A dichotomy result for locally compact sober dcpos

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Xiaodong Jia, Achim Jung[*] and Qingguo Li[†]

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Abstract

The second author proved in [7] that each cartesian closed category of pointed domains and Scott-continuous functions is contained in either the category of Lawson-compact domains or that of L-domains, and this result eventually led to a classification of continuous domains with respect to cartesian closedness, as laid out in [8].

In this paper, we generalise this result to the category **LcS** of pointed locally compact sober dcpos and Scott-continuous functions, and show that any cartesian closed full subcategory of **LcS** is contained in either the category of stably compact dcpos or that of L-dcpos. (Note that for domains Lawson-compactness and stable compactness are equivalent.) As we will show, this entails that any candidate for solving the Jung-Tix problem in **LcS** must be stably compact.

To prove our dichotomy result, we first show that any dcpo with a core-compact function space must be meet-continuous; then we prove that a function space in **LcS** is meet-continuous only if either its input dcpo is coherent or its output dcpo has complete principal ideals.

1 Introduction

Domain theory was initially introduced by Dana Scott in the late sixties for the purpose of modelling functional programming languages. In Scott's framework, programming languages are modelled by categories of (continuous) domains, and due to the syntactical structure of functional programming languages, these categories are required to be *cartesian closed*. For this very purpose, different domain structures have been proposed and investigated. Examples are SFP-domains, Scott-domains, bi-finite domains, RB-domains, et cetera, [2, 12, 10]. Moreover, a systematic study of cartesian closed subcategories of continuous domains was carried out by the second author, [7, 8]. To wit, one has two

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maximal cartesian closed subcategories of pointed domains, FS-domains and L-domains. The key ingredient of his proof is a lemma which shows that in the category of pointed continuous domains a function space is continuous only if the input domain is Lawson-compact or the output dcpo is an L-domain; then he showed that the categories of L-domains and FS-domains are cartesian closed and a dcpo with a pointed Lawson-compact continuous function space itself is an FS-domain.

In this paper, we will see that a similar but generalised investigation can be carried out in the category **LcS** of pointed locally compact sober dcpos and Scott-continuous functions. More precisely, we show that every cartesian closed full subcategory of **LcS** is included in either the category of stably compact dcpos or that of L-dcpos. To prove this, first of all we derive *meet-continuity* and *bi-completeness* from the core-compactness of function spaces, a result that illustrates deep interplay between topology and order-theoretic properties. Then we give characterisations of *coherence* and *L-dcpos*, respectively. By using these new characterisations we show that a function space in **LcS** is meet-continuous only if its input dcpo is coherent or the output dcpo has complete principal ideals. Since the objects of **LcS** are all compact, locally compact and sober, coherence implies stable compactness.

2 Preliminaries

We refer to [1, 2] for the standard definitions and notations of order theory and domain theory, and to [4] for topology.

For a dcpo L, the Scott topology $\sigma(L)$ on L is given by all the upper subsets of L that are inaccessible by joins of directed subsets of L. We take *coherence* of a topological space to mean that the intersection of any two compact saturated subsets is compact. A *stably compact* space is a topological space which is compact, locally compact, sober and coherent. We call a dcpo L stably compact (respectively, compact, sober, coherent, locally compact) if L with its Scott topology $\sigma(L)$ is a stably compact (respectively, compact, sober, coherent, locally compact) space. Without further reference, we always equip L with the Scott topology $\sigma(L)$. Finally, a dcpo L is said to be *core-compact* if its Scott topology $\sigma(L)$ is a continuous lattice in the inclusion order.

A function $f\colon L \to M$ between dcpos L and M is called Scott-continuous if it preserves sups of directed subsets. It is known that such an f is Scott-continuous if and only if it is continuous in the topological sense when L and M are equipped with the Scott topology. We use $[L \to M]$ to denote the set of all Scott-continuous functions from L to M. Given a compact set $K \subseteq L$ and a Scott open set $U \subseteq L$, $N(K, U)$ denotes the set of all Scott-continuous functions that send K into U. A Scott-continuous map $f : L \to M$ is called a *retraction* if there exists a Scott-continuous function g from M to L such that $f \circ g = \mathrm{id}_M$.

The category of stably compact dcpos and Scott-continuous functions will be denoted by **SCS**, while that of pointed locally compact sober dcpos and

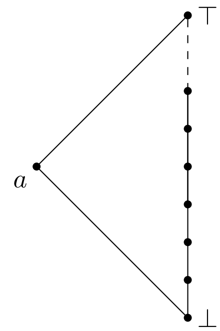

Figure 1: $M(\mathbb{N})_\perp$

Scott-continuous functions is denoted by **LcS**.

Finally, we use standard notations for ordinal numbers. In this paper, we will make use of both upward and downward well-ordered chains. For an upward well-ordered chain C and $c \in C$, we use $c+1$ to denote the least element of the set $\{x \in C \mid c < x\}$. Note that such elements are always compact in the sense of domain theory. Thus, C is an algebraic domain if a largest element is present. On the other hand, every downward well-ordered chain is always an algebraic domain, because it is a dcpo and every element is compact. Unless we say otherwise, well-ordered chains are meant to be upward.

3 Core-compactness entails meet-continuity

Following Kou [11], a dcpo L is called *meet-continuous* if for any $x \in L$ and directed subset D of L with $x \leq \sup D$, x is in $\overline{\downarrow x \cap \downarrow D}$, the Scott closure of $\downarrow x \cap \downarrow D$. It is easy to see that meet-continuity is preserved by Scott-continuous retractions.

We show in this section that for any dcpo L the core-compactness of its function space $[L \to L]$ entails that L is meet-continuous. To prove this, we first recall the following theorem from [5] which characterises meet-continuity via "forbidden substructures" $\mathcal{M}(C)$ and $\mathcal{M}(C)_\perp$.

Definition 3.1. *For every well-ordered chain C without a top element, we define the poset $\mathcal{M}(C) = C \cup \{\top, a\}$, where a and $\top$ are not in C and the order on $\mathcal{M}(C)$ is: $x \leq y$ iff $x = y = a$ or $y = \top$ or $x, y \in C$, $x \leq y$ in C. Define $\mathcal{M}(C)_\perp$ to be the lifting of $\mathcal{M}(C)$ by adding a least element $\perp$. Figure 1 shows $M(\mathbb{N})_\perp$ (where $\mathbb{N}$ is the ordered chain of natural numbers).*

Theorem 3.2. [5] *Let L be a dcpo which is not meet-continuous. Then L has some $\mathcal{M}(C)$ or $\mathcal{M}(C)_\perp$ as a Scott-continuous retract, where C is a well-ordered chain without a top element.*

The following lemma is useful in identifying core-compactness and local compactness.

Lemma 3.3. *Let L be a join-complete poset, that is, the supremum of any non-empty subset of L exists. If L is core-compact, then L is sober, hence locally compact.*

Proof. The proof is the same as that of [2, Corollary II-4.16] by noticing that a least element is not actually required in the argument. □

Although the following theorem is only needed for locally compact dcpos, we give the proof for a more general situation. Actually, it was shown in [5] that for any dcpo L quasicontinuity of the function space $[L \to L]$ implies that L must be meet-continuous. We will see in the following theorem that core-compactness of $[L \to L]$ suffices to guarantee meet-continuity of L. Since every quasicontinuous domain is locally compact, hence core-compact, the following result is a substantial generalisation of [5, Theorem 4.2]. Although the arguments are very much alike, in order to keep this paper self-contained, we spell out the proof in full.

Theorem 3.4. *Given a dcpo L, if the function space $[L \to L]$ is core-compact, then L must be meet-continuous.*

Proof. Assume that L is not meet-continuous, then by Theorem 3.2, L has some $\mathcal{M}(C)$ or $\mathcal{M}(C)_\perp$ as a Scott-continuous retract, where C is a well-ordered chain without a top element. So $[\mathcal{M}(C) \to \mathcal{M}(C)]$ or $[\mathcal{M}(C)_\perp \to \mathcal{M}(C)_\perp]$ is a Scott-continuous retract of $[L \to L]$ (see for example [7, Proposition 1.22]). Since core-compactness is preserved by Scott-continuous retractions, we reach a contradiction by showing that neither $[\mathcal{M}(C) \to \mathcal{M}(C)]$ nor $[\mathcal{M}(C)_\perp \to \mathcal{M}(C)_\perp]$ is core-compact.

To this end, let C be a well-ordered chain without a top element and c_0 its bottom element. We begin with $D_\perp := [\mathcal{M}(C)_\perp \to \mathcal{M}(C)_\perp]$ and assume for the sake of a contradiction that it is core-compact. First, one easily sees that $D_\perp$ is a complete lattice. It then follows from [2, Corollary II-4.16] that $D_\perp$ is sober, hence locally compact by [2, Theorem V-5.6]. Consider the function $a \searrow \perp$ that maps the element a to $\perp$ and keeps everything else fixed. It is clearly strictly less than the identity on $\mathcal{M}(C)_\perp$. By local compactness this implies that we should have a compact saturated neighbourhood K in $D_\perp$ such that $\mathrm{id}_{\mathcal{M}(C)_\perp}$ is in the interior of K and $a \searrow \perp \notin K$. Let $K' := \{f \in K \mid f \leq \mathrm{id}_{\mathcal{M}(C)_\perp}\}$. Clearly, K' is not empty (since $\mathrm{id}_{M(C)_\perp} \in K$), and for each $f \in K'$ we must have $f(a) = a$ as otherwise we would have $f \leq a \searrow \perp$ and $a \searrow \perp \in K$. Now $\top$ can only be mapped to a or to itself by such an f. In the former case, some $c \in C$ would also have to be mapped to a to ensure continuity but this would violate the condition $f \leq \mathrm{id}_{M(C)_\perp}$; so $f(\top) = \top$ is the only possibility that remains. In other words, each such f continuously maps the infinite well-ordered chain $C \cup \{\perp, \top\}$ into itself, keeping both $\perp$ and $\top$ fixed.

Note that K' is compact since it is the intersection of the compact set K and the closed set $\downarrow id_{\mathcal{M}(C)_\perp}$. We now show that K' does not "isolate" $\mathrm{id}_{\mathcal{M}(C)_\perp}$

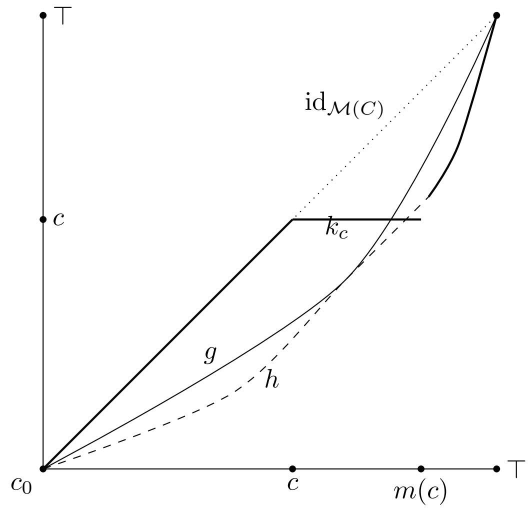

Figure 2: The situation in the proof of Theorem 3.4

against directed suprema from below. (For the argument that follows it may be useful to keep Figure 2 in mind.)

Consider the function $g\colon \mathcal{M}(C) \to \mathcal{M}(C)$ defined by $g(x) = \min\{f(x) \mid f \in K'\}$. As argued above it is easy to see that g is well-defined and monotone. We now show that it is in fact Scott-continuous. Note that a is fixed by g. We proceed by showing that g also continuously maps $C \cup \{\bot, \top\}$ into $C \cup \{\bot, \top\}$. To this end, let $x_0 \in C \cup \{\bot, \top\}$ and choose a basic Scott-open neighbourhood of $g(x_0)$ of the form $\uparrow c$, where $c \in C \cup \{\bot\}$. For every $f \in K'$, there is a neighbourhood U_f of x_0 and a neighbourhood V_f of f such that $f(x) \in \uparrow c$ for all $(f, x) \in V_f \times U_f$. This is because $\mathcal{M}(C)_\bot$ is core-compact and by [2, Theorem II-4.10] the evaluation mapping eval $(f, x) \mapsto f(x)\colon [\mathcal{M}(C)_\bot \to \mathcal{M}(C)_\bot] \times \mathcal{M}(C)_\bot \to \mathcal{M}(C)_\bot$ is continuous [1]. By compactness of K', a finite number of the V_f are covering K'. Let U be the intersection of the corresponding (finitely many) U_f. Then U is a neighbourhood of x_0 such that $f(x) \in \uparrow c$ for all $f \in K'$ and all $x \in U$. Hence, $g(x) \in \uparrow c$ for all $x \in U$. So we have proved that g is Scott-continuous.

Now we present a directed set of functions with supremum $\mathrm{id}_{\mathcal{M}(C)_\bot}$ but none of them is in K. This will be a contradiction to the assumption that K is a Scott-neighbourhood of $\mathrm{id}_{\mathcal{M}(C)_\bot}$. To this end, consider the Scott-continuous function $h\colon \mathcal{M}(C) \to \mathcal{M}(C)$ defined on a and the compact elements of $C \cup \{\top\}$

[1]Note that the Scott topology is finer than the Isbell topology on $[\mathcal{M}(C)_\bot \to \mathcal{M}(C)_\bot]$.

by

$$h(x) = \begin{cases} \bot, & x = \bot; \\ a, & x = a; \\ c_0, & x = c_0; \\ g(c), & x = c + 1. \end{cases}$$

It follows that g and h agree for limit ordinals and $h(\top) = g(\top) = \top$, but there are also many inputs where h is strictly less than g; more precisely, for any $e \in C$, there exists a $d \in C, d \geq e$ such that $h(d+1) < g(d+1)$. Indeed, suppose there exists some $e \in C$ such that $h(d+1) = g(d+1)$ for all $d \geq e$. Because $h(d+1) = g(d)$, it then follows that $g(d) = g(d+1)$ when $d \geq e$. Using transfinite induction and the fact that g is Scott-continuous, we get that $g(x) = g(y)$ for all $x, y \geq e$. In particular, we obtain $g(e) = g(\top) = \top$. However, g is below $\mathrm{id}_{\mathcal{M}(C)_\bot}$ and this implies $\top = g(e) \leq e$, which is not possible since C does not have a top element.

From $h(\top) = \top$ and Scott-continuity we get that for any $c \in C$, there exists $m > c$ such that $h(m) > c$. Define $m(c)$ to be the least element of $\{m \in C \mid h(m) > c\}$. We use this to define a family $\mathcal{K}$ of functions $k_c \colon \mathcal{M}(C)_\bot \to \mathcal{M}(C)_\bot$ indexed by the elements of C and defined by

$$k_c(x) = \begin{cases} x, & x \leq c; \\ c, & c < x \leq m(c); \\ h(x), & \text{otherwise.} \end{cases}$$

It is clear that each k_c is Scott-continuous as it is pieced together from Scott-continuous functions on Scott-closed subsets. It is also clear that the supremum of $\mathcal{K}$ is the identity on $\mathcal{M}(C)_\bot$, but unfortunately, $\mathcal{K}$ may not be directed. This is only a small hindrance, however, because $D_\bot$ is complete and we can enrich $\mathcal{K}$ with all finite suprema. Notice that for any non-empty finite subset $F \subseteq_{\text{fin}} C$, the supremum $\sup_{c \in F} k_c$ is equal to h on $\uparrow\max\{m(c) + 1 \mid c \in F\}$, hence from the last paragraph we know that $\sup_{c \in F} k_c$ cannot be greater than g. This, then, yields a directed set with supremum $\mathrm{id}_{\mathcal{M}(C)_\bot}$ no member of which is above g and therefore not above an element of K'. Since all of this takes place in $\downarrow\mathrm{id}_{\mathcal{M}(C)_\bot}$, none of them exceeds any of the other members of K either. Thus we have given a counterexample to the claim that K is a Scott neighbourhood of $id_{\mathcal{M}(C)_\bot}$ and this contradiction shows that the assumption that the function space $D_\bot$ is core-compact must have been wrong.

The argument for $D := [M(C) \to M(C)]$ is similar but easier because any order-preserving function below $\mathrm{id}_{M(C)}$ must map a to a and $\top$ to $\top$. Since D is join-complete, Lemma 3.3 suffices to bridge the gap between core-compactness and local compactness in this case. $\square$

4 Core-compactness entails bi-completeness

We call a dcpo L *bi-complete* if every filtered subset of L has an infimum, i.e., its dual L^{op} is also a dcpo. The second author proved in [7] that every dcpo that has

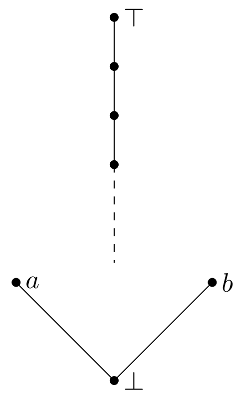

Figure 3: $\mathcal{K}(\mathbb{N}^{op})_\perp$

a continuous function space must be bi-complete. In this section, we generalise this result to core-compact dcpos. First, we give some general constructions of non bi-complete dcpos.

Definition 4.1. *For every downward well-ordered chain C without a bottom element, we define the poset $\mathcal{K}(C) = C \cup \{a, b\}$, where a and b are not in C and the order on $\mathcal{K}(C)$ is: $x \leq y$ iff $x = y = a$; $x = y = b$; $x \in \{a, b\}, y \in C$; or $x, y \in C$, $x \leq y$ in C. Define $\mathcal{K}(C)_\perp$ to be the lifting of $\mathcal{K}(C)$ by adding a least element $\perp$. Figure 3 shows $\mathcal{K}(\mathbb{N}^{op})_\perp$ (where $\mathbb{N}$ is the ordered chain of natural numbers).*

We now give characterisations of bi-completeness via these concrete order structures as defined above.

Lemma 4.2. *Let L be a sober dcpo. If every minimal element (if they exist) in L is compact, then the following statements are equivalent:*

1. *L is not bi-complete;*
2. *L has some C, $\mathcal{K}(C)$ or $\mathcal{K}(C)_\perp$ as a Scott-continuous retract, where C is a downward well-ordered chain without a bottom element.*

Proof. The interesting part is that 1 implies 2. Assume that L is not bi-complete, then we can find some chain C in L such that C does not have an infimum in L. Let A be the set of lower bounds of C in L. Then $A = \bigcap_{x \in C} \downarrow x$ is obviously a Scott-closed subset of L. Moreover, from the proof of [7, Theorem 1.37], this C can be chosen in such a way that it is downward well-ordered, and to satisfy that $C \cup A$ (with the induced order from L) is a Scott-continuous retract of L under the retraction map:

$$r(x) = \begin{cases} x, & x \in A; \\ \bigwedge\{c \in C \mid x \leq c\}, & \text{otherwise.} \end{cases}$$

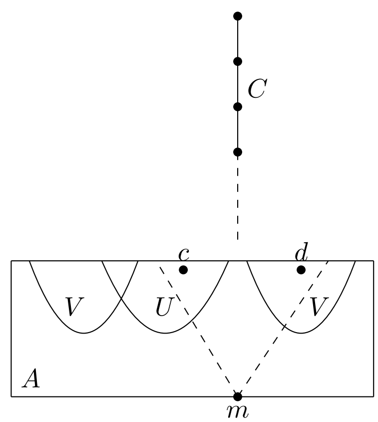

Figure 4: The situation in the proof of Lemma 4.2

We now distinguish two cases:

Case 1: A is empty. Then clearly C is a Scott-continuous retract of L.

Case 2: A is not empty. In this case we can assume that every element of A is above some minimal element in A since otherwise, we can find some descending chain in A without any lower bounds. Since every chain has a well-ordered cofinal subset, this enables us to find in A a downward well-ordered chain without any lower bounds as well, and this will lead us to Case 1.
Since A, the set of lower bounds of C, is a Scott-closed non-empty subset and C does not have an infimum, this means that A has at least two maximal elements, say a and b. We further consider two subcases:

Subcase 2.1: Every minimal element of A is below exactly one maximal element in A.
In this subcase, we define a function g on $C \cup A$ as:

$$g(x) = \begin{cases} x, & x \in C; \\ a, & x \in {\downarrow} a; \\ b, & \text{otherwise.} \end{cases}$$

It is easy to check that g is a Scott-continuous retraction on $C \cup A$ with image $\{a, b\} \cup C$, which is isomorphic to $\mathcal{K}(C)$. Then $g \circ r$ is the wanted Scott-continuous retraction and $\{a, b\} \cup C$ is a retract of L.

Subcase 2.2: There exists some minimal element $m \in A$ such that more than one maximal element of A is above it.

Now we consider set $\overline{{\uparrow} m \cap A}$, the Scott closure of ${\uparrow} m \cap A$. $\overline{{\uparrow} m \cap A}$ is Scott closed in L and has more than one maximal element, so it is not irreducible in the sober dcpo L. This implies that we have two Scott open subsets U, V of L such that they intersect with $\overline{{\uparrow} m \cap A}$ respectively, but $U \cap V \cap \overline{{\uparrow} m \cap A} = \emptyset$ [2]. Since

[2]Note that U and V may intersect in A.

U, V are Scott open and they intersect with $\overline{\uparrow m \cap A}$, then they also intersect with $\uparrow m \cap A$. Fix some points $c \in U \cap \uparrow m \cap A$ and $d \in V \cap \uparrow m \cap A$ (see Figure 4).

Now we can see that $\{C, c, d, m\}$ is a copy of $\mathcal{K}(C)_\perp$ inside L. Moreover, we show it is a Scott-continuous retract of $C \cup A$. Indeed, consider the function h defined on $C \cap A$ as follows:

$$h(x) = \begin{cases} x, & x \in C; \\ c, & x \in U \cap \uparrow m \cap A; \\ d, & x \in V \cap \uparrow m \cap A; \\ m, & \text{otherwise.} \end{cases}$$

Since in our assumption, the minimal element m is compact, $\uparrow m$ is Scott open. Now to check that h is a Scott-continuous retraction is just routine, and in this subcase, L has $\mathcal{K}(C)_\perp$ as a Scott-continuous retract witnessed by $h \circ r$. □

The following corollaries are straightforward consequences of the previous lemma.

Corollary 4.3. *Let L be a meet-continuous sober dcpo. If L is not bi-complete, then L has C, $\mathcal{K}(C)$ or $\mathcal{K}(C)_\perp$ as a Scott-continuous retract, where C is a downward well-ordered chain without a bottom element.*

Proof. Note that in a meet-continuous dcpo, every minimal element (if they exist) is compact. Then the statement follows from the previous lemma. □

Corollary 4.4. *Let L be a pointed sober dcpo. If L is not bi-complete, then L has $\mathcal{K}(C)_\perp$ as a Scott-continuous retract, where C is a downward well-ordered chain without a bottom element.* □

Proposition 4.5. *For any downward well-ordered chain C without a bottom element, none of the function spaces $[C \to C]$, $[\mathcal{K}(C) \to \mathcal{K}(C)]$ or $[\mathcal{K}(C)_\perp \to \mathcal{K}(C)_\perp]$ is core-compact.*

Proof. We first show that $[C \to C]$ is not core-compact. Since this function space is join-complete, by Lemma 3.3, we only need to show that it is not locally compact. More precisely, we prove that the identity map id_C does not have any compact neighbourhoods. By way of contradiction, suppose that W is a compact neighbourhood of id_C. Then for each $x \in C$, the set $\{g(x) \mid g \in W\}$ is compact since the evaluation function $\mathrm{eval}\colon [C \to C] \times C \to C$ is continuous and $\{g(x) \mid g \in W\}$ is the continuous image of the compact set $W \times \{x\}$ under eval. Moreover, $\{g(x) \mid g \in W\}$ has a least element since it is compact and C is a chain.

Consider the function $f\colon C \to C$ defined by $f(x) = \min\{g(x) \mid g \in W\}$. As argued above, f is well-defined. Obviously, f is monotone, and Scott-continuous since every element in C is compact. Since W is a Scott neighbourhood of id_C and $W \subseteq \uparrow f$, we have $f \ll \mathrm{id}_C$.

We proceed by showing that f cannot be way-below id_C. Consider the successor function τ on C, defined by $\tau(c) = c + 1$. Remember that C is downward well-ordered, so $c+1 < c$. The functions

$$g_c(x) = \begin{cases} \tau \circ f(x), & x \leq c; \\ x, & \text{otherwise.} \end{cases}$$

approximate id_C but none of them dominates f.

This contradiction shows that W is not a Scott neighbourhood of id_C. So $[C \to C]$ is not core-compact.

Note that all of the above also holds in $\downarrow \mathrm{id}_C$, so $\downarrow \mathrm{id}_C$ as a dcpo is not core-compact. Hence $[\mathcal{K}(C) \to \mathcal{K}(C)]$ is not core-compact since its Scott-continuous retract $\downarrow \mathrm{id}_{\mathcal{K}(C)}$, which is isomorphic to $\downarrow \mathrm{id}_C$, is not core-compact.

Finally, we prove $[\mathcal{K}(C)_\perp \to \mathcal{K}(C)_\perp]$ is not core-compact by showing that its principal ideal $\downarrow \mathrm{id}_{\mathcal{K}(C)_\perp}$ is not core-compact. To this end, consider the set $A := \{f \in \downarrow \mathrm{id}_{\mathcal{K}(C)_\perp} \mid f(a) = a \;\&\; f(b) = b\}$. One easily sees that A is Scott open in $\downarrow \mathrm{id}_{\mathcal{K}(C)_\perp}$ and A is isomorphic to $\downarrow \mathrm{id}_C$. So A is not core-compact. Hence $\downarrow \mathrm{id}_{\mathcal{K}(C)_\perp}$ is not core-compact, since in a core-compact dcpo every Scott open set is a core-compact dcpo in the induced order. □

We arrive at our main result in this section.

Theorem 4.6. *Let L be a sober dcpo with a core-compact function space $[L \to L]$. Then L is bi-complete.*

Proof. Suppose that L is not bi-complete. Since $[L \to L]$ is core-compact, L must be meet-continuous from Lemma 3.4. By Corollary 4.3, L has C, $\mathcal{K}(C)$ or $\mathcal{K}(C)_\perp$ as a Scott-continuous retract, where C is some downward well-ordered chain without a bottom element. Hence either $[C \to C]$, $[\mathcal{K}(C) \to \mathcal{K}(C)]$ or $[\mathcal{K}(C)_\perp \to \mathcal{K}(C)_\perp]$ is a Scott continuous retract of $[L \to L]$. This implies that one of these function spaces must be core-compact. However, this cannot be true as we see in Proposition 4.5 that none of them is core-compact. □

5 L-dcpos

In [7], the category of L-domains was introduced as one of the maximal cartesian closed subcategories of pointed domains. A domain is called an L-domain if every principal ideal is a complete lattice in the induced order. In general, this notion can be defined for arbitrary dcpos.

Definition 5.1. *A dcpo L is called an L-dcpo if every principal ideal $\downarrow x, x \in L$, is a complete lattice in the induced order. The dcpo $\mathbf{X}_\perp^\top$ in Figure 5 is a typical non L-dcpo.*

Theorem 5.2. *Let L be a pointed sober dcpo which is bi-complete. If L is not an L-dcpo, then L has $\mathbf{X}_\perp^\top$ (defined in Figure 5) as a Scott-continuous retract.*

Proof. Let L be a bi-complete pointed sober dcpo with a least element $\perp$. If L is not an L-dcpo, then we have some $e \in L$ such that $\downarrow e$ is not complete. Since

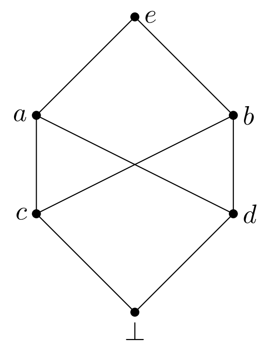

Figure 5: $\mathbf{X}_{\perp}^{\top}$

L is bi-complete, there exist elements $a, b \in {\downarrow}e$ such that a, b have no infimum in ${\downarrow}e$. Consider the closed set ${\downarrow}a \cap {\downarrow}b$. It is not empty since $\perp \in {\downarrow}a \cap {\downarrow}b$, then it has at least two maximal elements. The sobriety of L now tells us that ${\downarrow}a \cap {\downarrow}b$ is not irreducible, so there exist two Scott open sets U and V intersecting ${\downarrow}a \cap {\downarrow}b$, respectively, with $U \cap V \cap {\downarrow}a \cap {\downarrow}b = \emptyset$. Choose some element c in $U \cap {\downarrow}a \cap {\downarrow}b$ and some d in $V \cap {\downarrow}a \cap {\downarrow}b$, respectively. We define a function $r \colon L \to L$ as follows:

$$r(x) = \begin{cases} e, & x \notin {\downarrow}a \cup {\downarrow}b; \\ a, & x \in {\downarrow}a \setminus {\downarrow}b; \\ b, & x \in {\downarrow}b \setminus {\downarrow}a; \\ c, & x \in U \cap {\downarrow}a \cap {\downarrow}b; \\ d, & x \in V \cap {\downarrow}a \cap {\downarrow}b; \\ \perp, & \text{otherwise.} \end{cases}$$

It is clear that r is a Scott-continuous retraction on L with image $\{a, b, c, d, e, \perp\}$ which is a copy of $\mathbf{X}_{\perp}^{\top}$ inside L. □

6 Function spaces

Before we reach the main result of this paper, let us recall that in a topological space, given subsets A and B with $A \subseteq B$, A is said to be *relatively compact* in B if every open cover of B admits a finite subcover of A.

For locally compact sober dcpos, we have the following characterisation lemma for coherence.

Lemma 6.1. *Let L be a locally compact sober dcpo. Then L is coherent if and only if for any compact saturated subsets A, B and Scott open sets U, V with $A \subseteq U, B \subseteq V$, $A \cap B$ is relatively compact in $U \cap V$.*

Proof. "⇒": This is obvious since coherence implies that $A \cap B$ is compact.

"⇐": For compact saturated subsets A and B, consider the filter $\mathcal{F}$ of Scott open sets generated by the filter basis $\{U \cap V \mid A \subseteq U \ \&\ B \subseteq V,\ U, V \in \sigma(L)\}$.

We claim that $\mathcal{F}$ is a Scott open filter of $\sigma(L)$. Indeed, let $\{U_i \mid i \in I\}$ be a directed family of Scott open sets of L and $\bigcup_{i\in I} U_i \in \mathcal{F}$. This means that we have some open sets U, V with $A \subseteq U, B \subseteq V$, and $U \cap V \subseteq \bigcup_{i\in I} U_i$. Since L is locally compact, one has compact saturated subsets K, W and Scott open sets U', V' such that $A \subseteq U' \subseteq K \subseteq U$ and $B \subseteq V' \subseteq W \subseteq V$. From the assumption, $K \cap W$ is relatively compact in $U \cap V$, so we have some $i \in I$ such that $K \cap W \subseteq U_i$. Hence $U' \cap V' \subseteq U_i$, which implies that $\mathcal{F}$ is a Scott open filter of opens. Now the sobriety of L and the Hofmann-Mislove Theorem tell us that the intersection of $\mathcal{F}$, which equals $A \cap B$, is compact. □

In continuous domains, property M is introduced as an equivalent property with Lawson-compactness (see for example [2, Corollary III-5.13]). A continuous domain L is said to satisfy property M if for any $x_1, y_1, x_2, y_2 \in L$ with $y_1 \ll x_1$ and $y_2 \ll x_2$, there exists a finite set $F \subseteq L$ such that $\uparrow x_1 \cap \uparrow x_2 \subseteq \uparrow F \subseteq \uparrow y_1 \cap \uparrow y_2$. Such a property is useful in proving certain domains are Lawson-compact, for example, FS-domains, bi-finite domains, and also useful in constructing functions on domains from an element-level. The following observation is obvious from the previous lemma; we record it here as a rephrasing of property M.

Proposition 6.2. *A domain L satisfies property M if and only if for any $x, y \in L$ and any Scott open sets U, V with $x \in U, y \in V$, $\uparrow x \cap \uparrow y$ is relatively compact in $U \cap V$.* □

We now come to our classification theorem which is a generalisation of [7, Lemma 4.23].

Theorem 6.3. *Let D be a locally compact sober dcpo and E a pointed bicomplete sober dcpo. If D is not coherent and E is not an L-dcpo, then the function space $[D \to E]$ is not meet-continuous.*

Proof. Assume that $[D \to E]$ is meet-continuous although neither E is an L-dcpo nor D is coherent. From Theorem 5.2 we know that $[D \to \mathbf{X}_\perp^\top]$ (see Figure 5 for $\mathbf{X}_\perp^\top$) is also meet-continuous since it is a Scott-continuous retract of $[D \to E]$.

Since D is not coherent, Lemma 6.1 implies that there are compact saturated subsets A, B and Scott open sets U, V of D such that $A \subseteq U, B \subseteq V$, but $A \cap B$ is not relatively compact in $U \cap V$. Thus, there exists a directed family $\{U_i \mid i \in I\}$ of open sets such that $U \cap V = \bigcup_{i\in I} U_i$, but U_i fails to cover $A \cap B$ for every $i \in I$. Define a function f from D to $\mathbf{X}_\perp^\top$ as follows:

$$f(x) = \begin{cases} c, & x \in U \setminus V; \\ d, & x \in V \setminus U; \\ b, & x \in U \cap V; \\ \perp, & \text{otherwise.} \end{cases}$$

Moreover, for every $U_i, i \in I$, we define a function g_i as follows:

$$g_i(x) = \begin{cases} c, & x \in U \setminus V; \\ d, & x \in V \setminus U; \\ e, & x \in U_i; \\ a, & (U \cap V) \setminus U_i; \\ \perp, & \text{otherwise.} \end{cases}$$

It is easy to verify that f and $g_i, i \in I$, are Scott-continuous, and the set $G = \{g_i \mid i \in I\}$ is directed with its supremum above f. However, $f \in N(A, \uparrow c) \cap N(B, \uparrow d)$, and note that

$$N(A, \uparrow c) \cap N(B, \uparrow d) \cap \downarrow f \subseteq \{h \in [D \to \mathbf{X}_{\perp}^{\top}] \mid h(A \cap B) = \{b\}\}.$$

Moreover, since for every $i \in I$, $(A \cap B) \setminus U_i \neq \emptyset$, there is some $x \in (A \cap B) \setminus U_i$. Then $g_i(x) = a$, and therefore we have

$$N(A, \uparrow c) \cap N(B, \uparrow d) \cap \downarrow f \cap \downarrow G = \emptyset.$$

Note that $N(A, \uparrow c) \cap N(B, \uparrow d)$ is a Scott open neighbourhood of f, since c, d are compact in $\mathbf{X}_{\perp}^{\top}$. Hence f is not in $\overline{\downarrow f \cap \downarrow G}$, the Scott closure of $\downarrow f \cap \downarrow G$. This implies that the function space $[D \to \mathbf{X}_{\perp}^{\top}]$ is not meet-continuous. A contradiction. □

Finally, our dichotomy result for locally compact sober dcpos reads as follows:

Theorem 6.4. *Let* $\mathbf{C}$ *be a cartesian closed full subcategory in* **LcS**. *Then either* $\mathbf{C}$ *is included in* **SCS**, *or every object in* $\mathbf{C}$ *is an L-dcpo.*

Proof. Let L be any dcpo in $\mathbf{C}$. Then the function space $[L \to L]$ is in $\mathbf{C}$ from the cartesian closedness of $\mathbf{C}$. It is obvious that both L and $[L \to L]$ are compact, locally compact and sober in the Scott topology. Thus by Theorem 3.4 and Theorem 4.6 they are also meet-continuous and bi-complete.

If we assume that L is neither coherent nor has complete principal ideals, then from Theorem 6.3 the function space $[L \to L]$ is not meet-continuous. A contradiction. □

7 Closing remarks

The Jung-Tix problem [9, 3] asks for a nice category of dcpos and Scott-continuous functions in domain theory that is simultaneously cartesian closed and closed under the probabilistic powerdomain construction. Since lattice structures are destroyed by this powerdomain construction (see for example [6]), if we consider the Jung-Tix problem in the category **LcS**, from Theorem 6.4 and Theorem 3.4 it follows that one will always end up within a category consisting of meet-continuous stably compact dcpos. However, we do not know at this point of any new cartesian closed subcategories of **SCS** besides those included in the category of FS-domains. We would like to leave this as an open question.

For the general case in which a least element is not present, we want to mention that a similar investigation as in [7] can be conducted. One will have four subcategories of locally compact sober dcpos such that any cartesian closed subcategory is entirely contained in one of them. Analogous to the work in [7], the crucial step is to find conditions such that $[[L \to L] \to [L \to L]]$ is meet-continuous. This requirement implies that either L is well-rooted as defined in [7] or it is formed as a disjoint union of pointed dcpos. This work can be found in the last chapter of the first author's PhD thesis.

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