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Abstract

We identify yet another category equivalent to that of Kleene–Kreisel continuous functionals. Reasoning constructively and predicatively, all functions from the Cantor space to the natural numbers are uniformly continuous in this category. We do not need to assume Brouwerian continuity axioms to prove this, but, if we do, then we can show that the full type hierarchy is equivalent to our manifestation of the continuous functionals. We construct this manifestation within a category of concrete sheaves, called C-spaces, which form a locally cartesian closed category, and hence can be used to model system T and dependent types. We show that this category has a *fan functional* and validates the uniform-continuity principle in these theories. Our development is within informal constructive mathematics, along the lines of Bishop mathematics. However, in order to extract concrete computational content from our constructions, we formalized it in intensional Martin-Löf type theory, in Agda notation, and we discuss the main technical aspects of this at the end of the paper.

Keywords: Constructive mathematics, topological models, uniform continuity, fan functional, intuitionistic type theory, topos theory, sheaves, HA^ω, Gödel's system T, Kleene–Kreisel spaces, continuous functionals.

1 Introduction

In a cartesian closed category with a natural numbers object $\mathbb{N}$, define the *simple objects* to be the least collection containing $\mathbb{N}$ and closed under products and exponentials (function spaces). The simple objects of any such category give an interpretation of the simply typed lambda calculus and higher-type primitive recursion (the term language of Gödel's system T). The Kleene–Kreisel *continuous functionals*, or *countable functionals* [34, 30], form a category equivalent to the full subcategory on the simple objects of any of the following categories, among others: (1) compactly generated topological spaces [34, 16], (2) sequential topological spaces [16], (3) Simpson and Schröder's QCB spaces [3, 16], (4) Kuratowski limit spaces [23], (5) filter spaces [23], (6) Scott's equilogical spaces [4], (7) Johnstone's *topological topos* [25]. See Normann [35] and Longley [28, 29] for the relevance of Kleene–Kreisel spaces in the theory of higher-type computation. Counter-examples include Hyland's *effective topos* [24] and the hereditary effective operations (HEO) [28], which give a second simple-type hierarchy, not discussed in this paper (see [28] for a discussion). A third type hierarchy, discussed here in connection with the continuous functionals, is the *full type hierarchy*, which is the full subcategory on the simple objects of the category of sets [34].

We work with a category of sheaves, analogous to the topological topos, and with a full subcategory of concrete sheaves [2], here called C-*spaces*, analogous to the limit spaces (Section 2). The C-spaces can be described as sets equipped with a suitable continuity structure, and their natural transformations can be regarded as continuous maps. The main contributions of this work are summarized as follows:

1. The simple C-spaces form a category equivalent to that of Kleene–Kreisel continuous functionals (Section 3.1).

   The proof here is non-constructive (as are the proofs of the above equivalences). But we claim that the C-spaces form a good substitute of the above categories of spaces for the purposes of constructive reasoning.

2. If we assume the Brouwerian axiom that all set-theoretic functions $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous, then we can show constructively that the full type hierarchy is equivalent to the Kleene–Kreisel continuous hierarchy within C-spaces (Section 3.2).

3. Without assuming Brouwerian axioms, we show constructively that the category of C-spaces has a *fan functional* $(\mathbf{2}^{\mathbb{N}} \to \mathbb{N}) \to \mathbb{N}$ that continuously calculates moduli of uniform continuity of maps $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ (Section 4.1).

4. C-Spaces give a model of system T with a uniform-continuity principle (Section 4.2), expressed as the skolemization of

   $$\forall f \colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}.\ \exists m \in \mathbb{N}.\ \forall \alpha, \beta \in \mathbf{2}^{\mathbb{N}}.\ \alpha =_m \beta \implies f\alpha = f\beta,$$

   where $\alpha =_m \beta$ stands for $\forall i < m.\ \alpha_i = \beta_i$, with the aid of a fan-functional constant.

5. C-Spaces give a model of dependent types with a uniform-continuity principle, expressed as a type via the Curry–Howard interpretation (Section 4.3):

   $$\Pi(f \colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}).\ \Sigma(m \colon \mathbb{N}).\ \Pi(\alpha, \beta \colon \mathbf{2}^{\mathbb{N}}).\ \alpha =_m \beta \to f\alpha = f\beta$$

6. We give a constructive treatment of C-spaces (Section 2) suitable for development in a predicative intuitionistic type theory in the style of Martin–Löf [32], which we formalized in Agda notation [33, 8] for concrete computational purposes, and whose essential aspects are discussed in Section 5.

We stress, however, that in this paper we deliberately reason informally, along the lines of Bishop mathematics [6].

Among the above, (3) and (4), and part of (6), appeared in the preliminary conference version of this paper [40]. The other contributions, (1), (2), (5), regarding Kleene–Kreisel spaces and dependent types, and part of (6), regarding the formalization in predicative intuitionistic type theory, are new as far as we are aware, but of course there are connections with related work discussed below.

As mentioned above, our sheaf topos is closely related to Johnstone's topological topos. To build the topological topos, one starts with the monoid of continuous endomaps of the one-point compactification of the discrete natural numbers, and then takes sheaves for the canonical topology of this monoid considered as a category. The category of sequential topological spaces is fully embedded in the topological topos. The concrete sheaves are precisely the Kuratowski limit spaces, called *subsequential spaces* by Johnstone, because they are the subobjects of the sequential topological spaces. Working non-constructively, one can show that the topological topos has a fan functional and that it interprets uniform-continuity principles for both simple and dependent types.

The point of our contribution is that we can achieve this by working constructively instead, without assuming Brouwerian axioms, and remaining predicative. We replace Johnstone's monoid by that of uniformly continuous endofunctions of the Cantor space $\mathbf{2}^{\mathbb{N}}$, and we replace the canonical topology by a smaller, subcanonical, one, suitable for predicative, constructive reasoning. The category of concrete sheaves is equivalent to that of C-spaces, where a C-space is a set equipped with a collection of maps from the set $\mathbf{2}^{\mathbb{N}}$, called *probes*, subject to suitable axioms, corresponding to the axioms for limit spaces. A natural transformation, or continuous map, of C-spaces amounts to a function such that the composition with any probe is again a probe. Then, reasoning non-constructively, the limit spaces form a (full) reflective subcategory of that of C-spaces, and also an exponential ideal, which gives the connection with the Kleene–Kreisel continuous functionals. But, reasoning constructively, all maps $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ in C-**Space** are uniformly continuous, as discussed above, and this follows from the Yoneda Lemma. Moreover, if all set-theoretical functions $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous, then the discrete and indiscrete C-structures on $\mathbb{N}$ coincide, and because the indiscrete spaces form an exponential ideal, all simple C-spaces are indiscrete, and hence the simple C-spaces form a category equivalent to the full type hierarchy. This last result is analogous to Fourman's reflection theorem [20].

The work of van der Hoeven and Moerdijk [39] considers the monoid of continuous endomaps of the Baire space $\mathbb{N}^{\mathbb{N}}$ instead, inspired by [19, 18], to get a topos model of choice sequences, continuity principles and Bar induction. But this poses some problems when we consider the Curry–Howard interpretation. Consider, for instance, the continuity principle

$$\forall f \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}.\ \forall \alpha \in \mathbb{N}^{\mathbb{N}}.\ \exists m \in \mathbb{N}.\ \forall \beta \in \mathbb{N}^{\mathbb{N}}.\ \alpha =_m \beta \implies f\alpha = f\beta,$$

whose Curry–Howard interpretation is

$$\Pi(f\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\ \Pi(\alpha\colon \mathbb{N}^{\mathbb{N}}).\ \Sigma(m\colon \mathbb{N}).\ \Pi(\beta\colon \mathbb{N}^{\mathbb{N}}).\ \alpha =_m \beta \to f\alpha = f\beta.$$

But this type is always empty, even in intensional type theory [17]. One can, however, simultaneously have the type-theoretic and topos-theoretic quantifiers, as in homotopy type theory (HoTT) [38], by constructing the topos existential quantifier as the propositional truncation of the type-theoretic sum, and then formulate the continuity principle as

$$\Pi(f\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\ \Pi(\alpha\colon \mathbb{N}^{\mathbb{N}}).\ \|\Sigma(m\colon \mathbb{N}).\ \Pi(\beta\colon \mathbb{N}^{\mathbb{N}}).\ \alpha =_m \beta \to f\alpha = f\beta\|.$$

From this one cannot get a modulus-of-continuity function, because this time this would need topos-theoretic choice, which fails in this sheaf model. We plan to investigate this model from a type-theoretic point of view in future work, using such ideas from HoTT. In our category of C-spaces, the maps $f\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are precisely those that are "uniformly continuous on compact subsets of $\mathbb{N}^{\mathbb{N}}$", with uniform continuity expressed with the Σ quantifier rather than its propositional truncation, in the sense that for every uniformly continuous $p\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, the composite $f \circ p\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ is also uniformly continuous. With a non-constructive argument, this is of course equivalent to the continuity of f.

Our work is also related to Coquand and Jaber's forcing model [11, 12], which instead uses the semilattice of finite binary sequences under the prefix order as the underlying category of the site, modelling the idea of a generic infinite binary sequence. They iterate their construction in order to be able to model the fan functional, and our model can be regarded as accomplishing this iteration directly in a single step (personal communication with Coquand).

2 A variation of Johnstone's topological topos

In this section we define our sheaf topos. We also look at the full subcategory of concrete sheaves, and show how they can be regarded as spaces, and in particular we discuss discrete spaces and the natural numbers object.

2.1 Sheaves and natural transformations

Let C be the monoid of *uniformly continuous* endomaps of the Cantor space $\mathbf{2}^{\mathbb{N}}$, that is, the functions $t\colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ such that

$$\forall m \in \mathbb{N}.\ \exists n \in \mathbb{N}.\ \forall \alpha, \beta \in \mathbf{2}^{\mathbb{N}}.\ \alpha =_n \beta \implies t\alpha =_m t\beta.$$

We write 1 for the identity map of $\mathbf{2}^{\mathbb{N}}$ as it is the identity element of C. Notice that any continuous function $\mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ is uniformly continuous, assuming classical logic or the Fan Theorem. Because we do not assume such principles, we need to explicitly require uniform continuity in the definition of the monoid C.

Our *site* is the monoid C equipped with a countable *coverage* $\mathcal{J}$ consisting of the finite covering families

$$\{\mathrm{cons}_s \colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}} \mid s \in \mathbf{2}^n\}$$

for every $n \in \mathbb{N}$, where $\mathbf{2}^n$ is the set of binary sequences of length n and $\mathrm{cons}_s \colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ is the concatenation map:

$$\mathrm{cons}_s(\alpha) = s\alpha.$$

It is easy to verify that, for any $n \in \mathbb{N}$ and for any $s \in \mathbf{2}^n$, the map cons_s is uniformly continuous and thus an element of the monoid C.

The *coverage axiom* specialized to our situation amounts to saying that for all $t \in \mathrm{C}$,

$$\forall m \in \mathbb{N}.\ \exists n \in \mathbb{N}.\ \forall s \in \mathbf{2}^n.\ \exists t' \in \mathrm{C}.\ \exists s' \in \mathbf{2}^m.\ t \circ \mathrm{cons}_s = \mathrm{cons}_{s'} \circ t'. \qquad (\dagger)$$

It is routine to show that:

Lemma 2.1. *A map $t \colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ satisfies the coverage axiom (†) if and only if it is uniformly continuous.*

Thus, not only does the coverage axiom hold, but also it amounts to the fact that the elements of the monoid C are the uniformly continuous functions. By virtue of this view, we call $(\mathrm{C}, \mathcal{J})$ the *uniform-continuity site*.

Recall that presheaves on a one-object category, *i.e.* a monoid, can be formulated in terms of monoid actions [31, §I.1]: A *presheaf* on the monoid C amounts to a set P with an *action*

$$((p, t) \mapsto p \cdot t) \colon P \times \mathrm{C} \to P$$

such that for all $p \in P$ and $t, u \in \mathrm{C}$

$$p \cdot 1 = p, \qquad p \cdot (t \circ u) = (p \cdot t) \cdot u.$$

A *natural transformation* of presheaves $(P, \cdot)$ and $(Q, \cdot)$ amounts to a function $\phi \colon P \to Q$ that preserves the action, *i.e.*

$$\phi(p \cdot t) = (\phi\ p) \cdot t.$$

Because the maps in each covering family have disjoint images, we do not need to consider the compatibility condition in the definition of sheaf:

Lemma 2.2. *A presheaf $(P, \cdot)$ is a* sheaf *over $(\mathrm{C}, \mathcal{J})$ if and only if for every $n \in \mathbb{N}$ and every family $\{p_s \in P \mid s \in \mathbf{2}^n\}$, there is a unique* amalgamation *$p \in P$ such that, for all $s \in \mathbf{2}^n$,*

$$p \cdot \mathrm{cons}_s = p_s.$$

Notice also that, by induction, it is enough to consider the case $n = 1$:

Lemma 2.3. *A presheaf $(P, \cdot)$ is a* sheaf *over $(\mathrm{C}, \mathcal{J})$ if and only if for any $p_0, p_1 \in P$, there is a unique $p \in P$ such that*

$$p \cdot \mathrm{cons}_0 = p_0 \qquad \textit{and} \qquad p \cdot \mathrm{cons}_1 = p_1.$$

Our topos is the category $\mathbf{Shv}(\mathrm{C}, \mathcal{J})$ of sheaves over the uniform-continuity site $(\mathrm{C}, \mathcal{J})$.

2.2 C-Spaces and continuous maps

An important example of a sheaf is the monoid C itself with function composition as the action. Given $t_0, t_1 \in \mathrm{C}$, the amalgamation $t\colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ is simply

$$t(\alpha) = t_{\alpha_0}(\lambda n.\alpha_{n+1}).$$

We say a presheaf is *concrete* if its action is function composition. Then all the elements in a concrete presheaf $(P, \circ)$ must be maps from $\mathbf{2}^{\mathbb{N}}$ to some set X. Concrete sheaves admit a more concrete description as the set X with the additional structure given by the maps in P. We denote the full subcategory of concrete sheaves by $\mathbf{CShv}(\mathrm{C}, \mathcal{J})$.

Concrete sheaves can be regarded as spaces, and their natural transformations as continuous maps. More precisely, they are analogous to Spanier's quasi-topological spaces [37], which have the category of topological spaces and continuous maps as a *full* subcategory. One advantage of quasi-topological spaces over topological spaces, which is the main reason for Spanier's introduction of the notion of quasi-space, is that continuous maps of quasi-spaces form a cartesian closed category. This category serves as a model of system T and HA^ω that validates the uniform-continuity principle, assuming classical logic in the meta-language. Our concrete sheaves can be seen as analogues of quasi-topological spaces, admitting a constructive treatment.

A *quasi-topology* on a set X assigns to each compact Hausdorff space K a set $P(K, X)$ of functions $K \to X$ such that:

(1) All constant maps are in $P(K, X)$.

(2) If $t\colon K' \to K$ is continuous and $p \in P(K, X)$, then $p \circ t \in P(K', X)$.

(3) If $\{t_i\colon K_i \to K \mid i \in I\}$ is a finite, jointly surjective family and $p\colon K \to X$ is a map with $p \circ t_i \in P(K_i, X)$ for every $i \in I$, then $p \in P(K, X)$.

A *quasi-topological space* is a set endowed with a quasi-topology, and a *continuous map* of quasi-spaces (X, P) and (Y, Q) is a function $f\colon X \to Y$ such that $f \circ p \in Q(K, Y)$ whenever $p \in P(K, X)$. For example, every topological space X is a quasi-topological space with the quasi-topology P such that $P(K, X)$ is the set of continuous maps $K \to X$, and this construction gives the full embedding of topological spaces into quasi-topological spaces.

This definition can be modified by considering just one compact Hausdorff space, the Cantor space, rather than all compact Hausdorff spaces, and by restricting the jointly surjective finite families of continuous maps to the covering families $\{\mathrm{cons}_s\}$ considered in the previous section. We call the resulting objects C-*spaces*.

Definition 2.4. *A* C-space *is a set X equipped with a* C-topology P*, i.e. a collection of maps $\mathbf{2}^{\mathbb{N}} \to X$, called* probes*, satisfying the following conditions, called the* probe axioms*:*

(1) *All constant maps are in P.*

(2) (Presheaf condition) *If* $p \in P$ *and* $t \in \mathrm{C}$*, then* $p \circ t \in P$.

(3) (Sheaf condition) *For any* $n \in \mathbb{N}$ *and any family* $\{p_s \in P \mid s \in \mathbf{2}^n\}$*, the unique map* $p\colon \mathbf{2}^{\mathbb{N}} \to X$ *defined by* $p(s\alpha) = p_s(\alpha)$ *is in* P.

A continuous map *of* C*-spaces* (X, P) *and* (Y, Q) *is a map* $f\colon X \to Y$ *with* $f \circ p \in Q$ *whenever* $p \in P$*. We write* C-**Space** *for the category of* C*-spaces and continuous maps.*

Notice that the sheaf condition is equivalent to

($3'$) *For any* $p_0, p_1 \in P$*, the map* $p\colon \mathbf{2}^{\mathbb{N}} \to X$ *defined by* $p(i\alpha) = p_i(\alpha)$ *is in* P.

and

($3''$) *If* $p\colon \mathbf{2}^{\mathbb{N}} \to X$ *is a map such that there exists* $n \in \mathbb{N}$ *with* $p \circ \mathrm{cons}_s \in P$ *for all* $s \in \mathbf{2}^n$*, then* $p \in P$.

($3'$) is equivalent to (3) by induction on n, and is more convenient to work with when verifying that a given set is a C-space. And ($3''$) is the uncurried result of (3), and is more convenient to use if a given set is already known to be a C-space.

The idea is that we "topologize" the set X by choosing a designated set P of maps $\mathbf{2}^{\mathbb{N}} \to X$ that we want, and hence declare, to be continuous. For example, if X already has some form of topology, *e.g.* a metric, we can take P to be the set of continuous functions $\mathbf{2}^{\mathbb{N}} \to X$ with respect to this topology and the natural topology of the Cantor space. Of course we have to make sure the sheaf condition is satisfied.

As mentioned earlier, C-spaces provide a more concrete description of concrete sheaves in the following sense. Given a C-space (X, P), the C-topology P together with function composition is a concrete sheaf. Conversely, if $(P, \circ)$ is a concrete sheaf, then all maps in P should have the same codomain.

Proposition 2.5. *The two categories* C-**Space** *and* $\mathbf{CShv}(\mathrm{C}, \mathcal{J})$ *are naturally equivalent.*

By virtue of this equivalence, C-**Space** can also be viewed as a full subcategory of $\mathbf{Shv}(\mathrm{C}, \mathcal{J})$. Moreover, C-spaces are closed under products and form an exponential ideal.

The underlying set of a space X is written $|X|$ and its set of probes is written $\mathrm{Probe}(X)$, but we we often write X to mean $|X|$ by the standard abuse of notation.

2.3 The (local) cartesian closed structure of C-Space

Here we explore the cartesian closed structure of the category C-**Space**, in order to model simple types (Section 4.2), as well as its local cartesian closed structure, in order to model dependent types (Section 4.3).

Theorem 2.6. *The category* C-**Space** *is cartesian closed.*

Proof. Any singleton set $\mathbf{1} = \{\star\}$ with the unique map $\mathbf{2}^{\mathbb{N}} \to \mathbf{1}$ as the only probe is a C-space as well as a terminal object in C-**Space**.

Given C-spaces (X, P) and (Y, Q), their *product* is the cartesian product $X \times Y$ equipped with the C-topology R defined by the condition that $r \colon \mathbf{2}^{\mathbb{N}} \to X \times Y$ is in R iff $\mathrm{pr}_1 \circ r \in P$ and $\mathrm{pr}_2 \circ r \in Q$, where pr_1 and pr_2 are the projections. We skip the routine verifications of probe axioms and the required universal property.

Given C-spaces (X, P) and (Y, Q), their *exponential* is the set Y^X of continuous maps $X \to Y$ equipped with the collection R of probes defined by the condition that $r \colon \mathbf{2}^{\mathbb{N}} \to Y^X$ is in R iff for any $t \in \mathrm{C}$ and $p \in P$ the map $\lambda\alpha.r(t\alpha)(p\alpha)$ is in Q. Again, we have to verify that the probe axioms are satisfied and that this has the universal property of an exponential in C-**Space**, which involves some subtleties regarding the coverage axiom. $\square$

Colimits of sheaves are generally constructed as the sheafifications of the ones in the category of presheaves (see [31, §III.6]). Here we present a direct construction of finite coproducts of C-spaces.

Theorem 2.7. *The category* C-**Space** *has finite coproducts.*

Proof. The empty set equipped with the empty C-topology is clearly a C-space and an *initial object* in C-**Space**.

Binary coproducts can be constructed as follows: given C-spaces (X, P) and (Y, Q), their *coproduct* is the disjoint union $X + Y$ equipped with the C-topology R defined by the condition that $r \colon \mathbf{2}^{\mathbb{N}} \to X + Y$ is in R iff there exists $n \in \mathbb{N}$ such that for all $s \in \mathbf{2}^n$ either there exists $p \in P$ with $r(\mathrm{cons}_s\alpha) = \mathrm{inl}(p\alpha)$ for all $\alpha \in \mathbf{2}^{\mathbb{N}}$ or there exists $q \in Q$ with $r(\mathrm{cons}_s\alpha) = \mathrm{inr}(q\alpha)$ for all $\alpha \in \mathbf{2}^{\mathbb{N}}$, where inl and inr are the injections. Here we verify only the sheaf condition (3′): Given $r_0, r_1 \in R$, we get n_0 and n_1 from their witnesses of being a probe on $X + Y$. For the map $r \colon \mathbf{2}^{\mathbb{N}} \to X + Y$ defined by $r(i\alpha) = r_i(\alpha)$, one can clearly see that the maximum of n_0 and n_1 is the desired n that makes $r \in R$. $\square$

The category C-**Space** has all pullbacks, which are constructed in the same way as in **Set**. An exponential in a slice category C-**Space**$/X$ is constructed in the same way as in the slice category **Set**$/X$, with a suitable construction of the C-topology on its domain. The proof is available in [2, Proposition 43], in the generality of concrete sheaves on concrete sites. Here we present the construction, but skip the verification as it is similar to the one for exponentials of C-spaces (which is presented in our formalization).

Theorem 2.8. *The category* C-**Space** *is locally cartesian closed.*

Proof. We skip the easy constructions and verifications of a terminal object and products in a slice category, but give the construction of exponentials.

Given a continuous map $f \colon X \to Y$ and an element $y \in Y$, the *fiber*

$$f^{-1}(y) = \{x \in X \mid f(x) = y\}$$

is a C-space, whose C-topology is inherited from X.

Given objects $X \xrightarrow{f} Y$ and $Z \xrightarrow{g} Y$ in C-**Space**$/Y$, we construct the exponential g^f as follows: The underlying set of the domain of g^f is defined by

$$\mathrm{dom}(g^f) = \{(y, \phi) \mid y \in Y,\ \phi \colon f^{-1}(y) \xrightarrow{\text{cts}} g^{-1}(y)\}.$$

The C-topology on $\mathrm{dom}(g^f)$ is defined by the condition that a map $r \colon \mathbf{2}^{\mathbb{N}} \to \mathrm{dom}(g^f)$ is a probe iff

(i) the composite $\mathrm{pr}_1 \circ r \colon \mathbf{2}^{\mathbb{N}} \to Y$ is a probe on Y, and

(ii) for any $t \in \mathrm{C}$ and $p \in \mathrm{Probe}(X)$ such that $\forall \alpha \in \mathbf{2}^{\mathbb{N}}.\ \mathrm{pr}_1(r(t\alpha)) = f(p\alpha)$, the map $\lambda\alpha.\mathrm{pr}_2(r(t\alpha))(p\alpha)$ is a probe on Z.

Verifying the sheaf condition for $\mathrm{dom}(g^f)$ is similar to the one for exponentials in C-**Space**. The exponential $g^f \colon \mathrm{dom}(g^f) \to Y$ is then defined to be the first projection. Condition (i) amounts to the continuity of g^f. And the idea of (ii) is that the composite

$$\mathbf{2}^{\mathbb{N}} \times_Y X \xrightarrow{r \times 1_X} \mathrm{dom}(g^f) \times_Y X \xrightarrow{\mathrm{ev}} Z$$

is continuous, where evaluation map ev applies the second component of $(y, \phi) \in \mathrm{dom}(g^f)$ to $x \in f^{-1}(y) \subseteq X$. ☐

2.4 Discrete C-spaces and natural numbers object

We say that a C-space X is *discrete* if for every C-space Y, all functions $X \to Y$ are continuous. A map $p \colon \mathbf{2}^{\mathbb{N}} \to X$ into a set X is called *locally constant* iff

$$\exists m \in \mathbb{N}.\ \forall \alpha, \beta \in \mathbf{2}^{\mathbb{N}}.\ \alpha =_m \beta \implies p\alpha = p\beta.$$

We call m the *modulus of local constancy* of p.

Lemma 2.9. *Let X be any set.*

1. *The locally constant functions $\mathbf{2}^{\mathbb{N}} \to X$ form a* C*-topology on X.*
2. *For any* C*-topology P on X, every locally constant map $\mathbf{2}^{\mathbb{N}} \to X$ is in P.*

Proof. (1) Let P be a collection of all locally constant maps into X. The first two probe axioms are obviously satisfied. We only verify the sheaf condition $(3')$: If p_0 and p_1 are locally constant with moduli m_0 and m_1, then the unique map $p \colon \mathbf{2}^{\mathbb{N}} \to X$ defined by $p(i\alpha) = p_i(\alpha)$ is locally constant with the modulus $\max(m_0, m_1) + 1$.

(2) Let $p \colon \mathbf{2}^{\mathbb{N}} \to X$ be locally constant and n be its modulus of local constancy. Then, for each $s \in \mathbf{2}^n$, the composite $p \circ \mathrm{cons}_s$ is constant and thus a probe on X. Using the sheaf condition, we know that p is a probe on X. ☐

In other words, the locally constant maps $\mathbf{2}^{\mathbb{N}} \to X$ form the finest C-topology on the set X, in the sense of the smallest collection of probes. Moreover:

Lemma 2.10. *A* C*-space is discrete if and only if the probes on it are precisely the locally constant functions.*

Proof. ($\Rightarrow$) Let (X, P) be a discrete C-space. According to the previous lemma, all locally constant maps $\mathbf{2}^{\mathbb{N}} \to X$ form a C-topology, say Q, on X. Because (X, P) is discrete, the map $(X, P) \to (X, Q)$ which is identity on points is continuous. By the definition of continuity, all elements in P are also in Q, *i.e.* are locally constant.

($\Leftarrow$) Let P be the collection of all locally constant functions into X. Given a C-space (Y, Q) and a map $f : X \to Y$, we show that f is continuous: if $p : \mathbf{2}^{\mathbb{N}} \to X$ is locally constant whose modulus is n, then, for each $s \in \mathbf{2}^n$, the composite $f \circ p \circ \mathrm{cons}_s$ is constant and thus a probe on Y. By the sheaf condition, the map $f \circ p$ is a probe on Y. $\square$

We thus refer to the collection of locally constant maps $\mathbf{2}^{\mathbb{N}} \to X$ as the *discrete* C-topology on X. In particular, when the set X is $\mathbf{2}$ or $\mathbb{N}$, the locally constant functions amount to the uniformly continuous functions. Hence we have a discrete two-point space $\mathbf{2}$ and a discrete space $\mathbb{N}$ of natural numbers, which play an important role in our model:

Theorem 2.11. *In the category* C-**Space***:*

1. *The discrete two-point space* $\mathbf{2}$ *is the coproduct of two copies of the terminal space* $\mathbf{1}$*.*
2. *The discrete space* $\mathbb{N}$ *of natural numbers is the natural numbers object.*

Proof. The universal properties of $\mathbf{2}$ and $\mathbb{N}$ can be constructed in the same way as in the category **Set**, because the unique maps g and h in the diagrams below are continuous by the discreteness of $\mathbf{2}$ and $\mathbb{N}$:

$$\begin{tikzcd} \mathbf{1} \arrow[r, "\mathrm{in}_0"] \arrow[dr, "g_0"'] & \mathbf{2} \arrow[d, dashed, "g"] & \mathbf{1} \arrow[l, "\mathrm{in}_1"'] \arrow[dl, "g_1"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{1} \arrow[r, "0"] \arrow[dr, "x"'] & \mathbb{N} \arrow[r, "\mathrm{suc}"] \arrow[d, dashed, "h"] & \mathbb{N} \arrow[d, dashed, "h"] \\ & X \arrow[r, "f"'] & X. \end{tikzcd}$$

$\square$

2.5 The Yoneda lemma

The monoid C can be regarded as a one-object category with the object $\mathbf{2}^{\mathbb{N}}$ and the morphisms all uniformly continuous maps $\mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$. The *Yoneda embedding* $y : \mathrm{C} \to \mathbf{Shv}(\mathrm{C}, \mathcal{J})$ gives

$$y(\mathbf{2}^{\mathbb{N}}) = (\mathrm{C}, \circ),$$

where $(\mathrm{C}, \circ)$ is a concrete sheaf as discussed in Section 2.2 and hence can be seen as a C-space. Then the Yoneda embedding restricts to a functor

$$y : \mathrm{C} \to \mathrm{C}\text{-}\mathbf{Space}$$

This concrete sheaf, seen as a C-space, is the set $\mathbf{2}^{\mathbb{N}}$ equipped with all uniformly continuous maps $\mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ as the probes.

Lemma 2.12. *The* C*-space* $y(\mathbf{2}^{\mathbb{N}})$ *has the universal property of* $\mathbf{2}$ *to the power* $\mathbb{N}$ *in the category* C-**Space***, and hence also in the category* $\mathbf{Shv}(\mathrm{C}, \mathcal{J})$*.*

Proof. It is enough to verify

$$r \in \mathrm{C} \iff \forall t \in C.\ \forall p \in \mathrm{Probe}(\mathbb{N}).\ \lambda\alpha.r(t\alpha)(p\alpha) \in \mathrm{Probe}(\mathbf{2}),$$

i.e. the C-topologies on $y(\mathbf{2}^{\mathbb{N}})$ and on the exponential are the same. □

Hence we can write

$$y(\mathbf{2}^{\mathbb{N}}) = \mathbf{2}^{\mathbb{N}},$$

where, as already discussed, $\mathbf{2}^{\mathbb{N}}$ on the left-hand side stands for the only object of the monoid C, and, on the right-hand side, for an exponential in the category of C-spaces. This notational overloading should cause no confusion.

Using this, we conclude that the Yoneda Lemma amounts to saying that a function $\mathbf{2}^{\mathbb{N}} \to X$ into a C-space X is a probe iff it is continuous. More precisely:

Lemma 2.13 (Yoneda)**.** *A map of the set* $\mathbf{2}^{\mathbb{N}}$ *to the underlying set of a* C*-space* X *is a probe if and only if it is continuous when regarded as a map from the exponential* $\mathbf{2}^{\mathbb{N}}$ *to the space* X *in the category* C-**Space***.*

Proof. ($\Rightarrow$) Let $p\colon \mathbf{2}^{\mathbb{N}} \to X$ be a probe on X. By the presheaf condition of X, we have that $p \circ t$ is a probe on X for each $t \in \mathrm{C}$, which means that the map p is continuous. ($\Leftarrow$) Let $p\colon \mathbf{2}^{\mathbb{N}} \to X$ be a continuous map. The identity map 1 is uniformly continuous and thus a probe on $\mathbf{2}^{\mathbb{N}}$. Thus $p = p \circ 1$ is a probe on X by the continuity of p. □

3 The Kleene–Kreisel continuous functionals

There are two results in this section: The Kleene–Kreisel spaces are fully and faithfully embedded in the category C-**Space** (Section 3.1), and if we assume UC in a constructive set theory, then the full type hierarchy is equivalent to the category of Kleene–Kreisel spaces within C-**Space** (Section 3.2).

3.1 Kleene–Kreisel spaces as a full subcategory of C-Space

From a constructive point of view, the traditional treatment of the Kleene–Kreisel spaces is problematic, because the proofs available in the literature rely on either classical logic or constructively contentious principles such as the Fan Theorem or the Bar induction principle. An example is the fact that all functions $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ in the category are *uniformly* continuous. It is thus natural to ask whether it is possible to develop the theory of Kleene–Kreisel spaces constructively. It turns out that this is indeed the case, using our category C-**Space** to host the Kleene–Kreisel spaces as a full subcategory: again we start with the natural numbers object, and close under finite products and function spaces.

In this section we present a *classical* proof that the resulting full subcategory of C-**Space** is isomorphic to the category of Kleene–Kreisel spaces. But, as discussed above, we have *constructive* proof that this subcategory has a fan functional without assuming the Fan Theorem or Bar Induction (Section 4.1). Thus, this subcategory can be seen as a constructive, classically equivalent, substitute for the traditional manifestations of Kleene–Kreisel spaces. This section is the only part in this paper in which we employ non-constructive arguments.

As discussed in the introduction, limit spaces provide an approach to the Kleene–Kreisel continuous functionals via sequence convergence [23]. Therefore, instead of Kleene's notion of countable functional or Kreisel's notion of continuous functional, we relate our C-spaces to limit spaces to show that how Kleene–Kreisel spaces can be calculated within our model.

Let $\mathbb{N}_\infty$ be the set $\mathbb{N}\cup\{\infty\}$ (in a constructive development, a more careful definition of $\mathbb{N}_\infty$ is needed, but since other parts of this section use non-constructive arguments, this is not important here). Recall that a *limit space* consists of a set X together with a family of functions $x\colon \mathbb{N}_\infty \to X$, written as $(x_i) \to x_\infty$ and called *convergent sequences* in X, satisfying the following conditions:

1. The constant sequence (x) converges to x.
2. If (x_i) converges to x_∞, then so does every subsequence of (x_i).
3. If (x_i) is a sequence such that every subsequence of (x_i) contains a subsequence converging to x_∞, then (x_i) converges to x_∞.

A function $f\colon X \to Y$ of limit spaces is said to be *continuous* if it preserves convergent sequences, *i.e.* $(fx_i) \to fx_\infty$ whenever $(x_i) \to x_\infty$. We write **Lim** to denote the category of limit spaces and continuous maps.

The category **Lim** is cartesian closed and has a natural numbers object. Here we give the constructions of products and exponentials of limit spaces, omitting their verification: Let X and Y be limit spaces. The underlying set of the *product* $X \times Y$ consists of pairs of elements of X and Y; and, a sequence (x_i, y_i) converges to (x, y) in $X \times Y$ iff $(x_i) \to x$ in X and $(y_i) \to y$ in Y. The underlying set of the *exponential* Y^X consists of continuous maps $X \to Y$; and, a sequence (f_i) converges to f in Y^X iff $(f_i x_i) \to fx$ in Y whenever $(x_i) \to x$ in X. We recall the following facts without proof.

Lemma 3.1.

1. *Any topological space with all topologically convergent sequences forms a limit space.*
2. *Any continuous map of topological spaces is continuous in the sense of limit spaces.*

In particular, the one-point compactification $\mathbb{N}_\infty$ and the Cantor space $\mathbf{2}^{\mathbb{N}}$ (together with their topologically convergent sequences) are limit spaces. The following is analogous to the Yoneda Lemma 2.13.

Lemma 3.2. *Convergent sequences in any limit space X are in one-to-one correspondence with the (limit) continuous maps $\mathbb{N}_\infty \to X$.*

In this section, to avoid terminological confusion, we reserve the terminology *continuous function* for morphisms of topological spaces, *probe-continuous function* for morphisms of C-spaces, and *limit-continuous function* for morphisms of limit spaces.

We first prove the analogue of Lemma 3.5 for C-spaces.

Lemma 3.3.

1. *The continuous maps from the set $\mathbf{2}^{\mathbb{N}}$ with the usual Cantor topology to any topological space X form a* C*-topology on X.*

2. *Any continuous map of topological spaces is probe-continuous.*

Proof. We firstly show that the three probe axioms are satisfied.

(1) Clearly all constant maps are continuous.

(2) Let $p\colon \mathbf{2}^{\mathbb{N}} \to X$ be a continuous map and $t \in \mathrm{C}$. As t is uniformly continuous and thus continuous, the composite $p \circ t$ of two continuous maps is continuous.

(3) Let $p_0, p_1 \colon \mathbf{2}^{\mathbb{N}} \to X$ be continuous maps. Then it is clear that the map $p\colon \mathbf{2}^{\mathbb{N}} \to X$ defined by $p(i\alpha) = p_i(\alpha)$ is also continuous.

Now let $f\colon X \to Y$ be a continuous map of topological spaces. Since any probe p is a continuous map $\mathbf{2}^{\mathbb{N}} \to X$, the composite $f \circ p$ is continuous and thus a probe on Y. Hence f is probe-continuous. $\square$

In particular, $\mathbb{N}_\infty$ together with all continuous maps $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}_\infty$ forms a C-space. Now we define functors between the categories **Lim** and C-**Space**. By the above lemmas, the following holds for any of the three notions of continuity considered in this section:

Lemma 3.4. *The following maps $\mathfrak{r}$ and $\mathfrak{s}$ are continuous, and $\mathfrak{r}$ is a retraction with section $\mathfrak{s}$:*

$$\begin{array}{rcl} \mathfrak{r}\colon \mathbf{2}^{\mathbb{N}} & \to & \mathbb{N}_\infty \\ 1^n 0\alpha & \mapsto & n \\ 1^\omega & \mapsto & \infty, \end{array} \qquad \begin{array}{rcl} \mathfrak{s}\colon \mathbb{N}_\infty & \to & \mathbf{2}^{\mathbb{N}} \\ n & \mapsto & 1^n 0^\omega \\ \infty & \mapsto & 1^\omega. \end{array}$$

Lemma 3.5. *Any topological space with all (topologically) convergent sequences forms a limit space. And any continuous map is limit-continuous.*

By the above lemma, the Cantor space $\mathbf{2}^{\mathbb{N}}$, together with all convergent sequences (or, equivalently, continuous maps $\mathbb{N}_\infty \to \mathbf{2}^{\mathbb{N}}$), forms a limit space. Now we define functors between the categories **Lim** of limit spaces and C-**Space**.

For a limit space X, define the *limit probes* on X to be the limit-continuous maps $\mathbf{2}^{\mathbb{N}} \to X$ *w.r.t.* the limit structure on $\mathbf{2}^{\mathbb{N}}$ given in Lemma 3.5. The following shows that limit spaces can be regarded as a full subcategory of C-spaces.

Lemma 3.6 (The functor $G\colon \mathbf{Lim} \to \mathrm{C}\text{-}\mathbf{Space}$)**.**

1. *For any limit space X, the limit probes form a* C*-topology on X.*
2. *For any two limit spaces X and Y, a function $X \to Y$ is limit-continuous if and only if it is continuous* w.r.t. *the limit probes.*

This gives a full and faithful functor $G\colon \mathbf{Lim} \to \mathrm{C}\text{-}\mathbf{Space}$ which on objects keeps the same underlying set but replaces the limit structure by the C-topology given by limit probes, and is the identity on morphisms.

Proof. (1) We need to show that the three probe axioms are satisfied.

(i) It is clear that any constant map is limit-continuous and thus a probe.

(ii) Let $p\colon \mathbf{2}^{\mathbb{N}} \to X$ be limit-continuous and $t \in \mathrm{C}$. Given any convergent sequence $(x_i) \to x_\infty$ in $\mathbf{2}^{\mathbb{N}}$, the induced map $x\colon \mathbb{N}_\infty \to \mathbf{2}^{\mathbb{N}}$ is continuous. Because t is uniformly continuous, the composite $t \circ x$ is continuous and thus a convergent sequence. Then we have $(p(t(x_i))) \to p(t(x_\infty))$ by the limit-continuity of p, and thus the composite $p \circ t$ is limit-continuous.

(iii) Given probes $p_0, p_1\colon \mathbf{2}^{\mathbb{N}} \to X$, *i.e.* p_0, p_1 are limit-continuous, we define a map $\bar{p}\colon \mathbf{2} \to (\mathbf{2}^{\mathbb{N}} \to X)$ by $\bar{p}(0) = p_0$ and $\bar{p}(1) = p_1$. By the discreteness of $\mathbf{2}$, this map is limit-continuous. Since both the head function, $h(\alpha) = \alpha_0$, and the tail function, $t(\alpha) = \lambda n.\alpha_{n+1}$, are limit-continuous, the map $p\colon \mathbf{2}^{\mathbb{N}} \to X$, defined by
$$p = \lambda\alpha.\bar{p}(h\alpha)(t\alpha),$$
is also limit-continuous, by the cartesian closedness of $\mathbf{Lim}$. Clearly p is the unique amalgamation of p_0, p_1.

(2) Let X and Y be limit spaces. ($\Rightarrow$) Suppose that $f\colon X \to Y$ is a limit-continuous map. Then it is also probe-continuous *w.r.t.* the C-topologies given as above, because any probe p on X is a limit-continuous map, and the composite $f \circ p$ limit-continuous and hence a probe on Y. ($\Leftarrow$) Suppose that $f\colon X \to Y$ is a probe-continuous map, *w.r.t.* the C-topologies given as above. Given a convergent sequence $(x_i) \to x_\infty$, we know that the induced map $x\colon \mathbb{N}_\infty \to X$ is limit-continuous by Lemma 3.2. By (2($\Rightarrow$)), we know that x is probe-continuous, and hence so is the composite $f \circ x$. Since the retraction $\mathfrak{r}\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}_\infty$ is continuous and thus a probe on $\mathbb{N}_\infty$, the composite $f \circ x \circ \mathfrak{r}$ is a probe on Y, *i.e.* a limit-continuous map. As $(1^i 0^\omega) \to 1^\omega$ in $\mathbf{2}^{\mathbb{N}}$, the sequence $(f(x(\mathfrak{r}(1^i 0^\omega))))$ converges to $f(x(\mathfrak{r}(1^\omega)))$, which amounts to $(fx_i) \to fx_\infty$ by the definition of $\mathfrak{r}$. □

Lemma 3.7 (The functor $F\colon \mathrm{C}\text{-}\mathbf{Space} \to \mathbf{Lim}$)**.**

1. *For any* C*-space X, the probe-continuous maps $\mathbb{N}_\infty \to X$ form a limit structure on X.*
2. *For any two* C*-spaces X and Y, if a function $X \to Y$ is probe-continuous then it is limit-continuous* w.r.t. *the above limit structures.*

This gives a functor $F\colon \text{C-}\mathbf{Space} \to \mathbf{Lim}$ which on objects again keeps the same underlying set but replaces the C-topology by an appropriate limit structure, and is the identity on morphisms.

Proof. (1) We need to verify the three axioms of limit structure.

(i) Clearly any constant map $\mathbb{N}_\infty \to X$ is probe-continuous and thus a convergent sequence.

(ii) If $x\colon \mathbb{N}_\infty \to X$ is probe-continuous, *i.e.* $(x_i) \to x_\infty$, and (x_{fi}) is a subsequence of (x_i), then we extend the reindexing function $f\colon \mathbb{N} \to \mathbb{N}$ to $\tilde{f}\colon \mathbb{N}_\infty \to \mathbb{N}_\infty$ by defining $\tilde{f}(n) = f(n)$ for $n \in \mathbb{N}$ and $\tilde{f}(\infty) = \infty$. Once we show that $\tilde{f}$ is probe-continuous, then so is the composite $x \circ \tilde{f}$ and thus $(x_{fi}) \to x_\infty$. For endomaps $\mathbb{N}_\infty \to \mathbb{N}_\infty$, probe-continuity corresponds to continuity. Given an open set $U \subseteq \mathbb{N}_\infty$. If $\infty \notin U$ then U must be a finite subset of $\mathbb{N}$. Since f, as a subsequence-reindexing function, is injective (in fact, bijective) and strictly increasing, the set $\tilde{f}^{-1}(U)$ is also a finite subset of $\mathbb{N}$ and thus open. If $\infty \in U$ then its complement $\bar{U}$ is a finite subset of $\mathbb{N}$. The complement $\overline{\tilde{f}^{-1}(U)} = \tilde{f}^{-1}(\bar{U})$ is a finite subset of $\mathbb{N}$ and thus $\tilde{f}^{-1}(U)$ is open.

(iii) Suppose that (x_i) is a sequence such that every subsequence of (x_i) has a subsequence converging to x_∞. We need to prove the the induce map $x\colon \mathbb{N}_\infty \to X$ is probe-continuous. Let p be a probe on $\mathbb{N}_\infty$. By the assumption, we have a subsequence (x_{fi}) which converges to x_∞. We extend the reindexing function f as above and get a continuous map $\bar{f}\colon \mathbb{N}_\infty \to \mathbb{N}_\infty$. We know that f is bijective and thus the inverse $\bar{f}^{-1}$ is also continuous, because any continuous bijection of compact Hausdorff spaces is a homeomorphism. Since $(x_{fi}) \to x_\infty$, the map $x \circ \bar{f}$ is probe-continuous, *i.e.* $x \circ \bar{f} \circ q$ is a probe for any probe q on $\mathbb{N}_\infty$. If we choose $q = \bar{f}^{-1} \circ p$ which is a probe by the continuity of $\bar{f}^{-1}$, then $x \circ \bar{f} \circ \bar{f}^{-1} \circ p = x \circ p$ is a probe on X.

(Notice that in both (ii) and (iii) we have used non-constructive arguments.)

(2) Let X and Y be C-spaces, and let $f\colon X \to Y$ be a probe-continuous map. Given a convergent sequence $(x_i) \to x_\infty$, *i.e.* a probe-continuous map $x\colon \mathbb{N}_\infty \to X$, the composite $f \circ x$ is also probe-continuous and hence a convergent sequence on Y. □

Lemma 3.8. *Limit spaces form a reflective subcategory of* C*-spaces.*

Proof. It remains to show that $F\colon \text{C-}\mathbf{Space} \to \mathbf{Lim}$ is left adjoint to G, *i.e.* for any C-space X and limit space Y, we have $\mathbf{Lim}(FX, Y) \cong \text{C-}\mathbf{Space}(X, GY)$ naturally. As the underlying sets remain the same when we apply the functors, this is equivalent to saying that a map $f\colon X \to Y$ is limit-continuous iff it is probe-continuous.

($\Rightarrow$) Suppose f is limit-continuous, *i.e.* if $x\colon \mathbb{N}_\infty \to X$ is probe-continuous then $(fx_i) \to fx_\infty$. Given a probe $p\colon \mathbf{2}^\mathbb{N} \to X$ on X, we want to show that

$f \circ p$ is a probe on Y, *i.e.* $f \circ p$ is limit-continuous. Given a convergent sequence $\alpha \colon \mathbb{N}_\infty \to \mathbf{2}^{\mathbb{N}}$, *i.e.* α is probe-continuous, the composite $p \circ \alpha$ is also probe-continuous. Then by the limit-continuity of f we have that $(f(p\alpha^i))$ converges to $f(p\alpha^\infty)$.

($\Leftarrow$) Suppose f is probe-continuous. Given a probe-continuous function $x \colon \mathbb{N}_\infty \to X$ (a convergent sequence), we want to show that $(fx_i) \to fx_\infty$. Since the retraction $\mathfrak{r} \colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}_\infty$ defined in Lemma 3.4 is continuous and thus probe-continuous, so is the composite $x \circ \mathfrak{r}$. By the probe-continuity of f, we have that $f \circ x \circ \mathfrak{r}$ is a probe on Y and thus limit-continuous. Since $(1^i 0^\omega) \to 1^\omega$ in $\mathbf{2}^{\mathbb{N}}$, the sequence $(f(x(\mathfrak{r}(1^i 0^\omega))))$ converges to $f(x(\mathfrak{r}(1^\omega)))$, which amounts to $(fx_i) \to fx_\infty$ by the definition of $\mathfrak{r}$. □

Lemma 3.9. *The reflector* $F \colon$ C-**Space** $\to$ **Lim** *preserves finite products.*

Proof. It is trivial to verify that terminal objects are preserved by F. Now we show that F preserves binary products. Let X and Y be C-spaces. Both $F(X \times_{\text{C-}\mathbf{Space}} Y)$ and $F(X) \times_{\mathbf{Lim}} F(Y)$ have the same underlying set, the cartesian products of X and Y. We need to show that their limit structures are the same. ($\Rightarrow$) Given a convergent sequence $z \colon \mathbb{N}_\infty \to X \times Y$ in $F(X \times_{\text{C-}\mathbf{Space}} Y)$, *i.e.* a probe-continuous map, clearly the pair $(z \circ \mathrm{pr}_1, z \circ \mathrm{pr}_2)$ is a convergent sequence in $F(X) \times_{\mathbf{Lim}} F(Y)$. ($\Leftarrow$) Given a convergent sequence (x, y) in $F(X) \times_{\mathbf{Lim}} F(Y)$, where both $x \colon \mathbb{N}_\infty \to X$ and $y \colon \mathbb{N}_\infty \to Y$ are probe-continuous, we define a map $z \colon \mathbb{N}_\infty \to X \times Y$ by $z_i = (x_i, y_i)$. Clearly z is probe-continuous and thus a convergent sequence in $F(X \times_{\text{C-}\mathbf{Space}} Y)$. One can easily see that, if a convergent sequence is transferred by one of the above directions and then by the other, it remains the same. □

In view of the above, we can regard **Lim** as a full subcategory of C-**Space**.

Lemma 3.10. *[26, Corollary A.1.5.9] Let* $G \colon \mathcal{C} \to \mathcal{D}$ *be a functor between cartesian closed categories, and suppose* G *has a left functor* F*. If* G *is full and faithful and* F *preserves binary products, then* G *is cartesian closed (*i.e. *G preserves finite products and exponentials).*

Lemma 3.11. *[26, Proposition A.4.3.1] Let* $\mathcal{D}$ *be a cartesian closed category, and* $\mathcal{C}$ *a reflective subcategory of* $\mathcal{D}$*, corresponding to a reflector* F *on* $\mathcal{C}$*. Then* F *preserves finite products iff (the class of objects of)* $\mathcal{C}$ *is an exponential ideal in* $\mathcal{D}$ *(*i.e. *the exponential* C^D *is in* $\mathcal{C}$ *whenever* $C \in \mathcal{C}$ *and* $D \in \mathcal{D}$*).*

The above two general categorical lemmas give the following:

Theorem 3.12.

1. *The functor* $G \colon$ **Lim** $\to$ C-**Space** *is cartesian closed.*

2. *Limit spaces form an exponential ideal of* C-**Space**.

Moreover, the discrete objects in these two categories coincide.

Lemma 3.13. *If* X *is a discrete* C-*space, then* $G(F(X)) = X$.

Proof. It suffices to prove that $p\colon \mathbf{2}^{\mathbb{N}} \to X$ is a probe on X iff it is a probe on $G(F(X))$.

($\Rightarrow$) This direction holds for any C-space X. Let p be a probe on X. Then p is a probe-continuous map by the Yoneda Lemma 2.13. We need to show that p is a probe on $G(F(X))$, *i.e.* p is limit-continuous. Given a convergent sequence $x\colon \mathbb{N}_\infty \to \mathbf{2}^{\mathbb{N}}$, we know that x is also a probe-continuous map. Thus the composite $p \circ x$ is probe-continuous and thus a convergent sequence on X.

($\Leftarrow$) Let p be a probe on $G(F(X))$. According to the definitions of G and F, this means that, for any continuous maps $x\colon \mathbb{N}_\infty \to \mathbf{2}^{\mathbb{N}}$ and $q\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}_\infty$, the composite $p \circ x \circ q$ is locally constant. We need to show that p is a probe on X, *i.e.* p is locally constant. For the sake of contradiction, assume that p fails to be locally constant. By classical logic, this amounts to

$$\forall m \in \mathbb{N}.\ \exists \alpha, \beta \in \mathbf{2}^{\mathbb{N}}.\ \alpha =_m \beta \wedge p\alpha \neq p\beta$$

which, by countable choice, defines two sequences $(\boldsymbol{\alpha}^i)$ and $(\boldsymbol{\beta}^i)$ such that (i) $\boldsymbol{\alpha}^m =_m \boldsymbol{\beta}^m$ and (ii) $p\boldsymbol{\alpha}^m \neq p\boldsymbol{\beta}^m$ for all $m \in \mathbb{N}$. Because of the compactness of $\mathbf{2}^{\mathbb{N}}$ and (i), there are subsequences $(\boldsymbol{\alpha}^{fi})$ and $(\boldsymbol{\beta}^{fi})$, both of which converge to the same point $\gamma \in \mathbf{2}^{\mathbb{N}}$ and satisfy $\boldsymbol{\alpha}^{fi} =_{fi} \boldsymbol{\beta}^{fi}$ for all $i \in \mathbb{N}$. Because the composites $\boldsymbol{\alpha} \circ f$ and $\boldsymbol{\beta} \circ f$, being convergent, are continuous, we know that both $p \circ (\boldsymbol{\alpha} \circ f) \circ \mathfrak{r}$ and $p \circ (\boldsymbol{\beta} \circ f) \circ \mathfrak{r}$ are locally constant, where $\mathfrak{r}$ is the retraction defined in Lemma 3.4. Let m to be the maximum of their moduli. By their local constancy and the fact $1^m0^\omega =_m 1^\omega$, we have

$$p\boldsymbol{\alpha}^{fm} = p\left(\boldsymbol{\alpha}^{f(\mathfrak{r}(1^m0^\omega))}\right) = p\left(\boldsymbol{\alpha}^{f(\mathfrak{r}(1^\omega))}\right) = p\gamma$$

$$p\boldsymbol{\beta}^{fm} = p\left(\boldsymbol{\beta}^{f(\mathfrak{r}(1^m0^\omega))}\right) = p\left(\boldsymbol{\beta}^{f(\mathfrak{r}(1^\omega))}\right) = p\gamma$$

and thus $p\boldsymbol{\alpha}^{fm} = p\boldsymbol{\beta}^{fm}$ which contradicts (ii) and hence shows that p must be locally constant. □

Recall that Kleene-Kreisel spaces can be obtained within **Lim** by starting with the natural numbers object, and closing under finite products and function spaces. By Theorem 3.12 and Lemma 3.13, the full subcategory of C-**Space** generated by the same process is isomorphic to the above one of **Lim**, which leads to the following conclusion:

Theorem 3.14. *The Kleene–Kreisel spaces can be calculated within* C-**Space** *by starting from the natural numbers object and iterating products and exponentials.*

An outline of a more direct proof of this theorem is as follows: (0) $F(GX) = X$ for any limit space X. (1) If a map $p\colon \mathbf{2}^{\mathbb{N}} \to X$ is a probe on a C-space X, then it is also a probe on $G(FX)$. (2) If $X = G(FX)$ and $Y = G(FY)$, then $X \times Y = G(F(X \times Y))$. (3) If $Y = G(FY)$ then $Y^X = G(F(Y^X))$. The proofs of (0) and (1) are easy and those of (2) and (3) use (1). The advantage of the more abstract approach we have chosen is that it gives additional information.

3.2 The Kleene–Kreisel and full type hierarchies

The *full type hierarchy* is the smallest full subcategory of **Set** containing the natural numbers and closed under exponentials. If we work in a constructive set theory (or type theory) with the Brouwerian axiom UC, it turns out that the full type hierarchy is equivalent to the Kleene–Kreisel hierarchy calculated within C-**Space**. It is interesting that other Brouwerian axioms such as more general forms of continuity or Bar Induction are not needed to prove the equivalence. In the proofs below, we explicitly assume UC whenever it is needed.

For a set X, one can take all maps $\mathbf{2}^{\mathbb{N}} \to X$ as probes on X. The resulting space is called *indiscrete*, and refer to the collection of all maps $\mathbf{2}^{\mathbb{N}} \to X$ as the *indiscrete* C-topology on X. It is clear that X is indiscrete iff for any C-space Y, all maps $Y \to X$ are continuous. This is equivalent to saying that the functor $\nabla \colon \mathbf{Set} \to \text{C-}\mathbf{Space}$ that endows a set with the indiscrete C-topology is right adjoint to the forgetful functor C-**Space** $\to$ **Set**. Moreover, the adjunction becomes an equivalence when restricted to indiscrete spaces:

Lemma 3.15. *The category of indiscrete* C*-spaces is equivalent to* **Set**.

Lemma 3.16. *Indiscrete* C*-spaces form an exponential ideal.*

Proof. Let X be a C-space and Y an indiscrete C-space. Given $r \colon \mathbf{2}^{\mathbb{N}} \to Y^X$, for any $t \in \mathrm{C}$ and $p \in \mathrm{Probe}(X)$, the map $\lambda\alpha.r(t\alpha)(p\alpha)$ has codomain Y and thus a probe on Y. Therefore, all maps $\mathbf{2}^{\mathbb{N}} \to Y^X$ are probes on Y^X. $\square$

The crucial, but easy, observation is this:

Lemma 3.17. *If UC holds in* **Set**, *then the discrete space* $\mathbb{N}$ *is also indiscrete.*

Proof. By definition, the discrete topology consists of all *uniformly continuous* functions $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$. But if UC holds in **Set**, this amounts to *all* functions $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$, which, by definition, constitute the indiscrete topology, and hence $\mathbb{N}$ is indiscrete (and the discrete and indiscrete topologies are the same). $\square$

The desired result follows directly from the previous lemmas:

Corollary 3.18. *If* UC *holds in* **Set**, *then the full type hierarchy is equivalent to the Kleene–Kreisel hierarchy.*

This can be strengthened so that its converse also holds:

Theorem 3.19. *The forgetful functor from the Kleene–Kreisel hierarchy to the full type hierarchy is an equivalence if and only if UC holds in* **Set**.

Proof. By the above lemmas, if UC holds in **Set**, then the adjunction restricts to an equivalence of the two hierarchies. Conversely, if it is an equivalence, then UC holds in **Set**, because, as we have seen, it always holds in C-**Space**. $\square$

Two larger full subcategories of **Set** and C-**Space** are equivalent if UC holds.

Lemma 3.20.

1. *Finite products of indiscrete* C*-spaces are indiscrete.*

2. *If* UC *holds in the category of sets, then finite coproducts of indiscrete* C*-spaces are indiscrete.*

Proof. The first claim is trivial. If UC holds, then the space $\mathbf{2}$ is both discrete and indiscrete. If X and Y are indiscrete spaces, we construct a coproduct $X+Y$ as in the proof of Theorem 2.7. We also define, by cases, a map $i \colon X + Y \to \mathbf{2}$ which maps $\mathrm{in}_0\, x$ to 0 and $\mathrm{in}_1\, y$ to 1 for any $x \in X$ and $y \in Y$. As $\mathbf{2}$ is indiscrete, the map i is continuous. Given any map $r \colon \mathbf{2}^{\mathbb{N}} \to X + Y$, the composite $i \circ r$ is a probe on $\mathbf{2}$ and thus locally constant, *i.e.* there is a natural number n such that for all $s \in \mathbf{2}^n$ the composite $i \circ r \circ \mathrm{cons}_s$ is constant. If its value is 0, then $r \circ \mathrm{cons}_s$ maps all $\alpha \in \mathbf{2}^{\mathbb{N}}$ to $\mathrm{in}_0\, x$ for some $x \in X$, *i.e.* there is a map $p \colon \mathbf{2}^{\mathbb{N}} \to X$ such that $r(\mathrm{cons}_s \alpha) = \mathrm{in}_0(p\alpha)$; otherwise, there is a map $q \colon \mathbf{2}^{\mathbb{N}} \to Y$ such that $r(\mathrm{cons}_s \alpha) = \mathrm{in}_1(q\alpha)$. By the definition, the map r is a probe on $X + Y$. □

Define *extended hierarchies* by closing under finite products and coproducts, in addition to exponentials.

Theorem 3.21. *If* UC *holds in* **Set***, then the extended full type hierarchy is equivalent to the extended Kleene–Kreisel hierarchy.*

4 Modelling uniform continuity with C-spaces

We first show that there is a *fan functional* in C-**Space** that continuously calculates moduli of uniform continuity of maps $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$. We then use the cartesian closed structure of C-spaces to model simple types and system T, and their local cartesian closed structure to model dependent types. In both cases we show that the uniform-continuity principle UC is validated. In the case of system T, in which quantifiers are absent, we skolemize UC using the fan functional. In the case of dependent types, UC is represented by a type rather than a logical formula, via the Curry–Howard interpretation, and we show that this type is inhabited. This second case is also reduced to the fan functional. In the case of system T, we also prove an additional, but well known, result, namely that a definable function $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ in the full type hierarchy is uniformly continuous. We show this by establishing a logical relation between the full type hierarchy and the Kleene–Kreisel functionals.

4.1 The fan functional

According to the Yoneda lemma 2.13, the continuous maps from the Cantor space in C-**Space** to the natural numbers object are in natural bijection with the uniformly continuous maps $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ of the meta-language used to define the model. Moreover, the topology on the set of continuous maps $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ is discrete:

Lemma 4.1. *The exponential* $\mathbb{N}^{\mathbf{2}^{\mathbb{N}}}$ *is a discrete* C*-space.*

Proof. Given a probe $p\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}^{\mathbf{2}^{\mathbb{N}}}$, we want to show that it is locally constant. By the construction of exponentials in Section 2, we know that for all $t, r \in \mathrm{C}$,

$$\lambda\alpha.p(t\alpha)(r\alpha) \in \mathrm{Probe}(\mathbb{N}),$$

i.e. $\lambda\alpha.p(t\alpha)(r\alpha)$ it is uniformly continuous. In particular, we can take

$$t(\alpha)(i) = \alpha_{2i} \qquad \text{and} \qquad r(\alpha)(i) = \alpha_{2i+1},$$

which are both uniformly continuous, and define $q(\alpha) = p(t\alpha)(r\alpha)$. From the proof of uniform continuity of q, we get its modulus n. (NB. Here we are implicitly using choice, but this is not a problem in intensional type theory, provided we interpret the existential quantifier as a Σ type. In a setting without choice, we would need to define uniform continuity by explicitly requiring a modulus.) Now define a map $\mathrm{join}\colon \mathbf{2}^{\mathbb{N}} \times \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ by

$$\begin{aligned} \mathrm{join}(\alpha,\beta)(2i) &= \alpha_i \\ \mathrm{join}(\alpha,\beta)(2i+1) &= \beta_i. \end{aligned}$$

Given $\alpha, \alpha', \beta \in \mathbf{2}^{\mathbb{N}}$ with $\alpha =_n \alpha'$, we have

$$\begin{array}{cll} & p(\alpha)(\beta) & \\ = & p(t(\mathrm{join}(\alpha,\beta)))(r(\mathrm{join}(\alpha,\beta))) & (\text{by the definitions of } t, r, \mathrm{join}) \\ = & q(\mathrm{join}(\alpha,\beta)) & (\text{by the definition of } q) \\ = & q(\mathrm{join}(\alpha',\beta)) & (\mathrm{join}(\alpha,\beta) =_{2n} \mathrm{join}(\alpha',\beta), 2n \geq n) \\ = & p(\alpha')(\beta). & \end{array}$$

Hence p is locally constant and therefore $\mathbb{N}^{\mathbf{2}^{\mathbb{N}}}$ is discrete. □

Theorem 4.2. *There is a* fan functional

$$\mathrm{fan}\colon \mathbb{N}^{\mathbf{2}^{\mathbb{N}}} \to \mathbb{N}$$

in C-**Space** *that continuously calculates moduli of uniform continuity.*

Proof. Given a continuous map $f\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}$, *i.e.* an element of $\mathbb{N}^{\mathbf{2}^{\mathbb{N}}}$, we know f is uniformly continuous because the identity map 1 is a probe on $\mathbf{2}^{\mathbb{N}}$ and hence $f = f \circ 1 \in \mathrm{Probe}(\mathbb{N})$ by the continuity of f. Then we can get a modulus m_f from the witness of its uniform continuity. From this modulus we can calculate the smallest modulus of f as follows. We define a function $\mathrm{lmod}\colon (\mathbf{2}^{\mathbb{N}} \to \mathbb{N}) \to \mathbb{N} \to \mathbb{N}$ by induction on its second argument:

$$\begin{array}{lcl} \mathrm{lmod}\ f\ 0 & = & 0 \\ \mathrm{lmod}\ f\ (n+1) & = & \text{if } (\forall s \in \mathbf{2}^n.\ f(s0^\omega) = f(s1^\omega)) \quad \text{then} \quad (\mathrm{lmod}\ f\ n) \\ & & \qquad\qquad\qquad\qquad\qquad\qquad\quad\ \text{else} \quad (n+1). \end{array}$$

With a proof by induction, we can show that $\mathrm{lmod}\ f\ n$ is the smallest modulus if n is a modulus of f. Hence, we define

$$\mathrm{fan}(f) = \mathrm{lmod}\ f\ m_f.$$

According to the previous lemma, the space $\mathbb{N}^{\mathbf{2}^{\mathbb{N}}}$ is discrete and hence this functional is continuous. □

4.2 Gödel's System T

We firstly recover a well-known result, using a logical relation between the set-theoretical and the C-**Space** models of the term language of system T. Then we extend the theory with a constant for the fan functional so that it becomes expressive enough to formulate the principle UC, and show how C-**Space** validates UC.

The *term language* of system has a ground type N of natural numbers, binary product type $\times$ and function type $\to$. For our purposes, it is convenient (although not strictly necessary) to add a ground type $\mathsf{2}$ of booleans. The constants and equations associated to the ground types are

- the natural number $\mathsf{0} \colon \mathsf{N}$,
- the successor function $\mathsf{suc} \colon \mathsf{N} \to \mathsf{N}$, and
- the recursion combinator

$$\begin{array}{lcl} \multicolumn{3}{l}{\mathsf{rec} \colon \sigma \to (\mathsf{N} \to \sigma \to \sigma) \to \mathsf{N} \to \sigma} \\ \mathsf{rec}\ x\ f\ \mathsf{0} & = & x \\ \mathsf{rec}\ x\ f\ (\mathsf{suc}\ n) & = & f\ n\ (\mathsf{rec}\ x\ f\ n). \end{array}$$

- booleans $\mathsf{f}, \mathsf{t} \colon \mathsf{2}$,
- the case-distinction function:

$$\begin{array}{lcl} \multicolumn{3}{l}{\mathsf{if} \colon \mathsf{2} \to \sigma \to \sigma \to \sigma} \\ \mathsf{if}\ \mathsf{f}\ x\ y & = & x \\ \mathsf{if}\ \mathsf{t}\ x\ y & = & y, \end{array}$$

The atomic *formulas* in system T consist of equations between terms of the same type, and more complex *formulas* are obtained by combining these with the propositional connectives $\wedge$ and $\Rightarrow$ (the negation of a formula ϕ can of course be defined as $\phi \Rightarrow \mathsf{0} = \mathsf{suc}\ \mathsf{0}$).

The term language of system T can be interpreted in any cartesian closed category with a natural numbers object $\mathbb{N}$ and a coproduct $\mathbf{2}$ (or $\mathbf{1} + \mathbf{1}$) of two copies of the terminal object [27]. Specifically, types are interpreted as objects: the ground type N is interpreted as the object $\mathbb{N}$, the ground type $\mathsf{2}$ as the coproduct $\mathbf{2}$, product types as products, and function types as exponentials. Contexts are interpreted inductively as products. And a term in context is interpreted as a morphism from the interpretation of its context to the one of its type. The constant rec and if are interpreted using the universal properties of $\mathbb{N}$ and $\mathbf{2}$ in the standard way [27].

Throughout this section (and the next), we use σ, τ to range over types, bold lower case letters $\boldsymbol{t}$, $\boldsymbol{u}$, $\boldsymbol{x}$, $\boldsymbol{f}$, $\boldsymbol{m}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ to range over terms, and ϕ, ψ to range over formulas.

Uniform continuity of T-definable functions. As we have seen, both the categories **Set** and C-**Space** are cartesian closed and have a natural numbers object and a coproduct $\mathbf{1}+\mathbf{1}$, and hence serve as models of system T. We now recover a well-known result (Theorem 4.5), using a logical relation between these two models. In the following we use the semantic brackets $[\![-]\!]$ for the interpretation, and add **Set** and C-**Space** as subscripts to distinguish which model we are working with.

Definition 4.3. *The* logical relation R *over the set-theoretical and* C*-space models is defined by*

1. *If σ is a* T *type, then $\mathrm{R}_\sigma \subseteq [\![\sigma]\!]_{\mathbf{Set}} \times [\![\sigma]\!]_{\text{C-}\mathbf{Space}}$ is defined by induction on type σ as follows:*
   - *(a)* $\mathrm{R}_\iota(a,a')$ *iff $a=a'$, where ι is the ground type* $\mathsf{2}$ *or* N*;*
   - *(b)* $\mathrm{R}_{\sigma\to\tau}(f,f')$ *iff, for any $a \in [\![\sigma]\!]_{\mathbf{Set}}$ and any $a' \in [\![\sigma]\!]_{\text{C-}\mathbf{Space}}$, if $\mathrm{R}_\sigma(a,a')$ then $\mathrm{R}_\tau(f(a), f'(a'))$.*
2. *If $\Gamma \equiv x_1{:}\sigma_1,\ldots,x_n{:}\sigma_n$ is a context, then $\mathrm{R}_\Gamma \subseteq [\![\Gamma]\!]_{\mathbf{Set}} \times [\![\Gamma]\!]_{\text{C-}\mathbf{Space}}$ is defined by $\mathrm{R}_\Gamma(\vec{a},\vec{a'})$ iff $\mathrm{R}_{\sigma_i}(a_i,a'_i)$ for all $i \le n$.*
3. *Given $f \equiv [\![\Gamma \vdash \boldsymbol{t} : \tau]\!]_{\mathbf{Set}}$ and $f' \equiv [\![\Gamma \vdash \boldsymbol{t} : \tau]\!]_{\text{C-}\mathbf{Space}}$, $\mathrm{R}(f,f')$ iff, for any $\vec{a} \in [\![\Gamma]\!]_{\mathbf{Set}}$ and any $\vec{a'} \in [\![\Gamma]\!]_{\text{C-}\mathbf{Space}}$, if $\mathrm{R}_\Gamma(\vec{a},\vec{a'})$ then $\mathrm{R}_\tau(f(\vec{a}), f'(\vec{a'}))$.*

With a proof by induction on terms as usual, we can easily show that the interpretations of any T term in these two models are related.

Lemma 4.4. *If $\Gamma \vdash \boldsymbol{t} : \tau$, then $\mathrm{R}([\![\Gamma \vdash \boldsymbol{t} : \tau]\!]_{\mathbf{Set}}, [\![\Gamma \vdash \boldsymbol{t} : \tau]\!]_{\text{C-}\mathbf{Space}})$.*

We say that an element $x \in [\![\sigma]\!]_{\mathbf{Set}}$ in the set-theoretical model is T-*definable* if it is the interpretation of some closed T term, *i.e.* there exists a closed term $\boldsymbol{t} : \sigma$ such that $x = [\![\boldsymbol{t}]\!]_{\mathbf{Set}}$.

Theorem 4.5. *Any* T*-definable function $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous.*

Proof. If $f \colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ interprets the term $\boldsymbol{f} : (\mathsf{N}\to\mathsf{2})\to\mathsf{N}$, then f is related to the continuous map $[\![\boldsymbol{f}]\!]_{\text{C-}\mathbf{Space}} \colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ according to the above lemma. By the definition of the logical relation, we can easily show that f is uniformly continuous. $\square$

Validating the uniform-continuity principle in system T. The above shows that the definable functions $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ in the full type hierarchy are uniformly continuous, with uniform continuity formulated *externally* to the theory, in the model. We now show how to validate the *internal* principle of uniform continuity, working with C-spaces.

In a theory with quantifiers, such as HA^ω, the principle UC is formulated as follows:

$$\vdash \forall \boldsymbol{f}{:}(\mathsf{N}\to\mathsf{2})\to\mathsf{N}.\ \exists \boldsymbol{m}{:}\mathsf{N}.\ \forall \boldsymbol{\alpha},\boldsymbol{\beta}{:}\mathsf{N}\to\mathsf{2}.\ \boldsymbol{\alpha} =_{\boldsymbol{m}} \boldsymbol{\beta} \Rightarrow \boldsymbol{f}\boldsymbol{\alpha} = \boldsymbol{f}\boldsymbol{\beta}.$$

In order to express this in system T, which lacks quantifiers, we first treat $\Gamma \vdash \forall(\boldsymbol{x}:\sigma).\phi$ as $\Gamma, \boldsymbol{x}:\sigma \vdash \phi$, and then we add a constant

$$\mathsf{fan}\colon ((\mathsf{N}\to 2)\to \mathsf{N})\to \mathsf{N}$$

to remove the existential quantifier by skolemization, so that we get the purely equational formulation

$$\boldsymbol{f}:(\mathsf{N}\to 2)\to \mathsf{N}, \boldsymbol{\alpha}:\mathsf{N}\to 2, \boldsymbol{\beta}:\mathsf{N}\to 2 \ \vdash\ \boldsymbol{\alpha} =_{\mathsf{fan}(\boldsymbol{f})} \boldsymbol{\beta} \Rightarrow \boldsymbol{f}\boldsymbol{\alpha} = \boldsymbol{f}\boldsymbol{\beta}.$$

To formulate $\boldsymbol{\alpha} =_{\boldsymbol{m}} \boldsymbol{\beta}$, we define a term $\mathsf{agree}\colon (\mathsf{N}\to 2)\to(\mathsf{N}\to 2)\to \mathsf{N}\to 2$ by

$$\mathsf{agree}\ \boldsymbol{\alpha}\ \boldsymbol{\beta} \ \stackrel{\text{def}}{=\!=}\ \mathsf{rec}\ \mathsf{t}\ (\lambda \boldsymbol{i}.\ \lambda \boldsymbol{x}.\ \mathsf{min}\ (\mathsf{eq}\ \boldsymbol{\alpha i}\ \boldsymbol{\beta i})\ \boldsymbol{x}),$$

where $\mathsf{min}\colon 2\to 2\to 2$ gives the minimal boolean and $\mathsf{eq}\colon 2\to 2\to 2$ has value t iff its two arguments are the same, both of which can be defined using if. The idea is that

$$\mathsf{agree}\ \boldsymbol{\alpha}\ \boldsymbol{\beta}\ \boldsymbol{m} = \mathsf{t}\ \text{ iff }\ \boldsymbol{\alpha} \text{ and } \boldsymbol{\beta} \text{ are equal up to the first } \boldsymbol{m} \text{ positions.}$$

Then the formula of (UC) that we are working with becomes

$$\boldsymbol{f}:(\mathsf{N}\to 2)\to \mathsf{N}, \boldsymbol{\alpha}:\mathsf{N}\to 2, \boldsymbol{\beta}:\mathsf{N}\to 2 \ \vdash\ \mathsf{agree}\ \boldsymbol{\alpha}\ \boldsymbol{\beta}\ (\mathsf{fan}\ \boldsymbol{f}) = \mathsf{t} \Rightarrow \boldsymbol{f}\boldsymbol{\alpha} = \boldsymbol{f}\boldsymbol{\beta}.$$

We interpret types and terms of this theory in C-**Space** as before, while the meaning of the constant fan is given by the functional $\mathrm{fan}\colon \mathbb{N}^{\mathbf{2}^{\mathbb{N}}} \to \mathbb{N}$ constructed in Chapter 4.1. Here and in the next section, semantic brackets without explicit decorations refer to the C-**Space** interpretation. Formulas are interpreted inductively as follows. Given $\vec{\rho} \in [\![\Gamma]\!]$,

(1) $[\![\Gamma \vdash \boldsymbol{t} = \boldsymbol{u}]\!](\vec{\rho}) \ = [\![\Gamma \vdash \boldsymbol{t} : \sigma]\!](\vec{\rho}) = [\![\Gamma \vdash \boldsymbol{u} : \sigma]\!](\vec{\rho})$,

(2) $[\![\Gamma \vdash \phi \wedge \psi]\!](\vec{\rho}) \ = [\![\Gamma \vdash \phi]\!](\vec{\rho}) \times [\![\Gamma \vdash \psi]\!](\vec{\rho})$,

(3) $[\![\Gamma \vdash \phi \Rightarrow \psi]\!](\vec{\rho}) \ = [\![\Gamma \vdash \phi]\!](\vec{\rho}) \to [\![\Gamma \vdash \psi]\!](\vec{\rho})$,

where, in the right-hand side, $=$ represents equality (or identity type), $\times$ binary product, and $\to$ function space in the meta-theory. We then say that C-**Space** *validates* $\Gamma \vdash \phi$ iff $[\![\Gamma \vdash \phi]\!](\vec{\rho})$ is inhabited for any $\vec{\rho} \in [\![\Gamma]\!]$.

Theorem 4.6. *The model of* C-*spaces validates* UC.

Proof. If the interpretation of the formula $\mathsf{agree}\ \boldsymbol{\alpha}\ \boldsymbol{\beta}\ \boldsymbol{m} = \mathsf{t}$ is inhabited, then we have $[\![\boldsymbol{\alpha}]\!] =_{[\![\boldsymbol{m}]\!]} [\![\boldsymbol{\beta}]\!]$, with a proof by induction on $[\![\boldsymbol{m}]\!]$. In particular, the inhabitedness of the interpretation of $\mathsf{agree}\ \boldsymbol{\alpha}\ \boldsymbol{\beta}\ (\mathsf{fan}\ \boldsymbol{f}) = \mathsf{t}$ implies $[\![\boldsymbol{\alpha}]\!] =_{\mathrm{fan}[\![\boldsymbol{f}]\!]} [\![\boldsymbol{\beta}]\!]$. According to the definition of fan, we have $[\![\boldsymbol{f}\boldsymbol{\alpha}]\!] = [\![\boldsymbol{f}\boldsymbol{\beta}]\!]$. □

4.3 Dependent types

In the previous section, we showed that C-spaces give a model of system T that validates the uniform-continuity principle (UC), namely

$$\forall (f \colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}).\ \exists (m \in \mathbb{N}).\ \forall (\alpha, \beta \in \mathbf{2}^{\mathbb{N}}).\ \alpha =_m \beta \implies f\alpha = f\beta.$$

For this, we used the cartesian closedness of C-spaces to interpret simple types and formulas in system T, and we gave a skolemized version of (UC) using the fan functional in order to remove the quantifiers, which are absent from system T. In this section, we exploit the local cartesian closedness of C-spaces to model dependent types. In this case, the uniform-continuity principle is formulated as a closed type, via the Curry–Howard interpretation, rather than as a logical formula, namely

$$\Pi(f \colon (\mathsf{N} \to 2) \to \mathsf{N}).\ \Sigma(m \colon \mathsf{N}).\ \Pi(\alpha \colon \mathsf{N} \to 2).\ \Pi(\beta \colon \mathsf{N} \to 2).\ \alpha =_m \beta \to f\alpha = f\beta,$$

where 2 denotes the type of binary digits f and t, N denotes the type of natural numbers. Here $\alpha =_m \beta$ stands for $\Pi(i \colon \mathbb{N}).\ i < m \to \alpha_i = \beta_i$. For this, we have to introduce the less-than relation $<$ as a ground type, or equivalently define it as a Σ-type. Another way is to define a term $\mathsf{agree} \colon (\mathsf{N} \to 2) \to (\mathsf{N} \to 2) \to \mathsf{N} \to 2$ using the primitive recursor as in the previous section. It is provable in type theory that

$$\mathsf{agree}\ \alpha\ \beta\ m = \mathsf{t} \quad \text{iff} \quad \Pi(i \colon \mathbb{N}).\ i < m \to \alpha_i = \beta_i.$$

Because all these definitions are equivalent, they would have equivalent interpretations in any model. Therefore, it does not matter which definition of $\alpha =_m \beta$ that we are working with. Our objective is to show that the type UC is inhabited in the locally cartesian closed category of C-spaces.

As is well known, locally cartesian closed categories [36] and variations, such as categories with attributes [21] and categories with families [15], give models of dependent type theories. Seely's interpretation in locally cartesian closed categories [36] has a coherence issue with type substitution, as pointed out by Curien [13]. This problem can be addressed by changing the syntax to work with explicit substitutions [13], or by changing the semantics to work with categories with attributes [21] or categories with families [15]. A more recent discussion of categorical models of MLTT, relating Curien's [13] and Hofmann's [21] approaches can be found in [14].

Clairambault and Dybjer show that locally cartesian closed categories and categories with families are biequivalent [9]. Using one direction of this biequivalence, which amounts to Hofmann's construction [21], one can translate a locally cartesian closed category $\mathbf{C}$ to a category with families $(\mathbf{C}, T_{\mathbf{C}})$. Then one can show, by induction on types, that $\Gamma \vdash A$ is inhabited in $\mathbf{C}$ iff it is inhabited in $(\mathbf{C}, T_{\mathbf{C}})$. Therefore, we can ignore the coherence issue and work directly with the locally cartesian closed structure of C-**Space** to model the uniform-continuity principle.

Recall that, in Seely's model, a *type* in context $\Gamma \vdash A$ is interpreted as an object, *i.e.* a morphism $\bar{A} \to \bar{\Gamma}$, in the slice category $\mathbf{C}/\bar{\Gamma}$, and a *term* $\Gamma \vdash a : A$ is interpreted as a section of the interpretation of its type:

$$\begin{tikzcd} \bar{A} \arrow[rr, "\llbracket \Gamma \vdash A \rrbracket"] \arrow[dr, equal] & & \bar{\Gamma} \arrow[dl, "\llbracket \Gamma \vdash a : A \rrbracket"] \\ & \bar{A}. & \end{tikzcd}$$

The right adjoint to the pullback functor interprets Π-types, its left adjoint interprets Σ-types, and equalizers interpret identity types. We say the model $\mathbf{C}$ *validates* $\Gamma \vdash A$ iff the interpretation $\llbracket \Gamma \vdash A \rrbracket$ has sections.

Theorem 4.7. *The locally cartesian closed category of* C*-spaces validates the Curry–Howard formulation of the uniform-continuity principle.*

Proof. It is enough to show that the domain of $\llbracket \vdash \mathrm{UC} \rrbracket$ is inhabited. The proof is essentially the same as that of Theorem 4.6, which is carried out using the fan functional.

Since the space $\mathrm{dom}(\llbracket \vdash (\mathsf{N} \to 2) \to \mathsf{N} \rrbracket)$ is equivalent to the exponential $\mathbb{N}^{\mathbf{2}^{\mathbb{N}}}$ in C-**Space**, the underlying set of $\mathrm{dom}(\llbracket \vdash \mathrm{UC} \rrbracket)$ is equivalent to the set of continuous functions $\mathbb{N}^{\mathbf{2}^{\mathbb{N}}} \to \mathrm{dom}(u)$, where u is the interpretation of

$$f : (\mathsf{N} \to 2) \to \mathsf{N} \vdash \Sigma(m : \mathsf{N}).\ \Pi(\alpha : \mathsf{N} \to 2).\ \Pi(\beta : \mathsf{N} \to 2).\ \alpha =_m \beta \to f\alpha = f\beta.$$

The underlying set of $\mathrm{dom}(u)$ is equivalent to the set of pairs (f, m), where $f \in \mathbb{N}^{\mathbf{2}^{\mathbb{N}}}$ and $m \in \mathbb{N}$, such that $f\alpha = f\beta$ whenever $\alpha =_m \beta$. By the definition of the fan functional, the pair $(f, \mathrm{fan}(f))$ is clearly in $\mathrm{dom}(u)$. Therefore, we have a map

$$(f \mapsto (f, \mathrm{fan}(f))) : \mathbb{N}^{\mathbf{2}^{\mathbb{N}}} \to \mathrm{dom}(u)$$

which is continuous because $\mathbb{N}^{\mathbf{2}^{\mathbb{N}}}$ is discrete by Lemma 4.1. $\square$

Notice that the space $\mathrm{dom}(u)$ in the above proof consists of tuples (f, m, ϕ) which satisfy certain conditions. These conditions, together with the continuous map ϕ, amount to saying that m is a modulus of uniform continuity of the map f. Notice that this holds in various flavours of (extensional and intensional) MLTT, and that system T can be regarded as a subsystem of MLTT, and that the proof given here is essentially the same as the one given above for system T, in a slightly different language.

5 Construction of the model in type theory

In the previous sections, we constructed a model *of* type theory in informal set theory. In this section, we discuss the construction of the model *in* type theory, which we formalized in Agda notation. The main purpose of this formalization is to extract computational content from our model of C-spaces, rather than merely certify the correctness of our constructions and proofs.

The above results have been deliberately developed in such a way as to be routinely formalizable in predicative intensional type theory in Martin-Löf's style (MLTT). The main difficulties of formulating the constructions and of proving the theorems involve the absence of function extensionality and the presence of proof relevance in MLTT. We discuss the issues in Section 5.1 and the approaches to address them in Section 5.2.

5.1 Function extensionality and proof relevance

As mentioned above, the first difficulty of the type-theoretic development of the model is the lack of *function extensionality* (funext), that is, for any type X and any type family $x : X \vdash Y(x)$,

$$\Pi(f, g \colon \Pi(x : X).Y(x)).\ (\Pi(x \colon X).fx = gx) \to f = g.$$

Another difficulty is that MLTT is *proof-relevant*. Some issues related to these difficulties arise in the type-theoretic rendering of

1. discrete C-spaces,
2. exponentials of C-spaces, and
3. the fan functional.

(1) and (2) are in a similar situation related to the lack of (funext) for infinite sequences, that is, functions $\mathbb{N} \to \mathbf{2}$. In the informal proofs of (1) and (2), we take a prefix of a sequence, concatenate it to the suffix and then consider this the same sequence as the original one. However, this is not the case in MLTT unless (funext) is available.

(3) is subtler and more interesting: (funext) is seemingly necessary, but also not sufficient for constructing the fan functional and for proving its desired property, due to the presence of proof relevance in MLTT. For instance, uniform continuity is formulated as a Σ-type, *i.e.* a uniformly continuous map is a pair consisting of a underlying map $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ and a witness of uniform continuity. When formalizing the proof that the domain $\mathbb{N}^{\mathbf{2}^{\mathbb{N}}}$ of the fan functional is a discrete space (Lemma 4.1), if we attempt to prove an equality of two uniformly continuous maps, *i.e.* two pairs, we would be able to only obtain an equality of their underlying maps, which is not sufficient, because even for the same map there could be many different witnesses of uniform continuity. By requiring the existence of a *minimal* modulus of uniform continuity, the type that expresses that a map is uniformly continuous can have at most one inhabitant. However, even with this refinement, we still need (funext) to complete the type-theoretic implementation of the fan functional.

5.2 Dealing with the lack of function extensionality

To address the above issues caused by the lack of (funext), we developed the following approaches, all of which have been implemented in Agda:

1. *Use setoids.* This well known approach [22], which is also at the heart of Bishop's approach to constructive analysis [7], consisting of the use of types equipped with equivalence relations, works here with no surprises. But the drawback, as usual, is that it gives a long formalization that obscures the essential aspects of the constructions and proofs.

2. *Simply postulate* (funext). This is of course the easiest approach, but would potentially destroy the computational content of formal proofs, because then (funext) becomes a constant without a computational rule. Thus, although we obtain a clean formalization, we potentially lose computational content.

3. *Postulate* (funext) *within a computationally irrelevant field.* After the previous approach (2) was completed, we observed that our uses of (funext) do not really have computational content, and we used a feature of Agda, called irrelevant fields [1], to formulate and prove this observation. In practice, we actually needed to slightly modify approach (2) to make this idea work. In any case, the drawback is that it requires the extension of type theory with such irrelevant fields, which we would prefer to avoid.

4. *Postulate the double negation of* (funext). In turn, after we completed approach (3), we observed that it does not really depend on the nature of irrelevant fields, but only on the fact that irrelevant fields form a monad T with $T\emptyset \to \emptyset$. As is well known, double negation is the final such monad. There are two advantages with this approach: (i) we do not need to work with a non-standard extension of MLTT, and (ii) postulating negative, consistent axioms does not destroy computational content [10].

With the last approach (4), we achieve our main aim of extracting computational content from type-theoretic proofs that use the principle UC, in a relatively clean way, avoiding the usual bureaucracy associated with setoids.

5.3 Summary and relevance of the formalization in Agda

We formalized all the results of this paper in Agda, except Section 3.1, which anyway is non-constructive and plays mainly the role of convincing ourselves that we are working in the right category. The equivalence of the full type hierarchy and the Kleene–Kreisel hierarchy assuming UC seems to unavoidably rely on (funext), but for the other developments of this paper, $\neg\neg$funext is enough, and, moreover, postulating this does not break the computational content of the type-theoretic proofs [10]. We also developed a formalization of the conference version [40] of this paper using setoids rather than $\neg\neg$funext, which is much more cumbersome.

The formalization of Section 4.3 gives a practical illustration that it is possible to extract computational content from type-theoretic proofs that use the Brouwerian principle UC, without non-constructive axioms in the correctness of the computational extraction process. This is justified by the fact that the extraction can be done and proved in intensional Martin-Löf type theory without any added axiom (if we use setoids) or just with the double negation of function extensionality as a postulate (which gives a more direct and transparent approach). At the moment, our model does not address universes, and we regard this as an (important and) interesting open problem.

From a practical points of view, we are also interested in sharper information about uniform continuity, in the sense that a finite part of the input, not necessarily an initial segment, decides the output. For this, we may consider an alternative coverage based on overwriting maps

$$\mathrm{overwrite}_{(n,b)} \colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}},$$

where $n \in \mathbb{N}$ and $b \in \mathbf{2}$ indicate that the nth bit of the input sequence is to be overwritten to value b. This is also expected to reduce some computations from exponential to linear time, but this is left for future investigation.

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