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Flavell, Paul

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Paul Flavell

Abstract

Let R be a group of prime order r that acts on the r'-group G, let RG be the semidirect product of G with R, let $\mathbb{F}$ be a field and V a faithful completely reducible $\mathbb{F}[RG]$-module. Trivially, $C_G(R)$ acts on $C_V(R)$. Let K be the kernel of this action. What can be said about K? This question is considered when G is soluble. It turns out that K is subnormal in G or r is a Fermat or half-Fermat prime. In the latter cases, the subnormal closure of K in G is described. Several applications to the theory of automorphisms of soluble groups are given.

Let R be a group of prime order r that acts on the finite r'-group G and suppose V is a faithful completely reducible module for RG, the semidirect product of G with R, over some field. Trivially $C_V(R)$ is a module for $C_G(R)$. Let

$$K = \ker\left(C_G(R) \text{ on } C_V(R)\right).$$

A natural question to ask is:

What can be said about K?

If the underlying field has characteristic r then a simple argument, see Lemma 6.1, forces $K = 1$. In the contrary case, it may be that $K \neq 1$ and it is not *a priori* clear what form the answer should take. A special case resolves the issue:

If G is soluble of odd order then K is subnormal in G.

The question now becomes:

Describe the subnormal closure of K in G.

This is the smallest subnormal subgroup of G that contains K.

The main result, Theorem A, accomplishes this when G is soluble. Roughly speaking, it shows that the subnormal closure of K is not much more complex than K – so K is almost subnormal in G. As can be seen from the corollaries, Theorem A unifies and extends previous results of Glauberman and Thompson.

Throughout this paper, all groups considered are finite.

Theorem A. *Let R be a group of prime order r that acts on the r'-group G. Assume $[G,R]$ is soluble. Let V be an RG-module, possibly of mixed characteristic, with $V_{[G,R]}$ faithful and completely reducible.*

Suppose

$$K \leq \ker\left(C_G(R) \text{ on } C_V(R)\right) \text{ with } K \trianglelefteq C_G(R)$$

and let L be the subnormal closure of K in G. Then

$$L = K[L,R]$$

and $[L,R]$ is nilpotent.

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Write $[L,R]=S\times P$ with S a 2-group and P a $2'$-group. Then

$$V=C_V([L,R])\oplus[V,S]\oplus[V,P]$$

and all three summands are RL-submodules.

(a) *Assume $S\neq 1$. Then $r=2^n+1$ for some $n\in\mathbb{N}$; S is a special 2-group; $S=[S,R]=[S,K]$; $S'=C_S(R)$; $C_K(S')=C_K(S)$; $K/O(K)$ is not a 2-group and hence K is not nilpotent. Moreover, $[V,S]_{RS}$ is completely reducible. If U is an irreducible submodule of $[V,S]_{RS}$ then $C_U(R)=0$ and $S/C_S(U)\cong 2^{1+2n}$.*

(b) *Assume $P\neq 1$. Then $r=(1/2)(p^m+1)\neq 2$ for some prime p and $m\in\mathbb{N}$; P is a special p-group; $P=[P,R]=[P,K]$; $P'=C_P(R)=C_K(P)$; and $K/C_P(K)$ is an elementary abelian 2-group. Moreover, $[V,P]_{RP}$ is completely reducible. If U is an irreducible submodule of $[V,P]_{RP}$ then $C_U(R)\neq 0$; U is an RKP-submodule; $P/C_P(U)\cong p^{1+2m}$ and RK induces a semiregular cyclic group of order $2r=p^m+1$ on the Frattini quotient of $P/C_P(U)$.*

Finally,

$$O(K)\trianglelefteq\trianglelefteq G.$$

Remarks.

– The phrase *Let V be an RG-module, possibly of mixed characteristic,* means $V=V_1\oplus\cdots\oplus V_n$ where for each i there exists a field $\mathbb{F}_i$ such that V_i is an $\mathbb{F}_i[RG]$-module.
– Since K is R-invariant, so is its subnormal closure.
– Since $[G,R]\trianglelefteq RG$, if V is irreducible then $V_{[G,R]}$ is completely reducible by Clifford's Theorem.

Corollary B (Thompson [**13**]). *Let R be a group of prime order r that acts on the soluble r'-group G. Let q be a prime.*

(a) *$O_q(C_G(R))\leq O_{q,q',q}(G)$.*
(b) *$F(C_G(R))\leq F_4(G)$.*
(c) *At least one of the following holds:*
   – *$O_q(C_G(R))\leq O_{q,F,q}(G)$.*
   – *$q=2$; $2r-1$ is a power of a prime p and $O_q(C_G(R))\leq O_{q,F,p,q}(G)$.*

The next corollary concerns the action of a direct product $R\times K$ on a group G. The basic question being:

Suppose K acts trivially on $C_G(R)$. What can be said about the action of K on G?

Thompson's $P\times Q$-Lemma was the first such result. This considers the case where R and G are p-groups for some prime p and K is a p'-group. The conclusion is that K acts trivially on G. Glauberman used his Character Correspondence Theorem to prove an analogous result in the case that G is soluble with order coprime to $|RK|$, [**6**, Theorem 6].

Corollary C. *Suppose $R\times K$ acts on the soluble group G where R has prime order r and K and G are r'-groups. Assume that $[C_G(R),K]=1$.*

(a) *K acts nilpotently on $G/F_2(G)$ and trivially on $G/F_3(G)$.*
(b) *K^2 acts nilpotently on $G/F(G)$ and trivially on $G/F_2(G)$.*
(c) *Assume that K does not act nilpotently on $G/F(G)$. Set $P=[G/F(G),K;\infty]$. Then $r\neq 2$; $2r-1$ is a power of a prime p; P is a special p-group; $P=[P,R]$ and $[P',RK]=1$.*
(d) *Set $K^*=K/C_K(G/F(G))$. Then $K^*/F(K^*)$ is an elementary abelian 2-group. In particular, K^* is soluble.*

Remarks. In §1 there is a discussion of nilpotent action. The connection with Glauberman's Theorem is as follows:

– Glauberman has the restriction $(|K|, |G|) = 1$. We have removed this restriction but weakened the conclusion to nilpotent action rather than trivial action. Note that if $(|K|, |G|) = 1$ then nilpotent action implies trivial action.
– Glauberman's restriction that K be cyclic of prime power order has been removed.

Corollary D. *Suppose $R \times K$ acts on the soluble group G where R has prime order r, K and G are r'-groups and $(|K|, |G|) = 1$. Assume that $C_G(R) = C_G(K)$. Then $[G, K] = [G, R] \leq F(G)$.*

Corollary E. *Let R be a group of prime order r that acts on the soluble r'-group G. Let p be a prime and $P \leq O_p(C_G(R))$. Assume that $[C_G(P), R] = 1$. Then $[G, R] \leq O_{p'}(F(G))$.*

Although this paper is concerned with automorphisms of soluble groups, the original motivation came from the author's work on automorphisms of insoluble groups and the Signalizer Functor Theorem. Indeed, it was Corollaries C(d) and E which were discovered first and have applications in these areas.

An obvious goal for further work is to remove the solubility assumption in Theorem A. There is evidence that this is attainable. Indeed, an important special case of Theorem A is when $C_V(R) = 0$. This leads to the configuration described in Theorem A(a). The same conclusion follows from [**5**] without any solubility assumption.

We close the introduction with a generic example. First recall the following construction. Let R be a subgroup of the group X and suppose L is a group on which R acts. Then there exists a group $\widetilde{L}$ on which X acts and enjoys the following properties: $\widetilde{L}$ contains subgroups $L_1, \ldots, L_{|X:R|}$ with

$$\widetilde{L} = L_1 \times \cdots \times L_{|X:R|};$$

the subgroups L_i are isomorphic to L and permuted transitively by X; $N_X(L_1) = R$ and L_1 is R-isomorphic to L. The group $\widetilde{L}$ is the base group of the twisted wreath product of X with L.

Let R be a group of prime order r and let K be an r'-group on which R acts trivially. Let H be an r'-group on which R acts fixed point freely, let $\mathbb{F}$ be a field with $\operatorname{char} \mathbb{F} \neq r$ that contains a primitive r^{th}-root of unity and let U be a $\mathbb{F}[K]$-module. Then U is in fact an $\mathbb{F}[R \times K]$-module with R acting as scalar multiplication.

Let $L = UK$, so R acts on L. Put $X = RH$, $h = |H|$ and let $\widetilde{L}$ and $L_1, \ldots, L_h$ be as defined previously. Then $L_1 = U_1K_1$ with $K_1 = C_{L_1}(R) \cong K$ and $U_1 = [L_1, R] \cong U$. Let $\{K_1, \ldots, K_h\}$ be the X-conjugates of K_1. Put

$$G = H(K_1 \times \cdots \times K_h) \quad \text{and} \quad V = U_1 \times \cdots \times U_h.$$

Note that R is semiregular on $\{K_2, \ldots, K_h\}$ and on $\{U_2, \ldots, U_h\}$. Hence $C_V(R) \leq U_2 \oplus \cdots \oplus U_r$. Since $C_H(R) = 1$ we have $C_G(R) = K_1 \times C_{K_2 \times \cdots \times K_n}(R)$ and $C_V(R) = C_{U_2 \oplus \cdots \oplus U_n}(R)$. It follows that

$$K_1 = \ker\left(C_G(R) \text{ on } C_V(R)\right).$$

It will be clear from the proof, how to construct examples that realize the exceptional configurations described in the conclusions (a) and (b) of Theorem A.

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1. *Preliminaries – groups*

Let G be a group. Then $F(G)$, the *Fitting subgroup* of G is the largest nilpotent normal subgroup of G. The higher Fitting subgroups of G are defined by $F_1(G) = F(G)$ and $F_{n+1}(G) = F(G \bmod F_n(G))$, the inverse image of $F(G/F_n(G))$ in G.

Let q be a prime. Then $O_q(G)$ is the largest normal p-subgroup of G and $O_{q'}(G)$ is the largest normal q'-subgroup of G. Moreover, $O_{q,q'}(G) = O_{q'}(G \bmod O_q(G))$ and $O_{q,F}(G) = F(G \bmod O_q(G))$.

Define G^2 by

$$G^2 = \langle g^2 \mid g \in G \rangle.$$

Then G^2 is the smallest normal subgroup of G whose quotient is an elementary abelian 2-group. Thus $G' \leq G^2$. Recall that every group of exponent 2 is abelian.

If X and Y are subgroups of some group then $[X, Y] \trianglelefteq \langle X, Y \rangle$. The higher commutators are defined by

$$[X, Y; 1] = [X, Y] \text{ and } [X, Y; n+1] = [[X, Y; n], Y].$$

Then

$$\ldots\ldots \trianglelefteq [X, Y; 2] \trianglelefteq [X, Y; 1] \trianglelefteq \langle X, Y \rangle$$

and we define

$$[X, Y; \infty] = \bigcap_{n=1}^{\infty} [X, Y; n].$$

Suppose that the group A acts on the group G. We abuse notation and let AG denote the semidirect product of G with A. In particular, the commutator subgroup $[G, A]$ is defined and $[G, A] \trianglelefteq AG$. By definition, *$A$ acts nilpotently on G* if

$$[G, A; \infty] = 1,$$

equivalently, since G is finite, if $[G, A; n] = 1$ for some $n \geq 1$.

Lemma 1.1. *Suppose the group A acts on the group G. Then $A[G, A; \infty]$ is the subnormal closure of A in AG.*

Proof. Use the fact that $\langle X^Y \rangle = Y[X, Y]$. □

Theorem 1.2 [**10**, 4.24, p. 135 and 4.27, p. 137]. *Suppose the group A acts nilpotently on the group G. Then $A/C_A(G)$ and $[G, A]$ are nilpotent.*

We say that A acts *coprimely* on G if A acts on G; the orders of A and G are coprime; and A or G is soluble. We use $*$ to denote a central product.

Theorem 1.3 (Coprime Action). *Suppose the group A acts coprimely on the group G.*

(a) *Let N be an A-invariant normal subgroup of G and set $\overline{G} = G/N$. Then $C_{\overline{G}}(A) = \overline{C_G(A)}$.*

(b) *$G = C_G(A)[G, A]$ and $[G, A] = [G, A, A]$.*

(c) *If G is abelian then $G = C_G(A) \times [G, A]$.*
(d) *If $[G', A] = 1$ then $G = C_G(A) * [G, A]$.*
(e) *Suppose G is an extraspecial p-group and $[G', A] = 1$. Then $G = C_G(A) * [G, A]$. If $C_G(A) \neq G'$ then $C_G(A)$ is extraspecial with $C_G(A)' = G'$. If $[G, A] \neq 1$ then $[G, A]$ is extraspecial with $[G, A]' = G'$.*
(f) *If A acts nilpotently on G then A acts trivially on G.*
(g) *$C_{[G,A]}(A) \leq [G, A]'$.*
(h) *Suppose N is an A-invariant normal subgroup of G with $C_G(N) \leq N$ and $[N, A] = 1$. Then $[G, A] = 1$.*

Proof. (a), (b) and (c) are [**10**, 3.28, 4.28 and 4.34].

(d). We have $[C_G(A), G, A] \leq [G', A] = 1$. Trivially $[A, C_G(A), G] = 1$. The Three Subgroups Lemma forces $[G, A, C_G(A)] = 1$. Apply (b).

(e). Since G is extraspecial we have $G' = \Phi(G) = Z(G) \cong \mathbb{Z}_p$. By (d), $G = C_G(A) * [G, A]$. Let $H = C_G(A)$ or $[G, A]$. Then

$$H' \leq \Phi(H) \leq \Phi(G) \cap H = Z(G) \cap H \leq Z(H) \leq Z(G) = G' \cong \mathbb{Z}_p.$$

Thus if $H' \neq 1$ then H is extraspecial and $H' = G'$. Suppose $H' = 1$. Then $H = Z(H) \leq G'$. If $H = C_G(A)$ then $C_G(A) = G'$. If $H = [G, A]$ then $[H, A] = 1$ and (b) implies $[G, A] = 1$.

(f) follows from (b) and (g) follows from (a) and (c).

(h). We have $A \leq C_{AG}(N) = AC_G(N) = A \times Z(N)$. Then A is a characteristic subgroup of $C_{AG}(N)$ because $(|A|, |G|) = 1$. As $C_{AG}(N) \trianglelefteq AG$ we obtain $A \trianglelefteq AG$, so $[G, A] \leq G \cap A = 1$. $\square$

Lemma 1.4. *Suppose $A \times K$ acts on the p-group P with A a p'-group, $P = [P, A]$ and $[P', A] = 1$. Set $\overline{P} = P/P'$.*
(a) *$C_{\overline{P}}(K) = \overline{C_P(K)}$.*
(b) *$C_K(P) = C_K(\overline{P})$.*

Proof. (a). Let Q be the inverse image of $C_{\overline{P}}(K)$. Now $[K, Q, A] \leq [P', A] = 1$. Trivially $[A, K, Q] = 1$. The Three Subgroups Lemma forces $[Q, A, K] = 1$, so $[Q, A] \leq C_P(K)$. Now Q is A-invariant so Coprime Action implies $Q = [Q, A]C_Q(A)$. Moreover, as $P = [P, A]$ we have $C_P(A) \leq P'$. Thus $Q \leq C_P(K)P'$ and then $C_{\overline{P}}(K) \leq \overline{C_P(K)}$. The opposite inclusion is trivial.

(b). Let $K_0 = C_K(\overline{P})$. Now (a), with K_0 in the role of K, implies $P = C_P(K_0)P' = C_P(K_0)\Phi(P)$, whence $P = C_P(K_0)$ and $C_K(\overline{P}) \leq C_K(P) \leq C_K(\overline{P})$. $\square$

Suppose the group A acts on the set X. The action is *semiregular* if $xa = x$ implies $a = 1$ whenever $a \in A$ and $x \in X$. The following elementary result will be used without reference.

Lemma 1.5. *Suppose the group A acts on the set X. Assume that $A = BC$ where $B, C \leq A$; $(|B|, |C|) = 1$; and B and C act semiregularly on X. Then A is semiregular on X.*

Suppose the group A acts on the group G. We abuse notation and say that the action is semiregular if A acts semiregularly on $G^\#$, the set of nonidentity elements of G. Equivalently, $C_G(a) = 1$ for all $a \in A^\#$. Equivalently, AG is a Frobenius group with complement A and kernel G.

Lemma 1.6. *Suppose the group A acts on the group $G \neq 1$.*

(a) *A is semiregular on G if and only if $G = \{[g, a] \mid g \in G\}$ for all $a \in A^{\#}$.*
(b) *If N is a proper A-invariant normal subgroup of G and A is semiregular on G then A is semiregular on G/N.*
(c) *Suppose A is a cyclic q-group for some prime q, A is nontrivial on G and G is an abelian q'-group. Then A is semiregular on $[G, \Omega_1(A)]$.*
(d) *Suppose G is a p-group for some prime p and A is semiregular on G/G'. Then $G = [G, A]$.*

Proof. (a). Let $a \in A^{\#}$. The map $g \mapsto [g, a]$ is a bijection $G \longrightarrow G$ if and only if $C_G(a) = 1$.
(b). The property $G = \{[g, a] \mid g \in G\}$ is inherited by G/N.
(c). This follows from Coprime Action. Note that $[G, \Omega_1(A)]$ is A-invariant and recall that $\Omega_1(A)$ is the subgroup of A generated by elements of prime order.
(d). By (a) we have $G = [G, A]G'$. Since G is a p-group we have $G' \le \Phi(G)$, whence $G = [G, A]$. $\square$

2. *Preliminaries – modules*

The reader is assumed to be familiar with the rudiments of Representation Theory. Let $\mathbb{F}$ be a field and G a group. Then $\mathbb{F}[G]$ denotes the group algebra of G over $\mathbb{F}$. All $\mathbb{F}[G]$-modules will be finite dimensional right $\mathbb{F}[G]$-modules. Let V be an $\mathbb{F}[G]$-module and $H \le G$. Then

$$V_H$$

denotes V considered as an $\mathbb{F}[H]$-module. If $\mathbb{K}$ is an extension field of $\mathbb{F}$ then the $\mathbb{K}[G]$-module $V^{\mathbb{K}}$ is defined by

$$V^{\mathbb{K}} = V \otimes_{\mathbb{F}} \mathbb{K}.$$

The following is elementary and will frequently be used without reference.

Lemma 2.1. *Let $\mathbb{F}$ be a field, G a group and V an $\mathbb{F}[G]$-module. Assume that $G = AB$ where $A, B \le G$; $(|A|, |B|) = 1$; and V_A and V_B are faithful. Then V is faithful.*

Lemma 2.2. *Let $\mathbb{F} \subseteq \mathbb{K}$ be a field extension, G a group and V an $\mathbb{F}[G]$-module.*
(a) *$C_{V^{\mathbb{K}}}(G) = C_V(G) \otimes_{\mathbb{F}} \mathbb{K}$.*
(b) *$C_{V^{\mathbb{K}}}(G) = 0$ if and only if $C_V(G) = 0$.*
(c) *Suppose V is faithful and irreducible. Then every irreducible submodule of $V^{\mathbb{K}}$ is faithful.*

Proof. Let $e_1, \ldots, e_n$ be a basis for V. Then $e_1 \otimes 1, \ldots, e_n \otimes 1$ is a basis for $V^{\mathbb{K}}$. Let $v \in C_{V^{\mathbb{K}}}(G)$. Then $v = \lambda_1(e_1 \otimes 1) + \ldots + \lambda_n(e_n \otimes 1)$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$. Let $k_1, \ldots, k_m$ be an $\mathbb{F}$-basis for the $\mathbb{F}$-subspace of $\mathbb{K}$ spanned by $\lambda_1, \ldots, \lambda_n$. Set

$$W = V \otimes k_1 + \ldots + V \otimes k_m.$$

Then $v \in W$. The sum is direct because $k_1, \ldots, k_m$ are $\mathbb{F}$-linearly independent. Each $V \otimes k_i$ is an $\mathbb{F}[G]$-module, whence

$$\begin{aligned} v \in C_W(G) &= C_{V \otimes k_1}(G) \oplus \ldots \oplus C_{V \otimes k_n}(G) \\ &\le C_V(G) \otimes_{\mathbb{F}} \mathbb{K}. \end{aligned}$$

Then (a) holds and (b) follows trivially.
Assume the hypotheses of (c) and let U be an irreducible submodule of $V^{\mathbb{K}}$. Set $N = C_G(U)$ and suppose that $N \ne 1$. Now $N \trianglelefteq G$ and V is faithful and irreducible so $C_V(N) = 0$. By (b), $C_{V^{\mathbb{K}}}(N) = 0$, a contradiction. We conclude that U is faithful. $\square$

THEOREM 2.3 (Generalized Maschke Theorem) [**2**, (12.6),(12.8),p. 39]. *Let $\mathbb{F}$ be a field, G a group and V an $\mathbb{F}[G]$-module. Suppose $H \leq G$, $\operatorname{char}\mathbb{F}$ does not divide $|G:H|$ and V_H is completely reducible. Then V is completely reducible.*

COROLLARY 2.4. *Let $\mathbb{F}$ be a field, G a group and V an $\mathbb{F}[G]$-module. Assume $\operatorname{char}\mathbb{F}$ does not divide $|G|$.*
(a) $V = C_V(G) \oplus [V, G]$.
(b) *Let U be a submodule of V and set $\overline{V} = V/U$. Then $C_{\overline{V}}(G) = \overline{C_V(G)}$.*

The preceding corollary is not valid without the assumption on the characteristic of $\mathbb{F}$. However, the following is true:

LEMMA 2.5. *Let A be a cyclic group, $\mathbb{F}$ a field, V an $\mathbb{F}[A]$-module and U a submodule of V. Then*

$$\dim C_V(A) \geq \dim C_{V/U}(A) \geq \dim C_V(A) - \dim U.$$

Proof. Since $C_{V/U}(A) \geq (C_V(A) + U)/U$, the second inequality is clear. Let W be the inverse image of $C_{V/U}(A)$ in V. Then $[W, A] \leq U$. The Rank-Nullity Formula implies $\dim C_W(A) = \dim W - \dim[W, A] \geq \dim W - \dim U = \dim C_{V/U}(A)$. □

Let $\mathbb{F}$ be a field, G a group and V an $\mathbb{F}[G]$-module. A *system of imprimitivity* for V is a collection $\{V_1, \ldots, V_n\}$ of nonzero subspaces of V such that $V = V_1 \oplus \ldots \oplus V_n$ and $V_i g \in \{V_1, \ldots, V_n\}$ for all i and $g \in G$. This gives a permutation representation of G on $\{V_1, \ldots, V_n\}$. We say V is *primitive* if $\{V\}$ is the only system of imprimitivity for V.

Suppose that $N \trianglelefteq G$ and V is irreducible. Let $\{V_1, \ldots, V_n\}$ be the set of homogeneous components of V_N. Clifford's Theorem asserts that $\{V_1, \ldots, V_n\}$ is a system of imprimitivity for V. Moreover, the permutation action of G on $\{V_1, \ldots, V_n\}$ is transitive.

Recall that $\mathbb{F}[G]$ is itself an $\mathbb{F}[G]$-module. If $n \in \mathbb{N}$ then

$$n \times \mathbb{F}[G]$$

denotes the direct sum of n copies of $\mathbb{F}[G]$. Let $V \neq 0$ be an $\mathbb{F}[G]$-module. The following are equivalent:
– V is free.
– $V \cong n \times \mathbb{F}[G]$ for some $n \in \mathbb{N}$.
– V possesses a G-invariant basis on which G acts semiregularly.
– V possesses a system of imprimitivity on which G acts semiregularly.

Visibly, if V is free then

$$\dim C_V(H) = \frac{1}{|H|} \dim V$$

for all $H \leq G$. In particular, $C_V(G) \neq 0$.

We require a detailed knowledge of modules for cyclic groups. The following lemma is useful.

LEMMA 2.6. *Suppose I and J are nonzero ideals of the principal ideal domain D. Then*

$$\operatorname{Hom}_D(D/I, D/J) \cong_D D/(I + J).$$

Proof. We remark that we are regarding D/I and D/J as D-modules. Let i and j be generators for I and J respectively. Let h be a GCD of i and j, which exists since $I, J \neq 0$.

For each $a \in D$ define $f_a : D/I \longrightarrow D/J$ by

$$(I + d)f_a = J + a\frac{j}{h}d.$$

Trivially the map $a \mapsto f_a$ is a D-homomorphism $D \longrightarrow \mathrm{Hom}_D(D/I, D/J)$ with kernel (h).

Now suppose $f \in \mathrm{Hom}_D(D/I, D/J)$. Choose $x \in D$ with $(I + 1)f = J + x$. We have $0 = (I + i)f = (I + 1)fi = J + xi$ so $xi \in J$. Then $j \mid xi$ and so $(j/h) \mid x(i/h)$. Now j/h and i/h are coprime so $(j/h) \mid x$. Choose $a \in D$ with $x = a(j/h)$. Then $f = f_a$. We deduce that $\mathrm{Hom}_D(D/I, D/J) \cong_D D/(h)$. Since D is a principal ideal domain we have $I + J = (h)$ and the proof is complete. $\square$

Theorem 2.7. *Let $\mathbb{F}$ be a field, $A = \langle a \rangle$ a cyclic group and V an $\mathbb{F}[A]$-module.*
(a) *There exists $l \geq 0$ and uniquely determined proper ideals $I_1 \subseteq \ldots \subseteq I_l$ of $\mathbb{F}[A]$ such that*

$$V \cong \mathbb{F}[A]/I_1 \oplus \ldots \oplus \mathbb{F}[A]/I_l.$$

(b) *V is free if and only if $I_1 = \ldots = I_l = 0$.*
(c) *V has a free direct summand if and only if $I_1 = 0$.*
(d) *V has a free direct summand if and only if the minimal polynomial for a is $X^{|A|} - 1$.*
(e) $\dim \mathrm{End}_{\mathbb{F}[A]}(V) = \sum_{i=1}^{l}(2i - 1) \dim \mathbb{F}[A]/I_i$.

Proof. Let X be an indeterminate. The map $X \mapsto a$ extends to an $\mathbb{F}$-algebra epimorphism $\mathbb{F}[X] \longrightarrow \mathbb{F}[A]$ with kernel $(X^{|A|} - 1)$. This endows V with the structure of an $\mathbb{F}[X]$-module. The Structure Theorem for Modules over a Principal Ideal Domain [**11**, Theorem14, p. 299] implies there exists $l \geq 0$ and uniquely determined proper ideals $J_1 \subseteq \ldots \subseteq J_l$ of $\mathbb{F}[X]$ such that

$$V \cong \mathbb{F}[X]/J_1 \oplus \ldots \oplus \mathbb{F}[X]/J_l.$$

Now $X^{|A|} - 1$ annihilates V so $(X^{|A|} - 1) \subseteq J_i$ for all i. Also, $\mathbb{F}[A] \cong \mathbb{F}[X]/(X^{|A|} - 1)$ and then (a) follows. Then (b),(c) and (d) are trivial consequences of the uniqueness assertion in (a). To prove (e) we apply Lemma 2.6 to give

$$\begin{aligned}
\mathrm{End}_{\mathbb{F}[X]}(V) &\cong \bigoplus_{i,j} \mathrm{Hom}_{\mathbb{F}[X]}(\mathbb{F}[X]/J_i, \mathbb{F}[X]/J_j) \\
&\cong \bigoplus_{i,j} \mathbb{F}[X]/(J_i + J_j) \\
&\cong \bigoplus_{i,j} \mathbb{F}[X]/J_{\max(i,j)}.
\end{aligned}$$

For each i there are $2i - 1$ pairs (s, t) with $i = \max(s, t)$. Hence

$$\dim \mathrm{End}_{\mathbb{F}[X]}(V) = \sum_{i=1}^{l}(2i - 1) \dim \mathbb{F}[X]/J_i.$$

The definition of the $\mathbb{F}[X]$-module structure of V implies that $\mathrm{End}_{\mathbb{F}[X]}(V) = \mathrm{End}_{\mathbb{F}[A]}(V)$. Moreover, $\mathbb{F}[X]/J_i \cong \mathbb{F}[A]/I_i$ for each i. The proof is complete. $\square$

Corollary 2.8. *Let $\mathbb{F} \subseteq \mathbb{K}$ be a field extension, A a cyclic group and V an $\mathbb{F}[A]$-module.*
(a) *V is free if and only if $V^{\mathbb{K}}$ is free.*
(b) *V has a free direct summand if and only if $V^{\mathbb{K}}$ has a free direct summand.*

The following well known result illustrates many of the preceding ideas.

THEOREM 2.9. *Let A be a cyclic group that acts semiregularly on the abelian group N. Let $\mathbb{F}$ be a field and V an $\mathbb{F}[AN]$-module. Assume* $\operatorname{char}\mathbb{F}$ *does not divide $|N|$ and $C_V(N) = 0$. Then V_A is free.*

Proof. By Lemma 2.2 and Corollary 2.8 we may suppose that $\mathbb{F}$ is algebraically closed. Let Ω be the set of homogeneous components of V_N. By Maschke's Theorem, V_N is completely reducible so $V_N = \oplus\Omega$ and Ω is a system of imprimitivity for V. It suffices to show that the action of A on Ω is semiregular.

Let $U \in \Omega, a \in A$ and suppose $Ua = U$. Now N is abelian, $\mathbb{F}$ is algebraically closed and U is a homogeneous $\mathbb{F}[N]$-module. It follows that N acts on U by scalar multiplication. More precisely, the image of N in $\mathrm{GL}(U)$ is contained in $Z(\mathrm{GL}(U))$. Thus $[N, a]$ is trivial on U. If $a \neq 1$ then $N = [N, a]$ by Lemma 1.6 and then $U \leq C_V(N) = 0$, a contradiction. Thus $a = 1$ and the proof is complete. □

3. *Primitive modules*

We develop some techniques that are useful in the study of primitive modules. No great originality is claimed. Recall that a p-group P is *special* if

$$1 \neq P' = \Phi(P) = Z(P).$$

Note that special groups are nonabelian. If in addition P' is cyclic then P is *extraspecial*. We write $P \cong p^{1+2n}$ to indicate that P is extraspecial with order p^{1+2n}.

The following elementary fact will be used frequently: suppose x is an element of the group G and $[x, g] \in Z(G)$ for all $g \in G$. Then the maps $g \mapsto [x, g]$ and $g \mapsto [g, x]$ are homomorphisms.

LEMMA 3.1. *Let P be a group with $\Phi(P) \leq Z(P)$. Then P' is elementary abelian. In particular, if P is extraspecial then $|P'| = p$.*

Proof. Note that $P' \leq \Phi(P) \leq Z(P)$ so P' is abelian. Let $x, y \in P$. Now $P' \leq Z(P)$ and $y^p \in \Phi(P) \leq Z(P)$ so $[x, y]^p = [x, y^p] = 1$. Hence P' is elementary abelian. □

LEMMA 3.2. *Let P be a p-group and suppose that $Z(\Phi(P)) \leq Z(P)$. Then $\Phi(P) \leq Z(P)$.*

Proof. Let $\overline{P} = P/Z(P)$ and let N be the inverse image of $\Omega_1(Z(\overline{P}))$. Let n and g denote elements of N and P respectively. Since $[N, P] \leq Z(P)$ the map $n \mapsto [n, g]$ is a homomorphism $N \longrightarrow Z(P)$. Then

$$[n, g]^p = [n^p, g] \in [Z(P), g] = 1.$$

Also the map $\theta : g \mapsto [n, g]$ is a homomorphism $P \longrightarrow Z(P)$. Then $\operatorname{Im}\theta \leq \Omega_1(Z(P))$, whence $\Phi(P) \leq \ker\theta$ and we obtain

$$[N, \Phi(P)] = 1.$$

In particular, $N \cap \Phi(P) \leq Z(\Phi(P))$, so by hypothesis, $N \cap \Phi(P) \leq Z(P)$. Then $\Omega_1(Z(\overline{P})) \cap \overline{\Phi(P)} = 1$. As $\overline{\Phi(P)} \trianglelefteq \overline{P}$ and $\overline{P}$ is a p-group, this implies $\overline{\Phi(P)} = 1$ and completes the proof. □

We obtain an improvement on the well known fact [**2**, (24.7), p. 114].

Corollary 3.3. *Suppose A acts coprimely on the p-group P. Assume $P = [P, A] \neq 1$ and $[Z(P), A] = [Z(\Phi(P)), A] = 1$. Then P is special and $P' = C_P(A)$.*

Proof. Since A is trivial on $Z(\Phi(P))$ so is $[P, A] = P$. Hence $Z(\Phi(P)) \leq Z(P)$. The lemma implies $\Phi(P) \leq Z(P)$. By Coprime Action, $C_P(A) \leq P'$. Then

$$Z(P) \leq C_P(A) \leq P' \leq \Phi(P) \leq Z(P)$$

completing the proof. □

Lemma 3.4. *Let $\mathbb{F}$ be a field, G a group and V a faithful primitive $\mathbb{F}[G]$-module. Assume $\mathbb{F}$ contains an $|F(G)|^{\mathrm{th}}$-root of unity. Then every abelian normal subgroup of G is cyclic and contained in $Z(G)$.*

Proof. Since V is primitive it is irreducible, so $Z(G)$ is cyclic. Let N be an abelian normal subgroup of G. Clifford's Theorem implies V_N is homogeneous. Now $N \leq F(G)$ so $\mathbb{F}$ contains an $|N|^{\mathrm{th}}$-root of unity. Then N acts as scalar multiplication, whence $N \leq Z(G)$. □

Lemma 3.5. *Let G be a group, p a prime and $P \trianglelefteq G$ a nonabelian p-group. Assume that every abelian subgroup of P that is normal in G is cyclic and contained in $Z(G)$.*

(a) *$P' \leq \Phi(P) \leq Z(P)$ and $\mathbb{Z}_p \cong P' = \Omega_1(Z(P)) \leq \Omega_1(O_p(Z(G)))$.*

(b) *If T is a p'-subgroup of G with $[P, T] \neq 1$ then*

$$P = C_P(T) * [P, T]$$

and $[P, T]$ is extraspecial with

$$[P, T]' = C_{[P,T]}(T) = P' = Z(P) \cap [P, T].$$

Proof. (a). We have $Z(P), Z(\Phi(P)) \trianglelefteq G$ so $Z(P), Z(\Phi(P)) \leq Z(G)$. Then $Z(\Phi(P)) \leq Z(P)$ and Lemma 3.2 implies $\Phi(P) \leq Z(P)$. The first assertion holds. Lemma 3.1 implies P' is elementary abelian, so as $Z(P)$ is cyclic, the second assertion follows.

(b). By (a), $[P', T] = 1$ so Coprime Action implies $P = C_P(T) * [P, T]$. Then

$$Z([P, T]) \leq Z(P) \cap [P, T] \leq Z(G) \cap [P, T] \leq C_{[P,T]}(T).$$

Note that this implies $[P, T]$ is nonabelian so as $|P'| = p$ we have $P' \leq \Phi([P, T])$ By Coprime Action,

$$\begin{aligned} C_{[P,T]}(T) &\leq [P, T]' \leq P' \leq \Phi([P, T]) \\ &\leq \Phi(P) \cap [P, T] \leq Z(P) \cap [P, T] \leq Z([P, T]). \end{aligned}$$

Equality follows, forcing $[P, T]$ to be special with $[P, T]' = C_{[P,T]}(T) = P' \leq Z(P)$. Now $Z(P)$ is cyclic, so $[P, T]$ is extraspecial. □

Lemma 3.6. *Let P be a p-group. Suppose that*

$$1 \neq P' \leq \Phi(P) \leq Z(P)$$

and that P' is cyclic. Set $\overline{P} = P/Z(P)$, let z be a generator for P' and define $(\ ,\ ) : \overline{P} \times \overline{P} \longrightarrow \mathrm{GF}(p)$ by

$$[x, y] = z^{(\overline{x}, \overline{y})}.$$

(a) $\overline{P}$ *is a* GF(p)*-vectorspace,* (,) *is a symplectic form on* $\overline{P}$ *and particular,* $\dim \overline{P}$ *is even.*
(b) *Any automorphism of* P *that centralizes* P' *induces a symplectic transformation on* $\overline{P}$.
Let $Q \leq P$.
(c) $\overline{Q}^{\perp} = \overline{C_P(Q)}$.
(d) $\mathrm{Rad}(\overline{Q}) = \overline{Z(Q)}$.
(e) Q *is abelian if and only if* $\overline{Q}$ *is totally singular.*
(f) $|C_P(Q)| = |P : Q||Q \cap Z(P)|$.
(g) *The following are equivalent:* $\overline{Q}$ *is nondegenerate;* $\overline{P} = \overline{Q}^{\perp} \oplus \overline{Q}$; $P = C_P(Q) * Q$; $Z(Q) \leq Z(P)$.

Proof. Since $\Phi(P) \leq Z(P)$ it follows that $\overline{P}$ is elementary abelian and hence a GF(p)-vectorspace. Lemma 3.1 implies $P' \cong \mathbb{Z}_p$ so (,) is well defined. A commutator calculation shows that (,) is an alternating bilinear form. Note that if $x, y \in P$ then $[x, y] = 1$ if and only if $(\overline{x}, \overline{y}) = 0$. Then (c), (d) and (e) hold. Moreover $\mathrm{Rad}(\overline{P}) = \overline{Z(P)} = 1$ so (,) is nondegenerate. Then (a) holds and (b) follows readily.

Let $Q \leq P$. Now $\dim \overline{Q}^{\perp} = \operatorname{codim} \overline{Q}$ whence $|\overline{C_P(Q)}| = |\overline{P} : \overline{Q}|$ and (f) follows. The verification of (g) is elementary. □

As a simple application we have the following:

Corollary 3.7. *Assume the hypotheses of Lemma 3.5. Then* $P/Z(P)$ *is a completely reducible* GF(p)[G]*-module. Each irreducible summand possesses a* G*-invariant symplectic form.*

Proof. Adopt the notation of Lemma 3.6. Let $\overline{Q}$ be an irreducible submodule of $\overline{P}$ and let Q be the inverse image of $\overline{Q}$ in P. Then $Q \trianglelefteq G$. Moreover, $\overline{Q}$ is either totally singular or nondegenerate. In the former case, Q is abelian so, by hypothesis, $Q \leq Z(G) \cap P \leq Z(P)$ and $\overline{Q} = 1$, a contradiction. Thus $\overline{Q}$ is nondegenerate. Then $\overline{P} = \overline{Q}^{\perp} \oplus \overline{Q}$. Now $\overline{Q}^{\perp}$ is G-invariant, so $\overline{Q}$ has a complement. It follows that $\overline{P}$ is completely reducible. □

Lemma 3.8. *Let* G *be a group and* $N \trianglelefteq G$. *Assume that every abelian subgroup of* N *that is normal in* G *is cyclic and contained in* $Z(G)$.
(a) $F(N)/Z(N)$ *is a completely reducible* G*-module, possibly of mixed characteristic.*
(b) *Suppose* $C_N(F(N)) \leq F(N)$. *Then* $C_N(F(N)/Z(N)) = F(N)$.
(c) *Suppose* $C_N(F(N)) \leq F(N)$ *and* N *is not nilpotent. Then there exists a prime* p *and a nonabelian* p*-subgroup* $P \leq N$ *with* $P \trianglelefteq G$,

$$\mathbb{Z}_p \cong P' \leq \Phi(P) \leq Z(P) \leq Z(G),$$

G *acts irreducibly on* $P/Z(P)$ *and* N *acts nontrivially on* $P/Z(P)$.

Proof. (a). Note that $F(N)/Z(N)$ is G-isomorphic to the direct product of the groups $O_p(N)/Z(O_p(N))$ as p ranges over the primes for which $O_p(N)$ is nonabelian. Apply Corollary 3.7.

(b). Let $C = C_N(F(N)/Z(N)) \trianglelefteq N$. By (a), $F(N)/Z(N)$ is abelian, so $F(N) \leq C$. Now $[F(N), C] \leq Z(N) \leq Z(G)$ whence $[F(N), C, C] = 1$. Similarly $[C, F(N), C] = 1$. The Three Subgroups Lemma forces $[C', F(N)] = 1$. Then $C' \leq C_N(F(N)) = Z(F(N)) \leq Z(G)$. In particular, $C' \leq Z(C)$ and C is nilpotent. Hence $C \leq F(N)$.

(c). By (b), N is nontrivial on $F(N)/Z(N)$. Apply (a) and Lemma 3.5. □

Lemma 3.9. *Let P be an extraspecial p-group.*

(a) *Suppose $1 \neq Q \trianglelefteq P$ is elementary abelian. There are precisely $p^{-1}|Q|$ hyperplanes of Q that do not contain P'. The conjugation action of $P/C_P(Q)$ on these hyperplanes is regular.*

(b) *Let $\mathbb{F}$ be a field with $\operatorname{char}\mathbb{F} \neq p$ and V an $\mathbb{F}[P]$-module with $V = [V, P']$. Let $T \leq P$. Then*

$$\dim C_V(T) = \begin{cases} 0 & \text{if } P' \leq T \\ |T|^{-1}\dim V & \text{if } P' \not\leq T. \end{cases}$$

Proof. (a). Let $\mathcal{H}$ be the set of hyperplanes of Q that do not contain P'. A counting argument shows that $|\mathcal{H}| = p^{-1}|Q|$. Let $H \in \mathcal{H}$, so $Q = P' \times H$. Now $[H, N_P(H)] \leq H \cap P' = 1$ whence $N_P(H) = C_P(H) = C_P(Q)$. Using Lemma 3.7(f) we have $|H^P| = |P : C_P(Q)| = |Q : P'| = |\mathcal{H}|$ whence $H^P = \mathcal{H}$ and $P/C_P(Q)$ is regular on $\mathcal{H}$.

(b). Since $V = [V, P']$ we have $C_V(P') = 0$. Hence if $P' \leq T$ then $\dim C_V(T) = 0$. Suppose $P' \not\leq T$. Then $P' \cap T = 1$ since $P' \cong \mathbb{Z}_p$. Moreover $\Phi(T) \leq \Phi(P) \cap T = P' \cap T = 1$, so T is elementary abelian. Set $Q = P'T = P' \times T$ and adopt the notation of (a).

Now $\operatorname{char}\mathbb{F} \neq p$ so V_Q is completely reducible. If U is an irreducible submodule of V_Q then $Q/C_Q(U)$ is cyclic. As $C_V(P') = 0$ we have $C_Q(U) \in \mathcal{H}$. Consequently

$$V = \bigoplus_{H \in \mathcal{H}} C_V(H).$$

Since $C_V(P') = 0$, a simple argument shows that this sum is direct. Now P is transitive on $\mathcal{H}$ whence $\dim C_V(H) = \dim C_V(T)$ for all $H \in \mathcal{H}$. Since $|\mathcal{H}| = p^{-1}|Q| = |T|$, the conclusion follows. □

4. *Hall-Higman theory*

A fundamental configuration that arises in group theory is the following: A is a cyclic group that acts on a p-group P; $\mathbb{F}$ is a field; V is a faithful irreducible $\mathbb{F}[AP]$-module and one of the following holds:

- P is abelian and A is semiregular on P.
- P is extraspecial, A is semiregular on P/P' and $[P', A] = 1$.

The main issue being:

Describe the structure of V_A.

The abelian case has already been considered. Theorem 2.9 asserts that V_A is free.

The extraspecial case is more difficult. Let $q = \operatorname{char}\mathbb{F}$. This problem was first encountered by Hall and Higman [**8**]. They considered the *modular* case, that is when A is a q-group. Subsequently Shult [**12**] and Dade [**9**, Satz V.17.13, p. 574] considered the *nonmodular* case when A is a q'-group. Carlip [**4**], generalizing the Hall-Higman argument removes the distinction between these cases and requires only that A be cyclic and act irreducibly on P/P'.

The result soon to be stated is a slight extension of Carlip's. The proof differs from those of Carlip and Hall-Higman in two respects. Firstly, Theorem 2.7 is used in place of Jordan Normal Form. Secondly, an argument involving Theorem 2.9 replaces an argument involving enveloping algebras and detailed properties of representations of extraspecial groups. The remainder of the argument is similar to that of Hall-Higman and avoids the technical difficulties encountered by Carlip. As a bonus, the argument is matrix free.

THEOREM 4.1. *Let A be a cyclic group that acts on the extraspecial p-group $P \cong p^{1+2n}$. Assume that A is semiregular on P/P' and trivial on P'. Let $\mathbb{F}$ be a field and V an irreducible $\mathbb{F}[AP]$-module with $V_{P'}$ faithful. Assume at least one of the following holds:*

(i) *$\mathbb{F}$ is algebraically closed.*
(ii) *$\mathbb{F}$ is a splitting field for P and $\mathrm{End}_{\mathbb{F}[AP]}(V) = \mathbb{F} \cdot 1$.*
(iii) *$\mathbb{F}$ is a splitting field for P and V_P is irreducible.*

Then V is faithful, V_P is irreducible, $\dim V = p^n$ and there exists a 1-dimensional $\mathbb{F}[A]$-module U such that at least one of the following holds:

(a) *$|A|$ divides $p^n + 1$ and*

$$V_A \cong \left(\frac{p^n+1}{|A|} - 1\right) \times \mathbb{F}[A] \oplus \mathbb{F}[A]/U.$$

(b) *$|A|$ divides $p^n - 1$, A does not act irreducibly on P/P' and*

$$V_A \cong \left(\frac{p^n-1}{|A|}\right) \times \mathbb{F}[A] \oplus U.$$

Before proving this result, we derive some straightforward consequences.

COROLLARY 4.2. *Assume the hypotheses of Theorem 4.1 and that $\mathbb{F}[A]$ is not a direct summand of V_A. Then $|A| = p^n + 1$ and there exists a 1-dimensional $\mathbb{F}[A]$-module U such that*

$$V_A \cong \mathbb{F}[A]/U.$$

Proof. Now $\dim V = p^n > 1$ so Theorem 4.1(b) cannot hold. Then Theorem 4.1(a) holds and $(p^n + 1)/|A| - 1 = 0$. □

COROLLARY 4.3. *Let the group $A \times K$ act on the extraspecial group $P \cong p^{1+2n}$. Assume that A is semiregular on P/P'; $[P', AK] = 1$; and $(|A|, |K|) = 1$.*

(a) *$|A|$ divides $p^n + 1$ or $p^n - 1$. In the latter case, A is not irreducible on P/P'.*
(b) *Suppose $|A| = p^n + 1$. Then A is irreducible on P/P' and $[P, K] = 1$.*

Proof. (a). Since A is semiregular on P/P' and $P' = Z(P) \cong \mathbb{Z}_p$ it follows that P' is the unique minimal normal subgroup of AP. Let V be an irreducible submodule of $\mathbb{C}[AP]$ on which P' is nontrivial. Then $V_{P'}$ is faithful. Apply Theorem 4.1.

(b). Since A is semiregular on P/P' it follows that A is a p'-group and that if $U \neq 0$ is a $\mathrm{GF}(p)[A]$-submodule of P/P' then $|U| > p^n + 1$. In particular $\dim U > (1/2) \dim P/P'$. Maschke's Theorem implies that A is irreducible on P/P'.

Let K be a minimal counterexample to the assertion that $[P, K] = 1$. Then $C_K(P) = 1$ and K has prime order q. Lemma 1.6(d) implies $P = [P, A]$ and Lemma 1.4 implies $C_K(P/P') = 1$. Now $[A, K] = 1$ so $[P/P', K]$ is A-invariant. Irreducibility forces $P/P' = [P/P', K]$. Since P/P' is a p-group, it follows that $q \neq p$. Then K is semiregular on P/P'. Now $(|A|, |K|) = 1$ so $A \times K$ is semiregular on P/P' Applying (a) with $A \times K$ in the role of A, we have $|AK| \leq p^n + 1$. But $|A| = p^n + 1$, whence $K = 1$, a contradiction. □

In order to prove Theorem 4.1, a number of preliminary results are required. The first is often proved using the cumbersome theory of Projective Representations. We prefer Yoshida's elegant and concise proof [**15**].

THEOREM 4.4. *Let G be a group, $N \trianglelefteq G$, $\mathbb{F}$ a field and V an $\mathbb{F}[G]$-module. Assume that:*
– *V_N is homogeneous and $\mathrm{End}_{\mathbb{F}[G]}(V) = \mathbb{F} \cdot 1$.*
– *G/N is cyclic.*
– *$\mathbb{F}$ is a splitting field for N.*
Then V_N is irreducible.

THEOREM 4.5 [**2**, (34.0), p.180]. *Let P be an extraspecial group of order p^{1+2n}. Let $\mathbb{F}$ be a field with $\mathrm{char}\, \mathbb{F} \neq p$ that is a splitting field for P. The faithful irreducible $\mathbb{F}[P]$-modules have dimension p^n. If U and V are two such modules then*

$$U \cong_P V \text{ if and only if } U \cong_{P'} V.$$

LEMMA 4.6 [**8**, Lemma 2.5.3]. *Let D and A be positive integers. Suppose (λ_i) is a sequence of integers that satisfies*

$$A \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq 0 \text{ and } D = \sum \lambda_i. \tag{4.1}$$

Write $D = mA + r$ with $m, r \in \mathbb{Z}$ and $0 \leq r < A$. Then

$$\sum (2i-1)\lambda_i \geq m^2 A + (2m+1)r \tag{4.2}$$

with equality if and only if

$$\lambda_1 = \ldots = \lambda_m = A, \lambda_{m+1} = r \text{ and } \lambda_{m+2} = \ldots = 0. \tag{4.3}$$

Proof. We expand the argument given by Hall and Higman. Define a sequence (λ_i) satisfying (4.1) to be *mutable* if there exists a sequence (λ_i') also satisfying (4.1), with $\lambda_i' = \lambda_i$ for all except two values j and k of i. These must satisfy $j < k$, $\lambda_j' = \lambda_j + 1$ and $\lambda_k' = \lambda_k - 1$. Observe that

$$\sum (2i-1)\lambda_i > \sum (2i-1)\lambda_i'.$$

Trivially, if (λ_i) satisfies (4.3) then equality holds in (4.2). Thus it suffices to assume (λ_i) is immutable and show that (4.3) holds.

If possible, choose M maximal such that

$$\lambda_1 = \ldots = \lambda_M = A,$$

otherwise set $M = 0$. Then (4.1) and the choice of M imply

$$A > \lambda_{M+1}.$$

Suppose $\lambda_s > 0$ for some $s \geq M+2$. Choose s maximal with this property. Define (λ_i) by

$$\lambda_i' = \lambda_i \text{ for all } i \notin \{M+1, s\},\ \lambda_{M+1}' = \lambda_{M+1} + 1 \text{ and } \lambda_s' = \lambda_s - 1.$$

then (λ_i') satisfies (4.1), contrary to (λ_i) being immutable. We deduce that

$$\lambda_i = 0 \text{ for all } i \geq M+2.$$

Then $D = \sum \lambda_i = MA + \lambda_{M+1}$. As $0 \leq \lambda_{M+1} < A$ it follows that $M = m$ and $\lambda_{M+1} = r$, completing the proof. □

Proof of Theorem 4.1. Suppose (i) holds. Schur's Lemma implies (ii) holds. Suppose (ii) holds. Let W be an irreducible submodule of V_P. Note that $C_V(P') = 0$ since V is irreducible and $V_{p'}$ is faithful. As $\mathbb{Z}_p \cong P' = Z(P)$ it follows that W is faithful. Let $a \in A$. Then Wa is also a faithful irreducible submodule of V_P. As $[P', a] = 1$, Theorem 4.5 implies $W \cong_P Wa$. By irreducibility, $V = \langle Wa \mid a \in A \rangle$ so V_P is homogeneous. Theorem 4.4 implies V_P is irreducible, so (iii) holds. Hence it suffices to prove the theorem under the hypotheses of (iii).

Since $\mathbb{Z}_p \cong P' = Z(P)$ and A is semiregular on P/P', it follows that P' is the unique minimal normal subgroup of AP. Now $V_{P'}$ is faithful, hence V is faithful. As $O_p(AP) \neq 1$, irreducibility implies $\operatorname{char} \mathbb{F} \neq p$ and then Theorem 4.5 implies $\dim V = p^n$. Moreover, as $[P', A] = 1$ we have $Z(AP) = P'$.

Let $\mathcal{E} = \operatorname{End}_{\mathbb{F}}(V)$. Without loss, $AP \subseteq \mathcal{E}$ and then the conjugation action of AP on $\mathcal{E}$ endows $\mathcal{E}$ with the structure of an $\mathbb{F}[AP]$-module. For each $H \leq AP$ we have

$$C_{\mathcal{E}}(H) = \operatorname{End}_{\mathbb{F}[H]}(V).$$

Since V_P is irreducible and $\mathbb{F}$ is a splitting field for P we have

$$C_{\mathcal{E}}(P) = \mathbb{F} \cdot 1 = Z(\mathcal{E}). \tag{4.4}$$

Then $P' = Z(P) \leq C_{AP}(\mathcal{E})$. Also $C_{AP}(\mathcal{E}) \leq Z(AP) = P'$ whence $P' = C_{AP}(\mathcal{E})$ and AP/P' acts faithfully on $\mathcal{E}$.

We calculate $\dim C_{\mathcal{E}}(A)$ in two different ways. Since $\operatorname{char} \mathbb{F} \neq p$, Maschke's Theorem implies $\mathcal{E} = C_{\mathcal{E}}(P) \oplus [\mathcal{E}, P]$. Theorem 2.9 implies $[\mathcal{E}, P]$ is a free $\mathbb{F}[A]$-module. Using (4.4) we obtain

$$\dim C_{\mathcal{E}}(A) = 1 + \frac{1}{|A|} \dim[\mathcal{E}, P] = 1 + \frac{1}{|A|} \left((\dim V)^2 - 1\right). \tag{4.5}$$

Theorem 2.7 implies there exist ideals $I_1 \subseteq \ldots \subseteq I_l$ of $\mathbb{F}[A]$ such that

$$V_A \cong \mathbb{F}[A]/I_1 \oplus \ldots \oplus \mathbb{F}[A]/I_l. \tag{4.6}$$

For $1 \leq i \leq l$ set $\lambda_i = \dim \mathbb{F}[A]/I_i$ and for $i > l$ set $\lambda_i = 0$. Then

$$|A| \geq \lambda_1 \geq \ldots \geq 0 \text{ and } \dim V = \sum \lambda_i \tag{4.7}$$

Theorem 2.7 implies

$$\dim C_{\mathcal{E}}(A) = \sum (2i - 1)\lambda_i. \tag{4.8}$$

Choose $m, r \in \mathbb{Z}$ with

$$\dim V = m|A| + r \text{ and } 0 \leq r < |A|. \tag{4.9}$$

Lemma 4.6, (4.5) and (4.8) imply

$$1 + \frac{1}{|A|} \left((\dim V)^2 - 1\right) = \sum_i (2i - 1)\lambda_i \geq m^2|A| + (2m + 1)r. \tag{4.10}$$

Multiplying by $|A|$ and using (4.9) yields

$$|A| + m^2|A|^2 + 2m|A|r + r^2 - 1 \geq m^2|A|^2 + (2m + 1)r|A|$$

and then

$$0 \geq (r - 1)(|A| - (r + 1)). \tag{4.11}$$

Now A is semiregular on the p-group P/P' so p does not divide $|A|$. Since $\dim V = p^n$ it follows from (4.9) that $r \geq 1$. Then (4.9) and (4.11) imply $r = 1$ or $|A| = r + 1$.

Suppose $|A| = r + 1$. Then equality holds in (4.11). This forces equality in (4.10) and Lemma 4.6 implies $\lambda_1 = \ldots = \lambda_m = |A|$, $\lambda_{m+1} = |A| - 1$ and $\lambda_{m+2} = \ldots = 0$. Put $U = I_{m+1}$, so $\dim U = 1$. Then $V_A \cong m \times \mathbb{F}[A] \oplus \mathbb{F}[A]/U$ and, since $\dim V = p^n$, (a) holds.

Suppose $r = 1$. Again we have equality and Lemma 4.6 implies $\lambda_1 = \ldots = \lambda_m = |A|$, $\lambda_{m+1} = 1$ and $\lambda_{m+2} = \ldots = 0$. Put $U = \mathbb{F}[A]/I_{m+1}$, so $\dim U = 1$. Then $V_A \cong m \times \mathbb{F}[A] \oplus U$. Now $\dim V = p^n$ so $|A|$ divides $p^n - 1$. Then (b) will hold provided we can show that A is not irreducible on P/P'.

Assume that $|A|$ divides $p^n - 1$ and A is irreducible on P/P'. Set $W = P/P'$. Then W is a faithful irreducible $\mathrm{GF}(p)[A]$-module and $\dim W = 2n$. Let $\mathcal{F}$ be the subring of $\mathrm{End}_{\mathbb{F}}(W)$ generated by A. Then $A \subseteq \mathcal{F}$. Irreducibility implies $\mathcal{F}$ is a field and $\dim_{\mathcal{F}} W = 1$. Then $\mathcal{F} \cong \mathrm{GF}(p^{2n})$. Let $\mathcal{F}_0$ be the subfield of $\mathcal{F}$ with $\mathcal{F}_0 \cong \mathrm{GF}(p^n)$. The multiplicative group of $\mathcal{F}$ is cyclic. It follows that every subgroup of $\mathcal{F}^\times$ with order dividing $p^n - 1$ is contained in $\mathcal{F}_0$, so $A \subseteq \mathcal{F}_0$. But A generates $\mathcal{F}$, a contradiction. This completes the proof of Theorem 4.1. □

5. *Free direct summands*

Let R be a group of prime order r that acts on the r'-group G, let $\mathbb{F}$ be a field and V a faithful $\mathbb{F}[RG]$-module. An important special case of Theorem A is when $C_V(R) = 0$. Indeed, we then have $\ker(C_G(R) \text{ on } C_V(R)) = C_G(R)$. Trivially, if $C_V(R) = 0$ then $\mathbb{F}[R]$ is not a direct summand of V_R. It turns out that determining the consequences of this latter condition present no further difficulty. The goal of this section is to prove:

Theorem 5.1. *Let R be a group of prime order r that acts on the r'-group G, let $\mathbb{F}$ be a field and V an $\mathbb{F}[RG]$-module. Assume the following:*
– *$\mathbb{F}[R]$ is not a direct summand of V_R.*
– *$V_{[G,R]}$ is faithful and completely reducible.*
– *$[G, R]$ is soluble.*
Set $P = [G, R]$ and $C = C_G(R)$. Then either:
(a) *$P = 1$; or*
(b) *$r = 2^n + 1$ for some $n \in \mathbb{N}$; P is a special 2-group; $C_V(P') = C_V(P)$; $P' = C_P(R)$ and $C_C(P') = C_C(P)$. Moreover if U is an irreducible submodule of V_{RP} with P nontrivial on U then $P/C_P(U) \cong 2^{1+2n}$.*

We remark that in [**5**], the condition $C_V(R) = 0$ is investigated without assuming $[G, R]$ to be soluble. It is possible to extend that work to remove the solubility hypothesis in Theorem 5.1. The work of Berger [**3**], Güloğlu and Ercan [**7**] and Turull [**14**] must also be mentioned in this context.

Before proceeding with the proof of Theorem 5.1 we state a corollary which is a very slight variation of known results, see for example [**1**, §36].

Corollary 5.2. *Let R be a group of prime order r that acts on the soluble r'-group G. Suppose that H is an $RC_G(R)$-invariant subgroup of G with $H = [H, R]$.*
(a) *Let p be a prime. If $p = 2$ and r is a Fermat prime assume that the Sylow 2-subgroups of G are abelian. Then*
$$O_p(H) \le O_p(G).$$
(b) *If $H = O^2(H)$ then*
$$O_2(H) \le O_2(G).$$

Lemma 5.3. *Let A be a cyclic group, $\mathbb{F}$ a field and V an $\mathbb{F}[A]$-module. The following are equivalent.*
(a) *$\mathbb{F}[A]$ is a direct summand of V.*
(b) *The minimal polynomial of a generator of A is $X^{|A|} - 1$.*
(c) *There exist submodules $U \le W \le V$ such that $\mathbb{F}[A]$ is a direct summand of W/U.*
(d) *For any extension $\mathbb{K} \supseteq \mathbb{F}$, $\mathbb{K}[A]$ is a direct summand of $V^{\mathbb{K}}$.*

(e) *There exists an extension* $\mathbb{K} \supseteq \mathbb{F}$ *such that* $\mathbb{K}[A]$ *is a direct summand of* $V^{\mathbb{K}}$.
(f) *There exists a system of imprimitivity for* V *on which* A *has a regular orbit.*

Proof. The nontrivial implications follow from Theorem 2.7 and Corollary 2.8. □

Throughout the remainder of this section we assume the hypotheses of Theorem 5.1.

LEMMA 5.4. *Assume* $P \neq 1$.
(a) V_{RP} *is faithful.*
(b) *Every abelian normal subgroup of* RP *is contained in* $Z(RP) \cap P$.

Proof. (a). By Coprime Action, $P = [P, R] \neq 1$. By hypothesis, V_P is faithful so R is nontrivial on V. Now R has prime order so V_R is faithful. Since $(|R|, |P|) = 1$ we deduce that V_{RP} is faithful.

(b). Suppose $1 \neq N \trianglelefteq RP$ is abelian. We may assume N is a p-group for some prime p. If $p = r$ then as P is an r'-group and $|R| = r$, we have $R = N \trianglelefteq RP$ and so $P = [P, R] \leq P \cap R = 1$, a contradiction. Thus $p \neq r$ and $N \leq P$. Now V_P is faithful and completely reducible so it follows that $\operatorname{char} \mathbb{F} \neq p$.

Assume $[N, R] \neq 1$. Corollary 2.4 implies $V = C_V([N, R]) \oplus [V, [N, R]]$. Lemma 1.6(c) implies R is semiregular on $[N, R]$. Theorem 2.9 implies $[V, [N, R]]_R$ is free, contrary to hypothesis. Thus $[N, R] = 1$. Since $P = [P, R]$ it follows that $[N, P] = 1$, whence $N \leq Z(RP) \cap P$. □

LEMMA 5.5. *Assume* P *is a nontrivial* p*-group for some prime* p.
(a) P *is special and* $P' = C_P(R)$.
(b) $C_V(P') = C_V(P)$.
(c) $r = 2^n + 1$ *for some* $n \in \mathbb{N}$ *and* $p = 2$.
(d) *If* P *is extraspecial then* $P \cong 2^{1+2n}$ *and* $[P, C] = 1$.
(e) $C_C(P') = C_C(P)$.

Proof. Since V_P is completely reducible it follows that $p \neq \operatorname{char} \mathbb{F}$. By Lemma 5.3 we may assume that $\mathbb{F}$ is algebraically closed. By Coprime Action, $P = [P, R]$ so Lemmas 5.4(b) and Corollary 3.3 imply (a).

Let $\mathcal{H}$ be the set of homogeneous components of $V_{P'}$. Now $P' = Z(P) \leq Z(RC_C(P')P)$ so each element of $\mathcal{H}$ is in fact an $\mathbb{F}[RC_C(P')P]$-module. Moreover, $V = \oplus\mathcal{H}$.

Let $W \in \mathcal{H}$ and set $P^* = P/C_P(W)$. Then $P^* = [P^*, R]$ and $\mathbb{F}[R]$ is not a direct summand of W_R. The hypotheses of Theorem 5.1 are satisfied with P^* in the role of G. Trivially, $P^{*\prime} = P'^*$ and $R \times C_C(P')$ acts on P^*.

Suppose P' is trivial on W. Then $W = C_V(P')$ and P^* is abelian and so not special. Hence $P^* = 1$ by (a). Consequently $C_V(P') = C_V(P)$ and (b) holds. Trivially, $[P, C_C(P')] \leq C_P(W)$.

Suppose P' is nontrivial on W. Then $1 \neq P'^* = P^{*\prime}$. Since $W_{P'}$ is homogeneous and P' is elementary abelian we have $P'^* \cong \mathbb{Z}_p$. By (a), P^* is special, so P^* is extraspecial. As $P^* = [P^*, R]$ it follows that $P^{*\prime}$ is the unique minimal normal subgroup of RP^*. Let W_0 be an irreducible submodule of W_{RP^*}. Then $P^{*\prime}$ is nontrivial on W_0 and so W_{0RP^*} is faithful. Lemma 5.3 implies $\mathbb{F}[R]$ is not a direct summand of W_{0R}. Using Corollary 4.2 and the fact that R has prime order r, we obtain

$$r = 2^n + 1 \text{ for some } n,\ p = 2 \text{ and } P^* \cong 2^{1+2n}.$$

Corollary 4.3(b) implies $[P^*, C_C(P')] = 1$, whence $[P, C_C(P')] \leq C_P(W)$.

We have shown that

$$[P, C_C(P')] \leq \bigcap_{W \in \mathcal{H}} C_P(W).$$

Since $V = \oplus\mathcal{H}$ and V_P is faithful we have $[P, C_C(P')] = 1$ and (e) holds.

By (a), P is special so $P' \neq 1$ and there exists $W \in \mathcal{H}$ with P' nontrivial on W. Then (c) holds. Suppose P is extraspecial. Then P' is the unique minimal normal subgroup of RP so $C_P(W) = 1$ and $P \cong P^* \cong 2^{1+2n}$. Also, $P' \cong \mathbb{Z}_2$ so $C_C(P') = C$. Then (d) holds. □

Corollary 5.6.

(a) $[F(G), R]$ *is a 2-group.*
(b) *If* $[G, R]$ *is nilpotent then the conclusion of Theorem 5.1 holds.*

Proof. (a). Let p be a prime and suppose $[O_p(G), R] \neq 1$. Then $O_p([G, R]) \neq 1$ so complete reducibility implies $p \neq \operatorname{char} \mathbb{F}$. The hypotheses of Theorem 5.1 hold with $O_p(G)$ in place of G. Apply Lemma 5.5

(b). Note that $P = [G, R] \trianglelefteq RG$ so by Coprime Action, $[G, R] = [G, R, R] \leq [F(G), R]$ so (a) implies that P is a 2-group. Apply Lemma 5.5. Note that if U is an irreducible constituent of V_{RP} with P nontrivial on U then as $P' \leq Z(RP)$ it follows that $P/C_P(U)$ is extraspecial. □

Proof of Theorem 5.1. Assume false and consider a counterexample with $|G| + \dim V$ minimal. By Corollary 5.6, $[G, R]$ is not nilpotent. Coprime Action and minimality imply $G = [G, R]$, so $G = P$ and V_G is completely reducible.

Let $Z = Z(RG) \cap G$. Clifford's Theorem implies V_Z is completely reducible. Each homogeneous component of V_Z is RG-invariant. Minimality implies V is indecomposable since otherwise, $[G, R]$ would be nilpotent. Hence V_Z is homogeneous. Now Z is abelian, so Z is cyclic. Lemma 5.4 implies every abelian normal subgroup of RG is contained in Z.

Lemma 3.8(c), with RG and G in the roles of G and N respectively, implies there exists a prime p and a nonabelian p-subgroup $Q \leq G$ with $Q \trianglelefteq RG$ and $Q/Z(Q)$ irreducible for RG and nontrivial for G. Moreover $\mathbb{Z}_p \cong Q' \leq Z(Q) \leq Z(RG)$.

Let $Q_0 = [Q, R]$. Now G is nontrivial on Q and $G = [G, R]$ so $Q_0 \neq 1$. Lemma 3.5(b) implies Q_0 is extraspecial. Note that Q_0 is $C_G(R)$-invariant. Set $G_0 = C_G(R)Q_0$, so $[G_0, R] = Q_0$. Now $Q_0 \trianglelefteq\trianglelefteq G$ and V_G is completely reducible so Clifford's Theorem implies V_{Q_0} is completely reducible. Lemma 5.5, with G_0 in the role of G, implies $r = 2^n + 1$ for some n, $p = 2$, $Q_0 \cong 2^{1+2n}$ and $[Q_0, C_G(R)] = 1$.

Let $\overline{Q} = Q/Z(Q)$ so $\overline{Q}$ is an irreducible GF(2)[RG]-module. Note that $\dim \overline{Q}$ is even by Lemma 3.6. Set $G^* = G/C_G(\overline{Q}) \neq 1$. Irreducibility implies $O_2(G^*) = 1$. Let G_1 be the inverse image of $G^{*\prime}$ in G. Then $G_1 \neq G$ since G is soluble. Moreover $[G_1, R] \trianglelefteq G_1 \trianglelefteq G$ so as V_G is completely reducible, so is $V_{[G_1,R]}$ by Clifford's Theorem. The minimality of G implies $[G_1, R]$ is a 2-group. Consequently $[G_1, R]^* \leq O_2(G^*) = 1$, whence $[G^{*\prime}, R] = 1$. By Coprime Action, $C_{G^*}(R) = C_G(R)^*$ so the previous paragraph implies

$$1 \neq \overline{Q_0} \leq C_{\overline{Q}}(C_{G^*}(R)) \leq C_{\overline{Q}}(G^{*\prime}).$$

Since $G^{*\prime} \trianglelefteq G^*$, irreducibility forces $C_{\overline{Q}}(G^{*\prime}) = \overline{Q}$, whence $G^{*\prime} = 1$ and G^* is abelian.

Theorem 2.9 implies $\overline{Q}_R$ is free. Then

$$r \mid \dim \overline{Q} \quad \text{and} \quad \dim[\overline{Q}, R] = \left(1 - \frac{1}{r}\right) \dim \overline{Q}.$$

Recall that $\overline{Q} = Q/Z(Q)$. Lemma 3.5(b) implies $Q_0 \cap Z(P) = Q_0'$ so as $[Q,R] = Q_0 \cong 2^{1+2n}$ we have $\dim[\overline{Q},R] = 2n$. Also, $\dim \overline{Q}$ is even and $r = 2^n + 1$ is odd. Then $2r \mid \dim \overline{Q}$ and so

$$2n = \dim[\overline{Q},R] \geq \left(1 - \frac{1}{r}\right) 2r,$$

whence $n \geq r - 1 = 2^n$. This contradiction completes the proof of Theorem 5.1. □

Proof of Corollary 5.2. Assume the result to be false and let G be a minimal counterexample. If proving (b) then set $p = 2$. Let V be a minimal R-invariant normal subgroup of G, so V is an elementary abelian q-group for some prime q. Let N be the inverse image of $O_p(G/V)$ in G. The minimality of G implies that $O_p(H) \leq N$. Then $q \neq p$. In particular, $O_p(G) = 1$ since otherwise we could have chosen $V \leq O_p(G)$. Choose $S \in \mathrm{Syl}_p(N)$, so $N = VS$. Then $C_S(V) \leq O_p(N) \leq O_p(G) = 1$. Consequently $V = C_N(V)$.

Note that $O_p(HV) \leq C_G(V)$ so the minimality G forces $G = HV$. Then $C_V(O_p(H)) \trianglelefteq G$ so the choice of V implies that $C_V(O_p(H)) = 1$ or V. The latter is impossible as $C_N(V) = V$, whence $C_V(O_p(H)) = 1$. Note that $V \cap H \leq C_V(O_p(H))$ so $V \cap H = 1$. This implies that H acts faithfully on V since otherwise we could replace V with a minimal R-invariant normal subgroup of G contained in $C_H(V)$.

Since H is $C_G(R)$-invariant, so is $O_p(H)$. Then $[C_V(R), O_p(H)] \leq V \cap O_p(H) = 1$ so $C_V(R) \leq C_V(O_p(H)) = 1$. Regard V as an irreducible $\mathrm{GF}(q)[H]$-module. Since G is a counterexample, we have $H \neq 1$. Also $H = [H,R]$ by hypothesis. Theorem 5.1 implies that r is a Fermat prime and H is a nonabelian special 2-group. This is contrary to the assumptions of (a). Also, $O^2(H) = 1$, contrary to the assumption of (b). □

6. *The proof of Theorem A*

Lemma 6.1. *Let r be a prime, $\mathbb{F}$ a field of characteristic r, R an r-group, K an r'-group and V an $\mathbb{F}[R \times K]$-module. Assume that $[C_V(R), K] = 0$. Then $[V,K] = 0$.*

Proof. Corollary 2.4 implies

$$V = C_V(K) \oplus [V,K].$$

Note that $[V,K]$ is a submodule. By hypothesis $C_V(R) \leq C_V(K)$ so $C_{[V,K]}(R) = 0$. This forces $[V,K] = 0$ because R is an r-group and $\mathrm{char}\, \mathbb{F} = r$. □

The next two lemmas are used when analyzing the imprimitive case of Theorem A.

Lemma 6.2. *Let G be a group, $\mathbb{F}$ a field, V an $\mathbb{F}[G]$-module and Ω a system of imprimitivity for V. Suppose $R, K \leq G$ with $R \cong \mathbb{Z}_r$ for some prime r, K an r'-group and $K \leq \ker(C_G(R)$ on $C_V(R))$. Then*

$$[\oplus \mathrm{Mov}_\Omega(R), K] = 0.$$

Proof. Let x be a generator for R and suppose $V_0 \in \mathrm{Mov}_\Omega(R)$. For each $i \geq 0$ set $V_i = V_0 x^i$. Set $W = V_0 + \ldots + V_{r-1} = V_0 \oplus \ldots \oplus V_{r-1}$. Choose $0 \neq v_0 = V_0$ and set

$$v = v_0 + v_0 x + \ldots + v_0 x^{r-1} \in C_V(R).$$

Note that $v_0 x^i \in V_i$ for each i, so $v \neq 0$. Now $[v,K] = 0$ so K permutes the members of Ω into which v projects nontrivially. Hence K permutes $\{V_0, \ldots, V_{r-1}\}$. An r-cycle in $\mathrm{Sym}(r)$ is self centralizing, so as K is an r'-group it follows that K normalizes each V_i. Then for each $k \in K$,

$$0 = [v, k] = [v_0, k] + \ldots + [v_0 x^{r-1}, k]$$

and $[v_0 x^i, k] \leq V_i$. Consequently $[v_0, k] = 0$. □

LEMMA 6.3. *Let G be a group that acts faithfully and primitively on the set Ω. Assume $F(G) \neq 1$.*

(a) *Suppose $1 \neq K \leq G$. Then $|\operatorname{Fix}_\Omega(K)| \leq (1/2)|\Omega|$. If equality holds then K is a 2-group.*
(b) *Suppose $1 \neq K, R \leq G$ and that $\operatorname{Mov}_\Omega(R) \subseteq \operatorname{Fix}_\Omega(K)$. Then K and R are 2-groups.*

Proof. (a). If $\operatorname{Fix}_\Omega(K) = 0$ there is nothing to prove, so assume $\operatorname{Fix}_\Omega(K) \neq 0$. Let N be a minimal normal subgroup of G that is contained in $F(G)$. Then N is elementary abelian and acts regularly on Ω. In particular, $|N| = |\Omega|$ and $|\Omega|$ is a prime power.

Trivially $C_N(K)$ acts on $\operatorname{Fix}_\Omega(K)$. We claim this action is regular. Let $\alpha, \beta \in \operatorname{Fix}_\Omega(K)$. Choose $n \in N$ with $\alpha n = \beta$. Then $K, K^n \leq \operatorname{Stab}_G(\beta)$ whence $[K, n] \leq \operatorname{Stab}_G(\beta) \cap N = 1$. The claim follows. In particular

$$|\operatorname{Fix}_\Omega(K)| = |C_N(K)|.$$

Now $C_N(K) \neq N$ since $K \neq 1$ whence $|C_N(K)| \leq (1/2)|N|$. This proves the inequality.

Suppose $|\operatorname{Fix}_\Omega(K)| = (1/2)|\Omega|$. Then 2 divides $|\Omega|$ so $|\Omega|$ is a power of 2. Consider the action of K on $\operatorname{Mov}_\Omega(K)$. Since $|\operatorname{Fix}_\Omega(K)| = (1/2)|\Omega|$, the inequality implies that this action is semiregular. Consequently $|K|$ divides $|\operatorname{Mov}_\Omega(K)| = (1/2)|\Omega|$, so K is a 2-group.

(b). We have $|\operatorname{Mov}_\Omega(R)| \leq |\operatorname{Fix}_\Omega(K)| \leq (1/2)|\Omega|$ whence $|\operatorname{Fix}_\Omega(R)| \geq (1/2)|\Omega|$. Then (a), with R in the role of K, forces equality. Another application of (a) implies R and K are 2-groups. □

The next four lemmas relate to the primitive case of Theorem A.

HYPOTHESIS 6.4.
- $R \times K$ acts on the extraspecial p-group $P \cong p^{1+2n}$.
- R has prime order r and $K \neq 1$ is an r'-group.
- $[P', RK] = 1$, $P = [P, R]$ and $[P, K] \neq 1$.
- V is a faithful $\mathbb{F}[RKP]$-module with $\mathbb{F}$ an algebraically closed field whose characteristic is not p or r.
- $V = [V, P']$.
- $[C_V(R), K] = 0$.

LEMMA 6.5. *Assume Hypothesis 6.4 and that K is cyclic and semiregular on P/P'. Then $|K| = 2 \neq p$, $r = (1/2)(p^n + 1)$ and $R \times K$ is irreducible on P/P'.*

Proof. Replacing V by an irreducible submodule of V, we may assume that V is irreducible. Note that R is semiregular on P/P' because R has prime order $r \neq p$ and $P/P' = [P/P', R]$. Since K is an r'-group it follows that $R \times K$ is cyclic and semiregular on P/P'.

Since $[C_V(R), K] = 0$ we have

$$C_V(RK) = C_V(R).$$

On the other hand, for each $A \leq RK$ we have

$$\dim C_{\mathbb{F}[RK]}(A) = |RK : A|.$$

Consequently, $\mathbb{F}[RK]$ is not a direct summand of V_{RK}. Corollary 4.2, with RK in the role of A, implies there exists a 1-dimensional submodule $U \le \mathbb{F}[RK]$ such that

$$V_{RK} \cong \mathbb{F}[RK]/U,\ r|K| = p^n + 1 \text{ and } RK \text{ is irreducible on } P/P'.$$

Lemma 2.6, with $\mathbb{F}[RK]$ in the role of V, implies

$$|RK : A| \ge \dim C_V(A) \ge |RK : A| - 1$$

for each $A \le RK$. Then

$$1 \ge \dim C_V(RK) = \dim C_V(R) \ge |K| - 1.$$

whence $|K| = 2$, $2r = p^n + 1$ and $p \ne 2$. □

Lemma 6.6. *Assume Hypothesis 6.4.*
(a) *$C_K(P) = 1$.*
(b) *Assume further that K is a p'-group. Then K is semiregular on P/P'.*

Proof. (a). Let $K_0 = C_K(P)$ and suppose $K_0 \ne 1$. Now $K_0 \trianglelefteq RKP$ and V is faithful so $C_V(K_0)$ is a proper submodule. Let $W = V/C_V(K_0)$ so W is an $\mathbb{F}[RKP]$-module. By hypothesis, $C_V(R) \le C_V(K) \le C_V(K_0)$ so as $\operatorname{char}\mathbb{F} \ne r$, we have $C_W(R) = 0$. Now $V = [V, P']$ so $\mathbb{Z}_p \cong Z(P) = P'$ is nontrivial on W. Thus W_P is faithful. By hypothesis, $[P', K] = 1$. Theorem 5.1 implies $[P, K] = 1$, contrary to hypothesis. We deduce that $C_K(P) = 1$.

(b). We may assume that K has prime order $q \ne p$. Since K is faithful on P, Coprime Action implies

$$P = C_P(K) * [P, K]$$

and $[P, K]$ is extraspecial with $[P, K]' = P'$. Set $P_0 = [P, K]$ and note that P_0 is $R \times K$-invariant. Now $P = [P, R]$ so $C_P(R) = P'$, whence $P_0 = C_{P_0}(R)[P_0, R] = P_0'[P_0, R] = \Phi(P_0)[P_0, R]$, so $P_0 = [P_0, R]$. Lemma 6.5, with P_0 in the role of P, implies $p \ne 2$.

Note that $C_V(K)$ is $RC_P(K)$-invariant. Set $W = V/C_V(K)$, so W is an $\mathbb{F}[RC_P(K)]$-module. By hypothesis, $C_V(R) \le C_V(K)$ so as $\operatorname{char}\mathbb{F} \ne r$ we have $C_W(R) = 0$. In particular, $\mathbb{F}[R]$ is not a direct summand of W_R.

Suppose $C_P(K) \not\le P'$. Coprime Action implies $C_P(K)$ is extraspecial with $C_P(K)' = P'$. As $V = [V, P']$ we have $W = [W, P']$ so $C_P(K)$ is faithful on W. The same argument that proved $P_0 = [P_0, R]$ also proves that $C_P(K) = [C_P(K), R]$. Theorem 5.1, or a direct application of Corollary 4.2, implies that $C_P(K)$ is a 2-group. But $p \ne 2$, a contradiction. We deduce that $C_P(K) \le P'$. Coprime Action implies $C_{P/P'}(K) = 1$, completing the proof. □

Lemma 6.7. *Assume Hypothesis 6.4 and that K is a p-group. Then either:*
(a) *$r = 2, p = 3$ and $[Z(C_P(K)), R] \cong \mathbb{Z}_3$; or*
(b) *$r = 3, p = 2$ and $[Z(C_P(K)), R] \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Set $\overline{P} = P/P'$ and recall that the commutator map on P induces a symplectic form on $\overline{P}$, see Lemma 3.6. Set $Q = C_P(K) \ne P$. Lemma 1.4 implies $\overline{Q} = C_{\overline{P}}(K)$. Since K is a p-group we have $1 < \overline{Q} < \overline{P}$. Now $\dim \overline{Q}^{\perp} = \operatorname{codim} \overline{Q}$ so $\overline{Q}^{\perp} \ne 0$. As $\overline{Q}^{\perp}$ is K-invariant and K is a p-group we have $C_{\overline{Q}^{\perp}}(K) \ne 0$, whence $\overline{Q} \cap \overline{Q}^{\perp} \ne 0$ and $\overline{Q}$ is degenerate. Since $Z(Q)$ is the inverse image of $\operatorname{Rad}(\overline{Q})$ we have $Z(Q) \not\le P'$. Set $T = [Z(Q), R]$. Since $C_P(R) = P'$ we have $Z(Q) = P' \times T$ and so $T \ne 1$.

Set $W = V/C_V(K)$. Then W is an $\mathbb{F}[RQ]$-module. As $C_V(R) \le C_V(K)$ and $\operatorname{char}\mathbb{F} \ne r$ we have $C_W(R) = 0$. By Coprime Action, $T = [T, R]$. Note that T is abelian. Theorem 2.9, or

Theorem 5.1, implies that T is trivial on W. Then $[T, V] \le C_V(K)$ so $[T, V, K] = 0$. Trivially $[K, T, V] = 0$ so the Three Subgroups Lemma forces $[V, K, T] = 0$. Lemma 3.9 implies

$$\dim[V, K] \le \frac{1}{|T|} \dim V. \tag{6.1}$$

Choose $x \in P \setminus Q$. Since $\overline{Q} = C_{\overline{P}}(K)$ we have $[K, x] \not\le P'$. Recall that $P' = Z(P) \cong \mathbb{Z}_p$. Hence there exists $y \in P$ with $P' = [K, x, y]$. Consequently $P' \le \langle K, K^x, K^y, K^{xy} \rangle$ and

$$V = [V, P'] \le [V, K] + [V, K^x] + [V, K^y] + [V, K^{xy}],$$

so $\dim V \le 4 \dim[V, K]$. Then (6.1) implies $|T| \le 4$. Now $1 \ne T = [T, R]$ and the conclusion follows. □

Lemma 6.8. *Assume Hypothesis 6.4. One of the following holds:*
(a) $|K| = 2 \ne p$, $r = (1/2)(p^n + 1)$ *and* RK *is semiregular and irreducible on* P/P'.
(b) K *is a* p*-group,* $r = 2, p = 3$ *and* $[Z(C_P(K)), R] \cong \mathbb{Z}_3$.
(c) K *is a* p*-group,* $r = 3, p = 2$ *and* $[Z(C_P(K)), R] \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. If K is a p-group then (b) or (c) hold by Lemma 6.7, so assume K is not a p-group. Let $K_0 \ne 1$ be a cyclic p'-subgroup of K. Lemmas 6.6 and 6.5 imply that $|K_0| = 2 \ne p$. Then $\pi(K) \subseteq \{2, p\}$. Choose $S \in \mathrm{Syl}_2(K)$. Then S has exponent 2 so S is elementary abelian.

Lemma 1.4 implies RK is faithful on P/P'. Lemma 6.5 also implies that RK_0 is irreducible on P/P', hence so is RS. Now RS is abelian so it follows that RS is cyclic. Then $|S| = 2$. Burnside's Normal Complement Theorem implies $K = SO_p(K)$. As RK is faithful and irreducible on the p-group P/P' we have $O_p(RK) = 1$. Then $K = S$ and (a) holds. □

Next we analyze the minimal configuration that arises in the proof of Theorem A.

Lemma 6.9. *Assume the following:*
– R *is a group of prime order* r *that acts on the* r'*-group* G *and* $[G, R]$ *is soluble.*
– $\mathbb{F}$ *is a field and* V *is a faithful irreducible* $\mathbb{F}[RG]$*-module.*
– $1 \ne K \le \ker(C_G(R)$ *on* $C_V(R))$ *and* $K \trianglelefteq C_G(R)$.
– $G = \langle K^G \rangle$.
Then $G = K[G, R]$ *and one of the following holds:*
(a) $G = K$ *and* $C_V(R) = 0$.
(b) $C_V(R) = 0$, $r = 2^n + 1$ *for some* $n \in \mathbb{N}$, $[G, R]$ *is a special* 2*-group and* K *is not nilpotent.*
(c) $C_V(R) \ne 0$, $r = (1/2)(p^n + 1)$ *for some prime* p *and* $n \in \mathbb{N}$, $[G, R] \cong p^{1+2n}$, $[G, R]' \le Z(RG)$ *and* $|K| = 2$.

Proof. Using Lemma 2.2 we may assume that $\mathbb{F}$ is algebraically closed. Since $K \ne 1$, Lemma 6.1 implies $\operatorname{char} \mathbb{F} \ne r$. By Coprime Action, $G = C_G(R)[G, R]$. Now $G = \langle K^G \rangle$ and $K \trianglelefteq C_G(R)$ so $G/[G, R]$ is equal to the image of K. Hence $G = K[G, R]$.

Claim 1. *Suppose* $C_V(R) = 0$ *or* $[G, R] = 1$. *Then (a) or (b) holds.*

Proof. Let $S = [G, R]$. If $S = 1$ then $G = K \le C_G(R)$, irreducibility forces $C_V(R) = 0$ and (a) holds. Hence we may assume that $S \ne 1$. Then $C_V(R) = 0$. Theorem 5.1 implies $r = 2^n + 1$ for some $n \in \mathbb{N}$, S is a special 2-group $C_K(S') = C_K(S)$ and $S' = C_S(R)$. Now $S \trianglelefteq RG$ so $S' \le O_2(C_G(R))$. As $K \trianglelefteq C_G(R)$ we have $[S', O(K)] = 1$. Then $[S, O(K)] = 1$.

Suppose that K is nilpotent. It follows from $G = KS$ and $[S, O(K)] = 1$ that G is nilpotent. But $G = \langle K^G \rangle$ so $G = K \leq C_G(R)$ and $S = 1$, a contradiction. Thus K is not nilpotent and (b) holds. $\square$

For the remainder of the proof, we assume

$$C_V(R) \neq 0 \text{ and } [G, R] \neq 1.$$

As $C_V(R) \leq C_V(K)$, irreducibility implies

$$K \cap Z(RG) = 1.$$

Claim 2. *V is primitive.*

Proof. Assume false and let Ω be a minimal nontrivial system of imprimitivity for V. Let $\overline{RG}$ be the image of RG in $\mathrm{Sym}(\Omega)$. Then $|\Omega| \geq 2$ and $\overline{RG}$ is primitive on Ω. Lemma 6.2 implies

$$[\oplus \mathrm{Mov}_\Omega(R), K] = 0.$$

In particular, $\mathrm{Mov}_\Omega(\overline{R}) \subseteq \mathrm{Fix}_\Omega(\overline{K})$.

If $\overline{K} = 1$ then $\overline{RG} = \overline{R}\langle \overline{K}^{\overline{G}} \rangle = \overline{R}$ whence $V = \oplus \mathrm{Mov}_\Omega(R)$ and then $[V, K] = 0$, a contradiction. Thus $\overline{K} \neq 1$. Since $(|\overline{R}|, |\overline{K}|) = 1$, Lemma 6.3 forces $\overline{R} = 1$. Then $\overline{RG} = \overline{K}[\overline{G}, \overline{R}] = \overline{K}$, so $\overline{K}$ is transitive on Ω. Moreover, R normalizes each element of Ω. As $V = \oplus\Omega$ and $C_V(R) \neq 0$, there exists $U \in \Omega$ with $C_U(R) \neq 0$. But $[C_U(R), K] = 0$ so K normalizes U, contradicting the transitivity of $\overline{K}$ on Ω. $\square$

Claim 3. *Suppose p is a prime and $P \trianglelefteq RG$ is a nonabelian p-group.*
(a) *$P = C_P(R) * [P, R]$, $[P, R]$ is an extraspecial p-group and $[P, R]' \leq Z(RG)$.*
(b) *$[C_P(R), K] = P \cap K = 1$.*
(c) *Hypothesis 6.4 is satisfied with $[P, R]$ in the role of P.*

Proof. Lemma 3.4 implies that every abelian normal subgroup of RG is cyclic and contained in $Z(RG)$. Then $Z(P) \leq Z(RG)$. Now P is nonabelian so $p \neq r$ and $P \leq G$. Lemma 3.5 implies

$$P = C_P(R) * [P, R]$$

whence $Z(C_P(R)) \leq Z(P) \leq Z(RG)$ and $K \cap Z(C_P(R)) \leq K \cap Z(RG) = 1$. Now $K \trianglelefteq C_G(R)$ so $K \cap P = K \cap C_P(R) \trianglelefteq C_P(R)$. As $K \cap P \cap Z(C_P(R)) = 1$ and $C_P(R)$ is a p-group, it follows that $K \cap P = 1$. Moreover $[K, C_P(R)] \leq K \cap P = 1$ so (b) holds.

Since P is a nonabelian normal subgroup of G and $G = \langle K^G \rangle$ we have $[P, K] \neq 1$. Then (b) implies $[[P, R], K] \neq 1$ and Lemma 3.5 implies $[P, R]$ is extraspecial with $[P, R]' \leq Z(RG)$. Then (a) holds. Irreducibility implies $V = [V, [P, R]']$ so (c) holds. $\square$

Claim 4. *Suppose $[G, R]$ is nilpotent. Then conclusion (c) holds.*

Proof. Choose $p \in \pi([G, R])$ and set $P = O_p([G, R]) \trianglelefteq RG$. Since $[G, R]$ is nilpotent, Coprime Action implies $P = [P, R] \neq 1$. Lemma 3.4 implies P is nonabelian. Suppose K is a p-group. Set $\overline{G} = G/O_{p'}([G, R])$, so $[\overline{P}, R] \neq 1$. Now $\overline{G} = \overline{K}[\overline{G}, R] = \langle \overline{K}^{\overline{G}} \rangle$ so $\overline{G}$ is a p-group and then $\overline{G} = \overline{K} \leq C_{\overline{G}}(R)$, a contradiction. Thus K is not a p-group.

We apply Claim 3(c) and Lemma 6.8. Recall that $P = [P, R]$. Choose n with $P \cong p^{1+2n}$. Then $P' \leq Z(RG), |K| = 2 \neq p$ and $r = (1/2)(p^n + 1)$. In particular, p is uniquely determined so $[G, R]$ is a p-group, $[G, R] = P$ and conclusion (c) holds. □

In order to complete the proof, it suffices to assume that $[G, R]$ is not nilpotent and derive a contradiction. Lemma 3.8(c), with RG and $[G, R]$ in the roles of G and N respectively, implies there exists a prime p and a nonabelian p-subgroup $P \leq [G, R]$ with $P \trianglelefteq RG$ and $P/Z(P)$ irreducible for RG and nontrivial for $[G, R]$. Moreover $Z(P) \leq Z(RG)$.

Claim 3 and Lemma 6.8 imply that one of the following holds:

$$|K| = 2 \neq p \text{ and } 2r - 1 \text{ is a power of } p. \tag{6.2}$$

$$K \text{ is a } p\text{-group}, r = 2, p = 3 \text{ and } [Z(C_{[P,R]}(K)), R] \cong \mathbb{Z}_3. \tag{6.3}$$

$$K \text{ is a } p\text{-group}, r = 3, p = 2 \text{ and } [Z(C_{[P,R]}(K)), R] \cong \mathbb{Z}_2 \times \mathbb{Z}_2. \tag{6.4}$$

In particular, K is nilpotent.

Set $\overline{P} = P/Z(P)$ and $G^* = G/C_G(\overline{P})$. By Claim 3, $[\overline{P}, R] \neq 1$ so $\overline{P}$ is a faithful irreducible GF$(p)[RG^*]$-module. Claim3, Lemma 1.4 and Lemma 6.6 imply that $K^* \cong K$. By Coprime Action, $C_{G^*}(R) = C_G(R)^*$ so $K^* \trianglelefteq C_{G^*}(R)$. Also by Claim 3, $[C_{\overline{P}}(R), K^*] = 0$. As $Z(P) \leq C_G(\overline{P})$ we have $|G^*| < |G|$ so we may apply induction, with $\overline{P}$ in the role of V. Note that $G^* \neq K^*$ since $[G, R]$ is nontrivial on $\overline{P}$. As K^* is nilpotent, we deduce that

$$2r - 1 \text{ is a power of a prime } q,$$
$$[G^*, R] \text{ is a special } q\text{-group},$$
$$[[G^*, R]', RG^*] = 1 \text{ and } |K^*| = 2.$$

Now RG^* is faithful and irreducible on $\overline{P}$ so $O_p(RG^*) = 1$ and then $q \neq p$. We deduce that (6.4) holds. Let $T = [Z(C_{[P,R]}(K)), R]$ so

$$T \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Note that $C_T(R) = 1$ so as $Z(P) \leq Z(RG)$ we have $\overline{T} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $C_{\overline{T}}(R) = 0$. Note that T is $C_G(R)$-invariant since $K \trianglelefteq C_G(R)$. Then $\overline{T}$ is $C_{G^*}(R)$-invariant. Since $(|R|, |C_{G^*}(R)|) = 1$, the structure of Aut$(\overline{T})$ implies $C_{G^*}(R)$ is trivial on $\overline{T}$. But $1 \neq [G^*, R]' \leq Z(RG^*)$ and $C_{\overline{P}}([G^*, R]') = 0$ by irreducibility. This contradiction completes the proof of Lemma 6.9. □

It is now straightforward to complete the proof of Theorem A. Before doing so, we prove a simple lemma.

Lemma 6.10. *Suppose the group R acts coprimely on the p-group $P \neq 1$ and that V is a faithful completely reducible RP-module, possibly of mixed characteristic. Assume that $P = [P, R]$ and that $P/C_P(U)$ is special with $(P/C_P(U))' = C_{P/C_P(U)}(R)$ whenever U is an irreducible submodule of V. Then P is special and $P' = C_P(R)$.*

Proof. Let U be an irreducible submodule of V. Now $\Phi(P)$ maps into $\Phi(P/C_P(U)) = Z(P/C_P(U))$ whence $[\Phi(P), P] \leq C_P(U)$. Moreover $Z(P)$ maps into $Z(P/C_P(U)) = C_{P/C_P(U)}(R)$ so $[Z(P), R] \leq C_P(U)$. Since V is faithful and completely reducible we obtain

$$P' \leq \Phi(P) \leq Z(P) \leq C_P(R).$$

Now $P = [P, R]$ so Coprime Action implies $C_P(R) \leq P'$, completing the proof. □

Proof of Theorem A. Since L is the smallest subnormal subgroup of G that contains K it follows that L is R-invariant and that $L = \langle K^L \rangle$. Set $D = [L, R]$. By Coprime Action, $L =$

$C_L(R)D$ so as $K \trianglelefteq C_G(R)$ we obtain $L = \langle K^D \rangle = K[D,K]$. Then $D = [L,R] \leq [D,K]$. To summarize:

$$D = [D,K] = [D,R] \text{ and } L = KD.$$

Now $V = V_r \oplus V_{r'}$ where V_r and $V_{r'}$ are the sums of RG-modules whose fields of definition have characteristics r and not r respectively. Lemma 6.1 implies K is trivial on V_r. Then so is L because $L = \langle K^L \rangle$. The theorem describes the structure of L, so we may assume $V = V_{r'}$.

Since $D = [L,R] \trianglelefteq L \trianglelefteq\trianglelefteq G$ we have $D \trianglelefteq\trianglelefteq [G,R]$. Clifford's Theorem implies V_D is completely reducible. As $V = V_{r'}$, Maschke's Theorem implies V_{RD} is completely reducible. Let $\mathcal{V}$ be the set of irreducible submodules of V_{RD} on which D is nontrivial. Set

$$\begin{aligned}
&\mathcal{V}_0 = \{U \in \mathcal{V} \mid C_U(R) = 0\},\\
&\mathcal{V}_1 = \{U \in \mathcal{V} \mid C_U(R) \neq 0\},\\
&V_0 = \sum \mathcal{V}_0 \quad \text{and} \quad V_1 = \sum \mathcal{V}_1.
\end{aligned}$$

Then D is faithful on $V_0 \oplus V_1$. Now $[K,R] = 1$ so K normalizes RD and hence permutes $\mathcal{V}_0$ and $\mathcal{V}_1$. As $L = KD$ we see that V_0 and V_1 are RL-modules. Moreover

$$V = C_V(D) \oplus V_0 \oplus V_1.$$

Suppose $U \in \mathcal{V}_1$. Now $0 \neq C_U(R) \leq C_U(K)$ and K permutes $\mathcal{V}_1$. It follows that K normalizes U. Then, as $L = KD$, that U is an RL-module. Lemma 6.9 implies $D/C_D(U)$ is nilpotent of odd order. Complete reducibility implies that $D/C_D(V_1)$ is nilpotent of odd order. Now $C_{V_0}(R) = 0$ so Theorem 5.1 implies $D/C_D(V_0)$ is a 2-group. It follows that $D = C_D(V_1) \times C_D(V_0)$, that $C_D(V_1)$ is a 2-group and that $C_D(V_0)$ has odd order.

Recall the definitions of S and P in the statement of Theorem A. Then $S = C_D(V_1)$, $P = C_P(V_0)$, $V_0 = [V_0,S] = [V,S]$, $V_1 = [V_1,P] = [V,P]$ and

$$V = C_V(D) \oplus [V,S] \oplus [V,P].$$

Suppose that $S \neq 1$. Since $D = [D,K] = [D,R]$ we have $S = [S,K] = [S,R]$. Note that S is faithful on V_0. Now $C_{V_0}(R) = 0$ so conclusion (a) is a restatement of Theorem 5.1 except for the assertion that $K/O(K)$ is not a 2-group. Recall that $K \trianglelefteq C_G(R)$ so $O(K) \trianglelefteq O(C_G(R))$. Also $D \trianglelefteq L$ so $C_S(R) = S' \leq O_2(C_L(R))$. Then $O(K) \leq C_K(S') = C_K(S)$. As $S = [S,K] \neq 1$ and S is a 2-group it follows that $K/C_K(S)$ is not a 2-group. The proof of (a) is complete.

Suppose that $P \neq 1$. Again, $P = [P,K] = [P,R]$ and P is faithful on V_1. Then $\mathcal{V}_1 \neq \emptyset$. We have already seen that every member of $\mathcal{V}_1$ is an RL-module and hence an RKP-module.

Let $U \in \mathcal{V}_1$. Lemma 6.9 implies there is a prime p and $m \in \mathbb{N}$ such that $r = (1/2)(p^m + 1)$, $P/C_P(U) \cong p^{1+2m}$, $[(P/C_P(U))', RK] = 1$ and $K/C_K(P/C_P(U)) \cong \mathbb{Z}_2$. Note that p and m are uniquely determined. Complete reducibility and Lemma 6.10 imply that P is a special p-group and $P' = C_P(R)$.

Let K_0 be the smallest normal subgroup of K such that K/K_0 is an elementary abelian 2-group. Then $K_0 \leq C_K(P/C_P(U))$, so $[P,K_0] \leq C_P(U)$. Complete reducibility forces $[P,K_0] = 1$, whence $K/C_K(P)$ is an elementary abelian 2-group. Now $P = [P,K]$ so applying Lemma 6.10, with a Sylow 2-subgroup of K in the role of R, we obtain $P' = C_P(K)$. Then (b) holds.

Finally, we have seen that $[O(K),S] = 1$. Now $O(K) \leq K_0$ so $[O(K),P] = 1$. Since $L = K(S \times P) \trianglelefteq\trianglelefteq G$ we have $O(K) \trianglelefteq\trianglelefteq G$. This concludes the proof of Theorem A. $\square$

7. *The corollaries*

The following is useful when translating results about modules into results about groups. It is a variation of a well known result of Gaschütz.

LEMMA 7.1. *Let X be a group and $Y \trianglelefteq X$. Set*

$$V = F(Y)/Y \cap \Phi(X).$$

(a) *V is a completely reducible X-module, possibly of mixed characteristic.*
(b) *$V = F(Y/Y \cap \Phi(X))$.*
(c) *If Y is soluble then $C_Y(V) = F(Y)$.*

Proof. Recall that $\Phi(X)$ is nilpotent and that $\Phi(F(X)) \leq \Phi(X)$. Then $Y \cap \Phi(X) \leq F(Y)$. Set

$$\overline{X} = X/Y \cap \Phi(X).$$

so $V = \overline{F(Y)}$.

(a). Now $F(Y) \leq F(X)$ so $\Phi(F(Y)) \leq \Phi(F(X)) \leq \Phi(X)$, hence V is a direct product of elementary abelian groups. Suppose $\overline{U}$ is an irreducible submodule of V. Let U be the inverse image of $\overline{U}$ and let M be a maximal subgroup of X with $U \not\leq M$. Then $X = UM$ and $\overline{M}$ is a complement to $\overline{U}$ in $\overline{V}$.

(b). Let p be a prime. Clearly $\overline{O_p(Y)} \leq O_p(\overline{Y})$. Let K be the inverse image of $O_p(\overline{Y})$ in X and choose $P \in \mathrm{Syl}_p(K)$. Then $(Y \cap \Phi(X))P = K \trianglelefteq X$ so $(Y \cap \Phi(X))N_X(P) = X$, whence $N_X(P) = X$ and $P \leq O_p(Y)$. We deduce that $\overline{O_p(Y)} = O_p(\overline{Y})$ and the result follows.

(c). Since $V = \overline{F(Y)}$ is abelian we have $F(Y) \leq C_Y(V)$. Now $\overline{Y}$ is soluble so $C_{\overline{Y}}(F(\overline{Y})) \leq F(\overline{Y})$. Apply (b). □

Proof of Corollary B. Note that (a) and (b) follow from (c). We may assume that $O_q(G) = 1$. Set

$$X = RG, V = F(G)/G \cap \Phi(X), \overline{G} = G/F(G) \text{ and } K = O_q(C_G(R)).$$

Lemma 7.1 implies that V is a completely reducible $R\overline{G}$-module and that $V_{\overline{G}}$ is faithful. As $[\overline{G}, R] \trianglelefteq R\overline{G}$, Clifford's Theorem implies $V_{[\overline{G},R]}$ is completely reducible. Now $[C_{F(G)}(R), K] \leq F(G) \cap K = 1$ because $O_q(G) = 1$. Coprime Action implies $[C_V(R), K] = 0$, whence $\overline{K} \leq \ker (C_{\overline{G}}(R)$ on $C_V(R))$.

Let $\overline{L}$ be the subnormal closure of $\overline{K}$ in $\overline{G}$. Let $\overline{S}$ and $\overline{P}$ have the meanings as defined in the conclusion of Theorem A, so $\overline{L} = \overline{K}(\overline{S} \times \overline{P})$. Since $\overline{K}$ is a q-group, Theorem A implies $\overline{S} = 1$, so $\overline{L} = \overline{KP}$.

Suppose $\overline{P} = 1$. Then $\overline{K} = \overline{L} \trianglelefteq\trianglelefteq \overline{G}$ whence $\overline{K} \leq O_q(\overline{G})$. Since $\overline{G} = G/F(G)$, this gives $K \leq O_{F,q}(G)$ and (c) holds. Suppose $\overline{P} \neq 1$. Theorem A implies $q = 2$, $2r - 1$ is a power of a prime p and $\overline{P}$ is a p-group. As $\overline{PK} = \overline{L} \trianglelefteq\trianglelefteq \overline{G}$, this implies $\overline{L} \leq O_{p,q}(\overline{G})$. Then $K \leq O_{F,p,q}(G)$ and once again (c) holds. □

Proof of Corollary C. We work in the semidirect product $X = RKG$, so $K \cap G = 1$. Set $V = F(G)/G \cap \Phi(X)$. Lemma 7.1 implies V is a completely reducible X-module and $C_G(V) = F(G)$. Set $\overline{KG} = KG/C_G(V)$. Then V is an $R\overline{KG}$-module. Using Clifford's Theorem, $V_{[\overline{G},R]}$ is faithful and completely reducible. Coprime Action implies $[C_V(R), \overline{K}] = 0$ and $[C_{\overline{G}}(R), \overline{K}] = 1$. Hence

$$\overline{K} \leq \ker (C_{\overline{KG}}(R) \text{ on } C_V(R)) \text{ and } \overline{K} \trianglelefteq C_{\overline{KG}}(R).$$

Let $\overline{L}$ be the subnormal closure of $\overline{K}$ in $\overline{KG}$, so $\overline{L} = \overline{K}[\overline{G}, \overline{K}; \infty]$.

Theorem A, with $\overline{KG}$ in the role of G, implies $\overline{L} = \overline{K}[\overline{L}, R]$ and $[\overline{L}, R] = \overline{S} \times \overline{P}$ with $\overline{S}$ a 2-group and $\overline{P}$ a $2'$-group. Now $[C_{\overline{G}}(R), \overline{K}] = 1$ so $[C_{\overline{S}}(R), \overline{K}] = 1$. Suppose $\overline{S} \neq 1$. Theorem A(a) implies $\overline{S} = [\overline{S}, \overline{K}]$, $C_{\overline{S}}(R) = \overline{S}'$ and $C_{\overline{K}}(\overline{S}') = C_{\overline{K}}(\overline{S})$. Then $[\overline{S}, \overline{K}] = 1$, a contradiction. Thus

$\overline{S} = 1$. By Theorem A, $\overline{P} = [\overline{P}, K]$. As $\overline{K} \cap \overline{G} = 1$ we obtain

$$[\overline{G}, \overline{K}; \infty] = \overline{P}.$$

(a). Theorem A(b) implies $\overline{P} \leq F(\overline{G})$. Now $\overline{G} = G/F(G)$ whence $[G, K; \infty] \leq F_2(G)$ and K acts nilpotently on $G/F_2(G)$. Theorem 1.2 implies that $[G, K]$ is a nilpotent normal subgroup of G modulo $F_2(G)$. Consequently $[G, K] \leq F_3(G)$ and K is trivial on $G/F_3(G)$.

(b). Theorem A(b) implies $[\overline{P}, \overline{K}^{\,2}] = 1$. Then $[\overline{G}, \overline{K}^{\,2}; \infty] = 1$ and K^2 acts nilpotently on $G/F(G)$. As previously it follows that K^2 is trivial on $G/F_2(G)$.

(c). Then $\overline{P} \neq 1$. Apply Theorem A(b).

(d). By (b), K^{*2} acts nilpotently on $G/F(G)$. Theorem 1.2 implies K^{*2} is nilpotent and the conclusions follow. □

Lemma 7.2. *Let X be a group and $Y \trianglelefteq X$. Assume Y is soluble but not nilpotent and that Y/N is nilpotent whenever $1 \neq N \leq Y$ with $N \trianglelefteq X$. Then there exists a prime p such that $F(Y)$ is an elementary abelian p-group, $F(Y)$ is irreducible as an X-module and $Y/F(Y)$ is a nilpotent p'-group.*

Proof. Set $V = F(Y)/Y \cap \Phi(X)$. Lemma 7.1 implies V is a completely reducible X-module, $V = F(Y/Y \cap \Phi(X))$ and $C_Y(V) = F(Y)$. If $Y \cap \Phi(X) \neq 1$ then $Y/Y \cap \Phi(X)$ is nilpotent and then $Y = F(Y)$, a contradiction. Thus $Y \cap \Phi(X) = 1$ and $V = F(Y)$.

Suppose V_1 and V_2 are distinct irreducible submodules of V. Then Y embeds into the nilpotent group $Y/V_1 \times Y/V_2$, a contradiction. We deduce that V is an irreducible X-module and hence an elementary abelian p-group for some prime p. Then $V = O_p(Y)$ and $O_p(Y/F(Y)) = 1$. Since $Y/F(Y)$ is nilpotent, it follows that it is a p'-group. □

Proof of Corollary D. We may assume $C_K(G) = 1$. Coprime Action implies $[G, K] = [G, R]$. Since $[G, K] \trianglelefteq G$, the conclusion is equivalent to the assertion that $[G, K]$ is nilpotent.

Assume the corollary to be false and let G be a minimal counterexample. By Coprime Action $[G, K, K] = [G, K]$, whence $G = [G, K]$. If $1 \neq N \leq RKG$ with $N \trianglelefteq G$ then Coprime Action implies $C_{G/N}(R) = C_{G/N}(K)$ and then the minimality of G implies that G/N is nilpotent. Set $V = F(G)$. Lemma 7.2, with RKG in the role of X, implies that V is elementary abelian and irreducible as an X-module. Since G is soluble we have $C_G(V) = V$. Set $\overline{G} = G/V$.

We apply Theorem A with $K\overline{G}$ in the role of G. Now $C_V(R) = C_V(K)$ so $K \leq \ker(C_{K\overline{G}}(R) \text{ on } C_V(R))$. By Coprime Action, $C_{\overline{G}}(R) = C_{\overline{G}}(K)$ so $K \trianglelefteq C_{K\overline{G}}(R)$. Since $\overline{G} = [\overline{G}, K]$ it follows that $K\overline{G}$ is the subnormal closure of K in $K\overline{G}$. Also $[K\overline{G}, R] = [\overline{G}, R] = \overline{G} \neq 1$.

Suppose $C_V(R) = 0$. Theorem A implies $\overline{G}$ is a special 2-group, $\overline{G}\,' = C_{\overline{G}}(R)$ and $C_{\overline{K}}(\overline{G}\,') = C_{\overline{K}}(\overline{G})$. But $C_{\overline{G}}(R) = C_{\overline{G}}(\overline{K})$ so $\overline{K} = C_{\overline{K}}(\overline{G}\,')$ and then $[\overline{G}, \overline{K}] = 1$, a contradiction. Thus $C_V(R) \neq 0$. In the notation of Theorem A, $\overline{S} = 1$ and $\overline{G} = [R\overline{G}, R] = \overline{P}$. Then Theorem A(b) is applicable.

Let $K_0 = C_K(\overline{G})$. Now $K_0 \trianglelefteq RK\overline{G}$ so $C_V(K_0)$ is an $RK\overline{G}$-submodule of V. As

$$0 \neq C_V(R) \leq C_V(K) \leq C_V(K_0)$$

we have $C_V(K_0) = V$. Then $[G, K_0, K_0] \leq [C_G(V), K_0] = [V, K_0] = 1$ and Coprime Action forces $[G, K_0] = 1$. Since $C_K(G) = 1$, we deduce that

$$C_K(\overline{G}) = 1.$$

Now V is irreducible so Theorem A(b) implies RK induces a cyclic group of order $2r$ on $\overline{G}/\overline{G}\,'$. Then $|K| = 2$. In particular, K has prime order. The preceding arguments, with the roles of R and K interchanged, imply that $|R| = 2$. This contradicts the hypothesis that $(|R|, |K|) = 1$ and completes the proof. □

LEMMA 7.3. *Suppose R acts coprimely on the soluble group G. Let p be a prime and suppose R centralizes a Sylow p-subgroup of G. Then $[G,R] \leq O_{p'}(G)$.*

Proof. Set $\overline{G} = G/O_{p'}(G)$. Then $F(\overline{G}) = O_p(\overline{G})$ so $C_{\overline{G}}(O_p(\overline{G})) \leq O_p(\overline{G})$. Now $[O_p(\overline{G}), R] = 1$ so Coprime Action implies $[\overline{G}, R] = 1$. □

Proof of Corollary E. Let $Q \in \mathrm{Syl}_p(C_G(R))$. Then $P \leq O_p(C_G(R)) \leq Q$ whence $C_G(Q) \leq C_G(P) \leq C_G(R)$. Set $N = N_G(Q)$. Now $R \leq C_{RG}(Q) \trianglelefteq N_{RG}(Q)$ whence $[N,R] \leq C_G(Q) \leq C_G(R)$. By Coprime Action, $N = C_N(R)[N,R] \leq C_G(R)$. Since $Q \in \mathrm{Syl}_p(C_G(R))$ it follows that $Q \in \mathrm{Syl}_p(G)$. Lemma 7.3 implies $[G,R] \leq O_{p'}(G)$.

Let $H = O_{p'}(G)$. Now $[C_H(R), P] \leq O_{p'}(G) \cap O_p(C_G(R)) = 1$ so $C_H(R) \leq C_H(P)$. By hypothesis $[C_H(P), R] = 1$, whence $C_H(R) = C_H(P)$. Corollary D implies $[H,R] \leq F(H) \leq F(G)$. By Coprime Action, $[G,R] = [H,R]$, completing the proof. □

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Paul Flavell
The School of Mathematics
University of Birmingham
Birmingham B15 2TT
England

P.J.Flavell@bham.ac.uk