. Hence $\beta_{\mathcal{G}} = 1+\beta_0$ by Proposition 18, and so $1+\beta_0 \in B_2$, as required. This completes the proof that $1+B \subseteq B_2$.

Now let us prove that $B_2 \subseteq B'$. Let $\mathcal{A}$ be an addable minor-closed family and let $\beta = \beta_{\mathcal{A}}$. Let $0 < \varepsilon < \beta$. It suffices to show that there is a minor-closed class $\mathcal{A}_1$ with $\beta - \varepsilon < \beta_{\mathcal{A}_1} < \beta$.

For each graph $G \in \mathcal{A}$ with at least one edge, we have $\rho(G) < \rho_1(G) \leqslant \beta$ by Proposition 18. Hence $\rho(G) < \beta$ for each $G \in \mathcal{A}$. Let $G_0 \in \mathcal{A}$ satisfy $\rho(G_0) > \beta - \varepsilon$; and let G_1 be a minor of G_0 with maximum density, so that G_1 is minor-balanced and $\beta - \varepsilon < \rho(G_1) < \beta$. Let $\mathcal{A}_1$ be the class of graphs such that each component is a minor of G_1. Then $\mathcal{A}_1$ is minor-closed and $\beta_{\mathcal{A}_1} = \rho(G_1)$, so $\mathcal{A}_1$ is as required.

Finally, let $\beta \in B'$ with $\beta \leqslant 2$. By (1) we have $\beta \in [1,2]$, and by Theorem 4, we see that $\beta = 2$ or $\beta = 2 - \frac{1}{k}$ for some positive integer k. Hence, by (1) again, $\beta - 1 \in B$, that is $\beta \in 1+B$. Thus $B' \cap [0,2] \subseteq (1+B) \cap [0,2]$. But we have already shown that $1+B \subseteq B_2 \subseteq B'$, and so the proof is complete. □

4.3 Excluding 3-connected minors

By Theorem 5 (and recalling that $B = B_1$) we have $1+B_1 \subseteq B_2$. In this subsection, we 'increment by 1': we show in Proposition 21 that $1+B_2 \subseteq B_3$, as mentioned near the end of Section 1. After that, we give Proposition 23, which is the corresponding result for the 3-connected case to Proposition 18 for the addable (2-connected) case. The following, slightly technical lemma, will be the key to our proof of Proposition 21.

Lemma 20. *Let $\mathcal{A}$ be a non-empty (finite or infinite) set of 2-minor-balanced graphs each of which is K_3 or is 3-connected, and let $\rho_2^* = \sup_{G\in\mathcal{A}} \rho_2(G)$. Let $\mathcal{G}$ be the closure of $\mathcal{A}$ under k-sums for $0 \leqslant k \leqslant 2$ and under taking minors. Then $\mathcal{G}$ is minor-closed and each excluded minor is 3-connected; and $\beta_{\mathcal{G}} = \rho_2^*$.*

Proof. The fact that $\mathcal{G}$ is minor-closed is immediate from the definition of $\mathcal{G}$. Also, since clearly $\mathcal{G}$ is not $\mathrm{Ex}(K_t)$ for some $t \leqslant 3$, it follows by Proposition 17 that every excluded minor M for $\mathcal{G}$ is 3-connected. It remains to show that $\beta_{\mathcal{G}} = \rho_2^*$; but let us first show that $\rho_2^* > 1$. Let $G_1 \in \mathcal{A}$: then $e(G_1) \geqslant v(G_1)$, and so

$$\rho_2^* \geqslant \rho_2(G_1) = \frac{e(G_1)-1}{v(G_1)-2} \geqslant \frac{v(G_1)-1}{v(G_1)-2} > 1.$$

Next we show $\beta_{\mathcal{G}} \leqslant \rho_2^*$. Let $H \in \mathcal{G}$. We want to show that $\rho(H) < \rho_2^*$. It suffices to consider the case when H is connected, and $\rho(H) > 1$ (since $\rho_2^* > 1$). Further, we may assume that H is 2-connected. Otherwise, taking u to be a cut-vertex of H and v, w to be neighbours of u in different components of $H - u$, by identifying v and w into one vertex we could form a graph that is in $\mathcal{G}$ (as every forbidden minor is 3-connected), and has one less vertex and one less edge, and thus a strictly larger ρ value. Now H may be obtained by starting with an edge and repeatedly forming a 2-sum with a minor of a graph in $\mathcal{A}$. Thus for each minor G' of some $G \in \mathcal{A}$ added, if we add t vertices then we add at most $\rho_2(G')t \leqslant \rho_2(G)t \leqslant \rho_2^* t$ edges (this follows from the assumption that every graph $G \in \mathcal{A}$ is 2-minor-balanced). Hence if we add a total of t vertices then we add at most $\rho_2^* t$ edges. Hence

$$e(H) \leqslant 1 + \rho_2^* t < \rho_2^* (t+2) = \rho_2^* v(H).$$

This gives $\rho(H) < \rho_2^*$, as desired. We have now seen that $\beta_{\mathcal{G}} \leqslant \rho_2^*$.

For each $G \in \mathcal{A}$, $k \geqslant 1$, if we form $G^k \in \mathcal{G}$ from k copies of G overlapping in a single edge then

$$\rho(G^k) = \frac{1 + k(e(G) - 1)}{2 + k(v(G) - 2)},$$

so $\rho(G^k)$ increases to $\rho_2(G)$ as $k \to \infty$. Recalling that $\rho_2^* = \sup_{G \in \mathcal{A}} \rho_2(G)$, we see that also $\rho_2^* = \sup_{G \in \mathcal{A}, k \geqslant 1} \rho(G^k)$. Hence $\beta_{\mathcal{G}} = \rho_2^*$, and this value is in B_3. ☐

Proposition 21. *For the sets B_2 and B_3 as defined in* (4) *we have*

$$1 + B_2 \subseteq B_3.$$

Proof. Let $\beta \in B_2$: we must show that $1 + \beta \in B_3$. Recall that $B_2 \cap [0, 1) = \emptyset$. Since K_4 is 3-connected and $\beta_{\mathrm{Ex}(K_4)} = 2$, we see that 2 is in B_3: so we may assume that $\beta > 1$. Let $\mathcal{A}$ be an addable minor-closed class of graphs with $\beta_{\mathcal{A}} = \beta$. By Proposition 18, β is the supremum of $\rho_1(G)$ over the 2-connected graphs in $\mathcal{A}$.

Assume first that there is a 2-connected graph $G_0 \in \mathcal{A}$ with $\rho_1(G_0) = \beta$. Observe that G_0 must be 1-minor-balanced. Let $G_1 = G_0^+$. Then G_1 is 3-connected and, by Lemmas 15 and 14, it is a 2-minor-balanced graph with $\rho_2(G_1) = 1 + \rho_1(G_0) = 1 + \beta$. Hence by Lemma 20, taking our set of 2-minor-balanced and 3-connected graphs to be $\mathcal{A} = \{G_1\}$ (which gives $\rho_2^* = \rho_2(G_1)$), we see that $1 + \beta$ is in B_3.

Now, assume that there is no 2-connected graph G in $\mathcal{A}$ with $\rho_1(G) = \beta$. Then there is a sequence of 2-connected and 1-minor-balanced graphs $\{H_i\}_{i \geqslant 1}$ with $\rho_1(H_i) \nearrow \beta$ as $i \to \infty$. We proceed again using Lemma 20, with $\mathcal{A} = \{H_1^+, H_2^+, \ldots\}$ as our set of 3-connected and 2-minor-balanced graphs: we obtain

$$1 + \beta = 1 + \sup_{i \geqslant 1} \rho_1(H_i) = \sup_{i \geqslant 1} \rho_2(H_i^+) = \rho_2^* \in B_3$$

and we are done. ☐

Perhaps we have the strict containment $1 + B_2 \subsetneq B_3$ above? We shall use the following lemma for strictly 2-minor-balanced graphs, corresponding to part (b) of Lemma 19, in the proof of Proposition 23.

Lemma 22. *Let $v(G) \geqslant 5$ and let G be strictly 2-minor-balanced. Then G is 3-connected.*

Proof. Clearly G has no isolated vertices, and G cannot be a matching (since $\rho_2(jK_2) = 1/2$ for each $j \geqslant 2$). Hence G has a connected component C with at least 3 vertices, and since G is (strictly) 2-minor-balanced we have

$$\rho_2(G) \geqslant \rho_2(C) = \frac{e(C) - 1}{v(C) - 2} \geqslant 1.$$

It follows that G cannot have a leaf v, for if so then

$$\rho_2(G - v) = \frac{(e(G) - 1) - 1}{(v(G) - 2) - 1} \geqslant \frac{e(G) - 1}{v(G) - 2} = \rho_2(G).$$

Suppose that U_1 and U_2 satisfy $U_1 \cup U_2 = V(G)$, $e(U_1 \backslash U_2, U_2 \backslash U_1) = \emptyset$ and $|U_1 \cap U_2| = i \leqslant 2$. Then $|U_1|, |U_2| \geqslant 3$ by the above. Let G_{U_1} denote the subgraph of G induced on U_1, and similarly for G_{U_2}. Assume first that i is 0 or 1. Then, since $\rho_2(G) \geqslant 1$,

$$\rho_2(G) = \tfrac{e(G)-1}{v(G)-2} \leqslant \tfrac{e(G_{U_1})+e(G_{U_2})-1}{|U_1|+|U_2|-3} \leqslant \tfrac{(e(G_{U_1})-1)+(e(G_{U_2})-1)}{(v(G_{U_1})-2)+(v(G_{U_2})-2)} \leqslant \max\{\rho_2(G_{U_1}), \rho_2(G_{U_2})\},$$

which contradicts G being strictly 2-minor-balanced. Now assume that $i = 2$, say $U_1 \cap U_2 = \{a, b\}$. Let H_{U_1} denote the minor of G obtained by contracting all of $U_2 \setminus \{a, b\}$ onto a, and similarly for H_{U_2}. Then a and b are adjacent in both H_{U_1} and H_{U_2}. Much as above we have

$$\rho_2(G) = \tfrac{e(G)-1}{v(G)-2} \leqslant \tfrac{e(H_{U_1})+e(H_{U_2})-2}{v(H_{U_1})+v(H_{U_2})-4} = \tfrac{(e(H_{U_1})-1)+(e(H_{U_2})-1)}{(v(H_{U_1})-2)+(v(H_{U_2})-2)} \leqslant \max\{\rho_2(H_{U_1}), \rho_2(H_{U_2})\},$$

which again contradicts G being strictly 2-minor-balanced. □

The final result in this subsection corresponds to Proposition 18, and singles out the class $\mathrm{Ex}(K_4)$ of series-parallel graphs. Recall that each graph in this class has a vertex of degree at most 2, so there are no 3-connected graphs in the class. (Equivalently, every 3-connected graph has a minor K_4.)

Proposition 23. *Let the minor-closed class $\mathcal{A}$ of graphs have 3-connected excluded minors. Then*

(a) $\beta_{\mathcal{A}} = \sup\{\rho_2(G) : G \in \mathcal{A}\}$*;*

(b) if $K_4 \notin \mathcal{A}$, then $\mathcal{A}$ is the class $\mathrm{Ex}(K_4)$ *of series-parallel graphs, $\beta_{\mathcal{A}} = 2$, and $\rho_2(G) = 2$ for each edge-maximal graph $G \in \mathcal{A}$ with $v(G) \geqslant 3$; and*

(c) if $K_4 \in \mathcal{A}$, then $\beta_{\mathcal{A}} = \sup\{\rho_2(G) : G \in \mathcal{A}, G$ is 3-connected$\}$.

Proof. We shall deduce part (a) from parts (b), (c) and their proofs.

(b) Suppose that $K_4 \notin \mathcal{A}$. Since every 3-connected graph has a minor K_4, it follows that $\mathcal{A}$ is $\mathrm{Ex}(K_4)$. We have already seen that $\beta_{\mathcal{A}} = 2$; and, for each edge-maximal graph $G \in \mathcal{A}$ with $v(G) = n \geqslant 3$, we have $\rho_2(G) = \frac{(2n-3)-1}{n-2} = 2$. Thus in particular part (a) holds for $\mathcal{A}$.

(c) Since $\mathcal{A}$ is closed under 2-sums by Proposition 17, we see by Lemma 16 that $\beta_{\mathcal{A}} \geqslant \sup\{\rho_2(G) : G \in \mathcal{A}\}$. Clearly $\beta_{\mathcal{A}} \geqslant 1$; and by (9), for a connected graph G with $v(G) \geqslant 3$ we have $\rho(G) \leqslant \rho_2(G)$. Hence

$$\beta_{\mathcal{A}} = \sup\{\rho(G) : G \in \mathcal{A},\ G \text{ connected}\} \leqslant \sup\{\rho_2(G) : G \in \mathcal{A}\},$$

and so part (a) holds for $\mathcal{A}$. In the supremum in part (a) we may clearly restrict either to $G = K_4$, or to graphs $G \in \mathcal{A}$ which are strictly 2-minor-balanced with $v(G) \geqslant 5$. Lemma 22 now completes the proof. □

Remark 24. Propositions 18 and 23 single out the class $\mathrm{Ex}(K_3)$ of forests for $t = 2$, and the class $\mathrm{Ex}(K_4)$ of series-parallel graphs for $t = 3$. An analogous result for $t = 1$, singling out the (unexciting) class $\mathrm{Ex}(K_2)$ of edgeless graphs, can be obtained immediately. However, there is no matching result for $t = 4$. To see this, let H be a 4-connected planar graph, for example $K_{2,2,2}$ (which may also be described as K_6 minus a perfect matching). If $\mathcal{A} = \mathrm{Ex}(\{K_5, H\})$, then $K_5 \notin \mathcal{A}$ but $\mathcal{A} \neq \mathrm{Ex}(K_5)$.

4.4 Excluding highly connected minors

Recall that the sets B_t were defined in (4). In previous subsections we considered the values $\beta_{\mathcal{A}}$ when each excluded minor for $\mathcal{A}$ is 2-connected or 3-connected, and the corresponding sets B_2 and B_3. In this subsection we consider larger values of t, and investigate the values $\beta_{\mathcal{A}}$ when each excluded minor for $\mathcal{A}$ is t-connected, and how the sets B_t behave. The first proposition gives a lower bound on $\beta_{\mathcal{A}}$ whenever each excluded minor is t-connected.

Proposition 25. *Let $t \geqslant 1$, and let $\mathcal{A}$ be a minor-closed class such that each excluded minor is t-connected and has at least $h \geqslant t+1$ vertices. Then*

$$\beta_{\mathcal{A}} \geqslant \frac{h+t-3}{2}.$$

Proof. By Proposition 17, $\mathcal{A}$ is closed under k-sums for each $k \leqslant t-1$. Hence the current proposition follows immediately from the following construction of a graph $G \in \mathcal{A}$. We take m copies of K_{h-1} that all overlap in a fixed set of $t-1$ vertices. Clearly G has $(h-t)m+t-1$ vertices and

$$\begin{aligned} e(G) &= m\left(\binom{h-t}{2} + (h-t)(t-1)\right) + \binom{t-1}{2} \\ &= (h-t)m\frac{h+t-3}{2} + \binom{t-1}{2} \end{aligned}$$

edges, so we have

$$\beta_{\mathcal{A}} = \sup_{G\in\mathcal{A}} \frac{e(G)}{v(G)} \geqslant \sup_{m\geqslant 1} \frac{(h-t)m\frac{h+t-3}{2} + \binom{t-1}{2}}{(h-t)m+t-1} = \frac{h+t-3}{2},$$

as required. □

Let $\mathcal{A}$ be a proper minor-closed class of graphs with set $\mathcal{H}$ of excluded minors, such that each graph in $\mathcal{H}$ is t-connected. By the last result with $h = t+1$, we have $\beta_{\mathcal{A}} \geqslant t-1$; and by the last result with $h = t+2$, if K_{t+1} is not in $\mathcal{H}$ then $\beta_{\mathcal{A}} \geqslant t-1/2$. Now consider the well-studied case when $\mathcal{H}$ contains just K_{t+1} (the only t-connected graph on $t+1$ vertices). For $3 \leqslant t \leqslant 8$, it was shown by Dirac [4] ($t = 3$), Wagner [23] ($t = 4$), Mader [12] ($t = 5, 6$), Jørgensen [8] ($t = 7$), and Song and Thomas [20] ($t = 8$), that the largest K_{t+1}-free graphs on n vertices contain $(t-1)n - \binom{t}{2} + \gamma_t$ edges, where $\gamma_t = 0$ for $t \leqslant 6$ and $\gamma_t = 1$ for $t = 7, 8$. Thus $\beta_{\mathrm{Ex}(K_{t+1})} = t-1$ for $t = 3, \dots, 8$. Hence, for $1 \leqslant t \leqslant 8$, we see that if $K_{t+1} \in \mathcal{H}$ then $\beta_{\mathcal{A}} = t-1$, while if $K_{t+1} \notin \mathcal{H}$ then $\beta_{\mathcal{A}} \geqslant t - \frac{1}{2}$. Some examples when the lower bound on $\beta_{\mathcal{A}}$ in Proposition 25 is sharp include:

1. $\mathrm{Ex}(C_4)$ and $\mathrm{Ex}(D)$, where D is the diamond $D = K_4 - e$ (note that $\mathrm{Ex}(C_4) \subseteq \mathrm{Ex}(D)$). In both cases, the lower bound on β from Proposition 25 is $(h+t-3)/2 = (4+2-3)/2 = 3/2$. Both bounds are tight; for the graphs in $\mathrm{Ex}(D)$ are those in which each block is an edge or a cycle, so $\beta_{\mathrm{Ex}(C_4)} = \beta_{\mathrm{Ex}(D)} = 3/2$.

2. A graph G has no $K_{2,3}$ minor if and only if every block of G is outerplanar or a K_4 (see, e.g., Theorem 4.3 in Seymour [19]). Consequently, we have $\beta_{\mathrm{Ex}(K_{2,3})} = 2$, and the lower bound from Proposition 25 is indeed $(h+t-3)/2 = (5+2-3)/2 = 2$.

3. Let W_5 be the wheel graph on 5 vertices (the cycle C_4 with a universal vertex). Then W_5 is 3-connected and $\beta_{\mathrm{Ex}(W_5)} = 5/2$ (see Theorem 2.1 and Lemma 2.2 in Chlebíková [2]). Also, the lower bound on $\beta_{\mathrm{Ex}(W_5)}$ from Proposition 25 is $(h+t-3)/2 = (5+3-3)/2 = 5/2$. Indeed, noting also Proposition 23 (b), we see that the least two values in B_3 are 2 and 5/2.

However, recall that we have $\beta_{\mathrm{Ex}(K_t)} \approx \alpha\, t\sqrt{\log t}$ as $t \to \infty$, where $\alpha \approx 0.319$, as shown by Thomason [21]; thus, when $\mathcal{A}$ is $\mathrm{Ex}(K_t)$ with t large, the lower bound on β in Proposition 25 is far from tight.

The following question asks about a natural extension of the results in Theorem 5 and Proposition 21, where we show that the inclusions $1 + B_1 \subseteq B_2$ and $1 + B_2 \subseteq B_3$ hold.

Question 1. *Do we have $1 + B_t \subseteq B_{t+1}$ for all $t \geqslant 1$?*

This does not appear to be an easy question, and in particular the proof of Proposition 21 does not seem to be directly adaptable for higher values of t: the problem is the lack of the 'only if' part in Proposition 17 for $t \geqslant 3$.

We list some further questions about the possible relations between the sets B_t is Section 6. To make the first step in the direction of answering Question 1, one could start by addressing the following problem.

Question 2. *Does $\min B_t = t - 1$ hold for all $t \geqslant 1$?*

Clearly, by Proposition 25, any family $\mathcal{H}$ of t-connected minors giving $\beta_{\mathrm{Ex}(\mathcal{H})} = t - 1$ must contain K_{t+1}: on the other hand, by Thomason's result, if t is large enough then $\mathcal{H}$ must also contain some other excluded minor.

Let $\mathcal{H}_t$ be the family of all t-connected graphs. Under the relation of taking graph minors, by the Robertson-Seymour theorem, $\mathcal{H}_t$ has a finite set $\mathcal{H}_t^*$ of minimal graphs. (It is immediate that $\mathcal{H}_t^* = \{K_{t+1}\}$ for $t = 1, 2$; Tutte [22] showed that $\mathcal{H}_3^* = \{K_4\}$, and Halin and Jung [7] proved that $\mathcal{H}_4^* = \{K_5, K_{2,2,2}\}$; for $t = 5$ Fijavž [6] conjectured that $\mathcal{H}_5^*$ is a specific set of six graphs, see also Theorem 6.1 and Conjecture G in Kriesell [10].) If $\mathcal{A}_t = \mathrm{Ex}(\mathcal{H}_t^*)$, then clearly $\min B_t = \beta_{\mathcal{A}_t}$. We have already seen that always $\min B_t \geqslant t - 1$, and that $\min B_t = t - 1$ for each $t \leqslant 8$.

Proposition 26. *For each $t \geqslant 9$ we have $\min B_t \in [t - 1, 2t - 2]$.*

Proof. We must prove the upper bound $\min B_t \leqslant 2t-2$. Let G be a graph with $\rho(G) \geqslant 2t-2$; that is, with average degree at least $4t - 4$. Mader proved (see Theorem 1.4.3 in Diestel [3]) that every such graph G has a t-connected subgraph, and thus has a t-connected minor. Hence $\rho(G) < 2t - 2$ for each graph $G \in \mathcal{A}_t$, so $\beta_{\mathcal{A}_t} \leqslant 2t - 2$, and the upper bound follows. □

It seems unlikely that the upper bound in Proposition 26 is anywhere near sharp. Mader conjectured, that $\rho(G) \geqslant 3t/2$ is always enough to force the existence of a t-connected subgraph. Moreover, the conjectured extremal examples are formed by a union of many cliques K_{2t-2}, all sharing a common set of $t-1$ vertices, from which all edges are removed. Hence, even though these graphs contains no t-connected subgraphs, they trivially have t-connected minors.

5 Further observations about the structure of B

In this section we begin the study of the structure of the set B above the value 2. We give a construction and a proposition which allow us to resolve some further questions asked in [5] by Eppstein.

The first of these questions which we address here is: if $\beta \in B'$, must we have $\beta - 1 \in B$? This was repeated as Question 8.7 in the survey article [16]. The question is equivalent to asking if $B' \subseteq 1 + B$, and a positive answer would have implied that all values in B are rational. (This implication was noted in [5]. To see why it holds, observe first that any irrational value in B must be in B', since $B \backslash B' \subseteq A$ by Theorem 1. Thus we could use $B' \subseteq 1 + B$ to show that if β is an irrational value in B then so is $\beta - 1$; and repeating this step we would find an irrational value in $B \cap [0, 1)$ – but by (1) there are no such values.) Recall that Theorem 5 states that $1 + B \subsetneq B_2 \subseteq B'$; and so far we have proved that $1 + B \subseteq B_2 \subseteq B'$, and it remains only to prove that $1 + B \neq B_2$. In this section we complete the proof of Theorem 5, and thus show that the answer to Eppstein's question is negative. We do this by proving that $\frac{25}{11} \in B_2$, while by Remark 11 we know that $\frac{25}{11} - 1 = \frac{14}{11} \notin B$, as we have $\frac{14}{11} \in (\frac{5}{4}, \frac{9}{7})$.

To formulate Eppstein's other question, we need the following definition. We say that $\beta \in B$ is an order-1 cluster point if $\beta \in B'$, and that β is an order-i cluster point if β is a cluster point of order-$(i-1)$ cluster points in B. In [5] it was observed that the order type of B is at least ω^ω since B contains cluster points of all orders. Eppstein asked whether i is the smallest order-i cluster point for all i, which would imply that the order type of B is indeed ω^ω. Again, in what follows we prove that the answer to this question is negative; the construction we present in this section shows that $i - 1/2$ as an order-i cluster point for all $i \geqslant 3$. This does not rule out the possibility that the order type of B is ω^ω. However, even though $5/2$ might well be the smallest order-3 cluster point, we suspect that $i - 1/2$ is not always the smallest order-i cluster point for all $i \geqslant 3$. Thus it seems that B is not as 'well-behaved' as one might hope, and that further claims about its structure may not follow by any 'easy means'.

For the construction, let $k \geqslant 2$ and let P_k be the path on k vertices and $k-1$ edges. Let P_k^+ be the complete one-vertex extension of P_k (known also as the fan graph). We have $v(P_k^+) = k+1$ and $e(P_k^+) = 2k - 1$. For every edge of P_k we take $t \geqslant 1$ disjoint copies of K_4 and identify one edge in every copy with the chosen edge of P_k; we do the same for the two edges connecting the universal vertex to the endpoints of the path. Let us denote the resulting graph by $H_{k,t}$ (the graph $H_{3,2}$ is shown in Figure 3). We have

$$v(H_{k,t}) = k + 1 + (k+1)2t = (k+1)(2t+1)$$

and

$$e(H_{k,t}) = 2k - 1 + 5t(k+1) = (k+1)(5t+2) - 3.$$

Also, let $H_{0,t} = K_4$ for all $t \geqslant 1$ and let $H_{1,t}$ be the union of $2t$ copies of K_4 all sharing one common edge. Observe that the formulae for $v(H_{k,t})$ and $e(H_{k,t})$ hold also for $k = 1$.

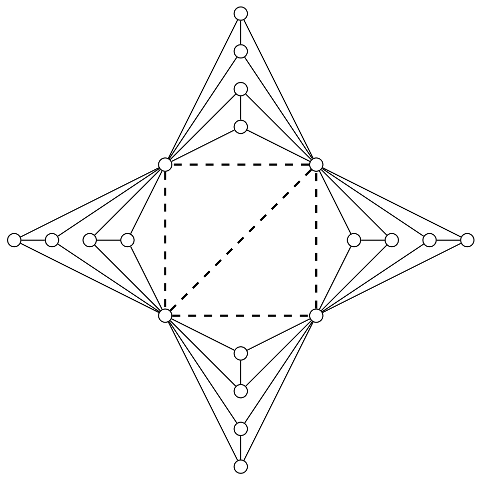

Figure 3: The graph $H_{3,2}$ obtained by placing two copies of K_4 on each of the 'external' edges of the fan P_3^+; the edges of P_3^+ are dashed.

For all $k \geqslant 1$, the 1-density of $H_{k,t}$ satisfies

$$\begin{aligned}\rho_1(H_{k,t}) &= \frac{(5t+2)(k+1)-3}{(k+1)(2t+1)-1} \\ &= \frac{(5t+2)(k+1-\frac{1}{2t+1})-\frac{t+1}{2t+1}}{(k+1)(2t+1)-1} \\ &= \frac{5t+2}{2t+1} - \frac{t+1}{(2t+1)((k+1)(2t+1)-1)} < \frac{5t+2}{2t+1} < \frac{5}{2}.\end{aligned} \tag{12}$$

Observe also that for all $t \geqslant 1$ we have $\rho_1(H_{0,t}) = \rho_1(K_4) = 2 < \rho_1(H_{k,t})$ for all $k \geqslant 1$.

Proposition 27. *Let H be a proper minor of $H_{k,t}$ for some $k \geqslant 0$ and $t \geqslant 1$. Then $\rho_1(H) < \rho_1(H_{k,t})$.*

Proof. We prove the Proposition by induction on k. Since every graph $H_{k,t}$ is connected, it is enough to show the inequality for all 2-connected minors H obtained from $H_{k,t}$ by a series of edge-contractions.

The proposition is immediate for $k = 0$, as the only such 2-connected minor of K_4 is a triangle with $\rho_1(K_3) = 3/2$.

Hence, let $k \geqslant 1$ and let H be a 2-connected minor of $H_{k,t}$ obtained by a series of edge-contractions. If any of the contracted edges is an edge of the original fan P_k^+ then H is also a minor of $H_{k-1,t}$, so by induction and (12) we have

$$\rho_1(H) \leqslant \rho_1(H_{k-1,t}) < \rho_1(H_{k,t}).$$

Otherwise, if H is obtained without contracting any of the edges of the original P_k^+, i.e., if all contractions are inside the copies of K_4 placed on the edges of P_k^+, then we immediately see that the ratio of the number of edges to the number of vertices that are removed is at least $5/2$. Since $\rho_1(H_{k,t}) < \frac{5t+2}{2t+1} < \frac{5}{2}$, this again gives $\rho_1(H) < \rho_1(H_{k,t})$ and completes the proof of the Proposition. $\square$

Let us now return to Eppstein's questions. For the first one, observe that since $\rho_1(H_{3,1}) = \frac{25}{11}$, by Proposition 27 the addable class of graphs with every block being a minor of $H_{3,1}$ has critical density $\frac{25}{11}$. Thus $\frac{25}{11} \in B_2$, and $B_2 \subseteq B'$ by Theorem 5, so we have answered question 1 as planned.

Moreover, let us demonstrate that $\frac{25}{11}$ belongs to the set A of achievable densities. First, take the cycle C_{10} on $\{1, \ldots, 10\}$ and for i odd, connect the vertices $i, i+2$ by an edge (including the edge $\{9, 1\}$). The resulting graph G on 10 vertices and 15 edges is minor-balanced. Now, add a universal vertex, to form G^+ consisting of 11 vertices and 25 edges. By Lemma 15 G^+ is 1-minor-balanced, and hence by case (c) of Lemma 19, it is (strictly) minor-balanced, so $\frac{25}{11} \in A$.

Moving to the question about the value of the smallest order-i cluster point, observe first that for fixed t, the values of $\rho_1(H_{k,t})$ in (12) converge to $\frac{5t+2}{2t+1}$ as $k \to \infty$. Since all the individual values of $\rho_1(H_{k,t})$ are in B', this makes $\frac{5t+2}{2t+1}$ an order-2 cluster point. Clearly, these values converge to $5/2$ as $t \to \infty$, making $5/2$ an order-3 cluster point. The statement about cluster points of higher orders then follows from the fact that $1 + B \subseteq B'$ (in Theorem 5).

6 Concluding remarks and open problems

We first discuss briefly the one remaining question from those listed as open by Eppstein [5] (the others have already been discussed), and then propose four new open questions. The question from Eppstein is whether each $\beta \in B$ is achievable (that is, if $B = A$) – which of course would imply that each β is rational. This question remains open and interesting. However, it is easy to see that if we allow infinite graphs in a natural way, then each $\beta \in B$ is achievable (though this does not for example tell us that β must be rational). To discuss this, we need some definitions. For an infinite graph G define its density $\rho(G)$ to be the supremum of the densities of its finite subgraphs. Define $\beta_{\mathcal{A}}$ for an arbitrary class $\mathcal{A}$ of graphs as the supremum of the densities of the graphs in $\mathcal{A}$. Given a class $\mathcal{A}$ of finite graphs, let $\mathcal{A}^+$ be the class of countable graphs such that each finite subgraph is in $\mathcal{A}$ and each component is finite. If $\mathcal{A}$ is decomposable, then the graphs in $\mathcal{A}^+$ are exactly the countable disjoint unions of (connected) graphs in $\mathcal{A}$.

Now let us see that each $\beta \in B$ is achievable when we allow infinite graphs. Recall that $B = B_1$, so there is a decomposable minor-closed class $\mathcal{A}$ of graphs such that $\beta = \beta_{\mathcal{A}}$. Let $G_1, G_2, \ldots$ be a sequence of disjoint connected graphs in $\mathcal{A}$ such that $\rho(G_k) \to \beta_{\mathcal{A}}$ as $k \to \infty$. Let G be the countable graph with components the graphs G_k. Then $G \in \mathcal{A}^+$ and $\rho(G) = \beta$.

Now we propose four new open questions, concerning densities just above 2, the limit points in B, and how quickly the gaps between consecutive critical densities vanish. It is not currently clear to us what the structure of the set B above 2 might be. In Remark 11 we listed the first few values in B above 1: the natural first question to ask here is the following.

Question 3. *What are the minimum values larger than* 2 *in* B *and in* B_2*?*

(Recall that these minimum values exist by Theorem 1.) We conjecture that the answer to Question 3 is

$$\min(B \cap (2, \infty)) = \tfrac{33}{16}, \text{ and } \min(B_2 \cap (2, \infty)) = \tfrac{11}{5}.$$

In the following proposition we show that these values are indeed upper bounds on the minima. (For $\min(B \cap (2, \infty)) \leqslant \frac{33}{16}$ consider the case $k = 3$ in the first statement of the proposition.)

Proposition 28. *For all $k \geqslant 1$ any graph obtained from k copies of the graph G_1 in Figure 4 by identifying vertex v in each copy into one vertex is strictly minor-balanced. Consequently, the following two statements hold.*

1. *For all $k \geqslant 1$ we have $2 + \frac{k-2}{5k+1} \in B$.*
2. *$\frac{11}{5} \in B_2$.*

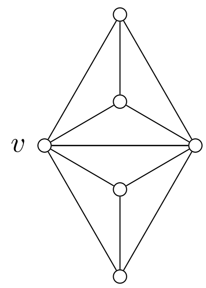

Figure 4: Graph G_1 on 6 vertices and 11 edges referred to in Proposition 28.

Proof. Let H_k, $k \geqslant 1$, be obtained from k copies of the graph G_1 by identifying vertex v in each copy into one vertex. Clearly we have

$$\rho(H_k) = \frac{e(H_k)}{v(H_k)} = \frac{11k}{5k+1} = 2 + \frac{k-2}{5k+1} < \frac{11}{5}.$$

Observe that for all $k \geqslant 1$, graph H_k is obtained from k disjoint copies of a bowtie graph (two triangles sharing one common vertex) by adding a universal vertex. Since k disjoint copies of a bowtie graph give a minor-balanced graph, by Lemma 15 and by case (c) of Lemma 19 we have that H_k is strictly minor-balanced.

Therefore the families $\mathcal{A}_k$ of graphs with every component being a minor of H_k are minor-closed with $\beta_{\mathcal{A}_k} = 2 + \frac{k-2}{5k+1} \in B$. The addable family of graphs with every block being a minor of G_1 has critical density 11/5. This completes the proof of the Proposition. □

By Theorem 5 we know that $1+B \subseteq B_2 \subseteq B'$, these three sets are equal on $[0, 2]$, and $1+B \neq B_2$. But what about B_2 and B'?

Question 4. *Do we have $B_2 = B'$?*

Since $B = B_1$, we may rephrase the above as noting that $B_2 \subseteq B_1'$, and asking if $B_2 = B_1'$. Perhaps $B_{t+1} \subseteq B_t'$ for each $t \geqslant 1$, but we do not have equality for $t = 2$. Let us show that $7/3 \in B_2' \setminus B_3$.

As we saw in Section 5, see (12), the addable class of graphs with every block being a minor of the graph $H_{k,1}$ has critical density $\frac{7}{3} - \frac{2}{3(3k+2)} \in B_2$. Since these values converge to 7/3, we have $7/3 \in B_2'$. But the smallest two values in B_3 are 2 and 5/2 (see note 3 following Proposition 25), so $7/3 \notin B_3$. Hence $7/3 \in B_2' \setminus B_3$, as desired.

Next, let us come back to Theorem 2, where we prove that $\delta_B(x) = O(x^{-2})$ as $x \to \infty$.

Question 5. *How fast does $\delta_B(x)$ approach 0 as $x \to \infty$? Can we improve on $O(x^{-2})$? Is the convergence polynomial in x^{-1}?*

Finally, one could ask if there is a pattern to the values in B that are the most difficult to "beat" by just a tiny margin. For example, are the integer densities followed by the largest gaps in B? More formally, given $n \in \mathbb{N}$, let

$$\delta_B^*(n) = \sup_{x \geqslant n} \delta_B(x).$$

Question 6. *Clearly for all n we have $\delta_B^*(n) \geqslant \delta_B(n)$, but is $\delta_B^*(n) = O(\delta_B(n))$? (See also the third question in Section 10 of [5].)*

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