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Triangle-tilings in graphs without large independent sets

József Balogh*, Andrew McDowell†, Theodore Molla‡, Richard Mycroft§

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Abstract

We study the minimum degree necessary to guarantee the existence of perfect and almost-perfect triangle-tilings in an n -vertex graph G with sublinear independence number. In this setting, we show that if $\delta(G) \geq n/3 + o(n)$ then G has a triangle-tiling covering all but at most four vertices. Also, for every $r \geq 5$, we asymptotically determine the minimum degree threshold for a perfect triangle-tiling under the additional assumptions that G is K_r -free and n is divisible by 3.

Mathematics Subject Classification Numbers: 05C35, 05C70, 05D40.

1 Introduction

A *triangle-tiling* in a graph G is a collection \mathcal{T} of vertex-disjoint triangles in G . We say that \mathcal{T} is *perfect* if $|\mathcal{T}| = n/3$, where n is the order of G . A trivial necessary condition for the existence of a perfect triangle-tiling is that 3 divides n . We let $V(\mathcal{T}) := \bigcup_{T \in \mathcal{T}} V(T)$ and say \mathcal{T} *covers* $U \subseteq V(G)$ (respectively $v \in V(G)$) when $U \subseteq V(\mathcal{T})$ (respectively $v \in V(\mathcal{T})$), so a perfect triangle-tiling covers every vertex of the host graph. Given disjoint sets A and B which partition $V(G)$, we say that a triangle T in G is an *A-triangle* if T contains two vertices of A and one vertex of B , and likewise that T is a

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B-triangle if T contains two vertices of B and one vertex of A . Observe that if $|A| \equiv 1 \pmod{3}$ and $|B| \equiv 2 \pmod{3}$, there are no B -triangles in G and also there is no pair of vertex-disjoint A -triangles in G , then G does not have a perfect triangle-tiling. In that case, we call the ordered pair (A, B) a *divisibility barrier* in G (note that order is important here). Similarly, if $A \subseteq V(G)$ has size $|A| \geq 2n/3 + r$ for some $r > 0$, but $G[A]$ has no triangles, then every triangle-tiling in G contains at most $n - |A| \leq n/3 - r$ triangles, and so leaves at least $3r$ vertices uncovered. We call such a set A a *space barrier*.

The classical Corrádi-Hajnal theorem [4] states that if G has minimum degree $\delta(G) \geq 2n/3$, and n is divisible by 3, then G contains a perfect triangle-tiling. The minimum degree condition of this result is easily seen to be best-possible by considering, for an arbitrary $m \in \mathbb{N}$, the complete tripartite graph $G_1(m)$ with vertex classes of size $m-1$, m and $m+1$. Indeed, $G_1(m)$ then has $n := 3m$ vertices and $\delta(G_1(m)) \geq 2m-1 = 2n/3 - 1$, but $G_1(m)$ has no perfect triangle-tiling, as the union of the two largest vertex classes is a space barrier. Observe, however, that $G_1(m)$ contains large independent sets. By proving the following theorem, Balogh, Molla and Sharifzadeh [2] recently showed that the minimum degree condition can be significantly weakened if we additionally assume that G has no large independent set. Throughout this paper we write $\alpha(G)$ to denote the independence number of G .

Theorem 1.1 ([2, Theorem 1.2]). *For every $\omega > 0$ there exist $n_0, \gamma > 0$ such that the following holds for every integer $n \geq n_0$ which is divisible by 3. If G is a graph on n vertices with $\delta(G) \geq n/2 + \omega n$ and $\alpha(G) \leq \gamma n$, then G contains a perfect triangle-tiling.*

For an arbitrary $m \in \mathbb{N}$, the graph $G_2(m)$ consisting of two copies of K_{3m+2} intersecting in a single vertex has $n := 6m + 3$ vertices, minimum degree $\delta(G_2(m)) = 3m + 1 = \lfloor n/2 \rfloor$ and independence number two. Moreover, $G_2(m)$ has a divisibility barrier (A, B) , where B is the vertex set of one of the copies of K_{3m+2} and $A = V(G_2(m)) \setminus B$, and so $G_2(m)$ does not contain a perfect triangle-tiling. This example demonstrates that the minimum degree condition of Theorem 1.1 is best-possible up to the ωn additive error term. Alon suggested that if one only wants a triangle-tiling that covers all but a constant number of vertices, then perhaps the condition $\delta(G) \geq (1/3 + o(1))n$ is sufficient. In this paper, we show that this is indeed the case, by proving that if $\delta(G) \geq (1/3 + o(1))n$ and $\alpha(G) = o(n)$, then G has a triangle-tiling covering all but at most four vertices. Furthermore, under the additional assumptions that G has no divisibility barrier and 3 divides n , we show that G contains a perfect triangle-tiling.

Theorem 1.2. *For every $\omega > 0$ there exist $n_0, \gamma > 0$ such that if G is a graph on $n \geq n_0$ vertices with $\delta(G) \geq n/3 + \omega n$ and $\alpha(G) \leq \gamma n$, then*

- (a) G contains a triangle-tiling covering all but at most four vertices of G , and
- (b) if 3 divides n and G contains no divisibility barrier, then G contains a perfect triangle-tiling.

Observe that for an arbitrary $m \in \mathbb{N}$, the graph $G_3(m)$ consisting of two disjoint copies of K_{3m+2} has $n := 6m + 4$ vertices, minimum degree $\delta(G_3(m)) = 3m + 1 = n/2 - 1$ and independence number two, but every triangle-tiling in $G_3(m)$ covers at most $n - 4$ vertices. This demonstrates that the conditions of Theorem 1.2 do not guarantee a triangle-tiling which leaves fewer than four vertices uncovered. Furthermore, a straightforward

construction demonstrates that the ωn error term in the minimum degree condition of Theorem 1.2 cannot be removed completely. For this we use the existence of triangle-free graphs on n vertices with independence number $o(n)$ and minimum degree $\omega(1)$, as exhibited by Erdős in [5]; we refer to such a graph as an *Erdős graph* and denote it by $\text{ER}(n)$. For an arbitrary $m \in \mathbb{N}$ we then form a graph $G_4(m)$ by taking the complete bipartite graph whose vertex classes U and V have sizes $2m + 1$ and $m - 1$ respectively, and then placing copies of $\text{ER}(|U|)$ and $\text{ER}(|V|)$ on U and V respectively. The graph $G_4(m)$ formed in this way has $n := 3m$ vertices, minimum degree $\delta(G_4(m)) \geq n/3 + \omega(1)$ and sublinear independence number. Moreover, since U is a space barrier, $G_4(m)$ has no perfect triangle-tiling.

The relationship between the results in this paper and the Corrádi-Hajnal theorem is clearly analogous to the relationship between Ramsey-Turán theory and Turán's theorem, as Ramsey-Turán theory is concerned with the maximum possible number of edges in an H -free graph on n vertices with some upper bound on $\alpha(G)$. More precisely, in classical Ramsey-Turán theory the principle object of study is the function $\mathbf{RT}(n, H, m)$, which is defined to be the maximum number of edges in an H -free, n -vertex graph with independence number at most m , whenever such a graph exists for n, H and m . The asymptotic value of $\mathbf{RT}(n, K_r, o(n))$ was established for odd r by Erdős and Sós [6] and for even r by Erdős, Hajnal, Sós and Szemerédi [7], giving the following theorem.

Theorem 1.3 ([6, Theorem 1] and [7, Theorem 1]). *For every $r \geq 3$, we define*

$$f_{RT}(r) := \begin{cases} \frac{r-3}{r-1} & \text{if } r \text{ is odd,} \\ \frac{3r-10}{3r-4} & \text{if } r \text{ is even.} \end{cases}$$

- (a) *For every $\omega > 0$, there exists $\gamma, n_0 > 0$ such that if G is a graph on $n \geq n_0$ vertices with $\alpha(G) \leq \gamma n$ and with at least $(f_{RT}(r) + \omega) \binom{n}{2}$ edges, then G contains a copy of K_r .*
- (b) *For every $\omega > 0$ and $\gamma > 0$, there exists $n_0 > 0$ such that for every $n \geq n_0$, there exists a K_r -free graph $G := G_{RT}(n, r, \omega, \gamma)$ on n vertices such that $\delta(G) \geq (f_{RT}(r) - \omega)n$ and $\alpha(G) \leq \gamma n$.*

Observe that for any $r \geq 3$, $\omega, \gamma > 0$ and each sufficiently large n divisible by 6, the graph $G_5(n)$ on n vertices consisting of the disjoint union of $G_{RT}(\frac{n}{2} - 1, r, \omega, \gamma)$ and $G_{RT}(\frac{n}{2} + 1, r, \omega, \gamma)$ is K_r -free, has minimum degree $\delta(G_5(n)) \geq \left(\frac{f_{RT}(r)}{2} - \omega\right)n$ and independence number at most γn . However, as $G_5(n)$ contains a divisibility barrier, it has no perfect triangle-tiling. Although the construction of $G_{RT}(n, r, \omega, \gamma)$ was given in [6] (when r is odd) and [7] (when r is even), for completeness, we describe $G_{RT}(n, r, \omega, \gamma)$ at the end of Section 5.

By combining Theorems 1.2 and 1.3 we determine, for every $r \geq 5$, the asymptotic minimum degree threshold for a perfect triangle-tiling in a K_r -free graph with sublinear independence number; this is the following corollary.

Corollary 1.4. *For every $r \geq 5$ and $\omega > 0$ there exist $n_0, \gamma > 0$ such that the following holds for every integer $n \geq n_0$ which is divisible by 3. If G is a K_r -free graph on n vertices with*

$$\delta(G) \geq \begin{cases} \frac{f_{RT}(r)}{2}n + \omega n & \text{if } r \geq 7 \\ \frac{n}{3} + \omega n & \text{if } r \in \{5, 6\} \end{cases}$$

and $\alpha(G) \leq \gamma n$, then G contains a perfect triangle-tiling.

Proof. Given $\omega > 0$, choose γ small enough and n_0 large enough to apply Theorem 1.2 with the same constants there as here and so that we may apply Theorem 1.3(a) with 3γ and $n_0/3$ in place of γ and n_0 respectively. We also insist that $\gamma n_0 + 2 \leq \omega n_0/2$. Since $\frac{f_{\text{RT}}(r)}{2} \geq \frac{1}{3}$ if and only if $r \geq 7$, by Theorem 1.2(b) it suffices to prove that no K_r -free graph on $n \geq n_0$ vertices with $\delta(G) \geq \frac{f_{\text{RT}}(r)}{2}n + \omega n$ and $\alpha(G) \leq \gamma n$ contains a divisibility barrier. So let G be such a graph, and suppose for a contradiction that (X, Y) is a divisibility barrier in G . Let A be the smaller of X and Y , and let B be the larger, so $|A| \leq n/2$. By definition of a divisibility barrier, if $A = Y$ then there is no pair of vertex-disjoint B -triangles in G , whilst if $A = X$ then there are no B -triangles in G at all. It follows that at most one vertex $a \in A$ has more than $\gamma n + 2$ neighbours in B , as given two such vertices $a, a' \in A$ we could use the fact that $\alpha(G) \leq \gamma n$ to choose an edge bc in $N(a) \cap B$ and then an edge $b'c'$ in $(N(a') \cap B) \setminus e$ to obtain a pair of vertex-disjoint B -triangles abc and $a'b'c'$ in G . So at least $|A| - 1$ vertices of A have at least $\delta(G) - \gamma n - 2 \geq \frac{f_{\text{RT}}(r)}{2}n + \frac{\omega}{2}n$ neighbours in A . So in particular $|A| \geq \frac{f_{\text{RT}}(r)}{2}n \geq \frac{n}{3}$. Moreover we have

$$e(G[A]) \geq \frac{1}{2}(|A| - 1) \left(\frac{f_{\text{RT}}(r)}{2} + \frac{\omega}{2} \right) n = \frac{n}{2|A|} (f_{\text{RT}}(r) + \omega) \binom{|A|}{2} \geq (f_{\text{RT}}(r) + \omega) \binom{|A|}{2},$$

so $G[A]$ contains a copy of K_r by Theorem 1.3(a). This contradicts our assumption that G was K_r -free and so completes the proof. \square

Observe that the graph $G = G_4(m)$ given by the construction following Theorem 1.2 has $n = 3m$ vertices, minimum degree at least $n/3 + \omega(1)$ and independence number $o(n)$, and that G contains a space barrier (and therefore does not contain a perfect triangle-tiling). Moreover, G is K_5 -free since $G[U]$ and $G[V]$ are each triangle-free. This demonstrates that the minimum degree condition in Corollary 1.4 is best-possible up to the ωn error term for $r \in \{5, 6, 7\}$ (and that the error term cannot be removed entirely in these cases). Furthermore, the graph $G_5(n)$ presented after Theorem 1.3 shows that the minimum degree condition in Corollary 1.4 is best-possible up to the ωn error term for $r \geq 8$ also.

In a K_4 -free graph, we can only construct space barriers when $\delta(G) < n/6$, so it may be true that, in a K_4 -free graph, the conditions $\delta(G) \geq (1/6 + o(1))n$ and $\alpha(G) = o(n)$ are sufficient to guarantee a perfect triangle-tiling when n is divisible by 3; we discuss this further in Section 5. Also in Section 5, we consider the problem of determining the minimum degree condition which guarantees a perfect K_k -tiling in a graph with sublinear independence number when $k \geq 4$.

1.1 Proof outline

To illustrate the proof ideas of this paper, we here outline the proof of Theorem 1.2(b). Let G be a graph on n vertices with sublinear independence number and minimum degree somewhat greater than $n/3$, where n is large and divisible by 3.

Our proof makes extensive use of the notion of a *regular pair* in G . Loosely speaking, this is a pair (A, B) of vertex-disjoint subsets of $V(G)$ such that the edges between A

and B are distributed in a ‘randomlike’ manner (see Section 2.1 for formal definitions). Now suppose that (A, B) is a regular pair in G of density d (*i.e.* there are $d|A||B|$ edges between A and B), for some not-too-small d and sets A and B of linear size. Most vertices $v \in A$ then have approximately $d|B|$ neighbours in B . Since G has sublinear independence number, there must be an edge in the neighbourhood of v , and this creates a triangle in G whose vertices are v and two neighbours of v in B . The same argument with A and B reversed allows us to find triangles with two vertices in A and one in B . It is not hard to see that, provided $|A|$ and $|B|$ differ by at most a factor of two, then we can construct a triangle-tiling covering almost all of the vertices of $A \cup B$ by greedily choosing and deleting triangles in this way (this is the first part of Lemma 3.1). Moreover, if (A, B) has density greater than $1/2$ and is super-regular, meaning that every vertex has neighbourhood of typical size, and $|A|$ and $|B|$ differ by at most a little less than a factor of two, then Lemma 3.1 shows that we can in fact construct a triangle-tiling covering every vertex of $A \cup B$ (so long as 3 divides $|A \cup B|$). The ability to find a spanning triangle-tiling in this setup is one way we may complete a perfect triangle-tiling in G at the end of the proof.

Another setup in which we can find a spanning triangle-tiling is where we have pairwise vertex-disjoint sets $A, B, C \subseteq V(G)$ whose sizes are linear and approximately equal to each other such that (A, B) , (B, C) and (A, C) are each super-regular pairs of not-too-small density and 3 divides $|A \cup B \cup C|$. Indeed, we first greedily find and remove triangles by the method described above so that equally many vertices remain in each of A , B and C , and then apply the Blow-up lemma to find a triangle-tiling covering all remaining vertices of A , B and C by triangles each using one vertex from each set. This argument is formalised by Lemma 3.2.

We begin the proof by a standard application of the Szemerédi regularity lemma to find a partition of G into a bounded number of *clusters* V_1, \dots, V_k of equal size and a small exceptional set V_0 , and define a reduced graph R whose vertices are the clusters of G and whose edges correspond to pairs of clusters which form regular pairs of not-too-small density in G . Then a straightforward counting argument shows that either

- (a) there is an edge $V_i V_j$ of R for which the pair (V_i, V_j) has density somewhat more than $1/2$, or
- (b) R has minimum degree at least $2k/3$. In particular, certainly there are clusters V_i, V_j and V_k which form a triangle in R .

In case (a), by removing a small number of vertices from V_i and V_j (and adding these to the exceptional set) we can make the pair (V_i, V_j) super-regular with density more than $1/2$, achieving the first setup described above. Similarly in case (b) we can remove a small number of vertices from each of V_i, V_j and V_k to achieve the second setup described above. These two or three clusters (according to which case we are in) form the ‘core’ of G . Our proof then proceeds by iteratively removing vertex-disjoint triangles so as to cover every vertex outside the core and only a small number of vertices within the core; we can then complete a perfect triangle-tiling in G by finding a triangle-tiling spanning the remaining vertices of the core as described above.

A key step in achieving this is the use of perfect fractional weighted matchings. The theory for these is developed in Section 2.4, with the key conclusion being that since R has minimum degree somewhat greater than $k/3$, we can partition all clusters outside

the core into *subclusters* of linear size, so that the subclusters form regular pairs (A_i, B_i) of not-too-small density and the sizes of A_i and B_i differ by at most a little less than a factor of two (the crucial ratio for being able to find a triangle-tiling covering almost all vertices of $A_i \cup B_i$ as described above). We then define an auxiliary reduced graph R^* with a vertex v_i corresponding to each pair (A_i, B_i) and a final vertex v^* corresponding to the core of G , with an edge of R^* indicating that the corresponding pairs (or perhaps triple, in the case of the core) include subsets of clusters of an edge of R .

Suppose for simplicity that the reduced graph R is connected; it follows that R^* is connected, and using a theorem of Win (Theorem 2.7) we find a spanning tree T in R^* of bounded maximum degree. We take v^* to be the root of T , and iteratively ‘work inwards’ from the leaves of T to v^* to construct a perfect triangle-tiling in G , as follows. First we choose a leaf v_i of T , and remove a triangle-tiling in the corresponding pair (A_i, B_i) covering almost all vertices of this pair. Writing v_j for the parent of v_i in T , we then remove a few more triangles to cover all uncovered vertices of $A_i \cup B_i$ as well as a small number of vertices in the pair (A_j, B_j) corresponding to v_j . We then delete the leaf v_i from T , and iterate. At the end of this iteration only the root v^* of T remains, at which point we have constructed a triangle-tiling covering all vertices of T outside the core as well as a small number of vertices of the core. We then find a perfect triangle-tiling within the remaining vertices of the core (recall that the core was chosen so as to permit this step) to complete the desired perfect triangle-tiling in G .

If instead R is not connected, then R has precisely two components (since $\delta(R) > k/3$). After allocating exceptional vertices appropriately, these components yield a partition of $V(G)$ into two parts, say X and Y . We may then use the fact that G contains no divisibility barrier to find and remove at most two triangles from G so that following these deletions both $|X|$ and $|Y|$ are divisible by 3. We then proceed exactly as above within each of $G[X]$ and $G[Y]$ (and the corresponding components of R) to obtain perfect triangle-tilings in each of these subgraphs; together with the removed triangles these form a perfect triangle-tiling in G , completing the proof.

2 Notation and preliminary results

In this section we introduce various results which we will use in the proof of Theorem 1.2, beginning with helpful notation. Given a graph G , we write $|G|$ and $e(G)$ for the number of vertices and edges of G respectively. We write $x = y \pm z$ to mean $y - z \leq x \leq y + z$, and $[n]$ to denote the set of integers from 1 to n . We omit floors and ceilings throughout this paper wherever they do not affect the argument. We write $x \ll y$ to mean that for every $y > 0$ there exists $x_0 > 0$ such that the subsequent statements hold for x and y whenever $0 < x \leq x_0$. Similar statements with more variables are defined similarly.

2.1 Regularity

In a graph G , for each pair of disjoint non-empty sets $A, B \subseteq V(G)$ we write $G[A, B]$ for the bipartite subgraph of G with vertex classes A and B and whose edges are all edges of G with one endvertex in A and the other in B , and denote the *density* of $G[A, B]$ by $d_G(A, B) := \frac{e(G[A, B])}{|A||B|}$. We say that $G[A, B]$ is (d, ε) -regular if $d_G(X, Y) = d \pm \varepsilon$ for

every $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$, and we write that $G[A, B]$ is $(\geq d, \varepsilon)$ -regular to mean that $G[A, B]$ is (d', ε) -regular for some $d' \geq d$. Also, we say that $G[A, B]$ is (d, ε) -super-regular if $G[A, B]$ is $(\geq d, \varepsilon)$ -regular, every vertex of A has at least $(d - \varepsilon)|B|$ neighbours in B and every vertex of B has at least $(d - \varepsilon)|A|$ neighbours in A . The following well-known results are elementary consequences of the definitions.

Lemma 2.1 (Slicing Lemma). *For every $d, \varepsilon, \beta > 0$, if $G[A, B]$ is (d, ε) -regular, and $X \subseteq A$ and $Y \subseteq B$ have sizes $|X| \geq \beta|A|$ and $|Y| \geq \beta|B|$, then $G[X, Y]$ is $(d, \varepsilon/\beta)$ -regular.*

Lemma 2.2. *For every $d, \varepsilon > 0$ with $\varepsilon < \frac{1}{2}$, if $G[A, B]$ is $(\geq d, \varepsilon)$ -regular, then there are sets $X \subseteq A$ and $Y \subseteq B$ with sizes $|X| \geq (1 - \varepsilon)|A|$, and $|Y| \geq (1 - \varepsilon)|B|$ such that $G[X, Y]$ is $(d, 2\varepsilon)$ -super-regular.*

We make use of Chernoff bounds on the concentration of binomial and hypergeometric distributions in the following form.

Theorem 2.3 ([8, Corollary 2.3 and Theorem 2.10]). *Suppose X has binomial or hypergeometric distribution and $0 < a < 3/2$. Then $\mathbb{P}(|X - \mathbb{E}X| \geq a\mathbb{E}X) \leq 2e^{-\frac{a^2}{3}\mathbb{E}X}$.*

The following lemma is similar to lemmas of Csaba and Mydlarz [3, Lemma 14] and Martin and Skokan [14, Lemma 10]. It states that if we randomly select a collection of disjoint subsets from each of the vertex classes of a super-regular pair, every pair of sets from different classes is super-regular with high probability.

Lemma 2.4 (Random Slicing Lemma). *Suppose that $1/n \ll \beta, \varepsilon \ll d$. Let $G[A, B]$ be (d, ε) -super-regular (respectively (d, ε) -regular) where $|A|, |B| \leq n$. Also let x_1, \dots, x_s and y_1, \dots, y_t be positive integers each of size at least βn such that $\sum_{i \in [s]} x_i \leq |A|$ and $\sum_{j \in [t]} y_j \leq |B|$. If $\{X_1, \dots, X_s\}$ is a collection of disjoint subsets of A and $\{Y_1, \dots, Y_t\}$ is a collection of disjoint subsets of B such that $|X_i| = x_i$ and $|Y_j| = y_j$ for all $i \in [s]$ and $j \in [t]$ selected uniformly at random from all such collections, then, with probability at least $1 - e^{-\Omega(n)}$, $G[X_i, Y_j]$ is (d, ε') -super-regular (respectively (d, ε') -regular) for all $i \in [s]$ and $j \in [t]$, where $\varepsilon' := (33\varepsilon)^{1/5}$.*

For completeness we present a proof of Lemma 2.4 in the Appendix. To make use of regularity properties, we apply the degree form of Szemerédi's Regularity Lemma (see [12, Theorem 1.10]).

Theorem 2.5 (Degree form of Szemerédi's Regularity Lemma). *For every $\varepsilon > 0$, real number $d \in [0, 1]$ and integers t and q there exists integers n_0 and T such that the following statement holds. Let G be a graph on $n \geq n_0$ vertices, and let U_1, \dots, U_q be a partition of $V(G)$ into q parts. Then there is a partition of $V(G)$ into an exceptional set V_0 and k clusters V_1, \dots, V_k , and a spanning subgraph $G' \subseteq G$ such that*

- (a) $t \leq k \leq T$,
- (b) $|V_1| = |V_2| = \dots = |V_k|$ and $|V_0| \leq \varepsilon n$,
- (c) for every $i \in [k]$ there exists $j \in [q]$ such that $V_i \subseteq U_j$,
- (d) $d_{G'}(v) \geq d_G(v) - (\varepsilon + d)n$ for all $v \in V(G)$,
- (e) $e(G'[V_i]) = 0$ for all $i \in [k]$, and

(f) for each distinct $i, j \in [k]$ either $G'[V_i, V_j]$ is $(\geq d, \varepsilon)$ -regular or $G'[V_i, V_j]$ is empty.

Theorem 2.5 as stated above is stronger than the form given in [12] in that it allows us to specify an initial partition of $V(G)$ and to insist that the clusters V_1, V_2, \dots, V_k are each a subset of some part of this partition (property (c) above). However, this statement follows from the same proof, which proceeds iteratively by alternately refining a partition of $V(G)$ and deleting some vertices of $V(G)$ (which are then placed in the exceptional set V_0). So to prove Theorem 2.5 we take our specified partition as the initial partition of this process.

2.2 Robustly-matchable sets

The following application of the regularity lemma is critical to the entire proof. Given a graph G , a small $A \subseteq V(G)$ and a small matching $B \subseteq E(G)$, we form an auxiliary bipartite graph F with vertex set $A \cup B$ in which there is an edge between $a \in A$ and $bc \in B$ if and only if abc is a triangle in G . So matchings in F correspond to triangle-tilings in G . In this setting, Lemma 2.6 allows us to choose subsets $X \subseteq A$ and $Y \subseteq B$ such that if we can find a triangle-tiling in G that covers every vertex of G except for the vertices incident to edges in Y and exactly $|Y|$ of the vertices in X , then we obtain a perfect triangle-tiling in G .

Lemma 2.6. *Suppose that $1/n \ll \phi \ll \varepsilon \ll d$. Let F be a bipartite graph with vertex classes A and B such that $n/10 \leq |A|, |B| \leq n$ and $d_F(A, B) \geq d$. Then there exist subsets $X \subseteq A$ and $Y \subseteq B$ of sizes $|X| = \phi n$ and $|Y| = (1 - \varepsilon)\phi n$ such that $F[X', Y]$ contains a perfect matching for every subset $X' \subseteq X$ with $|X'| = |Y|$.*

Proof. Let n_0 and T be the integers returned by Theorem 2.5 given inputs $\varepsilon, d' := d/200$ and $t = q = 2$. We may assume that $\phi \leq 1/4T$. We use Theorem 2.5 with initial partition $U_1 = A$ and $U_2 = B$ to obtain a spanning subgraph $F' \subseteq F$ and a partition of $V(F)$ into sets V_0, V_1, \dots, V_k which satisfy properties (a)–(f) of Theorem 2.5. In particular, by Theorem 2.5(d) at most $(\varepsilon + d/200)n^2$ edges of F are not edges of F' . Also, by Theorem 2.5(e) there are no edges in $F'[V_i]$ for any $i \in [k]$, and since $|V_0| \leq \varepsilon n$ by Theorem 2.5(b), at most εn^2 edges of F contain a vertex of V_0 . Since

$$e(F) = d_F(A, B)|A||B| \geq d \left(\frac{n}{10}\right)^2 > \left(\varepsilon + \frac{d}{200}\right)n^2 + \varepsilon n^2,$$

there must exist distinct $i, j \in [k]$ such that $F'[V_i, V_j]$ is non-empty, and since F is bipartite, by Theorem 2.5(c) we may assume without loss of generality that $V_i \subseteq A$ and $V_j \subseteq B$. Observe that $F'[V_i, V_j]$ is $(\geq d', \varepsilon)$ -regular by Theorem 2.5(f). Write m for the common size of V_i and V_j , so $m = |V(F) \setminus V_0|/k \geq n/2T \geq 2\phi n$ by Theorem 2.5(a) and (b). By Lemma 2.2 we may delete at most εm vertices from each of V_i' and V_j' to obtain subsets $V_i' \subseteq V_i$ and $V_j' \subseteq V_j$ such that $F[V_i', V_j']$ is $(d', 2\varepsilon)$ -super-regular. Having done so, choose $X \subseteq V_i'$ and $Y \subseteq V_j'$ uniformly at random with sizes ϕn and $(1 - \varepsilon)\phi n$ respectively (this is possible since $|V_i'|, |V_j'| \geq (1 - \varepsilon)m \geq \phi n$). Then Lemma 2.4 tells us that $F'[X, Y]$ is (d', ε') -super-regular with high probability, where $\varepsilon' := (66\varepsilon)^{1/5}$, so we may fix sets X and Y with this property. It then follows that every vertex of X has at least $(d' - \varepsilon')|Y| \geq \varepsilon'|X|$ neighbours in Y , whilst every set of at least $\varepsilon'|X|$ vertices of

X has at least $(1 - \varepsilon')|Y| \geq (1 - 2\varepsilon')|X|$ neighbours in Y (where we say that a vertex y is a neighbour of a set X' if y is a neighbour of some element of X'). Finally, since every vertex of Y has at least $(d' - \varepsilon')|X| > 2\varepsilon'|X|$ neighbours in X , every set of at least $(1 - 2\varepsilon')|X|$ vertices of X has every vertex of Y as a neighbour. So Hall's criterion is satisfied for every $X' \subseteq X$ of size $|X'| \leq |Y|$, so for every $X' \subseteq X$ with $|X'| = |Y|$ there is a perfect matching in $F[X', Y]$. \square

2.3 Spanning bounded degree trees

Our proof requires us to find a spanning tree of bounded maximum degree in the reduced graph R of G . For this, we use the following theorem of Win [16].

Theorem 2.7. *If $k \geq 2$ and R is a connected graph such that*

$$\sum_{v \in S} d(v) \geq |R| - 1 \text{ for every independent set } S \text{ of size } k,$$

then R contains a spanning tree T such that $\Delta(T) \leq k$. In particular, if R is a connected graph with $\delta(R) \geq (|R| - 1)/k$, then R contains a spanning tree T with maximum degree at most k .

2.4 Fractional weighted matchings via linear programming

Recall from the proof outline that we will consider regular pairs of clusters of vertices of G and use the regularity of each pair to find a triangle-tiling covering a given proportion of vertices from each cluster. We want to choose these proportions so that collectively these triangle-tilings cover (almost) all of the vertices of G . To do this we look for a generalized form of weighted matching in the reduced graph; the proportion of vertices to be covered by a triangle-tiling within a pair of clusters then corresponds to the weight in this matching of the corresponding edge of the reduced graph.

A *fractional matching* w in a graph G assigns a weight $w_e \geq 0$ to each edge $e \in E(G)$ such that for every vertex $u \in V(G)$ we have $\sum_{e \ni u} w_e \leq 1$. In other words, if we consider each edge uv to place weight w_{uv} at each of u and v , then the combined weight placed at each vertex is at most one. This is a relaxation of an integer matching M , in which we insist that for each $e \in E(G)$ we have $w_e = 1$ (meaning that $e \in M$) or $w_e = 0$ (meaning that $e \notin M$). Here we work with a more general notion of an (η, ξ) -*weighted fractional matching*, in which we consider each edge to place different weights at each end, subject to the restriction that the ratio of these weights is at most $\eta : \xi$. It is most natural to express these matchings in terms of directed graphs, as we can then consider a directed edge \vec{uv} of weight $w_{\vec{uv}}$ to place weight $\eta w_{\vec{uv}}$ on its tail u and weight $\xi w_{\vec{uv}}$ on its head v ; as before, we insist that the combined weight placed at each vertex is at most one.

Definition 2.8. Let Γ be a directed graph on n vertices and let η and ξ be positive real numbers. An (η, ξ) -*weighted fractional matching* w in Γ is an assignment of a weight $w_{\vec{uv}} \geq 0$ to each edge \vec{uv} of Γ such that for every vertex $u \in V(\Gamma)$ we have

$$\sum_{v \in N_{\Gamma}^{+}(u)} \eta w_{\vec{uv}} + \sum_{v \in N_{\Gamma}^{-}(u)} \xi w_{\vec{vu}} \leq 1. \tag{1}$$

The *total weight* of w is defined to be $W := \sum_{\vec{uv} \in E(\Gamma)} (\eta + \xi) w_{\vec{uv}}$. By (1) we have $W \leq n$; we say that w is *perfect* if $W = n$. Note that in this case we have equality in (1) for every vertex.

Given an undirected graph G , we consider (η, ξ) -weighted fractional matchings in the directed graph Γ formed by replacing every edge uv of G with both a directed edge \vec{uv} from u to v and a directed edge \vec{vu} from v to u . In particular, a $(\frac{1}{2}, \frac{1}{2})$ -weighted fractional matching w in Γ then corresponds to a fractional matching w' in G (in the standard notion of fractional matching as defined above). Indeed, given w , for each edge $e = uv \in E(G)$ we may take $w'_e = w_{\vec{uv}} + w_{\vec{vu}}$. In our proof we will instead consider (η, ξ) -weighted fractional matchings in Γ where ξ is close to twice as large as η . The advantage of this is shown by Lemma 2.10, which states that the minimum degree condition on G needed to guarantee the existence of a perfect (η, ξ) -weighted fractional matching in Γ is then approximately $n/3$, well below the $n/2$ threshold needed to guarantee the existence of a perfect fractional matching in G .

Let Γ be a directed graph on n vertices v_1, \dots, v_n , and fix $\eta, \xi > 0$. Then we define the (η, ξ) -*weighted characteristic vector* of an edge $\vec{v_i v_j} \in E(\Gamma)$ to be the vector $\chi_{\eta, \xi}(\vec{v_i v_j}) \in \mathbb{R}^n$ whose i th coordinate is equal to η , whose j th coordinate is equal to ξ , and in which all other coordinates are equal to zero. So an assignment w of non-negative weights to edges of Γ is an (η, ξ) -weighted fractional matching in Γ if and only if

$$\sum_{\vec{v_i v_j} \in E(\Gamma)} w_{\vec{v_i v_j}} \chi_{\eta, \xi}(\vec{v_i v_j}) \leq \mathbf{1}, \quad (2)$$

where $\mathbf{1}$ is the vector in \mathbb{R}^n with each coordinate equal to 1 and the inequality is treated pointwise. As before, w is perfect if and only if we have equality for each coordinate.

To prove the existence of a (η, ξ) -weighted fractional matching in a directed graph of high minimum indegree, we use the following version of Farkas' Lemma, for which we need the following definition; a vertex $\mathbf{v} \in \mathbb{R}^n$ is a *weighted sum* of vectors in $\mathcal{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_m\} \subseteq \mathbb{R}^n$ if

$$\mathbf{v} \in \left\{ \sum_{i=1}^m \lambda_i \mathbf{x}_i : \lambda_i \geq 0 \text{ for every } i \in [m] \right\},$$

otherwise \mathbf{v} is not a weighted sum of the vectors in \mathcal{X} .

Lemma 2.9 (Farkas' Lemma). *For every $\mathbf{v} \in \mathbb{R}^n$ and every finite $\mathcal{X} \subseteq \mathbb{R}^n$, if \mathbf{v} is not a weighted sum of the vectors in \mathcal{X} , then there exists $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{y} \cdot \mathbf{x} \geq 0$ for every $\mathbf{x} \in \mathcal{X}$ and $\mathbf{y} \cdot \mathbf{v} < 0$.*

We now give the main result of this section.

Lemma 2.10. *For every $\eta > 0$, every directed graph Γ on n vertices with $\delta^-(\Gamma) \geq \eta n$ admits a perfect fractional $(\eta, 1 - \eta)$ -matching. Furthermore, if $\eta = p/q$ for positive integers p and q , then we can assume that the weights of the matching are rational numbers with common denominator D bounded above by some function of p , q and n .*

Proof. Let v_1, \dots, v_n be an arbitrary ordering of the vertices of Γ . Then by (2), a perfect $(\eta, 1 - \eta)$ -weighted fractional matching in Γ corresponds to a weighted sum of the vectors in

$$\mathcal{X} := \{\chi_{\eta, 1-\eta}(\overrightarrow{v_i v_j}) : \overrightarrow{v_i v_j} \in E(\Gamma)\}$$

that equals $\mathbf{1}$.

If we assume that Γ does not have a perfect $(\eta, 1 - \eta)$ -weighted fractional matching, then, by Farkas' lemma (Lemma 2.9), as $\mathbf{1}$ is not a weighted sum of the vectors in \mathcal{X} , there exists a vector $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{y} \cdot \mathbf{1} < 0$ but $\mathbf{y} \cdot \chi_{\eta, 1-\eta}(\overrightarrow{v_i v_j}) \geq 0$ for every $\overrightarrow{v_i v_j} \in E(\Gamma)$. By reordering the vertices if necessary, we may assume that $y_1 \geq \dots \geq y_n$.

Let i be maximal such that $\overrightarrow{v_i v_n} \in E(\Gamma)$, so $i \geq \delta^-(\Gamma) \geq \eta n$. Then,

$$0 > \mathbf{y} \cdot \mathbf{1} = \sum_{j=1}^i y_j + \sum_{j=i+1}^n y_j \geq i y_i + (n - i) y_n \geq \eta n y_i + (1 - \eta) n y_n = n \mathbf{y} \cdot \chi_{\eta, 1-\eta}(\overrightarrow{v_i v_n}) \geq 0,$$

a contradiction.

The second statement is implied by basic linear programming theory, if we take the perfect fractional $(\eta, 1 - \eta)$ -matching to be one with the smallest possible number of non-zero weights, as then w is a basic feasible solution. \square

Note that if a directed graph Γ admits a perfect (η, ξ) -weighted fractional matching w with $\eta \leq \xi$ and $\eta + \xi = 1$, then $\alpha(\Gamma) \leq \xi n$, because for every independent set A in Γ we have

$$|A| = \sum_{a \in A} \left(\sum_{b \in N^+(a)} \eta w_{ab} + \sum_{b \in N^-(a)} \xi w_{ba} \right) \leq \xi \sum_{a \in A} \left(\sum_{b \in N^+(a)} w_{ab} + \sum_{b \in N^-(a)} w_{ba} \right) \leq \xi W \leq \xi n,$$

where the initial equality holds since we have equality in (1), and the penultimate inequality holds because (since A is an independent set) every edge of Γ contributes at most once to the sum. This shows that the minimum indegree condition of Lemma 2.10 is best possible for $\eta \leq 1/2$, since weaker conditions do not preclude the existence of independent sets of size greater than $(1 - \eta)n$.

3 Triangle-tilings in regular pairs and triples

As described in the proof outline, the proof of Theorem 1.2 proceeds by iteratively constructing a triangle-tiling in G which covers all of the vertices outside of a small 'core' subset of vertices but leaves most vertices inside this 'core' uncovered. This gives a perfect triangle-tiling in G , because the 'core' is robust in the sense that it has a perfect triangle-tiling after the removal of any sufficiently small set of vertices (provided that the number of vertices remaining is divisible by 3). Depending on the structure of the graph G , this 'core' will either consist of sets A and B which form a super-regular pair with density greater than $\frac{1}{2}$, or of sets A , B and C which form three super-regular pairs each with density bounded below by a small constant.

We begin with the case where the 'core' consists of a super-regular pair of density greater than $\frac{1}{2}$ (part (c) of Lemma 3.1). Let G be a graph whose vertex set is the disjoint

union of sets A and B . Recall that a triangle T in G is an A -triangle if T contains two vertices of A and one vertex of B , and likewise that T is a B -triangle if T contains two vertices of B and one vertex of A .

Lemma 3.1. *Suppose that $1/n \ll \gamma \ll \varepsilon \ll \phi, \varepsilon' \ll d \ll \omega$. Let A and B be disjoint sets of vertices with $n/3 + \omega n \leq |A|, |B| \leq 2n/3 - \omega n$ and $|A \cup B| = n$, and let G be a graph on vertex set $V := A \cup B$ with $\alpha(G) \leq \gamma n$. Then the following statements hold.*

- (a) *If $G[A, B]$ is $(\geq d, \varepsilon)$ -regular then G admits a triangle-tiling covering all but at most $2\varepsilon n$ vertices of G . Moreover, for every a and b with $2a + b \leq |A| - \varepsilon n$ and $a + 2b \leq |B| - \varepsilon n$ there is a triangle-tiling in G which consists of a A -triangles and b B -triangles.*
- (b) *If $G[A, B]$ is (d, ε) -super-regular then, for every $S \subseteq A$ of size $|S| = \phi n$ for which $|A \setminus S| + |B| + \lfloor \phi \varepsilon' n \rfloor$ is divisible by 3, there is a triangle-tiling in G which covers every vertex of $G[V \setminus S]$ and which covers precisely $\lfloor \phi \varepsilon' n \rfloor$ vertices of S .*
- (c) *If n is divisible by 3 and $G[A, B]$ is $(1/2 + d, \varepsilon)$ -super-regular then G contains a perfect triangle-tiling.*

Proof. For (a) the triangles may be chosen greedily. Indeed, suppose that we have already chosen a triangle-tiling \mathcal{T} consisting of at most a A -triangles and at most b B -triangles, then \mathcal{T} covers at most $2a + b$ vertices of A , and at most $a + 2b$ vertices of B . Taking $A' = A \setminus V(\mathcal{T})$ and $B' = B \setminus V(\mathcal{T})$, we find that $|A'|, |B'| \geq \varepsilon n$. Since $G[A, B]$ is $(\geq d, \varepsilon)$ -regular it follows that $d_G(A', B') \geq d - \varepsilon$, therefore some vertex $x \in A'$ has at least $(d - \varepsilon)|B'| \geq (d - \varepsilon)\varepsilon n > \gamma n$ neighbours in B' . Since $\alpha(G) \leq \gamma n$ it follows that some two of these neighbours are adjacent, giving a B -triangle which can be added to \mathcal{T} . The same argument with the roles of A' and B' reversed yields instead an A -triangle which may be added to \mathcal{T} . This proves the second statement of (a); the first follows by setting $a = \frac{1}{3}(2|A| - |B| - \varepsilon n)$ and $b = \frac{1}{3}(2|B| - |A| - \varepsilon n)$.

Next, for (b), let $z := \lfloor \phi \varepsilon' n \rfloor$, $t_4 := \lfloor z/2 \rfloor$ and $z' := z - 2t_4 \in \{0, 1\}$, so we will construct a triangle-tiling that covers all of $(A \setminus S) \cup B$ and exactly $z = 2t_4 + z'$ vertices of S . Let $B'_1 \subseteq B$ consist of all vertices in B with fewer than $(d - \frac{\varepsilon}{\phi})|S|$ neighbours in S ; since $G[S, B]$ is $(\geq d, \frac{\varepsilon}{\phi})$ -regular we have $|B'_1| \leq \frac{\varepsilon}{\phi}n$. Form B_1 by adding at most 2 arbitrarily selected vertices from $B \setminus B'_1$ to B'_1 so that $|B \setminus B_1| - t_4$ is divisible by 3. Since $G[A, B]$ is (d, ε) -super-regular, every vertex of B_1 has at least $(d - \varepsilon)|A| - |S| \geq \frac{dn}{3} > 2|B_1| + \gamma n$ neighbours in $A \setminus S$. Since $\alpha(G) \leq \gamma n$, we may greedily form a triangle-tiling \mathcal{T}_1 of A -triangles in G of size $|B_1|$ which covers every vertex of B_1 and does not use any vertex from S . We now select uniformly at random a subset $B_2 \subseteq B \setminus B_1$ of size $|B_2| = t_4$. Since every vertex in A has at least $(d - \varepsilon)|B| - |B_1| \geq \frac{dn}{3}$ neighbours in $B \setminus B_1$, Theorem 2.3 implies that, with probability $1 - o(1)$, every vertex of A has at least $\frac{\phi \varepsilon' d}{7}n$ neighbours in B_2 . Fix a choice of B_2 for which this event occurs. Let S' be an arbitrarily selected subset of S of size z' (so S' is either empty or a singleton) and let $A' := (A \setminus (S \cup V(\mathcal{T}_1))) \cup S'$ and $B' := B \setminus (B_1 \cup B_2)$. Recall that, by assumption, $|A \setminus S| + |B| + z$ is divisible by 3, so

$$|A'| + |B'| = |A \setminus S| + z' + |B| - |B_2| - |V(\mathcal{T}_1)| = (|A \setminus S| + |B| + z) - (3t_4 + |V(\mathcal{T}_1)|)$$

is divisible by 3. Since $|B'|$ is divisible by 3 by our selection of B_1 and B_2 , it follows that $|A'|$ is divisible by 3 as well. Let $t_3 = \lfloor \frac{\phi \varepsilon' d}{15}n \rfloor$, $a := \frac{2}{3}|A'| - \frac{1}{3}|B'|$ and $b := \frac{2}{3}|B'| - \frac{1}{3}|A'| - t_3$.

Since $G[A', B']$ is $(\geq d, \frac{\varepsilon}{2})$ -regular, (a) implies that there is a triangle-tiling \mathcal{T}_2 in $G[A' \cup B']$ such that $A'' := A' \setminus V(\mathcal{T}_2)$ and $B'' := B' \setminus V(\mathcal{T}_2)$ have sizes precisely $|A''| = |A'| - (2a + b) = t_3$ and $|B''| = |B'| - (a + 2b) = 2t_3$. Since by the choice of B_2 each vertex of A'' has at least $\frac{\phi \varepsilon' d}{7} n > 2|A''| + \gamma n$ neighbours in B_2 , we may greedily form a triangle-tiling \mathcal{T}_3 in $G[A'' \cup B_2]$ consisting of exactly t_3 B -triangles which covers every vertex of A'' and which covers precisely $2t_3$ vertices of B_2 . At this point we have obtained a triangle-tiling $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ in G which covers every vertex of A except for those in $S \setminus S'$ and every vertex of B except for the precisely $2t_3$ vertices in B'' and the precisely $t_4 - 2t_3$ vertices in $B_2 \setminus V(\mathcal{T}_3)$. Therefore, in total, precisely t_4 vertices of B remain uncovered, each of which has at least $(d - \frac{\varepsilon}{\phi})|S| - |S'| > 2|B_2| + \gamma n$ neighbours in $S \setminus S'$ by the choice of B_1 . We may therefore greedily form a triangle-tiling \mathcal{T}_4 of A -triangles in G which covers all the remaining uncovered vertices in B and precisely $2t_4$ vertices of $S \setminus S'$. Then $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4$ is the claimed triangle-tiling.

Finally, since none of the assumptions for (c) involve ϕ or ε' , we may assume that $\phi \ll \varepsilon'$. We also assume without loss of generality that $|B| \geq |A|$. Since $\alpha(G) \leq \gamma n$, we may greedily form a matching M of size at least $(|B| - \gamma n)/2 \geq n/10$ in $G[B]$. Fix such a matching M , and form an auxiliary bipartite graph H with vertex classes A and M where $a \in A$ and $e = xy \in M$ are adjacent if and only if xyz is a triangle in G . Note that for every edge $e = xy \in M$ we have that

$$\deg_H(e) = |N_G(x) \cap N_G(y) \cap A| \geq 2((1/2 + d) - \varepsilon)|A| - |A| \geq d|A|,$$

so H has density at least d . By Lemma 2.6, applied to H with ε' here in place of ε there, we may choose subsets $X \subseteq A$ and $M' \subseteq M$ such that $|X| = \phi n$, $|M'| = (1 - \varepsilon)\phi n$ and such that $H[X', M']$ contains a perfect matching for every subset $X' \subseteq X$ with $|X'| = |M'|$. Let $B' := B \setminus V(M')$ and $n' := |A| \cup |B'|$. Then, since we assumed that $|B| \geq |A|$, we have $n'/3 + \omega n' \leq |A|$, $|B'| \leq 2n'/3 - \omega n'$, so we can apply (b) to $G[A \cup B']$ with A , B' and X in place of A , B and S respectively to obtain a triangle-tiling \mathcal{T}_1 in G which covers every vertex of G except for the vertices of $V(M')$ and precisely $(1 - \varepsilon)\phi n$ vertices of X . So, taking X' to be the vertices of X not covered by \mathcal{T}_1 , we have $|X'| = |M'|$. By the choice of X and M' it follows that $H[X', M']$ contains a perfect matching, which corresponds to a perfect triangle-tiling \mathcal{T}_2 in $G[X' \cup V(M')]$. This gives a perfect triangle-tiling $\mathcal{T}_1 \cup \mathcal{T}_2$ in G . \square

We now turn to the case where the ‘core’ consists of three sets which form three super-regular pairs, for which the following lemma is analogous to Lemma 3.1.

Lemma 3.2. *Suppose that $1/n \ll \gamma, \varepsilon \ll d, \omega$, and that 3 divides n . Let V_1, V_2 and V_3 be disjoint sets of vertices with $|V_i| \geq n/6 + \omega n$ for each $i \in [3]$ such that $V := \bigcup_{i \in [3]} V_i$ has size $|V| = n$. Let G be a graph on vertex set V with $\alpha(G) \leq \gamma n$ such that $G[V_i, V_j]$ is (d, ε) -super-regular for each distinct $i, j \in [3]$. Then G contains a perfect triangle-tiling.*

To prove Lemma 3.2 we use the celebrated Blow-up Lemma of Komlós, Sárközy and Szemerédi [11] to obtain a perfect triangle-tiling. For simplicity, we state this only in the (very) special case that we use. Note that our definition of super-regularity differs slightly from theirs, but it is not hard to show that the two definitions are equivalent up to some modification of the constants involved (see, for example, [15, Fact 2]), so the validity of Theorem 3.3 is unaffected.

Theorem 3.3 (Blow-up Lemma for triangle-tilings). *Suppose that $1/n \ll \varepsilon \ll d$. Let A, B and C be disjoint sets of vertices with $|A| = |B| = |C| = n$, and let G be a graph on vertex set $V := A \cup B \cup C$ such that $G[A, B]$, $G[B, C]$ and $G[C, A]$ are each (d, ε) -super-regular. Then G contains a perfect triangle-tiling.*

The proof of Lemma 3.2 proceeds by iteratively deleting triangles from G with two vertices in one cluster and one in another cluster, until the same number of vertices remain in each cluster. We complete the proof by applying the Blow-up Lemma to obtain a perfect triangle-tiling covering all remaining vertices.

Proof of Lemma 3.2. Throughout this proof we perform addition on subscripts modulo 3. For each $i \in [3]$, the fact that $G[V_i, V_{i+1}]$ is (d, ε) -super-regular implies that each vertex $v \in V_i$ has $|N(v) \cap V_{i+1}| \geq (d - \varepsilon)|V_{i+1}| \geq dn/6$. So if we choose uniformly at random a set $Z_j \subseteq V_j$ of size ωn for each $j \in [3]$, then $|N(v) \cap Z_{i+1}|$ is hypergeometrically distributed with expectation at least $d\omega n/6$. By Theorem 2.3 the probability that v has fewer than $d\omega n/7$ neighbours in $|Z_{i+1}|$ declines exponentially with n , and likewise the same is true of the probability that v has fewer than $d\omega n/7$ neighbours in $|Z_{i+2}|$. Taking a union bound, with positive probability it holds that for each $i \in [3]$ every vertex $v \in V_i$ has at least $d\omega n/7$ neighbours in each of Z_{i+1} and Z_{i+2} . We fix such an outcome of our random selection of the sets Z_j , and define $X_i^0 = V_i \setminus Z_i$ for each $i \in [3]$. Without loss of generality we may assume that $\frac{n}{6} \leq |X_1^0| \leq |X_2^0| \leq |X_3^0| \leq \frac{2n}{3} - 3\omega n$.

We now proceed by an iterative process. At time step $t \geq 0$, if we have $|X_1^t| = |X_2^t| = |X_3^t|$ then we terminate. Otherwise, we choose a triangle xyz in G with $x \in X_2^t$ and $y, z \in X_3^t$ (we shall explain shortly why this will always be possible). We then set $Y_j^{t+1} := X_j^t \setminus \{x, y, z\}$ for $j \in [3]$ and define X_1^{t+1}, X_2^{t+1} and X_3^{t+1} such that $\{X_1^{t+1}, X_2^{t+1}, X_3^{t+1}\} = \{Y_1^{t+1}, Y_2^{t+1}, Y_3^{t+1}\}$ and $|X_1^{t+1}| \leq |X_2^{t+1}| \leq |X_3^{t+1}|$, before proceeding to the next time step $t + 1$.

Suppose that this procedure does not terminate prior to some time step T . Using the fact that 3 divides n it is easily checked that we must then have $|X_3^{t+2}| - |X_1^{t+2}| \leq |X_3^t| - |X_1^t| - 3$ for each $t \in [T - 2]$. In other words, the size difference between the smallest and largest set decreases by at least 3 over each two time steps. Similarly we find that $|X_1^t| - |X_1^{t+2}| \leq 1$ for each $t \in [T - 2]$, meaning that the smallest set size decreases by at most 1 over each two time steps. Furthermore, if at some time t we have $0 < |X_3^t| - |X_1^t| < 3$, then (since 3 divides n) we must have $|X_1^t| + 2 = |X_2^t| + 1 = |X_3^t|$, whereupon the procedure will terminate at time $t + 1$. It follows that the procedure must terminate at some time T , and moreover that

$$T \leq \frac{2}{3} (|X_3^0| - |X_1^0|) \leq \frac{2}{3} \left(\left(\frac{2n}{3} - 3\omega n \right) - \frac{n}{6} \right) = \frac{n}{3} - 2\omega n.$$

This implies that at each time $t < T$ we have $|X_3^t| \geq |X_2^t| \geq |X_1^t| \geq |X_1^0| - \lceil \frac{t}{2} \rceil \geq |X_1^0| - \frac{T}{2} \geq \omega n$, and so throughout the procedure it is always possible to pick a triangle as desired. Indeed, $G[X_2^t, X_3^t]$ is $(\geq d, \varepsilon/\omega)$ -regular by the Slicing Lemma (Lemma 2.1), so some vertex of X_2^t has at least $(d - \varepsilon/\omega)|X_3^t| \geq \omega dn/2$ neighbours in X_3^t . Since $\alpha(G) \leq \gamma n < \omega dn/2$ some two of these neighbours must be adjacent, giving the desired triangle.

After the procedure terminates, define $V_i' := X_i^T \cup Z_i$ for each $i \in [3]$. Then $|V_1'| = |V_2'| = |V_3'| \geq 2\omega n$, so by Lemma 2.1 and our choice of the sets Z_j it follows that $G[V_i', V_j']$

is $(d\omega/7, \varepsilon/2\omega)$ -super-regular for each distinct $i, j \in [3]$. By Theorem 3.3 there is a perfect triangle-tiling in $G[\bigcup_{i \in [3]} V_i']$; together with the triangles selected by the iterative procedure this gives a perfect triangle-tiling in G . \square

4 Proof of Theorem 1.2

In this section we prove Theorem 1.2. The following lemma is the central part of the proof, showing that if a graph G can be decomposed into clusters which form regular and super-regular pairs, indexed by a graph R which admits a bounded degree spanning tree, then by ‘working inwards’ from the leaves of the tree we can form a perfect triangle-tiling in G .

Lemma 4.1. *Suppose that $1/m \ll \gamma \ll 1/k \ll \varepsilon \ll d, \omega$. Let G be a graph whose vertex set is partitioned into k sets V_1, \dots, V_k , and let R be a graph with vertex set $[k]$ which admits a spanning tree T of maximum degree at most 10. Suppose also that the following statements hold.*

- (a) $|V_1| \geq (1 - \varepsilon)m$.
- (b) V_1 admits either a partition into parts A_1 and B_1 with $|A_1|, |B_1| \geq (1/3 + \omega)|V_1|$ such that $G[A_1, B_1]$ is $(1/2 + d, \varepsilon)$ -super-regular, or a partition into parts A_1, B_1 and C_1 with $|A_1|, |B_1|, |C_1| \geq (1/6 + \omega)|V_1|$ such that $G[A_1, B_1]$, $G[A_1, C_1]$ and $G[B_1, C_1]$ are each (d, ε) -super-regular.
- (c) For each $2 \leq i \leq k$, $(1 - \varepsilon)m \leq |V_i| \leq m$ and V_i admits a partition into parts A_i and B_i with $|A_i|, |B_i| \geq (1/3 + \omega)m$ such that $G[A_i, B_i]$ is (d, ε) -super-regular.
- (d) If $ij \in E(R)$, then at least $m/5$ vertices of V_i have at least $dm/5$ neighbours in V_j .
- (e) $\alpha(G) \leq \gamma m$.

Then G contains a triangle-tiling covering all but at most two vertices of G .

Proof. Introduce new constants ϕ and ε' with $\varepsilon \ll \phi \ll \varepsilon' \ll d$ and iterate the following process. Pick a leaf of T other than vertex 1, say vertex i , and let j be the neighbour of i in T . We will show that there exists a triangle-tiling in $G[V_i \cup V_j]$ that covers every vertex of V_i and at most $2\phi m$ vertices of V_j . We then delete the vertices covered by this tiling from G and delete vertex i from T . We proceed in this way until only vertex 1 of T remains. We then arbitrarily delete at most two further vertices of V_1 so that the number of remaining vertices in V_1 is divisible by three. Since, at this point, we have removed at most $2\phi m \cdot \Delta(T) + 2 \leq 21\phi m \leq \varepsilon' m/7$ vertices from V_1 , by (a), (b) and (e) there exists a bipartition or tripartition of the remaining vertices of V_1 which satisfies the conditions of Lemma 3.1(c) or Lemma 3.2 respectively (with $\omega/2$, ε' and 2γ in place of ω , ε and γ respectively). In either case there is a perfect triangle-tiling in the graph induced by the remaining vertices of V_1 , which together with the deleted triangle-tilings gives a triangle-tiling in G covering every vertex except for the at most two deleted vertices.

It therefore suffices to show that we can find the desired triangle-tiling in $G[V_i \cup V_j]$ at each step of this process. To this end, let S' be the set of vertices of V_i which have at least $dm/6$ neighbours in V_j . Observe that previous deletions can have removed at most $2\phi m \cdot \Delta(T) \leq dm/30$ vertices from each of V_i and V_j , so by (d) we have $|S'| \geq m/6$, and by (c) the remaining vertices of V_i can be partitioned into parts A_i and B_i with

$|A_i|, |B_i| \geq (1/3 + \omega/2)m$ such that $G[A_i, B_i]$ is (d, ε') -super-regular. Without loss of generality we may assume that $|S' \cap A_i| \geq |S' \cap B_i|$, so $|S' \cap A_i| \geq |S'|/2 \geq m/12$ and we can arbitrarily select $S \subseteq S' \cap A_i$ of size ϕn . Now we may use Lemma 3.1(b) (again with $\omega/2, \varepsilon'$ and 2γ in place of ω, ε and γ respectively) to find a triangle-tiling \mathcal{T} in $G[V_i]$ which covers every vertex of $V_i \setminus S$. Since each uncovered vertex has at least $dm/6 \geq 2\phi m + \gamma m$ neighbours in V_j , we may greedily extend \mathcal{T} to a triangle-tiling \mathcal{T}' in G which covers every vertex of V_i and which covers at most $2\phi m$ vertices of V_j . \square

It now suffices to show that for every graph G satisfying the conditions of Theorem 1.2, we can delete triangles and/or vertices from G to obtain a subgraph whose structure meets the conditions of Lemma 4.1. The following lemma shows how to do this under the additional assumption that G has no large sparse cut; this assumption is useful as it allows us to assume that the reduced graph R of G is connected, and so has spanning trees of bounded maximum degree. For this we make the following definition: given a graph G and a partition $\{A, B\}$ of $V(G)$, we say that an edge of G is (A, B) -crossing if it has one endvertex in A and one endvertex in B .

Lemma 4.2. *For every $\omega, \psi > 0$ there exist $n_0, \gamma > 0$ such that the following statement holds. Let G be a graph on $n \geq n_0$ vertices with $\delta(G) \geq n/3 + \omega n$ and $\alpha(G) \leq \gamma n$. Suppose additionally that for every partition $\{A, B\}$ of $V(G)$ with $|A|, |B| \geq n/3$ there are at least ψn^2 -many (A, B) -crossing edges of G . Then G contains a triangle-tiling covering all but at most two vertices of G (so in particular, if 3 divides n then G contains a perfect triangle-tiling).*

Proof. Introduce new constants satisfying the following hierarchy:

$$1/n \ll \gamma \ll 1/D \ll 1/T \ll 1/t \ll \varepsilon' \ll \varepsilon \ll d \ll \omega, \psi.$$

Then we may assume that n and T are large enough to apply Theorem 2.5 with constants $\varepsilon'/2, d, t$ and $q = 1$. We also assume without loss of generality that ω^{-1} is an integer, and define $D' := 30\omega^{-1}(D!)$. Let G be as in the statement of the lemma, and apply Theorem 2.5 to G to obtain a spanning subgraph $G' \subseteq G$, an integer k' with $t \leq k' \leq T$, an exceptional set U_0 of size at most $\varepsilon'n/2$ and clusters $U_1, \dots, U_{k'}$ of equal size. We now remove at most D' vertices from each cluster so that the number of remaining vertices in each cluster is divisible by D' , and add all removed vertices to the exceptional set U_0 . Since the total number of vertices moved in this way is at most $D'k' \leq 30\omega^{-1}(D!)T \leq \varepsilon'n/2$, and at most $D' \leq \varepsilon'n/2T \leq \varepsilon'/2|U_i|$ vertices are removed from each cluster U_i , by Lemma 2.1 the resulting partition of $V(G)$ into $U_0, U_1, \dots, U_{k'}$ has the following properties.

- (i) $|U_0| \leq \varepsilon'n$ and $|U_1| = |U_2| = \dots = |U_{k'}| =: m'$, where D' divides m' .
- (ii) $d_{G'}(v) \geq d_G(v) - (\varepsilon' + d)n \geq n/3 + 2\omega n/3$ for all $v \in V(G)$.
- (iii) $e(G'[U_i]) = 0$ for all $i \in [k']$.
- (iv) for each distinct $i, j \in [k']$ either $G'[U_i, U_j]$ is $(\geq d, \varepsilon')$ -regular or $G'[U_i, U_j]$ is empty.

In particular (i) implies that $(1 - \varepsilon')n/k' \leq m' \leq n/k'$. We form the reduced graph R on vertex set $[k']$ in the usual way, that is, with $ij \in E(R)$ if and only if $e(G'[U_i, U_j]) > 0$. For each $i \in [k']$ the number of edges of G' with an endvertex in U_i is at least $m'(n/3 + 2\omega n/3)$ by (ii). Also, by (iii) there is no edge in $G'[U_i]$, and by (i) there are at most at most $m'\varepsilon'n$

edges in $G'[U_0, U_i]$. Since for each $j \in [k']$ there are at most $(m')^2$ edges in $G'[U_i, U_j]$, it follows that

$$\delta(R) \geq \frac{m'(n/3 + 2\omega n/3) - m'\varepsilon'n}{(m')^2} \geq \left(\frac{1}{3} + \frac{2\omega}{3} - \varepsilon'\right) \frac{n}{m'} \geq \left(\frac{1}{3} + \frac{\omega}{2}\right) k'. \quad (3)$$

Now consider a partition $\{A_R, B_R\}$ of $[k']$ with $|A_R|, |B_R| \geq \delta(R)$, and define $A := U_0 \cup \bigcup_{i \in A_R} U_i$ and $B := \bigcup_{i \in [k'] \setminus B_R} U_i$. Then

$$|A|, |B| \geq \delta(R)m' \geq \left(\frac{1}{3} + \frac{\omega}{2}\right) k' \cdot \frac{(1 - \varepsilon')n}{k'} \geq \frac{n}{3},$$

so by assumption G has at least ψn^2 -many (A, B) -crossing edges. By (ii) at most $(d + \varepsilon')n^2$ edges of G are not in G' , and by (i) at most $\varepsilon'n^2$ edges of G intersect U_0 , so G' contains at least $\psi n^2 - (d + \varepsilon')n^2 - \varepsilon'n^2 > 0$ edges which are (A, B) -crossing but do not intersect U_0 . Let U_i and U_j be clusters containing the endvertices of some such edge; then ij is an (A_R, B_R) -crossing edge of R . In other words, for every partition $\{A_R, B_R\}$ of $[k']$ with $|A_R|, |B_R| \geq \delta(R)$ there is an (A_R, B_R) -crossing edge of R . Since every connected component of R has size at least $\delta(R)$, it follows that R is connected.

We now form a set V_1 from which we shall form the ‘core’ set of vertices mentioned in the proof overview at the beginning of Section 3. Suppose first that there exist $i, j \in [k']$ with $d(G'[U_i, U_j]) \geq 2/3$. Then $G'[U_i, U_j]$ is $(\geq 3/5, \varepsilon')$ -regular by (iv). In this case we define $V_1 := U_i \cup U_j$, and for convenience of notation later we define $X_1 := U_i$ and $Y_1 := U_j$. Now suppose instead that $d(G'[U_i, U_j]) < 2/3$ for every $i, j \in [k']$, that is, that each $G'[U_i, U_j]$ has at most $2(m')^2/3$ edges. Then we have an extra factor of $2/3$ in the denominator of the second term of (3), so we have $\delta(R) \geq k'/2$, and so R contains a triangle $ij\ell$ by Mantel’s theorem. In this case we take $V_1 := U_i \cup U_j \cup U_\ell$ and set $X_1 := U_i$, $Y_1 := U_j$, and $Z_1 := U_\ell$. We define an auxiliary graph R_0 to be the subgraph of R formed by deleting vertices i and j in the former case, and by deleting vertices i, j and ℓ in the latter case.

Since ω^{-1} is an integer, we may write $\eta := 1/3 + \omega/10$ as a rational number with denominator $L := 30 \cdot \omega^{-1}$. Let \vec{R}_0 be the directed graph formed from R_0 by replacing each edge by a pair of edges, one in each direction. Then by Lemma 2.10, we can find a perfect $(\eta, 1 - \eta)$ -weighted fractional matching w in \vec{R}_0 in which all weights are rational, and the least common denominator L' of all weights is bounded above by a function of $|V(R_0)|$ and L , that is, a function of k' and ω . Since $k' \leq 1/T$ and we assumed that $1/D \ll 1/T, \omega$, we may assume that $L' \leq D$, so L' divides $D!$, and so $D!w_{\vec{ij}}$ is an integer for every $\vec{ij} \in \vec{R}_0$. Define $m := m'/D!$, and observe that that since $D' = D!L$ divides m by (i), both m and ηm are integers.

We now partition each cluster not contained in V_1 into parts of size ηm and $(1 - \eta)m$ according to the weights in w , using the following probabilistic argument. For every $i \in V(R_0)$, we select a partition \mathcal{U}_i of U_i uniformly at random from all such partitions in which exactly $\sum_{j \in N^+(i)} D!w_{\vec{ij}}$ sets are of size ηm and exactly $\sum_{j \in N^-(i)} D!w_{\vec{ji}}$ sets are of size $(1 - \eta)m$. Since w is a perfect fractional $(\eta, 1 - \eta)$ -weighted matching, by (1) we have

$$\eta m \sum_{j \in N^+(i)} D!w_{\vec{ij}} + (1 - \eta)m \sum_{j \in N^-(i)} D!w_{\vec{ji}} = D!m = m' = |U_i|,$$

so we can indeed partition U_i in this way. We also consider the two or three clusters contained in V_1 to be partitioned into a single part. That is, for each $i \in [k'] \setminus V(R_0)$ we set \mathcal{U}_i to be the trivial partition $\{U_i\}$ of U_i . Now consider any edge $ij \in E(R)$, and recall that $G'[U_i, U_j]$ is $(\geq d, \varepsilon')$ -regular by (iv), so by Lemma 2.4¹, with probability at least $1 - e^{-\Omega(n)}$ we have that $G'[U'_i, U'_j]$ is $(\geq d, \varepsilon)$ -regular for every $U'_i \in \mathcal{U}_i$ and for every $U'_j \in \mathcal{U}_j$. Taking a union bound over all of the at most $\binom{k'}{2}$ edges of R we find that with positive probability this property holds for every edge of R . Fix a choice of partitions \mathcal{U}_i for $i \in [k']$ for which this is the case.

We now define another auxiliary graph R_1 with vertex set $\bigcup_{i \in [k']} \mathcal{U}_i$ in which, for each distinct $i, j \in [k']$, each $X \in \mathcal{U}_i$ and each $Y \in \mathcal{U}_j$, there is an edge XY if and only if $G'[X, Y]$ is $(\geq d, \varepsilon)$ -regular. Observe that by our choice of partitions \mathcal{U}_i the graph R_1 is then a blow-up of R , formed by replacing each vertex $i \in [k']$ by a set of $|\mathcal{U}_i|$ vertices and replacing each edge $ij \in E(R)$ by a complete bipartite graph between the corresponding sets. In particular, R_1 is connected. Also note that for each distinct $i, j \in [k']$ with $ij \notin E(R)$, each $X \in \mathcal{U}_i$ and each $Y \in \mathcal{U}_j$, the graph $G'[X, Y]$ is empty by (iv).

Next, for every edge $\vec{ij} \in E(\vec{R}_0)$, we define $s_{ij} := D! \cdot w_{\vec{ij}}$. We then label s_{ij} of the sets in \mathcal{U}_i of size ηm as $X_{ij}^1, \dots, X_{ij}^{s_{ij}}$ and label s_{ij} of the sets in \mathcal{U}_j of size $(1 - \eta)m$ as $Y_{ij}^1, \dots, Y_{ij}^{s_{ij}}$. Since \mathcal{U}_i has exactly $\sum_{j \in N^+(i)} s_{ij}$ sets of size ηm and exactly $\sum_{j \in N^-(i)} s_{ji}$ sets of size $(1 - \eta)m$, we may do this so that for each $i \in [k']$ each set in \mathcal{U}_i is uniquely labelled. We now relabel the sets X_{ij}^ℓ and Y_{ij}^ℓ for $\vec{ij} \in E(\vec{R}_0)$ and $\ell \in s_{ij}$ as X_2, \dots, X_k and Y_2, \dots, Y_k respectively, where $k - 1 := \sum_{\vec{ij} \in E(\vec{R}_0)} s_{ij} = D! \sum_{\vec{ij} \in E(\vec{R}_0)} w_{ij} = D! |V(R_0)|$ since w is perfect, so $k' \leq k \leq D!k'$. Then for each $2 \leq \ell \leq k$ our choice of partition implies that $G'[X_\ell, Y_\ell]$ is $(\geq d, \varepsilon)$ -regular; we define $V_\ell := X_\ell \cup Y_\ell$, and observe that $|V_\ell| = m$.

We now define a final auxiliary graph R^* with vertex set $[k]$ in which ij is an edge of R^* if and only if $e(G'[V_i, V_j]) > 0$. Observe that R^* is then a contraction of R_1 , in which the vertices of R_1 corresponding to the sets X_1 and Y_1 (and Z_1 if defined) are contracted to the single vertex 1 of R^* , and for $2 \leq i \leq k$ the vertices of R_1 corresponding to X_i and Y_i are contracted to the single vertex i of R^* . So, since R_1 is connected, R^* is connected also. Now suppose that ij is an edge of R^* . Since $G'[V_i, V_j]$ is nonempty there must exist sets $S \in \{X_i, Y_i, Z_i\}$ and $T \in \{X_j, Y_j, Z_j\}$ such that $G'[S, T]$ is non-empty (ignore Z_i unless $i = 1$ and Z_1 exists, and likewise for Z_j). We then have $S \in \mathcal{U}_{i'}$ and $T \in \mathcal{U}_{j'}$ for some $i', j' \in [k']$, so ST is an edge of R_1 , and so $G'[S, T]$ is $(\geq d, \varepsilon)$ -regular. Also, a similar calculation to (3) shows that we must have $\delta(R^*) \geq k/3$, so by Theorem 2.7 there is a spanning tree T in R^* with $\Delta(T) \leq 3$.

To recap, at this point we have a formed a partition $\{U_0, V_1, \dots, V_k\}$ of $V(G)$ and a graph R^* with vertex set $[k]$ which contains a spanning tree of maximum degree at most 3, such that the following statements hold.

- (v) V_1 admits either a partition $\{X_1, Y_1\}$ with $|X_1| = |Y_1| = m'$ such that $G'[X_1, Y_1]$ is $(\geq 3/5, \varepsilon')$ -regular, or a partition $\{X_1, Y_1, Z_1\}$ with $|X_1| = |Y_1| = |Z_1| = m'$ such that $G'[X_1, Y_1]$, $G'[X_1, Z_1]$ and $G'[Y_1, Z_1]$ are each $(\geq d, \varepsilon')$ -regular.

¹Note that m is much smaller than $\varepsilon' m'$ (since D is much larger than $1/\varepsilon'$) so we must use the random slicing lemma (Lemma 2.4) here, as opposed to, say, the standard slicing lemma (Lemma 2.1).

- (vi) For each $2 \leq i \leq k$, we have $|V_i| = m$ and V_i admits a partition $\{X_i, Y_i\}$ with $|X_i|, |Y_i| \geq \eta m = (1/3 + \omega/10)m$ such that $G'[X_i, Y_i]$ is $(\geq d, \varepsilon)$ -regular.
- (vii) If $ij \in E(R^*)$, then there are sets $S \subseteq V_i$ and $T \subseteq V_j$ with $|S| \geq |V_i|/3$ and $|T| \geq |V_j|/3$ such that $G'[S, T]$ is $(\geq d, \varepsilon)$ -regular.

If we are in the first case of (v), then by Lemma 2.2 we may choose subsets $A_1 \subseteq X_1$ and $B_1 \subseteq Y_1$ with $|A_1|, |B_1| \geq (1 - \varepsilon')m'$ such that $G'([A_1, B_1])$ is $(3/5, 2\varepsilon')$ -super-regular, and we then define $W_1 := A_1 \cup B_1$. If we are instead in the second case, by three applications of Lemma 2.2 we may choose subsets $A_1 \subseteq X_1$, $B_1 \subseteq Y_1$ and $C_1 \subseteq Z_1$ with $|A_1|, |B_1|, |C_1| \geq (1 - 2\varepsilon')m'$ such that $G'([A_1, B_1])$, $G'([B_1, C_1])$ and $G'([C_1, A_1])$ are each $(d, 3\varepsilon')$ -super-regular, and we then define $W_1 := A_1 \cup B_1 \cup C_1$.

Next, for each $2 \leq \ell \leq k$, by (vi) and Lemma 2.2 we may choose subsets $A_\ell \subseteq X_\ell$ and $B_\ell \subseteq Y_\ell$ with $|A_\ell| \geq (1 - \varepsilon)|X_\ell|$ and $|B_\ell| \geq (1 - \varepsilon)|Y_\ell|$ such that $G'[A_\ell, B_\ell]$ is $(d, 2\varepsilon)$ -super-regular, and define $W_\ell := A_\ell \cup B_\ell$. Finally, define $W_0 := U_0 \cup \bigcup_{i \in [k]} V_i \setminus W_i$. Then $\{W_0, W_1, \dots, W_k\}$ is a partition of $V(G)$ and, since $|W_i| \geq (1 - \varepsilon)|V_i|$ for each $i \in [k]$, we have $|W_0| \leq 2\varepsilon n$.

Write $W_0 := \{x_1, \dots, x_q\}$, so $q \leq 2\varepsilon n$. To complete the proof we greedily form a triangle-tiling $\mathcal{T} = \{T_1, \dots, T_q\}$ such that $x_i \in T_i$ for each $i \in [q]$ and $|V(\mathcal{T}) \cap W_j| \leq 20\varepsilon|W_j|$ for each $j \in [k]$. To see that this is possible, suppose that we have already chosen triangles T_1, \dots, T_{s-1} for some $s \in [q]$, let $X := \bigcup_{i \in [s-1]} V(T_i)$ be the set of vertices covered by these triangles, and let the set X' consist of all vertices in sets W_i with $|X \cap W_i| \geq 18\varepsilon|W_i|$ (that is, from which the previously-chosen triangles cover more than a 18ε -proportion of the vertices). Then we have $18\varepsilon|X'| \leq |X| \leq 3q \leq 6\varepsilon n$, so $|X'| \leq n/3$, and so x_s has at least $\delta(G) - |X| - |X'| - |W_0| \geq \omega n - 10\varepsilon n \geq \omega n/2$ neighbours not in X , X' or W_0 , so (since $\alpha(G) \leq \gamma n < \omega n/2$) two of these neighbours must be adjacent, giving the desired triangle T_s containing x_s . Having chosen T_s in this way for every $s \in [q]$ to obtain \mathcal{T} , observe that since we chose each T_s to avoid every set W_i from which at least $18\varepsilon|W_i|$ vertices were covered by previously-chosen triangles, we must have $|V(\mathcal{T}) \cap W_i| \leq 20\varepsilon|W_i|$ for each $i \in [k]$, as desired.

Finally, for each $i \in [k]$ define $A'_i := A_i \setminus V(\mathcal{T})$, $B'_i := B_i \setminus V(\mathcal{T})$, $V'_i := W_i \setminus V(\mathcal{T})$. Also define $V' := V(G) \setminus V(\mathcal{T})$ and $H := G[V']$. We claim that the graphs H and R^* and the partition $\{V'_1, \dots, V'_k\}$ of $V(H)$ meet the properties of Lemma 4.1 with $\varepsilon^* := 200\varepsilon$, $\omega' := \omega/20$ and $\gamma' := 2\gamma k'(D!)$ in place of ε , ω and γ respectively and with m, d and k playing the same role there as here. Indeed, our constant hierarchy allows us to assume that $1/m \ll \gamma' \ll 1/k \ll \varepsilon^* \ll d \ll \omega'$, as required. Also observe that for each $i \in [k]$ we have $|V'_i| \geq |V_i| - 20\varepsilon|V_i| - \varepsilon|V_i| = (1 - 21\varepsilon)|V_i|$, so certainly $|V'_i| \geq (1 - \varepsilon^*)m$ for each $i \in [k]$. So Lemma 4.1(a) holds, and Lemma 4.1(b) and (c) follow immediately from our choice of sets A_ℓ and B_ℓ (and possible C_1). Also, for each $ij \in E(R^*)$ by (vii) there exist sets $S \subseteq V'_i$ and $T \subseteq V'_j$ with $|S| \geq |V'_i|/4$ and $|T| \geq |V'_j|/4$ such that $G'[S, T]$ is $(\geq d, 2\varepsilon)$ -regular, which implies that at least $(1 - 2\varepsilon)|S| \geq m/5$ vertices in S have at least $(d - 2\varepsilon)|T| \geq dm/5$ neighbours in T , so Lemma 4.1(d) holds. Last of all, Lemma 4.1(e) holds since $\alpha(H) \leq \alpha(G) \leq \gamma n \leq \gamma(2k'm') = \gamma'm$. So we may apply Lemma 4.1 to obtain a triangle-tiling covering all but at most two vertices of H ; together with \mathcal{T} this yields a triangle-tiling in G covering all but at most two vertices. \square

Finally, to complete the proof of Theorem 1.2 it remains only to consider graphs G which admit a large sparse cut. In this case we show that can remove a small number

of vertices to obtain two vertex-disjoint subgraphs G_A and G_B of G whose vertex sets partition $V(G)$ and each of which satisfies a stronger minimum degree condition. We then apply Theorem 1.1 to obtain a perfect triangle-tiling in each of G_A and G_B (alternatively, one could note that the stronger minimum degree conditions preclude either G_A or G_B from having a large sparse cut and apply Lemma 4.2).

Proof of Theorem 1.2. Fix $\omega > 0$ and choose n_0 sufficiently large and γ sufficiently small for Lemma 4.2 with $\omega^2/40$ in place of ψ and also so that we can apply Theorem 1.1 with $\omega/2$, $n_0/3$ and 3γ in place of ω , n_0 and γ respectively. Now let G be a graph on $n \geq n_0$ vertices with $\delta(G) \geq n/3 + \omega n$ and $\alpha(G) \leq \gamma n$. If for every partition $\{A, B\}$ of $V(G)$ with $|A|, |B| \geq n/3$ there are at least $\omega^2 n^2/40$ -many (A, B) -crossing edges of G , then G contains a triangle-tiling covering all but at most two vertices by Lemma 4.2, so we are done. So we may assume that for some partition $\{A, B\}$ of $V(G)$ with $|A|, |B| \geq n/3$ there are fewer than $\omega^2 n^2/40$ -many (A, B) -crossing edges. Fix such a partition with the smallest number of (A, B) -crossing edges. Note that we cannot have $|A| \leq n/3 + 1$, as then there would be at least $|A|(\delta(G) - n/3 - 1) \geq (n/3) \cdot (\omega n - 1) \geq \omega n^2/4$ -many (A, B) -crossing edges. It follows that every vertex $x \in A$ lies in at most $\deg(x)/2$ -many (A, B) -crossing edges, as otherwise moving a from A to B would yield a partition of $V(G)$ with parts of size at least $n/3$ and with fewer (A, B) -crossing edges. So we must have $\delta(G[A]) \geq \delta(G)/2 \geq n/6 + \omega n/2$, and the same argument with B in place of A shows that $\delta(G[B]) \geq n/6 + \omega n/2$.

Our proof now diverges according to whether we are proving conclusion (a) or conclusion (b) of Theorem 1.2. For conclusion (a) we simply choose arbitrarily a set S of at most four vertices of G so that $|A \setminus S|$ and $|B \setminus S|$ are each divisible by 3. For conclusion (b) we instead use our additional assumptions that G has no divisibility barrier and that 3 divides n . Indeed, the latter implies that we must have one of the following three cases:

- (a) $|A| \equiv |B| \equiv 0 \pmod{3}$. In this case we take $S = \emptyset$.
- (b) $|A| \equiv 1 \pmod{3}$ and $|B| \equiv 2 \pmod{3}$. Since (A, B) is not a divisibility barrier, either G contains an B -triangle or a pair of vertex-disjoint A -triangles, and we take S to be the vertices covered by some such triangle or pair of triangles.
- (c) $|A| \equiv 2 \pmod{3}$ and $|B| \equiv 1 \pmod{3}$. Since (B, A) is not a divisibility barrier, either G contains an A -triangle or a pair of vertex-disjoint B -triangles, and we take S to be the vertices covered by some such triangle or pair of triangles.

Observe that in all cases we have $|S| \leq 6$ and that both $|A \setminus S|$ and $|B \setminus S|$ are divisible by 3. The remaining part of the proof is the same for both cases.

Let $X_A \subseteq A$ consist of all vertices of A with $\deg_{G[A]}(x) < n/3 + \omega n/2$. Then each vertex of X_A is contained in more than $\omega n/2$ -many (A, B) -crossing edges, and since there are at most $\omega^2 n^2/40$ -many (A, B) -crossing edges in total, each with one vertex in A , it follows that $|X_A| \leq \omega n/20$. Since $\alpha(G) \leq \gamma n$ and $\delta(G[A]) \geq n/6 \geq 2|X_A| + |S| + \gamma n$ we may greedily form a triangle-tiling \mathcal{T}_A of size at most $|X_A|$ in $G[A]$ which covers every vertex of X_A but which does not intersect S . We then define $A' := A \setminus (V(\mathcal{T}_A) \cup S)$, $G_A := G[A']$ and $n_A := |A'|$. Then $\delta(G_A) \geq n/3 + \omega n/2 - |V(\mathcal{T}_A)| - |S| \geq n/3 + \omega n/3$, so $n/3 + \omega n/3 \leq n_A \leq 2n/3$. It follows that G_A is a graph on n_A vertices with $\delta(G_A) \geq n_A/2 + \omega n_A/2$ and $\alpha(G_A) \leq \gamma n \leq 3\gamma n_A$. Also n_A is divisible by 3 (since 3 divides each of $|A \setminus S|$ and $|V(\mathcal{T}_A)|$), so G_A contains a perfect triangle-tiling \mathcal{T}'_A by Theorem 1.1.

By exactly the same argument with B in place of A we obtain a triangle-tiling \mathcal{T}_B in $G[B]$ and a graph G_B on vertex set $B' := B \setminus (V(\mathcal{T}_B) \cup S)$ which contains a perfect triangle-tiling \mathcal{T}'_B . Finally, for conclusion (a) observe that $\mathcal{T} := \mathcal{T}_A \cup \mathcal{T}_B \cup \mathcal{T}'_A \cup \mathcal{T}'_B$ is then a triangle-tiling in G covering all vertices outside S , that is, all but at most four vertices of G , and for conclusion (b) note that adding the triangle or triangles covering S to \mathcal{T} gives a perfect triangle-tiling in G . \square

5 Constructions and questions

Many of the ideas of this section are due to Balogh, Molla and Sharifzadeh [2], but we include them here for completeness.

We first consider the problem of finding perfect K_k -tilings instead of perfect triangle-tilings. By slightly modifying the construction of $G_4(m)$ given in the introduction we can give lower bounds for this question.

Question 5.1. *Let $k \geq 4$ and let G be an n -vertex graph with $\alpha(G) = o(n)$. What is the best-possible minimum degree condition on G that guarantees a perfect K_k -tiling in G ?*

The construction is slightly different depending on the parity of $k \geq 4$. We start with the odd case, so let $k = 2(\ell - 1) + 1$ for some integer $\ell \geq 3$. Consider the complete ℓ -partite graph with one part V_1 of size $n/k - 1$, another part V_2 of size $2n/k + 1$ and the remaining parts V_3, \dots, V_ℓ each of size $2n/k$, and place the Erdős graph $\text{ER}(|V_i|)$ on each of the parts V_i . When $k = 2\ell$ for some integer $\ell \geq 1$, the construction is essentially the same but we have one part of size $2n/k + 1$, one part of size $2n/k - 1$ and the remaining parts are each of size $2n/k$. In either case we obtain a graph G with $\delta(G) \geq (1 - \frac{2}{k})n + \omega(1)$, sublinear independence number and no K_k -factor. It is worth noting that in the odd case the graph G is K_{k+2} -free and in the even case G contains no K_{k+1} .

We feel that the following is another interesting related question.

Question 5.2. *Let G be an n -vertex K_4 -free graph with $\alpha(G) = o(n)$. What is the best-possible minimum degree condition on G that guarantees a perfect triangle-tiling in G ?*

We use a modified version of the Bollobás-Erdős graph [1] to construct a K_4 -free graph without a perfect triangle-tiling and with high minimum degree. For every large even n , the Bollobás-Erdős graph is an n -vertex, K_4 -free graph with sublinear independence number, which we denote by $\text{BE}(n)$. The vertex set of $\text{BE}(n)$ is the disjoint union of two sets V_1 and V_2 of the same order such that the graphs $G[V_1]$ and $G[V_2]$ are triangle-free and every vertex in V_1 has at least $(1/4 - o(1))n$ neighbors in V_2 and every vertex in V_2 has at least $(1/4 - o(1))n$ neighbors in V_1 . To construct our example, start with $\text{BE}(4n/3 + 2)$ and then remove a randomly selected subset of size $n/3 + 2$ from one of the two parts. Note that the two parts now have sizes $n/3 - 1$ and $2n/3 + 1$, the resulting graph clearly is K_4 -free and since the larger part is a space barrier, it has no perfect triangle-factor. Furthermore, with high probability, the minimum degree is $(1/6 - o(1))n$. We conjecture that $(1/6 + o(1))n$ is the proper minimum degree condition.

Conjecture 5.3. *For every $\omega > 0$ there exist $\gamma, n_0 > 0$ such that every K_4 -free graph on $n \geq n_0$ vertices with $\delta(G) \geq n/6 + \omega n$ and $\alpha(G) \leq \gamma n$ contains a perfect triangle-tiling.*

Using methods similar to those used in our proof of Theorem 1.2 we can show that every graph G which satisfies the conditions of Conjecture 5.3 has a triangle-tiling covering almost all of the vertices of G . More precisely, we can show that for $1/n \ll \gamma \ll \omega$, if G is a K_4 -free graph on n vertices with $\delta(G) \geq (1/6 + \omega)n$ and $\alpha(G) \leq \gamma n$, then G contains a triangle-tiling which covers all but at most ωn vertices. What follows is a brief sketch of the argument.

Apply Theorem 2.5 with $\gamma \ll \varepsilon \ll d \ll \omega$ to obtain a spanning subgraph $G' \subseteq G$, an exceptional set V_0 and clusters V_1, \dots, V_k of equal size m . Define the corresponding reduced graph R on vertex set $[k]$ in the usual way. The fact that G is K_4 -free implies the following two important facts about these clusters and the graph R . (These facts were first observed by Szemerédi in [13].)

- (a) there is no pair $i, j \in [k]$ for which $G'[V_i, V_j]$ is $(1/2 + d, \varepsilon)$ -regular, and
- (b) R is triangle-free.

Using a standard argument, it is not hard to see that (a) and the fact that $\delta(G) \geq (1/6 + \omega)n$ together imply that $\delta(R) \geq k/3$. So R must be connected, as otherwise Mantel's theorem would give a triangle in the smallest connected component of R , contradicting (b). By a result of Enomoto, Kaneko and Tuza [9], the fact that R is a connected graph on k vertices with $\delta(R) \geq k/3$ implies that R contains $\lfloor |R|/3 \rfloor$ vertex-disjoint copies of P_2 (the path on three vertices). In a manner similar to the proof of Lemma 3.1, for each such path ijk we can use the fact that $\alpha(G) \leq \gamma n$ to greedily construct a triangle-tiling covering all but at most $3.1\varepsilon m$ of the vertices of $G[V_i \cup V_j \cup V_k]$, where each triangle has one vertex in V_j , the central cluster in the path, and the other two vertices either both in V_i or both in V_k . The union of these $\lfloor |R|/3 \rfloor$ triangle-tilings is then a triangle-tiling in G which covers all but at most ωn vertices.

We can generalize Question 5.2 in the following way.

Question 5.4. *Let $k \geq 3$ and let G be an n -vertex K_{k+1} -free graph with $\alpha(G) = o(n)$. What is the best-possible minimum degree condition on G that guarantees a perfect K_k -tiling in G ?*

When k is even, we have previously shown that the minimum degree must be at least $\binom{k-2}{k} + o(1) n$. When $k = 2\ell + 1 \geq 5$, we form G by starting with the complete ℓ -partite graph that has one part V_1 of size $3n/k + 1$, one part V_2 of size $2n/k - 1$, and the remaining parts, V_3, \dots, V_ℓ , each of size $2n/k$. In V_1 , we place $\text{BE}(|V_1|)$ on V_1 , and, for every $2 \leq i \leq \ell$, we place a copy of $\text{ER}(|V_i|)$ on V_i . We then have $\delta(G) \geq \left(\frac{k-3}{k} + \frac{1}{4} \cdot \frac{3}{k} - o(1)\right) n = \left(\frac{4k-9}{4k} - o(1)\right) n$. Furthermore, G has sublinear independence number, is K_{k+1} -free, and has no perfect K_k -tiling, because each copy of K_k in G has at most 3 vertices in V_1 .

Finally, for $r \geq 3$, $\omega, \gamma > 0$ and sufficiently large n , we give the construction of $G := G_{\text{RT}}(n, r, \omega, \gamma)$ from Theorem 1.3(b). For odd r the construction was first given in [6] and for even r the construction is from [7]. We say that a partition V_1, \dots, V_ℓ of the vertices of a graph is *equitable* if $||V_i| - |V_j|| \leq 1$ for all $1 \leq i < j \leq \ell$.

When $r = 2\ell + 1$ is odd, we let V_1, \dots, V_ℓ be an equitable partition of $V(G)$ and form the complete ℓ -partite graph with vertex classes V_1, \dots, V_ℓ . For every $i \in [\ell]$, we then

place a copy of $\text{ER}(|V_i|)$ on V_i , so

$$\delta(G) \geq n - \left\lceil \frac{n}{\ell} \right\rceil \geq \left(\frac{r-3}{r-1} - \omega \right) n.$$

We can assume that n is large enough so that for each $i \in [\ell]$ the independence number of $G[V_i]$ is at most γn , which implies that $\alpha(G) \leq \gamma n$. Note that G is K_r -free, as $G[V_i]$ is K_3 -free for $i \in [\ell]$.

When $r = 2\ell$ is even, we let $U_1, \dots, U_{3\ell-2}$ be a equitable partition of $V(G)$, so $|U_i| \in \left\{ \lfloor \frac{2n}{3r-4} \rfloor, \lceil \frac{2n}{3r-4} \rceil \right\}$ for every $i \in [3\ell-2]$. Let

$$V_1 := U_1 \cup U_2 \cup U_3 \cup U_4 \quad \text{and} \quad V_i := U_{3i-1} \cup U_{3i} \cup U_{3i+1} \quad \text{for } 2 \leq i \leq \ell-1,$$

and form the complete $(\ell-1)$ -partite graph with vertex classes $V_1, \dots, V_{\ell-1}$. On V_1 , we then place a copy of $\text{BE}(|V_1|)$ and assume n is large enough so that $G[V_1]$ has minimum degree at least

$$\left(\frac{1}{4} - \omega \right) |V_1| \geq |V_1| - \left(\frac{6}{3r-4} + \omega \right) n$$

and independence number at most γn . For every $2 \leq i \leq \ell-1$, we place a copy of $\text{ER}(|V_i|)$ on V_i and we ensure that n is large enough so that the independence number of $G[V_i]$ is at most γn . Because every vertex in G is adjacent to all but at most $\left(\frac{6}{3r-4} + \omega \right) n$ vertices of G , we have that

$$\delta(G) \geq \left(\frac{3r-10}{3r-4} - \omega \right) n.$$

Furthermore, $\alpha(G) \leq \gamma n$ and G is K_r -free as $G[V_1]$ is K_4 -free and each of the subgraphs $G[V_2], \dots, G[V_{\ell-1}]$ is K_3 -free.

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6 Appendix

The purpose of this appendix is to prove Lemma 2.4. The lemma is essentially a corollary to the following two theorems of Kohayakawa and Rödl [10]. For this we use the following notation: let G be a bipartite graph with vertex classes A and B , and define $d := d(G[A, B])$. Then for any ε we define $D_{AB}(\varepsilon)$ to be the graph with vertex set A in which $x, x' \in A$ are adjacent if and only if

$$|N_G(x)|, |N_G(x')| > (d - \varepsilon)|B| \quad \text{and} \quad |N_G(x) \cap N_G(x')| < (d + \varepsilon)^2|B|.$$

Theorem 6.1 ([10, Theorem 45]). *Let $0 < \varepsilon < 1$, and let $G[A, B]$ be a bipartite graph with $|A| \geq 2/\varepsilon$. If $e(D_{AB}(\varepsilon)) > (1 - 5\varepsilon)|A|^2/2$, then $G[A, B]$ is $(d, (16\varepsilon)^{1/5})$ -regular, where $d := d(G[A, B])$.*

Theorem 6.2 ([10, Theorem 46]). *Let $0 < \varepsilon < 1$, and let $G[A, B]$ be a bipartite graph with $|B| \geq 1/d$, where $d := d(G[A, B])$. If $G[A, B]$ is (d, ε) -regular, then $e(D_{AB}(\varepsilon)) \geq (1 - 8\varepsilon)|A|^2/2$.*

The following two similar lemmas do most of the remaining work required to complete the proof.

Lemma 6.3. *Suppose that $1/n \ll \xi \ll \xi'$ and that $1/n \ll \beta$. Let $G[A, B]$ be a bipartite graph such that $|A|, |B| \leq n$, and let x_1, \dots, x_s and y_1, \dots, y_t be positive integers each of size at least βn such that $\sum_{i \in [s]} x_i \leq |A|$ and $\sum_{j \in [t]} y_j \leq |B|$. If $\{X_1, \dots, X_s\}$ is a collection of disjoint subsets of A and $\{Y_1, \dots, Y_t\}$ is a collection of disjoint subsets of B with $|X_i| = x_i$ and $|Y_j| = y_j$ for all $i \in [s]$ and $j \in [t]$ selected uniformly at random from all such collections, then, with probability at least $1 - e^{-\Omega(n)}$, for every $i \in [s]$, $j \in [t]$, $x, x' \in A$ and $y, y' \in B$ we have*

- (a) $|N_G(x) \cap Y_j|/y_j = |N_G(x)|/|B| \pm \xi$,
- (b) $|N_G(y) \cap X_i|/x_i = |N_G(y)|/|A| \pm \xi$,
- (c) $|N_G(x) \cap N_G(x') \cap Y_j|/y_j = |N_G(x) \cap N_G(x')|/|B| \pm \xi$,
- (d) $|N_G(y) \cap N_G(y') \cap X_i|/x_i = |N_G(y) \cap N_G(y')|/|A| \pm \xi$, and
- (e) $d(G[X_i, Y_j]) = d(G[A, B]) \pm \xi'$.

Proof. Note that the at most $t(|A| + |A|^2) + s(|B| + |B|^2) \leq 2\beta^{-1}(n + n^2)$ random variables of the form $|N_G(x) \cap Y_j|$, $|N_G(y) \cap X_i|$, $|N_G(x) \cap N_G(x') \cap Y_j|$, and $|N_G(y) \cap N_G(y') \cap X_i|$, where $i \in [s]$, $j \in [t]$, $x, x' \in A$ and $y, y' \in B$, are hypergeometrically distributed, so the fact that (a)-(d) hold with probability $1 - e^{-\Omega(n)}$ follows directly from Theorem 2.3 by taking a union bound. For (e), let $\ell := \xi^{-1}/2$ and define $D_k := \{v \in A : 2(k-1)\xi \leq |N(v)|/|B| < 2k\xi\}$ for each $k \in [\ell]$. Then, with probability $1 - e^{-\Omega(n)}$, for every $i \in [s]$ and $k \in [\ell]$, we have that

$$\frac{|D_k \cap X_i|}{x_i} = \frac{|D_k|}{|A|} \pm \xi^2.$$

Fix a choice of X_1, \dots, X_s and Y_1, \dots, Y_t , for which (a)-(d) hold and this event occurs. Note that for every $k \in [\ell]$, $v \in D_k$, and $j \in [t]$,

$$\frac{|N_G(v)|}{|B|} = (2k-1)\xi \pm \xi \quad \text{so} \quad \frac{|N_G(v) \cap Y_j|}{y_j} = (2k-1)\xi \pm 2\xi.$$

We compute $d(G[A, B])$ to be

$$\frac{1}{|A|} \sum_{k \in [\ell]} \sum_{v \in D_k} \frac{|N_G(v)|}{|B|} = \sum_{k \in [\ell]} \left(((2k-1)\xi \pm \xi) \cdot \frac{|D_k|}{|A|} \right) = \left(\sum_{k \in [\ell]} (2k-1)\xi \frac{|D_k|}{|A|} \right) \pm \xi.$$

Then for any $i \in [s]$ and $j \in [t]$ we have

$$\begin{aligned} d(G[X_i, Y_j]) &= \frac{1}{x_i} \sum_{k \in [\ell]} \sum_{v \in D_k \cap X_i} \frac{|N_G(v) \cap Y_j|}{y_j} = \sum_{k \in [\ell]} \left(((2k-1)\xi \pm 2\xi) \cdot \left(\frac{|D_k|}{|A|} \pm \xi^2 \right) \right) \\ &= \left(\sum_{k \in [\ell]} (2k-1)\xi \frac{|D_k|}{|A|} \right) \pm (\ell^2 \xi^3 + 2\xi + 2\ell \xi^3) = d(G[A, B]) \pm \xi', \end{aligned}$$

so (e) holds. \square

Lemma 6.4. *Suppose that $1/n \ll \xi \ll \xi'$ and $1/n \ll \beta$, and that x_1, \dots, x_s are positive integers each of size at least βn such that $\sum_{i \in [s]} x_i \leq n$. If G is a graph on n vertices and $\{X_1, \dots, X_s\}$ is a collection of disjoint subsets of $V(G)$ with $|X_i| = x_i$ for all $i \in [s]$ selected uniformly at random from all such collections, then, with probability at least $1 - e^{-\Omega(n)}$, for every $i \in [s]$ and $x, x' \in V(G)$ we have*

- (a) $|N_G(x) \cap X_i|/x_i = |N_G(x)|/n \pm \xi$,
- (b) $|N_G(x) \cap N_G(x') \cap X_i|/x_i = |N_G(x) \cap N_G(x')|/n \pm \xi$, and
- (c) $2e(G[X_i])/x_i^2 = 2e(G)/n^2 \pm \xi'$.

Proof. It is straightforward to modify the proof of Lemma 6.3 to prove this lemma; we omit the details. \square

Now we give the proof of Lemma 2.4.

Proof of Lemma 2.4. Introduce a new constant η with $1/n \ll \eta \ll \varepsilon$. Suppose that $G[A, B]$ is $(\geq d, \varepsilon)$ -regular, let $d^* := d(G[A, B])$, so $d^* = d \pm \varepsilon$, and define $D := D_{AB}(\varepsilon)$. Note that, by Theorem 6.2, we have that $2e(D)/|A|^2 \geq 1 - 8\varepsilon$. We apply Lemma 6.3 to $G[A, B]$ and Lemma 6.4 to D , with ξ' replaced by η in each case, to find that with probability $1 - e^{-\Omega(n)}$ our random selection satisfies the conclusions of each of these lemmas. We fix such an outcome of our random selection, and consider any $i \in [s]$ and $j \in [t]$. Define $d_{ij} := d(X_i, Y_j)$, so $d_{ij} = d^* \pm \eta$, and

$$d_{ij} = d \pm (\varepsilon + \eta). \quad (4)$$

We also have that

$$\frac{2e(D[X_i])}{x_i^2} \geq \frac{2e(D)}{|A|^2} - \eta \geq 1 - 8\varepsilon - \eta \geq 1 - 5(2\varepsilon).$$

Recall that, if $xx' \in E(D[X_i])$, then

$$\frac{|N_G(x)|}{|B|}, \frac{|N_G(x')|}{|B|} > d^* - \varepsilon \text{ and } \frac{|N_G(x) \cap N_G(x')|}{|B|} < (d^* + \varepsilon)^2,$$

so

$$\frac{|N_G(x) \cap Y_j|}{y_j}, \frac{|N_G(x') \cap Y_j|}{y_j} > (d^* - \varepsilon) - \eta > d_{ij} - 2\varepsilon,$$

and, as we can assume η is small enough so that $\eta^{1/2} + \eta < \varepsilon$,

$$\frac{|N_G(x) \cap N_G(x') \cap Y_j|}{y_j} < (d^* + \varepsilon)^2 + \eta < (d_{ij} + \eta + \varepsilon)^2 + (\varepsilon - \eta)^2 < (d_{ij} + 2\varepsilon)^2.$$

This proves that $xx' \in E(D_{X_i Y_j}(2\varepsilon))$, so D is a subgraph of $D_{X_i Y_j}(2\varepsilon)$. Therefore, by Lemma 6.1 with d and ε replaced by d_{ij} and 2ε , respectively, $G[X_i, Y_j]$ is $(d_{ij}, (32\varepsilon)^{1/5})$ -regular, and is therefore $(d, (32\varepsilon)^{1/5} + 2\varepsilon)$ -regular, because, by (4), $d = d_{ij} \pm 2\varepsilon$. Since we can assume that ε is small enough so that $(32\varepsilon)^{1/5} + 2\varepsilon \leq (33\varepsilon)^{1/5}$, it follows that $G[X_i, Y_j]$ is $(d, (33\varepsilon)^{1/5})$ -regular.

Clearly, if $G[A, B]$ is (d, ε) -super-regular, then, by (a) and (b) of Lemma 6.3, we can also ensure that $G[X_i, Y_j]$ is $(d, (33\varepsilon)^{1/5})$ -super-regular for each $i \in [s]$ and $j \in [t]$. \square