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Untyped Pluralism

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Abstract: In the semantic debate about plurals, pluralism is the view that a plural term denotes some things in the domain of quantification and a plural predicate denotes a plural property, i.e a property that can be instantiated by many things jointly. According to a particular version of this view, untyped pluralism, there is no type distinction between objects and properties. In this article, I argue against untyped pluralism by showing that it is subject to a variant of a Russell-style argument put forth by Timothy Williamson and that it clashes with a plural version of Cantor's theorem. I conclude that pluralists should postulate a type distinction between objects and properties.

1   Introduction

It is generally agreed that, even for the purpose of regimenting basic constructions involving natural language plurals, a first-order language is inadequate (see Boolos 1984, Lewis 1991, Schein 1993, Higginbotham 1998, Yi 1999, 2005, Oliver and Smiley 2001, Rayo 2002, and McKay 2006). A more suitable regimenting language can be obtained by expanding the standard language of first-order logic to include plural terms and quantifiers, plural predicates, and a relation of plural membership corresponding to the natural language 'being one of'. However, the mere choice of a regimenting language leaves a fundamental semantic question wide open: how should the semantic interpretations of that language be specified?

Pluralism is the view that, in any interpretation of the language, a plural term *tt* denotes some *things* in the domain of the interpretation, whereas a plural predicate *P* denotes a *plural property*. A plural property is one that can be instantiated by many things *jointly*. To use Frege's example, although he did not endorse pluralism, 'laid the foundations of spectral analysis' stands for a plural property *jointly* instantiated by Bunsen and Kirchhoff. In contrast, the property of being prime is *singular* in that it is instantiated *separately* by each prime number. So, for the pluralist, a plural predication of the form $P(tt)$ is true in an interpretation if and only if, relative to that interpretation, the things denoted by *tt* jointly instantiate the property denoted by *P*.

Among philosophers, though not among linguists, pluralism has now become the most prominent view about how to specify semantic interpretations for lan-

guages containing plural expressions (Yi 1999, 2005, 2006, Hossack 2000, Oliver and Smiley 2005, and McKay 2006). The main alternative to pluralism is standard set-theoretic semantics, which can be naturally extended to plurals by interpreting plural terms as denoting non-empty subsets of the domain of quantification and by interpreting plural predicates as denoting sets of subsets of the domain. But this set-theoretic approach and its variants have been found to be unsatisfactory on two main grounds. First, since plural terms are taken to denote *sets*, the semantics introduces ontological commitments which are arguably absent in ordinary discourse. (For a dissenting voice, see Resnik 1988.) Second, by requiring that the domains of quantification be set-sized, the set-theoretic approach rules out any interpretation whose domain is too big to form a set. As a result, the set-theoretic approach is unable to capture all the intuitive interpretations of the language, such as those in which the domain of quantification contains absolutely everything. This is especially problematic if, as in the case of plural logic with standard semantics, the completeness theorem fails, and thus there is no assurance via Kreisel's squeezing argument that the mere appeal to interpretations with a set-sized domain yields an extensionally adequate relation of logical consequence (see Kreisel 1967 and, among others, Rayo and Williamson 2003).

Pluralism seeks to remedy both problems. It dispenses with the idea that a plural term denotes a set of things by taking it to denote plurally the things themselves. Moreover, as will be clear from a more detailed presentation of the semantics, pluralism can capture intuitive interpretations in which the first-order quantifiers range over absolutely everything. However, pluralism is not a uniform camp. There are two versions arising from two alternative conceptions of property.

According to the first conception, properties are simply objects of a special kind, i.e. objects that can be instantiated by other objects. Thus there is no type distinction between objects and properties—properties are *untyped*. Some pluralists, such as Hossack (2000) and McKay (2006), have combined pluralism with this conception of property. We call their view *untyped pluralism*. According to the second conception, there is a type distinction between objects and properties. Properties are not objects but higher-order entities that can be predicated of objects. Other pluralists, such as Oliver and Smiley (2005) and Yi (2005, 2006), have combined pluralism with the second conception of property. We call their view *typed pluralism*.

Untyped pluralism has a lot of initial appeal. First, it forgoes higher-order logic, avoiding common misgivings about its legitimacy. In contrast, typed pluralism is unconventional: it requires the acceptance of higher-order logic in addition to plural logic. This means abandoning the proposal famously championed by Boolos to use plural logic to tame second-order logic (Boolos 1984, 1985, and Rayo and Uzquiano 1999). Second, the conception of property underlying untyped pluralism may seem metaphysically preferable. For instance, one may regard the postulation of a fundamental distinction between objects and properties (i.e. particulars and universals) as

a metaphysical dogma and thus regard the introduction of a type distinction as unjustified (MacBride 2005). Moreover, our ordinary way of talking about properties provides little evidence for such a distinction: we commonly nominalize predicates and take first-order quantifiers to range over the semantic values of those nominalizations. Furthermore, if properties are to combine with objects to form propositions or facts conceived as complex objects, then properties must be of the same type as objects. Finally, introducing a type distinction between objects and properties generates a hierarchy of entities or languages which threatens the view that we can quantify over absolutely everything (Linnebo 2006, Rayo 2006, and Linnebo and Rayo 2012). Since, as we have seen, this view plays an important role in motivating pluralism, untyped pluralists might be thought to be on safer ground.

The aim of this article is to show that, despite its appeal, untyped pluralism should be rejected. It is subject to a variant of a Russell-style argument put forth by Timothy Williamson (2003) and it clashes with a plural version of Cantor's theorem. The conclusion is that pluralists should postulate a type distinction between objects and properties.

2   The regimenting language and its semantics

Untyped pluralism is a semantic view concerning how to specify the interpretations of plural predication. To give it a precise formulation, we need to fix a suitable regimenting language. For our purposes, it is enough to expand the standard language of first-order logic by means of the vocabulary listed below. We call this language $\mathscr{L}_{PL}$.[1]

   A. Plural variables ($vv, vv_0, vv_1, ...$), roughly corresponding to the natural language pronoun 'they'. In $\mathscr{L}_{PL}$, plural variables exhaust the category of plural terms.

   B. A plural existential quantifier ('some things') binding plural variables ($\exists vv$, $\exists vv_0$, ...). The plural universal quantifier is defined from the existential quantifier in the usual way.

   C. Symbols for collective plural predicates, with or without numerical subscripts: $P, Q, R...$. Examples of collective plural predicates are 'cooperate', 'gather', 'meet', 'outnumber'.[2] Symbols for plural predicates are accompanied by a double superscript of the form $n[m]$ ($0 < m \leq n$) indicating that the predicate has n argument places of which exactly m are plural, i.e. reserved for plural

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[1]This is a version of the language known as PFO+. See Rayo 2002 and Linnebo 2003.

[2]Collective plural predicates are contrasted with distributive plural predicates, such as 'are prime', 'are students', 'visited Rome'. Here we can take distributive predicates to be those which license the following kind of inference involving their corresponding singular form:

terms.[3] It is convenient to stipulate that plural argument places always precede those occupied by singular terms. (For simplicity, I will often depart from these conventions by leaving the arity unmarked and by allowing the order of the arguments to reflect the order found in English.)

D. A distinguished binary predicate $\prec$ for plural membership, roughly corresponding to the natural language 'is one of' or 'is among'. Plural membership will be treated as logical.

The recursive clauses defining a well-formed formula are the obvious ones.

Here are some examples of regimentation in $\mathscr{L}_{PL}$.

(1)   Some shipmates gathered.

(1*) $\exists xx\ (S(xx)\ \&\ G(xx))$.

(2)   Russell and Whitehead cooperated.

(2*) $\exists xx\ (\forall y\ (y \prec xx \leftrightarrow (y = r\ \vee\ y = w))\ \&\ C(xx))$.

(3)   Some critics admire only one another.

(3*) $\exists xx\ (\forall x(x \prec xx \rightarrow Cx)\ \&\ \forall x\ \forall y\ ((x \prec xx\ \&\ A(x,y)) \rightarrow (x \neq y\ \&\ y \prec xx)))$.

It is useful to introduce abbreviations for plural inclusion ('are among') and for plural identity:

(4) $vv_1 \preccurlyeq vv_2 \leftrightarrow_{def} \forall v\ (v \prec vv_1 \rightarrow v \prec vv_2)$.

(5) $vv_1 \approx vv_2 \leftrightarrow_{def} (vv_1 \preccurlyeq vv_2\ \wedge\ vv_2 \preccurlyeq vv_1)$.

$$\frac{\text{They } \textit{are students} \qquad \text{she is one of them}}{\text{she is } \textit{a student}}$$

Distributive plural predicates in this sense can be obtained by paraphrase from their corresponding singular forms. Instead of saying 'they are students', one may simply say 'every one of them is a student'. That is why distributive plural predicates have been omitted from $\mathscr{L}_{PL}$.

[3]As presented above, $\mathscr{L}_{PL}$ draws a rigid distinction between singular and plural predicates or argument places. However, predication in natural language seems more flexible. Although some predicates can only be combined grammatically with plural terms (e.g. 'cooperate with one another' or 'are two in number'), other predicates can be combined grammatically with both singular and plural terms (e.g. Frege's example: 'laid the foundations of spectral analysis'). To avoid complications, we will focus on the first class of plural predicates. This will not affect the arguments presented below.

Now that a basic regimenting language is in place, we can turn to illustrate the main features of untyped pluralism, including its metaphysical underpinnings.

Metaphysically, the universe of the untyped pluralist is populated by entities belonging to a single ontological category: the objects. Properties are just objects of a certain kind. They can be singular ($p, q, ...$) or plural ($\alpha, \beta, ...$). Expressively, untyped pluralism can be formulated in a language that mirrors the regimenting one.[4] So the untyped pluralist can avail herself of plural resources and speak plurally of some or all objects.

On the traditional model-theoretic semantics, an interpretation is a thing, specifically a set-theoretic function mapping the non-logical vocabulary to its semantic values. In the new setting, it is quite natural to employ the plural resources available in the metalanguage and construe a semantic interpretation as *some things* rather than a single thing.[5] Some things are *collectively* an interpretation if they code the relevant semantic information about the domain of quantification and the interpretation of the non-logical vocabulary. In particular, if we postulate a pairing operation subject to the usual constraints, the coding can be done by ordered pairs whose first coordinate is an item of the non-logical vocabulary and whose second coordinate or coordinates specify the semantic value or values of the first coordinate relative to the given interpretation. An interpretation is then some ordered pairs satisfying certain conditions.

Any admissible domain of quantification can be coded by pairing the symbol for the existential quantifier with each element of the domain. For example, if we want the domain of a given interpretation to consist of the objects a and b, the pairs $(\exists, a)$ and $(\exists, b)$—and no other pair of the form $(\exists, x)$—will be among the things which make up the interpretation.

Similarly, the semantic value of a predicate can be coded by the pair whose first coordinate is the predicate itself and the second coordinate is the property assigned to it. For instance, if a given interpretation assigns the plural property α to the the plural predicate P, then the pair (P, α)—and no other pair of the form (P, x)—will be among the pairs that make up the interpretation.

Finally, a variable assignment can be coded by some pairs where each variable is paired with its semantic value or values. If we want a given variable assignment to assign a, b, and c to the plural variable vv, then the pairs (vv, a), (vv, b), and (vv, c)—and no other pair of the form (vv, x)—will be among those which make up the assignment.

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[4]In order to avoid potential confusion between object language and metalanguage, I will at least distinguish the variables of the object language from those of the metalanguage. The former will be exclusively $v, v_0, ..., vv, vv_0, ...$. All other variables (e.g. x, y, xx, yy, ii, and ss) will be used in the metalanguage. The logical or defined symbols of the object language will do double-duty.

[5]For convenience, I will occasionally refer to some things as a plurality. This is just a shorthand for a plural construction and should not be construed as involving any kind of set-like entity.

It is important to notice that this departure from traditional model-theoretic semantics allows the untyped pluralist to capture interpretations whose domain consists of absolutely everything. An interpretation of this sort will be some pairs containing, for every x whatsoever, the ordered pair $(\exists, x)$. The existence of these interpretations is secured by the principle of plural comprehension—a principle about what pluralities there are—expressed by the schema:

$$\exists x \, \varphi(x) \; \rightarrow \; \exists xx \, \forall y \, (y \prec xx \leftrightarrow \varphi(y)).$$

The details of the semantics will be provided in Appendix A. Here we introduce the basic notions and notation employed in our discussion. In general, for any interpretation I, any variable assignment S, and any non-logical expression E, $[\![E]\!]_I$ and $[\![E]\!]_S$ will stand for the denotation or denotations of E according to I and S respectively. A key model-theoretic notion is that of satisfaction of a formula φ by an interpretation I and a variable assignment S, written $I \vDash \varphi \, [S]$. As usual, satisfaction is characterized inductively on the basis of the interpretation of terms and predicates. In the specific case of untyped pluralism, an interpretation is given by some pairs and so is a variable assignment. This is why we denote an interpretation and a variable assignment, respectively, with the plural variables *ii* and *ss*. Crucially, for any predicate A, $[\![A]\!]_{ii}$ is an untyped property—singular or plural depending on whether A is a singular or a plural predicate.

3   Williamson's argument

As noted above, pluralism has an important advantage over the set-theoretic approach in that it is better suited to capture intuitive interpretations of the language in which the quantifiers range over absolutely everything. Call *absolute generality* the view that quantification over absolutely everything is possible. A Russell-style argument put forward by Timothy Williamson shows there is a tension between absolute generality, the assumption that interpretations are objects, and a natural principle about what interpretations there are. According to this principle, an atomic predicate of the object language should be interpretable by any formula of the metalanguage (see Williamson 2003 and, for discussion, Glanzberg 2004, Glanzberg 2006, Linnebo 2006, McKay 2006, and Parsons 2006).

One way to escape the argument—in effect the one recommended by Williamson—is to give up the assumption that an interpretation is an objects. And untyped pluralism does just that: it constructs an interpretation as some things. However, as I will argue in the next section, this move is ineffective when combined with the conception of property underlying untyped pluralism. Let us begin by rehearsing Williamson's argument.

Let F be a one-place singular predicate. For the moment, we want to leave undecided exactly how the notion of interpretation should be construed. If the formula

$F(v)$ is satisfied by an interpretation I when v denotes the object x in the domain of I, we say that F *applies to x according to I*. So 'F applies to x according to I' means that

$$I \vDash F(v)\ [V(v/x)],$$

where V is any variable assignment suitable for I and $V(v/x)$ is variant of V in which the singular variable v denotes x. Equivalently, 'F applies to x according to I' means that $[\![F]\!]_I(x)$.

An intuitively plausible view concerning interpretations is that an atomic predicate F of the object language can be interpreted by any formula in the metalanguage. This yields a principle—call it *the liberal principle of interpretations* (LPI)—according to which, for any formula of the metalanguage $\Phi(x)$ and any admissible domain D,

(LPI)   there is an interpretation I with domain D such that, for any x in D, F applies to x according to I if and only if $\Phi(x)$.

For consistency, we assume throughout that I does not occur free in the comprehension formula $\Phi(x)$.

Absolute generality sanctions that the all-inclusive domain, the domain containing absolutely everything, is admissible. So LPI and absolute generality jointly entail the principle—call it LPI*—that, for every formula of the metalanguage $\Phi(x)$,

(LPI*)   there is an interpretation I with an all-inclusive domain such that, for every x whatsoever, F applies to x according to I if and only if $\Phi(x)$.

Now LPI* is incompatible with the view that interpretations are objects. To see this, let $\Psi(x)$ be the metalanguage formula 'x is not an interpretation such that F applies to x according to x'. Suppose that semantic interpretations are objects, as opposed to pluralities or higher-order entities. Apply LPI* to $\Psi(x)$ and infer that

(i)   there is an interpretation i with an-all inclusive domain such that, for every x whatsoever, F applies to x according to x if and only if x is not an interpretation according to which F applies to x.

Since interpretations are objects, we can take x in (i) to be i. An inconsistency ensues:

(ii)   there is an interpretation i with an all-inclusive domain such that F applies to i according to i if and only if i is not an interpretation according to which F applies to i.

Williamson's own suggestion is that semantic interpretations should be taken to be second-order entities rather than objects. The liberal principle of interpretations would then be reformulated in the following, second-order way. For every admissible domain D and any formula of the metalanguage $\Phi(x)$,

(LPI2) there is a second-order interpretation *I* with domain *D* such that, for every *x* in *D*, *F* applies to *x* according to *I* if and only if $\Phi(x)$.

This blocks the argument by introducing a type distinction between *I* and *x* that makes the substitution of *I* for *x* illegitimate (a pioneering formulation of the semantics based on this notion of interpretation is given in Boolos 1985; see also Rayo and Uzquiano 1999 and Rayo and Williamson 2003).

Of course, other responses to Williamson's argument are possible. One may deny absolute generality. However, this is not an attractive option for a pluralist, since absolute generality plays an important role in motivating pluralism. Alternatively, one may reject LPI. But this principle expresses an intuitively plausible requirement on what interpretations there are: a given atomic predicate should be interpretable by any formula of the metalanguage. It is not obvious why the above formulas should be ruled out.

Does Williamson's argument threaten untyped pluralism? This view construes a semantic interpretation as some pairs rather than as an object. So one might think that untyped pluralism should block the argument for the same reason Williamson's second-order move blocks it (see McKay 2006, pp. 147-54). But that is not the case. If there is no type distinction between objects and properties, rejecting the view that interpretations are objects in favour of the view that interpretations are pluralities is not enough to avoid the argument, or a close variant of it. This is what I show next.

4   Extending the argument to plural predication

According to untyped pluralism, an atomic predication is true in an interpretation if and only if, relative to that interpretation, the property denoted by the predicate is instantiated by the object or objects denoted by the subject term. On this view, the liberal principle of interpretations becomes in effect a comprehension principle for properties. By asserting the existence of certain interpretations, it asserts the existence of the properties required for the existence of those interpretations. In the presence of absolute generality, however, the liberal principle of interpretations functions as a naïve comprehension principle for untyped properties, leading to contradiction. Burgess (2008) and Nicolas (2008) have already expressed worries about the consistency of the theory of untyped properties in connection with the semantics of plurals. Here I spell out a way in which theses worries may be substantiated.

In the context of untyped pluralism, the liberal principle of interpretations takes this form: for any admissible domain *dd* and any formula $\Phi(x)$ of the metalanguage,

(LPIUP) there are some pairs *ii* with domain *dd* such that, for every $x \prec dd$, *F* applies to *x* according to *ii* if and only if $\Phi(x)$.

Assuming absolute generality, LPIUP entails that, for any formula $\Phi(x)$ of the metalanguage,

(LPIUP*) there are some pairs *ii* with an all-inclusive domain such that, for every *x* whatsoever, *F* applies to *x* according to *ii* if and only if $\Phi(x)$.

We obtain a contradiction by replacing $\Phi(x)$ with '*x* does not instantiate *x*'. This substitution yields that there are some things *ii* with an all-inclusive domain such that, for every *x* whatsoever, *F* applies to *x* according to *ii*—i.e. $[\![F]\!]_{ii}(x)$—if and only if *x* does not instantiate *x*. Let *p* be $[\![F]\!]_{ii}$, i.e. the singular property denoted by *F* according to *ii*. Then, for every *x*, *x* instantiates *p* if and only if *x* does not instantiate *x*. Since *p* is an object in the all-inclusive domain of quantification, we infer that *p* instantiates *p* if and only if *p* does not instantiate *p*. This is Williamson's argument again.

A couple of remarks are in order. First, the argument can be blocked by banning the formula '*x* does not instantiate *x*' from the metalanguage. But this does not sit well with the notion of untyped property. If properties are objects, there is nothing problematic with the formula '*x* does not instantiate *x*', which is perfectly grammatical. Second, the problem cannot be avoided by merely giving up absolute generality. The weaker assumption that we can quantify over all untyped properties is enough to generate the argument.

Notice that, as stated, the argument does not involve plural predicates or plural properties. It turns exclusively on the semantics of singular predication based on untyped properties. This might suggest a response on behalf of the untyped pluralist. She could give up the idea that a *singular* predicate denotes a property, perhaps embracing the view that a singular predicate denotes a plurality of things, the things to which the predicate applies. On this view, $F(t)$ is true in an interpretation if and only if, relative to that interpretation, the object denoted by *t* is one of the *objects* denoted by *F*. Of course, this response seems ad hoc and introduces an awkward asymmetry between the semantics of singular predication and that of plural predication. But even if one is willing to pursue it, the prospects for untyped pluralism are bleak. Williamson's argument can be reformulated in terms of plural predication.

Let $R(vv, v)$ be a two-place plural predicate of the object language taking a plural and a singular argument. The intuitive considerations that motivate LPI also motivate the following principle. For any admissible domain *dd* and any formula $\Phi(xx, x)$ of the metalanguage,

(LPIP) there are some pairs *ii* with domain *dd* such that, for all $xx \preccurlyeq dd$ and for every $x \prec dd$, *R* applies to *xx* and *x* according to *ii* if and only if $\Phi(xx, x)$,

where saying that *R* applies to *xx* and *x* according to *ii* just means that

$$ii \models R(vv, v) \; [ss(vv/xx)(v/x)]$$

where *ss* are any assignment from the domain of *ii*.

So LPIP and absolute generality jointly entail that

(LPIP*) there are some pairs *ii* with an all-inclusive domain such that, for all *xx* and
for every *x*, *R* applies to *xx* and *x* according to *ii* if and only if $\Phi(xx,x)$.

We now obtain an inconsistency by replacing $\Phi(xx,x)$ with '*x* is not instantiated by *xx* and *x*'.[6]

Giving up a uniform account of predication does not help the untyped pluralist to avoid Williamson's argument. If we can quantify over absolutely everything, or at least over all untyped properties, untyped pluralism is at odds with the liberal principle of interpretations. At this point the untyped pluralist might simply decide to sacrifice the liberal principle of interpretations in spite of its intuitive plausibility. Unfortunately, more trouble lies ahead.

5   A plural version of Cantor's theorem

Even though it can capture interpretations with an all-inclusive domain of quantification, there are still some intuitive interpretations of $\mathscr{L}_{PL}$ that untyped pluralism is unable to capture. Indeed, in the context of untyped pluralism the existence of some intuitive interpretations is incompatible with a plural version of Cantor's theorem.

Consider an atomic plural predication in $\mathscr{L}_{PL}$, say $P(vv_1)$. For any things *xx*, we can conceive an assignment *ss* in which vv_1 denotes *xx* and an interpretation *ii* with an all-inclusive domain according to which *P* applies to *xx* and to no other plurality. That is, for any things *xx*, there is, or there should be, an interpretation *ii* with an all-inclusive domain such that

(6) $ii \models P(vv_1) \wedge \forall vv_2 \, (vv_2 \not\approx vv_1 \rightarrow \neg P(vv_2)) \, [ss(vv_1/xx)]$,

where *ss* are any assignment from the domain of *ii* and $ss(vv_1/xx)$ are a variant of *ss* in which vv_1 denotes *xx*.

If we introduce a plural constant in the language, the point can be made even more vividly. Add the plural constant *cc* to $\mathscr{L}_{PL}$, and extend the characterization of an interpretation *ii* in the obvious way, i.e by requiring that $[\![cc]\!]_{ii}$ be some things in the domain of *ii*. Then, for any things *xx*, there is, or there should be, an interpretation *ii* with an all-inclusive domain such that $[\![cc]\!]_{ii}$ denotes *xx* and

(7) $ii \models P(cc) \wedge \forall vv \, (vv \not\approx cc \rightarrow \neg P(vv)) \, [ss]$.

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[6]Suppose that there are some pairs *ii* with an all-inclusive domain such that, for every *xx* and *x*, *R* applies to *xx* and *x* according to *ii*—i.e. $[\![R]\!]_{ii}(xx,x)$—if and only if *x* is not instantiated by *xx* and *x*. Let α be the plural (relational) property denoted by *R* according to *ii*, i.e. $\alpha = [\![R]\!]_{ii}$. Then it follows that, for every *xx* and *x*, α is instantiated by *xx* and *x* if and only if α is not instantiated by *xx* and *x*. Let *rr* be any things. Since α is an object in the all-inclusive domain, we can infer that α is instantiated by *rr* and α if and only if α is not instantiated by *rr* and α. Contradiction.

where *ss* are, again, any assignment from the domain of *ii*.

Given the untyped pluralist's account of plural predication, the existence of these interpretations entails that, for any *xx*, there is an untyped property instantiated by *xx* and by no other plurality. In symbols:

(8) $\forall xx \, \exists \alpha \, \forall yy \, (\alpha(yy) \leftrightarrow yy \approx xx).$

In other words, admitting the existence of the interpretations described just above commits the untyped pluralist to the existence of a one-to-one mapping from the pluralities to the plural properties.

However, this clashes with a plural version of Cantor's theorem. The intuitive idea is that, if there is more than one thing, there are more pluralities of objects than objects. Therefore, there are more pluralities than untyped properties. As a result, there cannot be a one-to-one mapping from the pluralities to the plural properties.

The way I just described the idea behind the plural version of Cantor's theorem is hardly coherent. Pluralities were treated as things, and I invoked a cardinality comparison between pluralities and objects. The task now is to provide a suitable rendering of the idea. Specifically, we must to prove a version of Cantor's theorem which uses only resources that are deemed acceptable by the untyped pluralist.

Let us start by recalling the set-theoretic version of Cantor's theorem that will serve as the model for the plural one. If R is a relation (i.e. a set of pairs), its domain is the set $\{x : \exists y \, (x,y) \in R\}$. An element x of the domain of R is said to code a set X if and only if $X = \{y : (x,y) \in R\}$. Moreover, we say that R codes a set X if and only if X is coded by some element of the domain of R. In one of its formulations, Cantor's theorem states that there is no relation that codes every subset of its domain.

Moving to the plural setting, we can adapt the terminology as follows. For any plurality *xx*, the domain of *xx*—abbreviated as dom(*xx*)—are the things *yy* satisfying this condition:

$$\forall x \, (x \prec yy \leftrightarrow \exists y \, (x,y) \prec xx).$$

We then say of an object x in dom(*xx*) that it codes a plurality *zz* (or is the code of it) when

$$\forall y \, (y \prec zz \leftrightarrow (x,y) \prec xx).$$

Finally, *xx* is said to code *yy* if and only if there is an object in dom(*xx*) that codes *yy*.

Note that some pairs can code multiple pluralities at once. For instance, the pairs $(a,c), (a,d), (b,e), (b,f)$ simultaneously code two pluralities: c and d, and e and f. The domain of these pairs are a and b, where a is the code of c and d, and b is the code of e and f.

Which pluralities can be coded by some pairs? Our plural version of Cantor's theorem describes a constraint on coding. Here is the statement of the theorem. A proof is given in Appendix B.

Plural Cantor. *There are no things which code every subplurality of their domain if their domain contains at least two objects:*

$$\neg\ \exists xx\ (|dom(xx)| \geq 2\ \&\ \forall yy\ (yy \preccurlyeq dom(xx) \rightarrow \exists x\ \forall y\ ((x,y) \prec xx\ \leftrightarrow\ y \prec yy))).^{7}$$

An immediate corollary is the following.

Corollary. *There are no things which code every plurality.*

This spells trouble for the untyped pluralist.

　As we have seen, admitting the existence of some intuitive interpretations commits the untyped pluralist to

(8)　　$\forall xx\ \exists \alpha\ \forall yy\ (\alpha(yy)\ \leftrightarrow\ yy \approx xx).$

That is, for every plurality, there is a plural property uniquely associated with that plurality. A relatively straightforward consequence of (8) via plural comprehension is that there are some things which code every plurality (see Hossack forthcoming for a rejection of plural comprehension related to the present argument; see also the discussion in Linnebo 2010). In particular, if a property is instantiated by a unique plurality, then it can serve as the code of that plurality. But this violates our plural version of Cantor's theorem.[8] So the untyped pluralist cannot admit the existence of those interpretations.

　Construing properties as untyped comes at the cost of sacrificing the adequacy of the semantics. Since the ability to capture all intuitive interpretations of the language—including those with an all-inclusive domain—plays an important role

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[7]Second-order analogues of this result are proved in Bernays 1942 and Shapiro 1991. For discussion, see Rayo 2002. Other plural versions of Cantor's theorem are found in Yi 2006 and Hawthorne and Uzquiano 2011.

[8]*Proof.* Suppose that (8) holds. Then consider the open formula of the metalanguage

$$\varphi(x)\ \equiv\ \exists \alpha\ \exists z\ (x = (\alpha,z)\ \wedge\ \exists zz\ (z \prec zz\ \wedge\ \alpha(zz))),$$

which asserts that x is a pair whose second coordinate z is among some things zz which jointly instantiate its first coordinate. Clearly, there is some x such that $\varphi(x)$. For example, take two objects a and b (say two cooperative individuals), and let α be a plural property that they jointly instantiate (e.g. that of cooperating). Then the pair (α,a) satisfies the formula, i.e. $\varphi(x)$ for $x = (\alpha,a)$. Since the formula $\varphi(x)$ is instantiated, we can apply plural comprehension to it and obtain the plurality of objects satisfying the formula:

$$\exists xx\ \forall y\ (y \prec xx \leftrightarrow \varphi(y)).$$

Call these things *aa*. We want to show that *aa* code every plurality. Let *yy* be any plurality. It follows from (8) that there is a property α dependent on *yy* such that, for every *xx*, $\alpha(xx)$ if and only if $xx \approx yy$. By the characterization of *aa*, for every y, $(\alpha,y) \prec aa$ just in case $y \prec yy$. So *aa* code the plurality *yy*, and α is their code. Since *yy* was an arbitrary plurality, it follows that *aa* code every plurality, which contradicts the plural version of Cantor's theorem.

in motivating pluralism, the untyped version of the view should be rejected. By introducing a type distinction between objects and properties, both problems raised in this article can be sidestepped. The type distinction blocks Williamson's argument just as Williamson's own second-order response does. Moreover, if properties are no longer objects, it is consistent to hold that there is a one-to-one mapping from the pluralities to the plural properties *and* that there are more pluralities than objects. So the existence of the interpretations described at the beginning of this section does not clash with Cantor's theorem. I conclude that pluralists should embrace a type distinction between objects and properties.[9]

Appendix A: untyped pluralism

In this appendix, we provide a more detailed presentation of untyped pluralism. Some things *ii* are an interpretation if they meet the following conditions.

1. There is some x such that $(\exists, x) \prec ii$. (This means that the domain of the interpretation must be non-empty.) The domain of *ii*—which will be denoted by $d(ii)$—are exactly the *things* satisfying:

$$\forall x \, (x \prec d(ii) \leftrightarrow (\exists, x) \prec ii).$$

2. For any singular constant c, there is a unique x such that $(c, x) \prec ii$, and $x \prec d(ii)$. (A singular constant denotes one and only one object in the domain.)

3. For any singular predicate F^n, there is a unique singular property p such that $(F^n, p) \prec ii$. Moreover, for every $x_1, ..., x_n$, if $p(x_1, ..., x_n)$, then $x_1, ..., x_n \prec d(ii)$. (A singular predicate denotes one and only one singular property which, if instantiated, is instantiated by objects in the domain.)

4. For any plural predicate $P^{n[m]}$, there is a unique plural property α such that $(P^{n[m]}, \alpha) \prec ii$. Moreover, for every $xx_1, ..., xx_m, x_1, ..., x_{n-m}$, if

$$\alpha(xx_1, ..., xx_m, x_1, ..., x_{n-m}),$$

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then
$$xx_1, ..., xx_m \preccurlyeq d(ii) \text{ and } x_1, ..., x_{n-m} \prec d(ii).$$

(A plural predicate denotes one and only one plural property which, if instantiated, is instantiated by objects and pluralities of objects in the domain.)

Next one needs to characterize the notion of variable assignment. Some things *ss* are a variable assignments relative to a domain $d(ii)$ just in case:

5. For any singular variable v, there is a unique x such that $(v, x) \prec ii$, and $x \prec d(ii)$.

6. For any plural variable vv, there is an object x such that $(vv, x) \prec ii$ and $x \prec d(ii)$. Such an object need not be unique, as vv might denote many things, those appearing as the second coordinates of pairs among *ss* of the form (vv, y). However, those things must all be in $d(ii)$. That is, for every y, if $(vv, y) \prec ss$, then $y \prec d(ii)$.

Variants $ss(v/x)$ and $ss(vv/xx)$ of a variable assignment *ss* with respect to v and vv are defined in the usual way.

For any interpretation *ii* and variable assignment *ss*, and for any non-logical expression E, let $[\![E]\!]_{ii}$ and $[\![E]\!]_{ss}$ indicate the denotation or denotations of E according to *ii* and *ss*, respectively. If t is a singular term, we use the following abbreviation:

$$[\![t]\!]_{ii/ss} = \begin{cases} [\![t]\!]_{ii} & \text{if } t \text{ is a constant,} \\ [\![t]\!]_{ss} & \text{if } t \text{ is a variable.} \end{cases}$$

We are now are ready to characterize the central notion of satisfaction. This is defined as a relation $\mathbf{Sat}(\varphi, ii, ss)$—written as $ii \vDash \varphi \; [ss]$—among a formula φ of $\mathscr{L}_{PL}$, an interpretation *ii*, and a variable assignment *ss* from $d(ii)$. The following clauses implicitly define this relation:

(i) If φ is of the form $t = r$, $ii \vDash \varphi \; [ss]$ if and only if $[\![t]\!]_{ii/ss} = [\![r]\!]_{ii/ss}$.

(ii) If φ is of the form $t \prec vv$, $ii \vDash \varphi \; [ss]$ if and only if $[\![t]\!]_{ii/ss} \prec [\![vv]\!]_{ss}$.

(iiia) If φ is of the form $F^n(t_1, ..., t_n)$,

$$ii \vDash F^n(t_1, ..., t_n) \; [ss] \text{ if and only if } [\![F^n]\!]_{ii}([\![t_1]\!]_{ii/ss}, ..., [\![t_n]\!]_{ii/ss}).$$

(iiib) If φ is of the form $P^{n[m]}(vv_1, ..., vv_m, t_1, ..., t_{n-m})$,

$$ii \vDash P^{n[m]}(vv_1, ..., vv_m, t_1, ..., t_{n-m}) \; [ss] \quad \text{if and only if}$$
$$[\![P^{n[m]}]\!]_{ii}([\![vv_1]\!]_{ss}, ..., [\![vv_m]\!]_{ss}, [\![t_1]\!]_{ii/ss}, ..., [\![t_{n-m}]\!]_{ii/ss}).$$

(iv) If φ is of the form $\exists v\psi$, $ii \vDash \varphi$ $[ss]$ if and only if, for some $x \prec d(ii)$, $ii \vDash \psi$ $[ss(v/x)]$.

(v) If φ is $\exists vv\psi$, $ii \vDash \varphi$ $[ss]$ if and only if, for some $xx \preccurlyeq d(ii)$, $ii \vDash \psi$ $[ss(vv/xx)]$.

(vi-xi) The clauses for negation and for the binary connectives are the obvious ones.

Finally, we can characterize the notion of logical consequence and logical truth. For any set of sentences Γ and any sentence σ, σ is a logical consequence of Γ ($\Gamma \vDash \sigma$) just in case, for any interpretation ii and variable assignment ss from $d(ii)$, if $ii \vDash \gamma$ $[ss]$ for any member γ of Γ, then $ii \vDash \sigma$ $[ss]$. A sentence σ is a logical truth in the special case in which $\emptyset \vDash \sigma$. This completes the presentation of the untyped pluralist semantics.

Appendix B: proof of Plural Cantor

Plural Cantor. *There are no things which code every subplurality of their domain if their domain contains at least two objects:*

$$\neg\, \exists xx\, (|dom(xx)| \geq 2\, \&\, \forall yy\, (yy \preccurlyeq dom(xx) \rightarrow \exists x\, \forall y\, ((x,y) \prec xx\, \leftrightarrow\, y \prec yy))).$$

Proof. Suppose for *reductio* that there are some things xx such that $dom(xx)$ contains at least two objects and xx code every subplurality of $dom(xx)$.

Under this supposition, there is $z \prec dom(xx)$ such that $(z,z) \nprec xx$. This will be proved as a separate lemma below. Now, through plural comprehension using xx as parameter, we can reproduce the diagonal argument behind the traditional version of Cantor's theorem. That is, since there is $z \prec dom(xx)$ such that $(z,z) \nprec xx$, by plural comprehension we obtain:

$$\exists yy\, \forall y\, (y \prec yy\, \leftrightarrow\, (y \prec dom(xx)\, \&\, (y,y) \nprec xx)).$$

Call these things *bb*. Note that $bb \preccurlyeq dom(xx)$. Since xx code every subplurality of $dom(xx)$, xx code bb, i.e.

$$\exists x\, (x \prec dom(xx)\, \&\, \forall y\, ((x,y) \prec xx\, \leftrightarrow y \prec bb)).$$

By the characterization of *bb*, this entails in turn that, for some $r \prec dom(xx)$,

$$\forall y\, ((r,y) \prec xx\, \leftrightarrow (y \prec dom(xx)\, \&\, (y,y) \nprec xx)).$$

Thus, in particular,

$$r \prec dom(xx)\, \&\, ((r,r) \prec xx\, \leftrightarrow (r \prec dom(xx)\, \&\, (r,r) \nprec xx)).$$

Therefore,

$$(r,r) \prec xx \leftrightarrow (r,r) \not\prec xx.$$

Contradiction. We conclude that there are no things which code every subplurality of their domain if this contains at least two objects. □

Lemma. *If xx code every subplurality of dom(xx) and* $|dom(xx)| \geq 2$*, then there is* $z \prec dom(xx)$ *such that* $(z,z) \not\prec xx$*.*

Proof. As a preliminary, note that, for any object x, there are some things such that x and only x is among them:

$$\exists yy \, \forall y \, (y \prec yy \leftrightarrow y = x).$$

The existence of this 'degenerate' plurality follows from plural comprehension, using x as a parameter.[10] Let ee_x stand for the things such that x and only x is among them. Then, by definition, if $y \prec ee_x$, then $y = x$.

Now suppose that xx code every subplurality of dom(xx) and that $|dom(xx)| \geq 2$. Also, suppose for *reductio* that, for every $z \prec dom(xx)$, $(z,z) \prec xx$. The main idea for the proof is that these assumptions force every object x in the domain to be the code of ee_x. So there are no objects left in the domain to serve as the codes of the 'non-degenerate' pluralities, contrary to the fact that xx code every subplurality of dom(xx). Let us make this more precise.

Let x be any object in dom(xx). Since xx code every subplurality of dom(xx), xx code ee_x. Let c_x be the code of ee_x, i.e.

$$\forall y \, ((c_x, y) \prec xx \leftrightarrow y \prec ee_x).$$

Then, clearly, $(c_x, x) \prec xx$. However, we are assuming that for every $z \prec dom(xx)$, $(z,z) \prec xx$. Since $c_x \prec dom(xx)$, it follows in particular that $(c_x, c_x) \prec xx$. Thus $c_x \prec ee_x$. Hence $c_x = x$. This means that, for any object x in dom(xx), x is the code of ee_x.

Since $|dom(xx)| \geq 2$, there are a_1 and a_2, distinct objects in dom(xx). Plural comprehension entails the existence of the plurality comprising exactly a_1 and a_2. That is,

$$\exists yy \, \forall y \, (y \prec yy \leftrightarrow (y = a_1 \lor y = a_2)).$$

Call this plurality aa. Since xx code every subplurality of their dom(xx), xx code aa. Let c_{aa} be the code of aa. By the definition of a code, $(c_{aa}, a_1) \prec xx$, $(c_{aa}, a_2) \prec xx$, and, for every y, if $(c_{aa}, y) \prec xx$, then $y = a_1$ or $y = a_2$. But, recall, we are assuming

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[10] The proof could be easily revised to accommodate the requirement that a plurality contain always more than one thing (but see, e.g., Yi 2005 for an argument against this requirement).

that, for every $z \prec \text{dom}(xx)$, $(z,z) \prec xx$. So $(c_{aa}, c_{aa}) \prec xx$. Therefore, c_{aa} must be either a_1 or a_2.

However, we have shown above that an object x in $\text{dom}(xx)$ is the code of ee_x. Thus c_{aa} can be neither a_1 nor a_2. For, if $c_{aa} = a_1$, c_{aa} must be the code of ee_{a_1} and cannot be the code of aa. Likewise, if $c_{aa} = a_2$, c_{aa} is the code of ee_{a_2} and cannot be the code of aa. Contradiction. So there must be some z in $\text{dom}(xx)$ such that $(z,z) \nprec xx$. □

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