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Plural logic is widely assumed to have two important virtues: ontological innocence and determinacy. It is claimed to be innocent in the sense that it incurs no ontological commitments beyond those already incurred by the first-order quantifiers. It is claimed to be determinate in the sense that it is immune to the threat of non-standard (Henkin) interpretations that confronts higher-order logics on their more traditional, set-based semantics. We challenge both claims. Our challenge is based on a Henkin-style semantics for plural logic that does not resort to sets or set-like objects to interpret plural variables, but adopts the view that a plural variable has many objects as its values. Using this semantics, we also articulate a generalized notion of ontological commitment which enables us to develop some ideas of earlier critics of the alleged ontological innocence of plural logic.

1 Introduction

Plural logic is a form of higher-order logic which adds to first-order logic the plural quantifiers $\exists xx$ and $\forall xx$, interpreted respectively as 'there are some things xx such that ...' and 'whenever there are some things xx, then ...'. This logic has recently become an important component of the philosopher's toolkit. Interest in it is motivated in large part by two alleged virtues: ontological innocence and expressive power.

It is commonly assumed that plural logic is ontologically innocent in the sense that plural quantifiers do not incur ontological commitments beyond those incurred by the first-order quantifiers. This alleged virtue of plural logic is supported by an alternative approach to semantics for higher-order logic. Instead of letting the values of the higher-order variables be sets constructed from objects in the ordinary first-order domain, we help ourselves to higher-order resources in the metatheory and use these resources to represent the values of the higher-order variables of the object language. On the semantics for plural logic due to Boolos (1985)—which many philosophers now regard as its canonical semantics—the value of a plural variable is not a set (or any kind of set-like object) whose members are drawn from the ordinary, first-order domain. Rather, a plural variable has *many values* from this ordinary domain and thus *ranges plurally* over this domain. Of course, in ascribing to a plural variable many values, Boolos's semantics makes essential use of the plural resources of the metalanguage—this is why henceforth we will refer to it as *plurality-based* semantics. In a nutshell, on the traditional set-based semantics, a higher-order variable ranges in an ordinary way over a special domain reserved for variables of its type, whereas on the new kind of higher-order semantics, a higher-order variable ranges in a special, higher-order way over the ordinary domain.[1]

The second alleged virtue of plural logic is expressive power. To see this point, consider first the case of second-order logic with its two kinds of traditional set-based semantics. In standard semantics, the second-order quantifiers range over the full powerset of the first-order domain, whereas in Henkin semantics the second-order quantifiers may range over a subset of this powerset. This gives rise to an interesting debate about semantic determinacy.[2] Does our linguistic practice single out, relative to a given domain, the interpretation given by the standard semantics as the correct one? An important aspect of this question is that it is only on the standard semantics that second-order logic can truly be said to offer more expressive power than first-order logic. For second-oder logic on the Henkin semantics may be regarded as a version of first-order logic, namely a first-order system with two sorts of quantifiers. As such, it has all the main metalogical features of first-order logic: it is complete, compact, and has the Löwenheim-Skolem property. But, for the same reason, it fails with respect to the main accomplishments of second-order logic with the standard semantics. Chiefly, it does not discriminate between importantly different classes of structures, such as countable and uncountable ones, and it fails to ensure the categoricity of arithmetic and analysis, and the

quasi-categoricity of set theory.

In this respect, plural logic on the plurality-based semantics (as well as higher-order logic on the novel kind of semantics) is thought to provide a significant improvement over second-order logic on the set-based semantics. Indeed, one finds many claims to the effect that plural logic, on the plurality-based semantics, is immune to the threat of non-standard (Henkin) interpretations that confronts higher-order logics on their more traditional, set-based semantics. Nearly all writers who have embraced plural logic on the plurality-based semantics ascribe to this system metalogical properties which presuppose that the semantics is standard rather than Henkin, but without flagging this as a substantive presupposition as one would do as a matter of routine in the case of systems with a set-based semantics.[3]

A striking feature of the literature on this novel kind of semantics for higher-order logic is the near-absence of debate about the semantic determinacy of higher-order quantification thus interpreted.[4] Indeed, on the higher-order approach, the only interpretation of the higher-order quantifiers that has been articulated is the standard one. No analogue of Henkin semantics has been developed. The following diagram sums up the current situation:

kind of semantics	standard	Henkin
set-based	A. Tarski	L. Henkin
higher-order (e.g. plurality-based)[5]	G. Boolos	—

The apparent absence of a plurality-based Henkin semantics has no doubt influenced the ensuing debate. It has encouraged the thought that plural logic on the plurality-based semantics is immune from non-standard interpretation, and thus the thought that plural logic does better than higher-order logic on the set-based semantics in securing a gain in expressive power.

As appealing as this common picture of plural logic may be, we believe that it is far too optimistic. Our aim in this article is to develop an alternative picture, one in which both alleged virtues of plural logic—ontological innocence and expressive power—are much less significant than they are made out to be. We argue that set-based and plurality-based semantics are on a par with respect to worries about indeterminacy. Moreover, we articulate a generalized notion of ontological commitment according to which plural logic is not, after all, innocent. This provides, for the first time, a precise development of some ideas adumbrated by Parsons (1990: section 6), Hazen (1993), Shapiro (1993), and Linnebo (2003). Our focus

is on plural logic, though much of what we say would apply, *mutatis mutandis*, to second- and higher-order logics which quantify into predicate position.

Our pursuit of the mentioned aims uses as its main tool a semantics for plural logic that fills the gap in the above diagram. Accordingly, the first part of the article is devoted to the development and defense of a plurality-based Henkin semantics. (Technical details are relegated to an appendix.) In the second part of the paper, we reconsider the alleged virtues of plural logic in light of the new semantics. The resulting picture is one in which the role of plural logic as a philosophical tool appears substantially diminished.

2 A plurality-based Henkin semantics

As announced, our first step is to construct a plurality-based Henkin semantics for plural logic and thus populate the empty quadrant in the above diagram. Although from a technical standpoint this is largely a straightforward adaptation of the familiar set-based Henkin semantics, arguing for its philosophical legitimacy is all but straightforward. Once the resources needed to develop a plurality-based Henkin semantics are identified, they must be shown to be in good standing vis-à-vis the resources used to develop the plurality-based standard semantics.

We adopt an object language that expands the usual language of first-order logic with countably many plural terms, constants (aa, bb, ..., aa_1, aa_2, ..., bb_1, bb_2,...) and variables ($vv, vv_0, vv_1, ...$), plural quantifiers binding plural variables, and a distinguished relation of plural membership, $\prec$, which will be treated as logical.[6] The recursive clauses defining a well-formed formula are the obvious ones.

As with the set-based semantics, our plurality-based Henkin models consist of a domain for the first-order quantifiers, a representation of the range of the plural quantifiers, and an interpretation function that specifies the semantic values of the non-logical terminology of the language. The crucial difference is that, in our case, the first-order domain, the range of the plural quantifiers, and the interpretation functions will not be set-theoretic objects.

A domain dd for the first-order quantifiers will consists of *some things*—any things in the domain of the metatheory. Next, to represent the range of the plural quantifiers, we need a 'collection' **D** of pluralities. We will think of **D** as a plural property, i.e. a property (such as that of cooperating) that is instantiated jointly by many things. The notion of property we invoke is an abundant one. (An alternative interpretation of **D** will be mentioned below.) The

pluralities 'in' **D** will be exactly those that instantiate **D**. We require that the two domains be connected in the following way: for every xx such that $\mathbf{D}(xx)$ (i.e. xx instantiate **D**), xx are among dd. In symbols:

$$\forall xx(\mathbf{D}(xx) \rightarrow \forall x(x \prec xx \ \rightarrow \ x \prec dd)).^{7}$$

Finally, it is extremely convenient to assume that the metatheory is equipped with a pairing operation so that an interpretation function can consist of some ordered pairs ii specifying the semantic value or values of each non-logical item in the vocabulary of the object language. In particular, the first coordinate of each pair will be an item from the non-logical vocabulary of the object language (i.e. a singular term, a plural term, or a singular predicate), whereas the second coordinate or coordinates will be the semantic value or values of the first coordinate relative to the given interpretation. A variable assignment, covering both singular and plural variables, can be constructed in analogy with an interpretation, i.e. as some ordered pairs specifying the thing or things assigned to each variable. A more precise formulation of the semantics is provided in Appendix A.

As is well known, the standard deductive system for second-order logic is sound and complete with respect to set-based Henkin semantics. As one would expect, this result carries over to the case of plurality-based Henkin semantics for plural logic. A completeness proof is given in Appendix B.

Two aspects of our semantics deserve to be highlighted. First, as in Boolos's semantics, plural quantifiers in our plurality-based Henkin semantics do not range over any special kind of set-like objects. Rather, they range plurally over things in the domain of the first-order quantifiers. Second, the formulation of the semantics requires expressive resources that go beyond those of plural logic. The variable **D**, used to represent the non-standard interpretations for the plural quantifiers, introduces a form of third-order quantification. As presented above, **D** stands for a plural property. An alternative is to take **D** to stand for a 'superplurality', that is, a 'plurality of pluralities' or, more precisely, some things articulated into distinct subpluralities, such as: Russell and Whitehead, and Hilbert and Bernays, or: these things, those things, and these other things.[8] Either interpretation of **D** might raise worries about the legitimacy of the additional expressive resources required by our semantics. So let us address this issue next.

3 The legitimacy of ascending one level

Boolos's plurality-based semantics does not require expressive resources beyond those of plural logic. When describing standard interpretations of the object language, there is no need to invoke a variable **D**. This is only needed if we wish to 'select' a non-standard range for the plural quantifiers. In Boolos's semantics, a sentence of the form $\exists vv\ \varphi$ is true in a model of the language just in case *some things* among those in the first-order domain satisfy the formula φ. The formulation of this clause relies only on plural logic. In our plurality-based Henkin semantics, we want to impose the additional requirement that the things satisfying the formula also be among the pluralities represented by **D**.

The expressive economy of the plurality-based standard semantics may be thought to constitute an important advantage of that semantics over our Henkin alternative, especially when coupled with some skepticism about the legitimacy of expressive resources going beyond plural logic. However, we believe that this advantage of the plurality-based standard semantics over our Henkin alternative is not significant. For, as we will now argue, the additional expressive resources required by our semantics are available, and they are needed anyway for independent semantic reasons.

It is relatively straightforward to develop a formal system of third-order quantification suitable to develop the plurality-based Henkin semantics (see, e.g., Rayo 2006). Thus the expressive resources under discussion are available at least in the sense of belonging to the inventory of possible semantic mechanisms. Moreover, there is evidence from natural language that such resources are available also in the stronger sense of being actually in use. On the one hand, familiar arguments for the presence in natural language of quantification into predicate position extend to quantification into predicate position of plural predicates. For singular predicates, a treatment of simple examples such as 'John is everything we wanted him to be' seems to require variable binding of predicate positions (Higginbotham 1998, 251, but see also Rayo and Yablo 2001). The same conclusion vis-à-vis plural predicates is suggested by analogous examples involving plural predication, such as 'John and Mary are everything we wanted them to be'. This vindicates the interpretation of **D** in terms of plural properties introduced above. On the other hand, it has been argued that natural languages such as English contain superplural expressions (see Oliver and Smiley 2004, 2005, 2013; and Linnebo and Nicolas 2008), which provides at least *prima facie* support for the

superplural interpretation of **D**.

The reason why the expressive resources required by our semantics are needed anyway has to do with absolute generality. An attractive feature of the plurality-based standard semantics is that it allows us to capture models whose first-order domain of quantification contains absolutely everything. By means of the plural resources available in the metalanguage, one can define models in which the first-order quantifiers range over *all things*. But, if quantification over absolutely everything is possible, developing a model theory for plural logic requires the introduction of a new non-logical predicate. Specifically, it requires the introduction of a plural predicate functioning as a satisfaction predicate (see Rayo and Uzquiano 1999). However, once the original language of plural logic has been expanded to include plural predicates, ascending one level further becomes unavoidable. For it is now known that a model theory for the language expanded to include plural predicates will require a language that is one level higher than plural logic.[9] So, if one wants to do justice to the possibility of quantifying over absolutely everything, semantic considerations push the expressive resources up one level.

As we have already remarked, this higher-order quantification can be understood either as quantification over plural properties or as superplural quantification. In either case, semantic reflection will eventually lead the proponent of the plurality-based standard semantics to embrace the expressive resources needed to formulate the plurality-based Henkin semantics.

Since the additional resources needed to formulate our Henkin semantics are available and needed anyway for independent semantic reasons, we conclude that the expressive economy of plurality-based standard semantics does not constitute a significant advantage over our plurality-based Henkin semantics.

4 Does ontological innocence ensure determinacy?

The previous two sections establish that there exist plurality-based yet non-standard interpretations of a plural language. This is significant. For it is commonplace to maintain that plural logic on the plurality-based semantics is determinate. The view goes back at least to Boolos's famous argument that plural logic is non-firstorderizable. The argument is based on plural logic's alleged ability to distinguish standard from non-standard models of arithmetic (Boolos 1984a, Boolos 1984b, and Boolos 1985). But of course, if our plurality-based

non-standard interpretations are admitted, then plural logic is no better equipped to make such distinctions that, say, a first-order set theory. More generally, it is often held that, when formulated with the help of plural quantification, arithmetic and analysis are categorical, and set theory is quasi-categorical (see, for example, Hossack 2000, 439-41; Rayo and Uzquiano 1999, 315-18; McKay 2006, 139-43). Moreover, Yi holds that 'a system of logic that does justice to plurals [...] cannot be axiomatizable' (Yi 2006, 256-57). The same view is endorsed by Oliver and Smiley (Oliver and Smiley 2013, 236-39). To be perfectly clear: we are not claiming that all these authors deny or fail to recognize the existence of plurality-based non-standard interpretations. Our claim is that their remarks are potentially misleading because they suggest that the only plurality-based interpretation is the standard one.

It might be responded that, while we have shown that plurality-based non-standard interpretations exist, they can safely be set aside as unintended or illegitimate. Doing so would restore the determinacy of plural logic, which the views just referenced all presuppose. The key question, it seems to us, is whether this response is any better than the analogous response for traditional set-based interpretations. That is, does plural logic on a plurality-based semantics have a *better* claim to determinacy than plural logic on a set-based semantics? Let *Plural Robustness* be the view that this question should be answered in the affirmative. A defense of Plural Robustness would have to show that the plurality-based standard interpretations are in better standing vis-à-vis their (plurality-based) Henkin rivals than the set-based standard interpretations are vis-à-vis their (set-based) Henkin rivals. Our aim in this section is to articulate and reject a natural defense of Plural Robustness. In the next section, we argue that the two forms of standard semantics are equally well (or poorly) placed against their respective Henkin rivals and that Plural Robustness should therefore be rejected.

Plural Robustness has considerable initial plausibility, as is brought out nicely in the following passage by Keith Hossack.

> The singularist [a proponent of a set-based semantics] cannot solve the problem of indeterminacy, but the pluralist [a proponent of a plurality-based semantics] can. [...] Plural set theory has no non-standard models, so the indeterminacy problem does not arise for pluralism. [...] [P]lural variables range plurally over the very same particulars that the singular variables range over individually. *Therefore the pluralist does not confront an independent problem of identifying what the plural variables range over.* [...] Plural sentences therefore provide

> the missing additional constraint we were seeking on admissible interpretations. This is why the pluralist [a proponent of a plurality-based semantics] is able to solve the indeterminacy problem, though the singularist cannot do so. (Hossack 2000, 440–41, our emphasis)

As we understand it, the argument has as its point of departure the other virtue that plural logic is widely believed to enjoy, namely ontological innocence. According to this view—which we call *Plural Innocence*—plural quantification does not incur ontological commitments to entities beyond those in the first-order domain. In particular, plural quantification is not reducible to singular quantification over sets or mereological sums, nor does it involve reference to such entities. Rather, plural variables range plurally over objects in the ordinary, singular domain. And the use of such variables incurs ontological commitments only to objects in this ordinary domain, not to any sets or sums of such objects.

Of course, Plural Innocence is not uncontroversial (see Resnik 1988, Parsons 1990, Hazen 1993, and Linnebo 2003); we too take issue with it below. But if the thesis is false, so is an essential premise of the argument we wish to reject, and we are done. In the remainder of this section we therefore proceed on the assumption that the thesis is true.

It would be very natural to think that Plural Innocence supports Plural Robustness. Since the plural quantifiers do not range over any kind of 'plural objects', such as the subsets of the first-order domain, we do not—as Hossack puts it—'confront an independent problem of identifying what the plural variables range over.' Plural quantifiers just range plurally over the very same domain that the singular quantifiers range over. This is unlike second-order logic with set-based semantics, where the standard interpretation requires one to single out a range for the second-order quantifiers that contains *all* the subsets of the first-order domain. The possibility of failing to single out such a range gives rise to the possibility of non-standard interpretations in the set-based semantics. Since Plural Innocence ensures that no new range of entities needs to be singled out for the plural quantifiers to range over, this thesis renders plural logic on the plurality-based semantics immune to non-standard interpretation, or at least more immune than plural logic on the set-based semantics.

However, we contend that our plurality-based Henkin semantics is just as innocent as Boolos's plurality-based semantics. On both semantics, plural variables range plurally over objects in the ordinary, first-order domain. The only difference is that, on our semantics, the range of the plural variables can be so restricted as to make room for general interpretations

in addition to the standard one.

In fact, this notion of ontological innocence can be understood in a less and in a more demanding way. In the less demanding way what is required is, as specified above, the ontological innocence of the plural quantifiers. Then our claim that plural quantification is innocent on the plurality-based Henkin semantics is incontrovertible. Since the semantics is plurality-based, the plural quantifiers do not range over special kinds of objects. They range plurally over the objects in the first-order domain. This is the sense of ontological innocence operative in the argument from Plural Innocence to Plural Robustness spelled out above.

However, one might also want innocence in a more demanding form which includes the resources employed by the semantic theory itself. (For instance, the plurality-based semantics uses a pairing operation which is not ontologically innocent.) Our semantics may possess a high degree of innocence even in this more demanding sense. For there are arguments, akin to the one developed by Boolos himself, for the ontological innocence of the third-order quantification that binds the variable **D**. This is fairly straightforward in the case of the 'superplural' interpretation of **D**. As for the official interpretation of **D** as a plural property, one may argue for its innocence along the lines of Rayo and Yablo 2001 (see also Wright 2007). Moreover, in the more demanding sense of innocence the two semantics appear to be on equal footing. As argued above, an appeal to higher-order resources is unavoidable when the defender of the plurality-based standard semantics attempts to articulate a semantics for a language containing plural predicates (as she will have to do when doing the semantics for her own metalanguage). So, when seen from this perspective, the semantic machinery of the plurality-based standard semantics is no more innocent than that of its Henkin competitor.

We conclude that, no matter which understanding of Plural Innocence is assumed, the plurality-based Henkin semantics has as good of a claim to innocence as the standard semantics. This shows that Plural Innocence does not support Plural Robustness. For there is an innocent semantic option, namely the plurality-based Henkin semantics, for which Plural Robustness fails. This poses a challenge for defenders of Plural Robustness. If their claim is not supported by Plural Innocence, then what, if anything, does support it?

5 The semantic determinacy of plural quantification

The question of semantic determinacy, we recall, is whether the unique correct interpretation of our quantificational practice is the one associated with the standard interpretations. We contend that plural logic with the traditional set-based semantics and plural logic with plurality-based semantics are on a par with regard to semantic determinacy.

Two remarks about this *parity thesis*—as we shall call it—are in order. First, our contention is that the determinacy claims concerning set-based semantics stand or fall with the corresponding determinacy claim concerning plurality-based semantics. We do not take a stand on whether they stand together or fall together. Second, the parity thesis includes, but goes beyond, the claim that Plural Robustness is false. If Plural Robustness is false, then no *additional* assurance of determinacy is gained by switching from a set-based to a plurality-based semantics. Our parity thesis consists of this claim and its converse.

We submit that the parity thesis has a great deal of plausibility whenever the domain of quantification is set-sized, e.g. for higher-order quantification over the natural numbers or the reals. Assume that the domain is a set D, and let dd be its elements. (We will indicate this relationship by writing $D = \{dd\}$.) In the case of the set-based semantics, we need to single out a special object—the standard interpretation—from a large pool of other objects—the Henkin interpretations. In the case of the plurality-based semantics, we need to single out a special way of ranging over the domain dd—the standard way—from a large pool of other ways of ranging over dd—the Henkin ways. But why should it be any easier—or harder—to single out an object from a pool of objects than to single out a way from an isomorphic pool of ways? Since the two tasks are isomorphic, whatever can be said in one case, carries over to the other.

While these considerations capture the gist of our argument, some work remains to be done if we are to establish the parity thesis in full generality, i.e. independently of the assumption that the domains of the plurality-based semantics are set-sized. Consider first the possibility that plural logic is determinate on the plurality-based semantics while being *in*determinate on the set-based semantics. If plural logic is determinate on the plurality-based semantics, this means that no plurality-based Henkin interpretation whatsoever can be correct. *A fortiori*, no plurality-based Henkin interpretation can be countenanced in which the elements dd of the domain form a set D. But this is incompatible with the idea that set-based

Henkin interpretations are legitimate, since the legitimacy of an interpretation would then depend entirely on the way in which the interpretation is described. Henkin interpretations with set-sized domains would be legimate when described set-theoretically but illegitimate when described with the help of higher-order resources. So we must conclude that plural logic on the set-based semantics is determinate too, and thus Plural Robustness is false.

We now consider the converse. Might plural logic be determinate on the set-based semantics but not on the plurality-based semantics? We believe the answer is negative. The determinacy of plural logic on the set-based semantics rules out non-standard (i.e. Henkin) interpretations whenever the domain is set-sized. So, if plural logic admits non-standard interpretations on the plurality-based semantics, such interpretations could only arise when the domain is too large to form a set. As a result, the type of interpretation legitimate for the plural quantifiers would vary depending on the size of the domain. That is, the interpretation of the plural quantifiers would be standard whenever the domain forms a set but may be non-standard when the domain is too big to form a set. Why should that be so? If plural quantifiers are to be treated as logical, this asymmetry would be implausible. Thus it appears that if plural logic is determinate on the set-based semantics, it must also be determinate on the plurality-based semantics.

6 The metaphysical determinacy of plural quantification

We now briefly examine a different determinacy question pertaining to plural and other forms of higher-order quantification. This question is metaphysical and challenges a presupposition of the semantic determinacy question discussed above. Consider a domain $D = \{dd\}$. Is there a determinate maximal set of subsets of D or a determinate maximal property of being a subplurality of dd? Where the semantic question asks whether our practice uniquely singles out as correct a maximal interpretation of the plural and higher-order quantifiers, the metaphysical question asks whether the sort of thing we are attempting to uniquely single out even exists.

The metaphysical question is interesting in part because it might provide an argument in favor of the plurality-based semantics against the set-based semantics. Specifically, might metaphysical determinacy hold in the case of pluralities but fail in the case of sets? We think not, since metaphysical determinacy in the case of pluralities *supports* metaphysical

determinacy in the case of sets. Assume that there is a determinate and maximal property of being a subplurality of the things dd that serve as our domain. This provides strong support for the existence of a determinate maximal set of subsets of D. For every subplurality aa of dd, consider the corresponding set $A = \{aa\}$. (This is legitimate by Separation, because dd form a set D.) Now we wish to collect together all these sets A into a single set, which would give us the desired determinate and maximal set of subsets of D. Does such a set exist? We believe an affirmative answer follows from our assumption that the range of sets that we wish to collect into a set is determinate. Once it is granted that a range of objects is determinate, it is hard to see why these objects should not form a set. Indeed, skepticism about the existence of a determinate powerset of a given infinite set (such as the set of natural numbers) has typically resulted from the denial that there is a determinate notion of arbitrary subset of the given infinite set (see Dummett 1963 and Weyl 1918).

We turn lastly to the converse claim—that metaphysical determinacy in the case of sets supports metaphysical determinacy in the case of pluralities. Recall that we are restricting ourselves to the case where the domain dd forms a set D. Assume that D has a powerset $\wp(D)$. Then, by comprehension for plural properties using $\wp(D)$ as a parameter, we have that there is a property that applies exactly to those things that form a set in $\wp(D)$. In symbols:

$$\exists \mathbf{P}\, \forall xx\, (\mathbf{P}(xx) \;\leftrightarrow\; \exists y\, (y \in \wp(D)\ \&\ y = \{xx\})).$$

This is the property of being a subplurality of dd. In sum, on the assumption that the domain forms a set, there is good reason to believe that the assumptions of metaphysical determinacy underlying the two competing semantics—the set-based one and the plurality-based one—are on a par.

This leaves open whether there is a determinate and maximal property of being a subplutality of the domain dd when dd do not form a set. But any trouble here would only serve to limit the advantage that the plurality-based semantics is commonly taken to enjoy over its set-based rival.

7 A generalized notion of ontological commitment

Let us finally consider the debate about the ontological commitments of plural logic. According to Boolos and followers, plural languages appear to be ontologically innocent. For

instance, when I say that I had a bowl of Cheerios for breakfast, I am talking exclusively about the Cheerios, not about a set of them, their sum, or any kind of 'plural entity'. Call this the *narrow notion* of ontological commitment. It will be made precise below. We have seen how to develop a semantics for a plural object language in a plural metalanguage in which the semantic values of a plural variable is one or more objects from the ordinary first-order domain. This semantics preserves the appearance that the use of plural quantifiers incurs no new commitments to sets, sums, or any kind of 'plural entities' (Boolos 1985).

The opposite side responds by disputing the *prima facie* case for the ontological innocence of plural quantification. For instance, commenting on Boolos's example 'there are some sets which are all and only the non-self-membered sets', Charles Parsons writes:

> in a context of this kind a quantifier like 'there are some sets' is saying that there is a plurality of some kind. Cantor's notion of 'multiplicity' and Russell's of 'class as many' were more explicit versions of this intuitive notion, both attempting to allow that pluralities might fail to constitute sets. (Parsons 1990, 326)

(See also Hazen 1993, Shapiro 1993 and Linnebo 2003, as well as Resnik 1988 for a more 'singularizing' version of the view.) The semantics developed in a plural metalanguage cuts both ways. Both parties to the debate can agree that if the use of the plural quantifiers in the metalanguage is innocent, then so is their use in the object language. One party will assert the antecedent, while the other will deny the consequent. Thus there are two internally coherent views on the matter, and we appear to have reached a standoff.

The best way to make progress, it seems to us, is by considering two alternative construals of the notion of ontological commitment. If the notion is understood in the *narrow sense* (i.e. as concerned exclusively with the existence of objects), and if an object is understood as the value of a singular first-order variable, then the plurality-based semantics does indeed show that plural logic is ontologically innocent. For this semantics does not use singular first-order variables to ascribe values to the plural variables of the object language; rather, this ascription is made by means of plural variables of the metalanguage.

However, there is a *broad notion* of ontological commitment. According to this notion, ontological commitment is tied to the presence of existential quantifiers of *any logical category* in a sentence's truth conditions. If this notion is operative, then even the plurality-based

semantics shows that plural locutions incur additional ontological commitments. The resulting view is an analogue of that espoused by Frege when he held that quantification into predicate position incurs its own distinctive kind of commitment, not to objects but rather to (what he called) concepts.

But before a meaningful debate can take place about which notion of commitment is more interesting and appropriate, both notions need to be clearly articulated. We will now show that our plurality-based Henkin semantics is precisely the tool that we need in order to articulate the more inclusive notion.

Let us begin with the narrow notion, which ties ontological commitment to the values of singular first-order variables. Here is one of Quine's more helpful statements of the view.

> The ontology to which an (interpreted) theory is committed comprises all and only the objects over which the bound variables of the theory have to be construed as ranging in order that the statements affirmed in the theory be true. (Quine 1951, 11)

This suggests the following precise definition. A theory T is committed to κ objects each of which φs if and only if any model of T contains at least κ objects satisfying the formula φ.

In light of our work in earlier sections, it becomes straightforward to extend this criterion of commitment to plural variables. In both cases, the formulation of the criterion relies on the use of quantifiers that are assumed to be antecedently understood in the metatheory. A theory T is committed to κ pluralities which φ if and only if any *plurality-based Henkin model* of T has a range of the plural quantifiers $\mathbf{D}$ containing at least κ pluralities satisfying the formula φ. (Of course, the proper way to talk about many pluralities is by means of plural properties or 'super-pluralities', as discussed above.)[10] It is important to note that the appeal to plurality-based *Henkin* models is essential. If we had instead appealed to Boolos-style plurality-based *standard* models, then the ontological commitment of any theory involving plural quantifiers would be trivially determined by the ontological commitments of the first-order quantifiers of the theory. For any theory would incur commitments to all and only the pluralities based on the objects to which the theory is committed. By contrast, the definition of commitment to pluralities that we have proposed has the desirable feature that a theory's commitment to pluralities can add information over and above its commitment to objects.

The value of this information is most easily appreciated when it is denied that there is a

single maximal interpretation of the plural quantifiers, that is, when the metaphysical determinacy of these quantifiers is denied. When this is denied, there can be no hope of determining the theory's commitments to pluralities directly on the basis of its commitments to objects. Instead, one must assess the commitments to pluralities independently, using the generalized Quinean criterion set out above. In order to illustrate this point and, more generally, the value of our notion of commitment to pluralities, let us consider a puzzle due to Hazen (1993, 135). Consider the scheme of plural comprehension:

$$\exists x\, \varphi(x) \;\rightarrow\; \exists xx\, \forall x\, (x \prec xx \leftrightarrow \varphi(x)).$$

Which instances of the scheme should we accept? The traditionalist (whose position is enshrined in the standard semantics for plural logic) accepts *all instances*—with the obvious and uncontroversial proviso that $\varphi(x)$ not contain xx free. According to the predicativist, however, we should only accept the instances that are predicative in the sense that $\varphi(x)$ does not to contain any bound plural variable. As Hazen observes, there is a clear and intuitive sense in which the predicativist is committed to fewer pluralities than the traditionalist. Thus, if a notion of commitment is to be worth its salt, it must capture this sense. And this is exactly what our broad notion of ontological commitment enables us to do. Using this notion, we can maintain that the traditionalist, but not the predicativist, takes on commitments to impredicatively defined pluralities. By contrast, if had we assumed the plurality-based standard semantics, this conclusion would not have been available.

Our notion of commitment to pluralities is also useful when the metaphysical determinacy of plural quantification is granted. When this is granted, there is a notion of commitment to pluralities—namely the one associated with the maximal interpretation of the plural quantifiers—according to which these commitments supervene on the commitments to objects. Once the commitments to objects of a theory have been determined, so have the commitments to pluralities associated with the maximal interpretation. It must therefore be conceded that there is no *further* question concerning the theory's commitments to pluralities. However, the supervenience of one parameter on certain others does not mean that there is no genuine and theoretically interesting question as to the value of this parameter! In our case, even if the commitments to pluralities of a theory are uniquely determined by its commitments to objects, we want to know how many, and what kind of, pluralities the theory is committed to. Even if one believes in the metaphysical determinacy of plural quantification,

one may have views about how strong, or mathematically rich, one's notion of subplurality is (e.g. Shapiro 1993 and Parsons 2013). The notion of commitment to pluralities that we have articulated allows such views to be expressed.

An example might be helpful. Assume that the commitments to objects of a theory involve an omega-sequence, which we may think of as the natural numbers. If metaphysical determinacy holds, then there is a sense in which the commitments to pluralities are determined by the commitments to objects. Even so, we can ask *which* pluralities the theory is committed to. Different answers are possible. For instance, one theorist—who believes the axiom of constructibility, $V = L$—may answer that the only subpluralities of the 'natural numbers' to which the theory is committed are the ones that are constructible (in the sense that they correspond to sets in the constructible hierarchy L). Another theorist—who rejects the axiom of constructibility—may disagree and insist that the commitments to pluralities go beyond the constructible ones.

It may be objected to the broad notion of commitment that the commitments associated with plural and higher-order quantifiers is not a form of *ontological* commitment but perhaps, following Quine, of *ideological* commitment. We see little point in quarreling over terminology. A more interesting question is whether ideological commitments in this sense give rise to fewer philosophical problems, or whether they are philosophically less substantive, than ontological commitments narrowly understood. It is far from obvious that this is so. Indeed, it seems to us that questions involving the broad notion of commitment can be just as interesting and problematic as those involving the narrow ones. How are we to understand the values of different sorts of variables—in extensional or intensional terms? Which such values are there and which comprehension axioms should we therefore accept? How do we trace a value from one context (e.g. time or possible world) to another?

In light of these considerations, we are inclined to agree with Parsons when he writes that, on the narrow notion,

> ontological commitment may just not have the significance that both nominalists and many of their opponents attribute to it, or that Boolos seems to attribute to it in the case of proper classes. That might be a victory for the Innocence Thesis, but it would be a Pyrrhic victory. (Parsons 2013, 173)

Thus, if Parsons is right, then either Plural Innocence is false, or else it is true but not nearly

as interesting as one might have thought.

Our primary goal in this section has been less to adjudicate in this debate than to prepare the ground for a precise and well informed debate. We have done so by using our plurality-based Henkin semantics to provide a clear articulation of a generalized notion of commitment that is associated with the former horn. Still, on the picture emerging from our discussion, the role of plural logic as a philosophical tool appears substantially diminished. As we have shown, plural logic is not immune from the threat of non-standard interpretations and does not secure a gain in expressive power. Moreover, there is a precise and interesting sense in which plural logic may be said to be committing. Whether this commitment is ontological or ideological, it is a full-fledged commitment nonetheless.

Appendices

A Henkin semantics

Let us provide a more precise formulation of the semantics for plural logic outlined in section 2. Some pairs ii form an interpretation relative to a domain dd and a plural property $\mathbf{D}$ if

(i) for every singular constant c, there is a unique x such that $(c,x) \prec ii$, and $x \prec dd$;

(ii) for every plural constant cc, there is at least one x such that $(cc,x) \prec ii$, and for all xx such that

$$\forall y(y \prec xx \leftrightarrow (cc,y) \prec ii),$$

it holds that $\mathbf{D}(xx)$;

(iii) for every predicate S^n and object x, if $(S^n,x) \prec ii$, then x is an n-tuple of things from the domain dd.

The second condition captures the idea that, in any interpretation ii, a plural constant cc denotes some things that instantiate $\mathbf{D}$, specifically those appearing as second coordinates of pairs whose first coordinate is cc.

The characterization of a variable assignment mirrors that of an interpretation. That is, some ordered pairs ss form a variable assignment relative to dd and $\mathbf{D}$ if

(i) for every singular variable v, there is a unique x such that $(v,x) \prec ss$, and $x \prec dd$;

(ii) for every plural constant vv, there is at least one x such that $(vv,x) \prec ss$, and for all xx such that

$$\forall y(y \prec xx \leftrightarrow (cc,y) \prec ss),$$

it holds that $\mathbf{D}(xx)$.

Here too the second condition captures the idea that a plural variable vv is assigned some things that instantiate $\mathbf{D}$, specifically those appearing in the assignment as second coordinates of pairs whose first coordinate is vv. Variants of variable assignments $ss(v/x)$ and $ss(vv/xx)$ are, as usual, assignments just like ss, with the possible exception that they assign, respectively, x to v and xx to vv.

A model of the object language is given by the domain dd and $\mathbf{D}$, plus an interpretation ii relative to dd and $\mathbf{D}$. Given how these three components have been characterized, a model is not an object or the value of a single higher-order variable. However, such components can be 'merged' so as to be represented by a single variable $\mathbf{I}$, whose value is a plural property (or, alternatively, a superplurality) that codes the three components.[11] Quantifying over models then amounts to quantifying over plural properties (or superpluralities). For convenience, however, we speak of a model as a triple and represent it as (dd, $\mathbf{D}$, ii).

Before defining the notion of satisfaction, let us introduce some additional notation. For any model (dd, $\mathbf{D}$, ii) and any non-logical expression E, let $[\![E]\!]_{(dd,\mathbf{D},ii),ss}$—but, in fact, we will write $[\![E]\!]_{ii,ss}$ leaving the domains implicit—indicate the semantic value or values of the expression E relative to the model $(dd,\mathbf{D},ii)$ and to the variable assignment ss. That is,

(ia) for any singular constant c, $[\![c]\!]_{ii,ss}$ is the unique x such that $(c,x) \prec ii$;

(ib) for any singular variable v, $[\![v]\!]_{ii,ss}$ is the unique x such that $(v,x) \prec ss$;

(iia) for any plural constant cc, $[\![cc]\!]_{ii,ss}$ are the things xx such that

$$\forall y(y \prec xx \leftrightarrow (cc,y) \prec ii);$$

(iib) for any plural variable vv, $[\![vv]\!]_{ii,ss}$ are the things xx such that

$$\forall y(y \prec xx \leftrightarrow (vv, y) \prec ss);$$

(iii) for any predicate S^n, $[\![S^n]\!]_{ii,ss}$ are the n-tuples xx, if any, such that

$$\forall y(y \prec xx \leftrightarrow (S^n, y) \prec ii).$$

Satisfaction is characterized inductively. In the Henkin semantics, A model (dd, $\mathbf{D}$, ii) satisfies a formula φ with a variable assignment ss relative to dd and $\mathbf{D}$, written $(dd, \mathbf{D}, ii) \vDash_H \varphi\ [ss]$, just in case

(i) if φ is $t = s$, then $[\![t]\!]_{ii,ss} = [\![t]\!]_{ii,ss}$;

(ii) if φ is $S^n(t_1, ..., t_n)$, then $[\![S^n]\!]_{ii,ss}$ is non-empty and $([\![t_1]\!]_{ii,ss},...,[\![t_n]\!]_{ii,ss}) \prec [\![S^n]\!]_{ii,ss}$;

(iii) if φ is $\exists v\ \psi$, then there is x among dd such that $(dd, \mathbf{D}, ii) \vDash_H \psi\ [ss(v/x)]$;

(iv) if φ is $\exists vv\ \psi$, then there are xx such that $\mathbf{D}(xx)$ and $(dd, \mathbf{D}, ii) \vDash_H \psi\ [ss(vv/xx)]$;

(v) the clauses for the logical connectives are the obvious ones.

We say that a model (dd, $\mathbf{D}$, ii) is *faithful* if it satisfies every instance of the plural comprehension schema:

$$\exists v\varphi(v) \ \rightarrow\ \exists vv\, \forall v(v \prec vv \leftrightarrow\ \varphi(v)).$$

Logical consequence is defined with respect to faithful models only. A sentence σ is a consequence of a set of sentences Γ in the Henkin semantics ($\Gamma \vDash_H \sigma$) if, for every faithful model (dd, $\mathbf{D}$, ii), $(dd, \mathbf{D}, ii) \vDash_H \gamma$ for every member γ of Γ only if $(dd, \mathbf{D}, ii) \vDash_H \sigma$.

B Completeness of the Henkin semantics

Here we describe a standard proof system S for plural logic and we prove that it is sound and complete with respect to the plurality-based Henkin semantics formulated above. Just as the object language of plural logic expands the standard language of first-order logic, its proof system extends that of (classical) first-order logic. In addition to introduction and

elimination rules for the logical connectives and singular quantifiers, we have introduction and elimination rules for the plural quantifiers, plus every instance of plural comprehension. The introduction and elimination rules for the plural quantifiers mirror those of the singular quantifiers. Let us use the symbol $\vdash$ to denote the relation of provability is S.

We want to show that S is (sound and) complete with respect to the plurality-based Henkin semantics. That is, we want to show that, for any sentence σ and set of sentences Γ, $\Gamma \vDash_H \sigma$ (if and) only if $\Gamma \vdash \sigma$. The shortest and most elegant way of proving this is through a squeezing argument.

First, it is a routine exercise to verify that S is sound with respect to the plurality-based Henkin semantics, i.e.,

(1) if $\Gamma \vdash \sigma$, then $\Gamma \vDash_H \sigma$.

Now consider the familiar set-based Henkin semantics for second-order logic.[12] It is relatively straightforward to adapt this semantics to the object language introduced above. A model is given by a triple (d_1, d_2, I), where d_1 is a set, d_2 is a set of non-empty subsets of d_1—the range of the plural quantifiers—and I is interpretation function from the non-logical vocabulary of the language to elements of d_1 (for singular terms), elements of d_2 (for plural terms), and possibly empty sets of n-tuples from d_1 (for singular n-ary predicates). Plural membership ('is one of') is systematically interpreted as set-theoretic membership. Let us use the symbol $\vDash_h$ for the resulting relation of logical consequence when confined to *faithful* models, namely, those which satisfy every instance of plural comprehension. So $\Gamma \vDash_h \sigma$ means that σ is a logical consequence of Γ in the set-based Henkin semantics. In other words, for every faithful model (d_1, d_2, I), if $(d_1, d_2, I) \vDash_h \gamma$ for every member γ of Γ, then $(d_1, d_2, I) \vDash_h \sigma$.

It is evident that every set-theoretic model just described corresponds to a plurality-based Henkin model. Take any model (d_1, d_2, I). Then its corresponding plurality-based model $(dd, \mathbf{D}, ii)$ is one in which dd are the elements of d_1, $\mathbf{D}$ is the property of being a plurality that forms a set in d_2, and ii is an interpretation function that matches I.[13] This correspondence establishes the following:

(2) If $\Gamma \vDash_H \sigma$, then $\Gamma \vDash_h \sigma$.

Finally, the standard proof of completeness of second-order logic with respect to the set-based Henkin semantics can be run, *mutatis mutandis*, for the proof system S of plural logic

with respect to the set-based Henkin semantics outlined in the paragraph just above. This gives us that

(3) If $\Gamma \vDash_h \sigma$, then $\Gamma \vdash \sigma$.

Putting together (1), (2), and (3), we obtain the result we wanted to prove:

(4) $\Gamma \vDash_H \sigma$ (if and) only if $\Gamma \vdash \sigma$.

So S is complete with respect to the plurality-based Henkin semantics. Therefore, as captured by S, plural logic on the plurality-based Henkin semantics is complete, hence compact and axiomatizable.

Notes

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[1]For this alternative approach to the semantics of higher-order logic, see Boolos 1985, Rayo and Uzquiano 1999, Rayo and Yablo 2001, and Williamson 2003. Defenses of the innocence of plural logic are put forth, among others, by Boolos (1984b; 1985), Yi (1999; 2002; 2005; 2006), Hossack (2000), Oliver and Smiley (2001; 2013), Rayo (2002), and McKay (2006).

[2]See Shapiro 1991, chapter 8. A notable consequence of the view that second-order quantification is determinate is the thesis famously held by Kreisel and others that the Continuum Hypothesis is either true or false (for discussion, see Weston 1976).

[3] See, for instance, Yi 1999, 2006; Rayo and Uzquiano 1999; Hossack 2000; McKay 2006; and Oliver and Smiley 2013.

[4]Rayo and Yablo 2001 provides a rare exception.

[5]As explained above, a 'plurality-based' semantics for a system is a special case of a higher-order semantics. A semantics for plural logic can thus be either set-based, as in the traditional approach, or plurality-based, as in the novel approach initiated by Boolos.

[6]This is the language known as PFO. See Rayo 2002 and Linnebo 2003. Our semantics could be extended to PFO+, i.e. to the extension of PFO by means of plural predicates. However, to avoid unnecessary complications, we focus here on PFO.

[7]Hereafter we abbreviate $\forall x(x \prec xx \rightarrow x \prec dd)$ as $xx \preccurlyeq dd$.

[8]See Lewis 1991, Hazen 1993, Oliver and Smiley 2005, Rayo 2006, and Linnebo and Nicolas 2008.

[9]More details about the result are provided in Rayo and Uzquiano 1999, Rayo 2006, and Yi 2006, Appendix 1. Linnebo and Rayo 2012 extends the result into the transfinite.

[10]In a perfectly analogous way, we can define a notion of ontological commitment incurred by quantification into predicate position.

[11]Here is one way of doing it. For any things xx and object y, let ${}^{y}xx$ be the ordered pairs obtained by pairing y with each x in xx. Also, let $xx \approx yy$ abbreviate $\forall z(z \prec xx \leftrightarrow z \prec yy)$. Then, given dd, $\mathbf{D}$, and ii, there is $\mathbf{I}$ such that for all yy, $\mathbf{I}(yy)$ if and only if one of the following holds:

(1) $yy \approx {}^{a}dd$;

(2) there are zz such $\mathbf{D}(zz)$ and $yy \approx zz$;

(3) $yy \approx {}^{b}ii$;

where a and b are any two distinct objects. The plural property (or superplurality) $\mathbf{I}$ so characterized can be used as a surrogate for the triple (dd, $\mathbf{D}$, ii).

[12]See, for instance, Shapiro 1991 and Enderton 2002.

[13]Specifically, if $I(t) = x$, then $(t,x) \prec ii$. If $I(aa) = \{xx\}$, then $\forall y((aa,y) \prec ii \leftrightarrow y \prec xx)$. And, for any n-tuple $(x_1,...,x_n)$, $(S^n,(x_1,...,x_n)) \prec ii$ if and only if $(x_1,...,x_n) \in I(S^n)$.

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