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DOI:
10.1093/mind/fzu039

Set Theory, Type Theory, and Absolute Generality

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Abstract: In light of the close connection between the ontological hierarchy of set theory and the ideological hierarchy of type theory, Øystein Linnebo and Agustín Rayo have recently offered an argument in favour of the view that the set-theoretic universe is open-ended. In this paper, we argue that, since the connection between the two hierarchies is indeed tight, any philosophical conclusions cut both ways. One should either hold that both the ontological hierarchy and the ideological hierarchy are open-ended, or that neither is. If there is reason to accept the view that the set-theoretic universe is open-ended, that will be because such a view is the most compelling one to adopt on the purely ontological front.

It is now common to interpret ordinary, Zermelo-Fraenkel set theory (with or without urelements) as capturing the so-called iterative conception of set. The idea is that sets, or at least well-founded sets, are arranged in levels, with the levels indexed by ordinals. We 'begin' with a collection R_0 of urelements, which we take to be a set. For each ordinal α, the successor rank $R_{\alpha+1}$ is $\wp(R_\alpha) \cup R_0$, the powerset of R_α together with the urelements. If λ is a limit ordinal, then

$$R_\lambda = \bigcup_{\beta<\lambda} R_\beta.$$

With so-called 'pure' set theory, there are no urelements. So we 'begin' with the empty set V_0, and, for each ordinal α, the corresponding rank is dubbed V_α.

In the philosophical literature, there has been much discussion of late on the question of absolute generality, whether there is a definite totality, or plurality, of absolutely all objects (see, e.g., Rayo and Uzquiano 2006). For present purposes, the most interesting instance of this question is whether there is a definite totality, or plurality, of absolutely all sets, or absolutely all well-founded sets.[1] On the iterative conception, that question comes down to whether there is a definite totality, or plurality, of all ordinals, through which one does the above iteration (assuming that the powerset operation is well-defined, and definite).

Call someone who countenances a definite totality, or plurality, of all sets an *absolutist*. There are two kinds. The *non-austere absolutist* holds that there are (well-founded) collections of sets that are not themselves sets. These are the so-called 'proper classes'. One such class, sometimes called V, contains all pure, iterative sets. Another proper class is Ω, the class of all ordinals.[2] Accordingly,

$$V = \bigcup_{\beta \in \Omega} V_\beta .$$

A non-austere absolutism is more or less implicit in the treatment of higher-order set theory in Shapiro 1991, where proper classes are among the 'logical sets'.

In contrast, the *austere absolutist* holds that sets are the only (well-founded) collections that there are. Such a view was championed by George Boolos who responded to the proposal of proper classes with a quip: 'Wait a minute! I thought that set theory was supposed to be a theory about all, "absolutely" all, the collections that there were and that "set" was synonymous with "collection".' (Boolos 1998, p. 35)

A position that opposes both kinds of absolutism is that there just is no definite totality, or plurality, of absolutely all sets, or absolutely all well-founded 'pure' sets. The sets are, in effect, open-ended. Michael Dummett (1993, p. 441) defines an 'indefinitely extensible concept' to be 'one such that, if we can form a definite conception of a totality all of whose members fall under the concept, we can, by reference to that totality, characterize a larger totality all of whose members fall under it'. The set-theoretic relativist holds that the iterative hierarchy is indefinitely extensible (see Zermelo 1930, Shapiro and Wright 2006).

A recent paper by Øystein Linnebo and Agustín Rayo (forthcoming) includes an interesting argument against absolutism, based on Kurt Gödel's remark that

> the theory of aggregates, as presented by Zermelo, Fraenkel and von Neumann ... is nothing else but a natural generalization of the theory of types, or rather, it is what becomes of the theory of types if certain superfluous restrictions are removed. (Gödel 1933, pp. 45–6)

Linnebo and Rayo claim that, in light of the close connection between set theory and type theory, we should accept that the set-theoretic (ontological) hierarchy is

[1] As is customary, we do not take 'plurality' to refer to an entity, set-like or otherwise. It is merely a shorthand for a plural construction.

[2] We take ordinals here to be the order-types of well-orderings. It is an option to identify them with certain sets, such as the von Neumann ordinals.

indefinitely extensible *given that* is it natural to hold that the suitably idealized type-theoretic (ideological) hierarchy is similarly open-ended.

We dispute that line of argument here. The connection between set theory and a suitably liberalized type theory does suggest some arguments against absolutism, but those are analogues of arguments that have been brought against absolutism in set theory already, arguments that the absolutist has already confronted, or should have confronted, and somehow resolved, either by biting a bullet or by coming up with some compelling counter-considerations.

In effect, once type theory has been liberalized, in the way Linnebo and Rayo indicate, the connection between type theory and set theory is extremely tight. The two become all but notational variants on each other. So any philosophical conclusions cut both ways. One should either hold that both the set-theoretic hierarchy and the liberalized ideological hierarchy are open-ended, or that neither is.

In what follows, we briefly recapitulate the main positions and arguments on set-theoretic absolutism. After that, we sketch the Linnebo-Rayo connection between set theory and type theory, suitably liberalized. We note that the underlying semantic principles they invoke are remarkably similar to Cantor's two number principles, and give rise to the same problems. This sets the stage for our argument that the connection between set theory and liberalized type theory provides no *new* pressure on set-theoretic absolutism, but perhaps reinforces or, better, recapitulates earlier arguments on the issue.

1 Traditional arguments against absolutism

The most powerful arguments against absolutism about sets are tied somehow to the paradoxes of Russell and Burali-Forti. We begin with the former. According to a natural principle of set formation, any definite totality or plurality forms a set. The absolutist countenances a definite totality of all sets. So it follows from this natural principle that there is a set of all sets. But this is inconsistent with a separation principle. One can, of course, challenge the separation principle (see, e.g., Quine 1937 and Forster 1992), but we do not consider that option here. All parties to the present debate—austere absolutist, non-austere absolutist, and relativist—accept that there is no set of all sets. Therefore, both kinds of absolutists must reject the aforementioned natural principle that any definite totality forms a set. This leaves them with the hard challenge of explaining why some definite totalities, or pluralities, fail to form sets, and of saying something about which ones do and which ones do not.

The argument can be generalized further against the non-austere absolutist. Whatever collections he is willing to admit, an analogue of Russell's paradox would show that there is no collection of all such collections (assuming a generalization of the separation principle, anyway). So either he must explain why some definite totalities do not even form collections, or else he must agree that there is no definite totality of all collections. The second option is to give up on the absolutism at the next level. Such a combination might be consistent, but it looks unmotivated and ad hoc.

There is a semantic version of the Russell argument. There is, of course, no interpretation of the language of set theory whose domain is a set containing all sets as members (continuing to assume a separation principle). The non-austere absolutist can overcome this by expanding the ontology to include proper classes. But then the non-austere absolutist cannot produce an interpretation of the language of class theory whose domain consists of all classes. She might try to overcome this limitation by expanding the ontology even further, and introduce what we may call proper super-classes, collections of classes which are not themselves classes. The procedure here obviously iterates. For the non-austere absolutist, semantic reflection progressively motivates the introduction of more and more layers of classes. The introduction of such layers looks suspiciously like expanding the iterative hierarchy to more levels. And, of course, there is a lingering question concerning the totality, or plurality, of *all* such classes.

In the hierarchy of classes, there is a precise sense in which what appear to be proper classes at each stage in the process can be regarded as sets at the next stage. This is exactly the picture of the open-ended universe of sets proposed by Ernst Zermelo (1930). The idea is that any theory that purports to characterize all sets, or all classes, or whatever, can be reinterpreted as describing only a particular level in the iterative hierarchy, not the hierarchy itself. This is the position of the set-theoretic relativist. The absolutist, or at least the austere absolutist, insists that any such interpretation misrepresents the intended semantic content of set theory, which, to repeat the quip by Boolos, is 'supposed to be a theory about all, "absolutely" all, the collections that there are'. The relativist, however, is content with the limited semantic picture. He takes the austere absolutist's insistence to be symptomatic of her 'lack of a more comprehensive conception of set' (Parsons 1974, p. 11), and he finds the non-austere iteration of layers of classes to just be a description of the iterative hierarchy (of sets) itself.

A further threat to absolutism comes from the Burali-Forti paradox and, in particular, from apparent cases of well-orderings that are 'longer' than Ω, the supposed definite totality of all ordinals.[3] If quantification over all ordinals is legitimate, then it is straightforward to construct formulas that define well-orderings 'longer' than Ω. Consider, for example, the relation between pairs (α,β) of ordinals in lexicographic order:

$$(\alpha_1,\beta_1) \prec (\alpha_2,\beta_2) \text{ if and only if either } \alpha_1 < \alpha_2 \text{ or both } \alpha_1 = \alpha_2 \text{ and } \beta_1 < \beta_2.$$

[3]To avoid begging the question against the austere absolutist, we do not have to think of 'Ω' here as a singular term denoting a proper class. One can take it to be a plural term, such as 'the ordinals'. Or one can think of 'Ω' as a predicate, and read a phrase like 'α is in Ω' as 'α is an ordinal'.

This characterizes a well-ordering of type Ω^2 (so to speak). The well-ordering type of this well-ordering, of course, cannot be in Ω. However, Ω is supposed to be the totality of, to paraphrase Boolos, all, absolutely all, the ordinals that there are. And, it seems, 'ordinal' is supposed to be synonymous with 'well-ordering type'.

Occasionally, set theorists at least seem to carry out transfinite recursions and inductions on these 'long' well-orderings. In the spirit of non-austere absolutism, it would tempting to accommodate this phenomenon by expanding the ontology to include what we may call 'super-ordinals', well-ordering types longer than Ω. But this expansion immediately gives rise to the possibility of defining even more well-orderings. The upshot is that, if quantification over all ordinals is legitimate, as it is for the absolutist in defining the iterative hierarchy, then there is pressure in the direction of indefinite extensibility (see Shapiro 2003, Shapiro and Wright 2006). Of course, this is not to say that the pressure cannot be resisted.

2 The connection between set theory and type theory

Let us define *liberalized type theory* to be the result of extending standard type theory through the transfinite, by indexing the orders with ordinals. This means that there are, or can be, at least as many typed languages as there are ordinals. For each ordinal α, there is, or can be, an α^{th}-order language.

More details will be provided below. For now, we are interested in a broad justification of this liberalization. Gödel himself seemed to have had a pragmatic attitude: we develop a liberalized type theory just because we can. One of the main aims of Linnebo and Rayo forthcoming, however, is to provide some systematic considerations in favour of a liberalized type theory.

Define a *generalized semantics* for a given language to be a theory of all possible interpretations of that language. Suppose, now, that you are open to the possibility of embracing a typed language of a particular order. Suppose also that you espouse what we may call *Semantic Optimism*, namely the view that, if you embrace a language of a given order, it should be possible to provide a generalized semantics for that language. Linnebo and Rayo (forthcoming, Appendix, Theorem 2) show that, on the assumption that it is possible to quantify over absolutely everything, an analogue of Cantor's Theorem entails that a generalized semantics for a typed language of order α cannot be developed in any language of order α. They also show that for any level α, a generalized semantics for a language of order α can be developed in a language of order $\alpha+1$, or $\alpha+2$ if α is a limit ordinal (Linnebo and Rayo forthcoming, Appendix, Theorem 3).

These two results push the semantic optimist from any legitimate type-theoretic language to another language immediately above it, i.e., to what we may call its 'successor language'. By themselves, however, the results do not deliver a genuine liberalized type theory, as they are compatible with there being only finite orders: a generalized semantics for an n^{th}-order language is carried out in a language of order $n+1$. What is needed is a principle capable of justifying languages corresponding to limit orders.

An intuitive idea is that, for any legitimate languages, there is, or can be, a 'union' language that encompasses all of them. Call this the principle of *Naïve Union*. As noted, a language of each finite order is justified on the basis of Semantic Optimism (in light of the results reported just above). So Naïve Union justifies a language of order ω. Semantic Optimism would then give us languages of order $\omega+1$, $\omega+2$, etc. Naïve Union would yield a language of order 2ω. A simple transfinite induction yields the existence of a language at the level of each ordinal. This is a fully liberalized type theory.

However, as it stands, Naïve Union is self-defeating—it is inconsistent with Semantic Optimism (in light of the foregoing results). We apply Naïve Union to *all* of the legitimate languages generated by the transfinite induction, and generate a 'union' language that encompasses *all* of them. But then Semantic Optimism would yield a language distinct from all of these languages (again in light of the results from Linnebo and Rayo). In effect, this just is the Burali-Forti paradox—assuming that the orders of liberalized type languages are to be indexed by ordinals. Linnebo and Rayo opt for the following restriction of Naïve Union.

> *Set-theoretic Principle of Union*: For λ a limit ordinal, if one is prepared to countenance languages of order β, for every $\beta<\lambda$, then one should also countenance a language of order λ.

With the axiom of choice, this amounts to the restriction of Naïve Union to *sets* of languages. That is, Set-theoretic Union is the principle that for any set of legitimate languages, a 'union' language that encompasses all of them is also legitimate. This union principle succeeds in delivering the full liberalized hierarchy of languages without incurring the problem generated by Naïve Union, assuming the background set theory, of course.

Semantic Optimism and the various principles of Union are remarkably similar to Georg Cantor's two generating principles for transfinite numbers. The first licenses the addition of 'a unity to an already formed and existing number' (Cantor 1883, p. 907), a successor to that number. This is the analogue of Semantic Optimism. Cantor's second principle states that

> if any definite succession of defined ... [numbers] is put forward of which no greatest exists, a new number is created ... which is thought of as the *limit* of those numbers; that is, it is defined as the next number greater than all of them. (Cantor 1883, pp. 907-8)

It is easy to see that these two generating principles are inconsistent with the idea that there is a definite succession of *all* transfinite numbers. If there were, then, by the second principle, there would be a number greater than all numbers. But such a number would have to be greater than itself (not to mention being greater than its successor, as generated by the first principle).

Cantor must thus renounce the idea that there is a definite succession of all transfinite numbers and, of course, he did. However, at least at times, he took the transfinite numbers, and more generally, all mathematical entities, to belong to a changeless, fully determined universe. So he embraced what we call absolutism.[4] The talk of 'generation' and 'creation' is only a metaphor.

The compelling question, then, is this: what did Cantor mean by 'definite' (or, to be specific, 'definite succession')? Or, perhaps better, what should he have meant? Being part of a changeless, fully determined universe, is, presumably, not sufficient for definiteness. With a liberal dose of hindsight, we can 'interpret' a definite succession as one whose elements form a *set*. Under this 'interpretation', Cantor's second principle states that, for any set of transfinite numbers, without a greatest element, there is a least number that is greater than all of the numbers in the set. And, of course, this is true in contemporary set theory, where 'number' is either 'cardinal number' or 'von Neumann ordinal'. The analogy between this number principle and Linnebo and Rayo's Set-theoretic Principle of Union should be striking.

As noted above, Cantor took an absolutist stance toward the transfinite numbers, at least at times. If his two number principles are somehow consistent with his stance, once they are properly interpreted, then there is reason to suspect that the analogous principles in the context of type theory will not be able to adjudicate between absolutism and relativism. To be blunt, it would be surprising if Semantic Optimism and the Set-theoretic Principle of Union could undermine absolutism about sets in a way that Cantor's generating principles do not.

Let us now briefly recapitulate some details of Linnebo and Rayo's liberalized type theory. This will articulate the sense in which, as Gödel said, set theory just is type theory with 'certain superfluous restrictions removed'. If α is an ordinal, then let $\mathscr{L}^{\alpha}$ be an α^{th}-order language. It contains variables of type β, for every ordinal $\beta < \alpha$. The type of a variable is indicated by a superscript. Type predication is cumulative, in the sense that an entity of type β can apply to entities of any type $\gamma < \beta$. Moreover, a formula $x^{\beta}(x^{\gamma})$, indicating atomic predication, is always well-formed, even if $\gamma \geq \beta$. As usual, the entities in the type-theoretic hierarchy are assumed to be extensional. Finally, we assume that for each formula $\varphi(x^{\beta})$, of $\mathscr{L}^{\alpha}$, with x^{β} free, $\mathscr{L}^{\alpha}$ contains an abstraction term $\lambda x^{\beta}.\varphi(x^{\beta})$, of type $\beta+1$.

Let $\mathscr{L}_{\in}$ be the standard language of set theory, augmented with set-abstraction terms. If α is an ordinal, let $\mathscr{L}_{\in}^{\alpha}$ be the result of augmenting $\mathscr{L}_{\in}$ with primitive terms

[4] If 'inconsistent multitudes' count as collections, then Cantor's absolutism was non-austere. However, we will not engage this exegetical issue here. See Jané 2010 for a lucid account of the idealist and realist (or perhaps absolutist) elements in Cantor's thinking.

R_β, for each $\beta < \alpha$. The intended denotation of the term R_β, of course, is the β^{th} rank in the iterative hierarchy.[5] In every $\mathscr{L}_\in^\alpha$ it is required that each quantifier be bounded by one of the R_β's.

For each variable x^β in the typed language $\mathscr{L}^\alpha$, associate a (unique) variable x_β in $\mathscr{L}_\in$. There is a natural map $*^\alpha$ from $\mathscr{L}^\alpha$ to $\mathscr{L}_\in^\alpha$, defined by recursion as follows:

$$\begin{array}{rcl} x^\beta = y^\beta & \xmapsto{*^\alpha} & x_\beta = y_\beta \\ x^\gamma(x^\beta) & \xmapsto{*^\alpha} & x_\beta \in x_\gamma \\ \forall x^\beta \varphi(x^\beta) & \xmapsto{*^\alpha} & \forall x_\beta \in \mathrm{R}_\beta\ [\varphi(x^\beta)]^{*^\alpha} \\ \lambda x^\beta.\varphi(x^\beta) & \xmapsto{*^\alpha} & \{x_\beta \in \mathrm{R}_\beta :\ [\varphi(x^\beta)]^{*^\alpha}\} \end{array}$$

where $[\varphi(x^\beta)]^{*^\alpha}$ is the result of applying the map $*^\alpha$ to $\varphi(x^\beta)$. Of course, the variables of $\mathscr{L}_\in$ and $\mathscr{L}_\in^\alpha$ are not typed. The function of the subscripted indices is to track variables.

There is also a quite natural map in the opposite direction. Restrict attention to formulas of $\mathscr{L}_\in^\alpha$ whose variables are indexed with an ordinal $\beta < \alpha$. If a quantifier is bounded by R_β, then the variable it binds should be replaced with an appropriate variable whose index is β. For example, $\exists x \in \mathrm{R}_\beta\ \varphi(x)$ is to be re-written as $\exists y_\beta \in \mathrm{R}_\beta\ \varphi(y_\beta)$, where y_β does not occur in $\exists x \in \mathrm{R}_\beta \varphi(x)$.

Define $x^\beta \equiv x^\gamma$ as $\forall z^\delta(z^\delta(x^\beta) \leftrightarrow x^\delta(x^\gamma))$ with $\delta = \max(\beta,\gamma)+1$. Then predication, even when involving a mismatch of types, can then be reinterpreted as follows:

$$x^\gamma(x^\beta) \quad \longmapsto \quad \exists z^\delta\ (x^\gamma \equiv z^\delta\ \wedge\ z^\delta(x^\beta)),$$

where $\delta = \max(\beta,\gamma)+1$.

For each ordinal α, define a map $\dagger^\alpha$ from $\mathscr{L}_\in^\alpha$ to $\mathscr{L}^{\alpha+2}$, by recursion, as follows:

$$\begin{array}{rcl} x_\beta = x_\gamma & \xmapsto{\dagger^\alpha} & x^\beta \equiv x^\gamma \\ x_\beta \in x_\gamma & \xmapsto{\dagger^\alpha} & x^\gamma(x^\beta) \\ \forall x_\beta \in \mathrm{R}_\beta\ \varphi(x_\beta) & \xmapsto{\dagger^\alpha} & \forall x^\beta[\varphi(x_\beta)]^{\dagger^\alpha} \\ \{x_\beta \in \mathrm{R}_\beta : \varphi(x_\beta)\} & \xmapsto{\dagger^\alpha} & \lambda x^\beta.\ [\varphi(x_\beta)]^{\dagger^\alpha} \end{array}$$

What interesting properties are preserved under these translations? Linnebo and Rayo argue that *truth* is preserved:

[5]We indulge in some use-mention confusion here. At the outset of this paper, we *use* 'R_β' for the β^{th} rank in the iterative hierarchy, starting with a set R_0 of urelements. Here, 'R_β' is a primitive *symbol* in the language $\mathscr{L}_\in^\alpha$. As usual, we rely on context to disambiguate, although the presence or absence of italics should help.

> Assume a type theorist and a set theorist confront a domain of individuals. The type theorist is interested in the ideological hierarchy of concepts based on these individuals. By contrast, the set theorist is interested in the ontological hierarchy of sets based on these individuals regarded as urelements. ... *Assume that the individuals in question form a set*. ... Then ... the translations preserve truth value. For in both cases a limit level consists of the 'union' of the values of variables of all proceeding levels. And in both cases a successor level consists of the 'union' of the values of variables of the proceeding level plus all 'collections' of values of variables of that level. This ensures that the concepts of type γ are isomorphic to the sets in V_γ, for γ an arbitrary ordinal. It follows that $*[^\alpha]$ must map every truth of $\mathscr{L}^\alpha$ to a true sentence of $\mathscr{L}^\alpha_\in$, and that $\dagger[^\alpha]$ must map every truth of $\mathscr{L}^\alpha_\in$ to a true sentence of $\mathscr{L}^{\alpha+2}$. (Linnebo and Rayo forthcoming, italics added)

Let α be an ordinal and let $0 < \beta < \alpha$. In the intended interpretation of $\mathscr{L}^\alpha$, the range of a variable x^β is all β-order concepts of the individuals, construed extensionally. Given the assumption that the individuals form a set, and that the types are cumulative, the range of x^β just is the β-rank R_β in the iterative conception of set (as defined at the start of this paper), with the individuals as urelements.[6]

Notice that the map $\dagger^\alpha$ only provides a (truth-preserving) translation from *bounded* set theory to liberalized type theory, where the bounds are the R_β's (with $\beta < \alpha$). One may desire a translation from the language of ordinary, unbounded set theory. As Linnebo and Rayo point out, this can be accomplished if there is a reasonable way to restrict the quantifiers of ordinary set theory. Let $\star^\alpha$ be a map from $\mathscr{L}_\in$ to $\mathscr{L}^{\alpha+1}_\in$ which restricts the quantifiers to R_α. Notice the following:

(i) If λ is a limit ordinal greater than ω, then $\star^\lambda$ maps every theorem of Zermelo set theory to a truth of $\mathscr{L}^{\lambda+1}_\in$.

(ii) If κ is a strong inaccessible cardinal, then $\star^\kappa$ maps every theorem of ZFC to a truth of $\mathscr{L}^{\kappa+1}_\in$.

(iii) The absolutist, at least, holds that there is such a thing as truth in the iterative hierarchy (given a fixed set of urelements). By adopting a suitable reflection principle, there is a cardinal ξ such that the corresponding rank R_ξ satisfies all and only the true sentences of second-order ZFC (see Shapiro 1987 or Shapiro 1991, section 6.3). So $\star^\xi$ maps every truth of set theory (with the given urelements) to a truth of $\mathscr{L}^{\xi+1}_\in$.

[6]See note 5 above. To avoid begging a metaphysical question, perhaps we should say that the intended range of x^β just is tightly isomorphic to the β-rank R_β in the iterative conception of set.

By composing $\star^{\alpha}$ and $\dagger^{\alpha+1}$, we obtain a map from $\mathscr{L}_{\in}$ to the type-theoretic language $\mathscr{L}_{\in}^{\alpha+3}$. By choosing an appropriate ordinal α, the map can preserve interesting features of the language of set theory. In the case of (iii), involving a reflection principle, truth is preserved. Does that map preserve any more than truth? In particular, does it preserve the intended interpretation of the language of set theory? Not for the absolutist. To quote Boolos once again:

> that truth is preserved does not mean that anything whatsoever is. In particular, the speakers may have been talking about all sets, and reinterpreting what they say in such a way that it is not about all sets is changing the meaning of what is said if not the truth value; to report our speakers as having spoken only about the members of some set would be to misrepresent what they said. (Boolos 1998, p. 31)

The entire project of liberalizing type theory presupposes set theory in the background. Set theory just is the metatheory. When constructing the liberalized ideological hierarchy, the ordinals were given, or presupposed, from the 'outside'. Indeed, we defined a 'language' corresponding to each ordinal. As noted, we maintain consistency by adopting the Set-theoretic Principle of Union in place of Naïve Union. Notice that that principle yields what is, so to speak, a proper class of 'languages'. There is no set whose members are all of the languages in the liberalized hierarchy.

In the present dialectic, this much is not problematic, since all parties to the debate embrace set theory, in one form or another. A theorist might be more interested in alternate principles of Union which fall short of motivating the full liberalization of type theory. Specifically, if one is concerned about the amount of idealization implicit in thinking of the levels in the ideological hierarchy as *languages*—as tools useful for communication between humans—one might want to limit the ideological hierarchy to levels indexed by ordinals that are somehow accessible to us, say via some system of ordinal notation (see, e.g., Feferman 1988). In what follows, we put this concern aside and focus on the absolutist prepared to accept fully liberalized type theory.

3 Is the ideological hierarchy open-ended?

The connection between set theory and fully liberalized type theory is tight and, as Linnebo and Rayo point out, it does suggest some arguments against absolutism. The idea is that we should accept that the set-theoretic hierarchy is indefinitely extensible *given that* it is natural to hold that the hierarchy of liberalized type-theoretic languages is itself open-ended. In particular, the existence of tight translations between the ontological language(s) of (bounded) set theory and the ideological languages of type theory shows that the 'process' of expanding expressive resources by going to higher and higher types can be reproduced within set theory. So, the argument concludes, *if* the process of expanding expressive resources is open-ended, then so is the iterative hierarchy.

Is the ideological hierarchy of languages open-ended? No matter how one wants to characterize the notion of open-endedness, the following seems clear to us: since the levels in the ideological hierarchy are indexed by ordinals, the process of expanding one's expressive resources by moving to higher types is open-ended only to the extent that one's system of indices—the ordinals in this case—are open-ended. How could a definite stock of indices give rise to an open-ended hierarchy of languages? The absolutist does a modus tollens on the conditional at the end of the previous paragraph. Since there is a definite plurality of ordinals, the ideological hierarchy is not open-ended.

An observer, trying to adjudicate the matter in favour of relativism, might point out that, for the absolutist, the ideological hierarchy need not be confined to ordinals. It could be expanded beyond the ordinals. For example, new indices, and corresponding new types, could be introduced, by appealing to super-ordinals: $\Omega, \Omega+1, \Omega+2, ..., 2\Omega, ..., \Omega^2, ...$. A non-austere absolutist might construe these as proper classes. But this just pushes the problem up another level. The non-austere absolutist is in essentially the same position with respect to the realms of proper classes and super-ordinals as the austere absolutist is with respect to the original iterative hierarchy. If type theory is expanded beyond the ordinals, the resulting hierarchy is open-ended only to the extent that the totality of proper classes and super-ordinals is open-ended. The non-austere absolutist about sets is, or at least should be, an *absolutist about proper classes and super-ordinals*.[7] Therefore, she does not regard the ideological hierarchy as open-ended, even when extended beyond the ordinals. Moreover, there would be a similar absolute expansion of the ontological hierarchy. There are proper classes, proper super-classes, proper super-duper-classes, ... , what we may call proper ω-level classes, proper $\omega+1$-level classes, ...

Perhaps the idea of moving the ideological hierarchy beyond the ordinals could be made more palatable, even to the *austere* absolutist. Instead of resorting to proper classes, new indices and corresponding new types could be introduced by appealing to set-theoretic objects that even the austere absolutist accepts. In section 1 above, we showed how to define a relation, on pairs of ordinals, that, in effect, characterizes a well-ordering 'longer' than that of the ordinals:

$$(\alpha_1, \beta_1) \prec (\alpha_2, \beta_2) \text{ if and only if either } \alpha_1 < \alpha_2 \text{ or both } \alpha_1 = \alpha_2 \text{ and } \beta_1 < \beta_2.$$

The 'order-type' of this is Ω^2 (so to speak). So our absolutists might consider indexing the ideological hierarchy with pairs of ordinals, ordered in the way indicated.

[7]As remarked above, we suppose that someone could be a non-austere absolutist about sets, but a relativist about proper classes and super-ordinals. We will not comment on the extent to which that combination counts as a form of absolutism. It is certainly not in the spirit of absolutism.

The main problem with this proposal is that this extended ideological hierarchy is unmotivated. As we have seen, Linnebo and Rayo propose that the liberalized type theory gets its justification from Semantic Optimism and the principle of Set-theoretic Union, the thesis that for any set of languages, there is a language that encompasses all of them. For any ordinal α, Set-theoretic Union does yield a language indexed by the pair $(0,\alpha)$, but it does not deliver a language corresponding to even the pair $(1,0)$.

Linnebo and Rayo accept a similar conclusion:

> ABSOLUTE GENERALITY—together with the Principle of Semantic Optimism and the [Set-Theoretic] Principle of Union—can be used to motivate ascent to languages of type α for α an arbitrary ordinal. This entails that the type-theoretic hierarchy has at least as many levels as there are ordinals. But it does not, by itself, deliver the conclusion that the hierarchy is open-ended. (Linnebo and Rayo forthcoming)

Of course, it will not do to reinstate the original Naïve Principle of Union. That way lies madness (or at least contradiction). Linnebo and Rayo suggest that a strengthened Principle of Union (short of the Naïve one) might do the trick:

> *Strengthened Principle of Union*: Let C be any definite collection of type-theoretic languages. Then there is a 'union' language L_C which encompasses every language in C.

The analogy with Cantor's original second generating principle is now complete. Let us repeat it:

> if any definite succession of defined ... [numbers] is put forward of which no greatest exists, a new number is created ... which is thought of as the *limit* of those numbers; that is, it is defined as the next number greater than all of them.

Both principles invoke an unanalyzed notion of 'definite'. On the basis of the Strengthened Principle of Union, one might argue in favour of the open-endedness of the type-theoretic hierarchy as follows. If the totality of all type-theoretic languages were not open-ended, there would be a definite collection K of all such languages. The Strengthened Principle of Union would then yield a 'union' language L_K, a language that somehow encompasses all languages. By an extension of the above negative result for liberalized type theory, a generalized semantics for L_K could not be developed in L_K. Semantic Optimism would yield yet another language, contradicting that conclusion that L_K encompasses all languages.

However, if this argument is good, then an analogous (but somewhat simpler) argument would go from Cantor's second number principle to the conclusion that the ontological hierarchy of transfinite numbers is open-ended. Indeed, we have seen this argument before (section 2). Suppose that there were a definite totality of all transfinite numbers. Then the second principle would yield a number greater than all of them.

So which collections are definite? If the Strengthened Principle of Union is to go beyond the above liberalized type theory (where we have a language corresponding to each ordinal), then it must be stronger than the Set-Theoretic Principle of Union. So there must be definite collections of languages that are not set-sized. Of course, the austere absolutist will not accept this. For her, set theory is 'a theory about all, "absolutely" all, the collections that there' are, and 'set' is 'synonymous with "collection".' The situation of the non-austere absolutist is parallel to the one we outlined just above. It may be consistent to maintain a non-austere absolutism about sets, and to become a relativist about collections, or definite collections, but once again such a view seems highly unmotivated, and is certainly not in the spirit of absolutism. Whatever collections our non-austere absolutist is willing to countenance, he will be absolutist about them. And, of course, he will also deny that there is a collection of all collections. In sum, the non-austere absolutist will maintain that the collections are definite without admitting that there is a collection of all of them. This may be a predicament, but notice that it has little to do with the ideological hierarchy. The issue concerning what collections there are is one of the traditional problems for this kind of absolutism. Advocates of that view have already confronted it, or should have confronted it, and presumably made their peace with it. And the resolution will carry over, directly, to the ideological hierarchy via the Strengthened Principle of Union. Our non-austere absolutist will insist that the languages yielded by the principle are somehow definite, but that there is no collection of them.

Perhaps the issue of collections can be circumvented altogether if we reformulate a principle of union in plural terms, avoiding even apparent talk of collections or totalities (as Boolos preferred to do). That might even bring the austere absolutist back on board.

> *Plural Principle of Union*: For any definite plurality of type-theoretic languages, there is a 'union' language that encompasses all of them.

As with Strengthened Union, we restrict ourselves to 'definite' pluralities in order to avoid the collapse into Naïve Union. And the question of definiteness arises once again: which pluralities are definite? Consider an analogous principle for sets: for any definite plurality of things, there is a set of all and only those things. The absolutist, of either variety, holds that, say, the ordinals are definite. Indeed, this is just what it is to be an absolutist. Yet there is no set of all ordinals. So the absolutist simply rejects this plural principle of sets. And, presumably, the non-austere absolutist also rejects an analogous plural principle of collections. So, we conclude, the absolutist, of either stripe, would simply reject the Plural Principle of Union. Indeed, our absolutist would simply point out that, given the absolutism, the Plural Principle of Union just collapses into the Naïve Principle of Union.

A final point: recall that the tight connection between the ontological iterative hierarchy and the liberalized type theory assumed that the individuals/urelements form a set. In that case, the α^{th} rank, R_α, just is (or is strongly equivalent to) the range of the variables x^α of type α. What happens if the individuals do not form a set? For example, what if we start the ideological hierarchy with all ordinals, or all iterative sets, as individuals? We need not decide whether the individuals in question are somehow definite (e.g. are a definite plurality).

This is an important scenario, at least for the absolutist. It is one way to articulate the semantic argument from section 1 above. A generalized semantics for the language $\mathscr{L}_\in$ cannot be developed in $\mathscr{L}_\in$, for the usual reasons. But it can be provided in a second-order language of set theory. Of course, a generalized semantics for that language cannot be developed in that language. And so semantic reflection might take us to a third-order language. If the motivation for climbing up the ideological hierarchy is strictly semantic, however, we only need to embrace the finite orders. There is no motivation for taking this procedure, if that is what it is, into the transfinite.

An alternative to the expansion of ideology is to expand ontology. Instead of embracing concepts, concepts of concepts, etc. of sets or ordinals, we embrace classes, classes of classes, etc., again up through the finite orders. This, of course, is not available to the austere absolutist, but it is somewhat natural for the non-austere absolutist. Indeed, it seems to be essentially the same thing (or a closely isomorphic thing) as the given ideological expansion, to finite levels. Or one might embrace relativism, and hold that the iterative hierarchy is itself indefinitely extensible. What look like proper classes at one level are just interpreted as sets at the next level. It may be that this is the most compelling and least ad hoc picture to adopt. But, if so, that will be because the relativist picture is the most compelling one to adopt on the purely ontological front. The tight connection between set theory and liberalized type theory does not put new pressure on the absolutist positions.[8]

References

Boolos, George, 1998: 'Reply to Charles Parsons' "Sets and Classes".' In Boolos 1999, pp. 30-6.

[8]We wish to thank Simon Hewitt, Luca Incurvati, Nicholas Jones, Sam Roberts, Gabriel Uzquiano, Sean Walsh, and the audience at the conference 'Higher-Order Logic versus Set Theory' held at the Institute of Philosophy, London, in March 2011. We would especially like to thank Øystein Linnebo and Agustín Rayo for most helpful conversation and for offering comments on a draft of this paper. We appreciate the collegiality.

———, 1999: *Logic, Logic, and Logic*. Cambridge, MA: Harvard University Press.

Cantor, Georg, 1883: *Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen*. Leipzig: Teubner. Translated as 'Foundations of a General Theory of Manifolds: A Mathematico-Philosophical Investigation into the Theory of the Infinite' in Ewald 1996, pp. 878-919.

Dummett, Michael, 1993: *The Seas of Language*. Oxford: Oxford University Press.

Ewald, William (ed.), 1996: *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, vol. 2. Oxford: Oxford University Press.

Feferman, Solomon, 1988: 'Turing in the Land of 0(z).' In Herken 1988, pp. 113-47.

Forster, Thomas E., 1992: *Set Theory with a Universal Set*. Oxford: Oxford University Press.

Gödel, Kurt, 1933: 'The Present Situation in the Foundations of Mathematics.' In Gödel 1995, pp. 45-53.

———, 1995: *Collected Works: Volume III*. Oxford: Oxford University Press.

Herken, Rolf (ed.), 1988: *The Universal Turing Machine: A Half-Century Survey*. Oxford: Oxford University Press.

Jané, Ignasi, 2010: 'Idealist and Realist Elements in Cantor's Approach to Set Theory.' *Philosophia Mathematica*, 18, pp. 193–226.

Linnebo, Øystein and Rayo, Agustín, forthcoming: 'Hierarchies Ontological and Ideological.' Forthcoming in *Mind*.

Parsons, Charles, 1974: 'Sets and Classes.' *Noûs*, 8, pp. 1–12.

Quine, Willard V. O., 1937: 'New Foundations for Mathematical Logic.' *American Mathematical Monthly*, 44, pp. 70–80.

Rayo, Agustín and Uzquiano, Gabriel (eds.), 2006: *Absolute Generality*. Oxford: Oxford University Press.

Shapiro, Stewart, 1987: 'Principles of Reflection and Second-Order Logic.' *Journal of Philosophical Logic*, 16, pp. 309–33.

———, 1991: *Foundations without Foundationalism. A Case for Second-order Logic*. Oxford: Oxford University Press.

———, 2003: ‘All Sets Great and Small: And I do mean ALL.’ *Philosophical Perspectives*, 17, pp. 467–90.

Shapiro, Stewart and Wright, Crispin, 2006: ‘All Things Indefinitely Extensible.’ In Rayo and Uzquiano 2006, pp. 253-304.

Zermelo, Ernst, 1930: ‘Über Grenzzahlen und Mengenbereiche: Neue Untersuchungen über die Grundlagen der Mengenlehre.’ *Fundamenta Mathematicae*, 16, pp. 29–47. Translated as ‘On Boundary Numbers and Domains of Sets: New Investigations in the Foundations of Set Theory’ in Ewald 1996, pp. 1219-33.