Kripke semantics for full ground references (work in progress)

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“Full ground references” means references to integers and to other references, but not functions or thunks. Game semantics for full ground references was given in [1]. This work aims to give Kripke semantics.

**Language** We fix a set $S$ of sorts. We extend the call-by-push-value types with a reference type $\text{Ref}_s$ for each sort $s$. We then define full ground types $D ::= 0 \mid D + D \mid \sum_{i \in \mathbb{N}} D_i \mid 1 \mid D \times D \mid \text{Ref}_s \ (s \in S)$ and fix, for each sort $s$, a full ground type $D_s$, intended to be the type of values stored in a reference cell of sort $s$.

A world is a finite set over $S$. (Alternatively: a finite sequence of sorts.) The judgements are $w, \Gamma \vdash v : A$ for values and $w, \Gamma \vdash c : B$ for computations, where $w$ is a world and $\Gamma$ a typing context (finite set of identifiers with an associated value type). Term syntax and operational semantics is defined as usual. Evaluation terminates because of the restriction to full ground references.

**Denotational semantics** A value type $A$ will denote a functor from the category $\text{Inj}$ of worlds and injections to $\text{Set}$. Intuitively, $[A]_w$ is a semantic domain for closed values in world $w$, and such values can be renamed along an injection $w \to w'$. In particular $[\text{Ref}_s]_w$ is the set of $s$-sorted cells in $w$.

A $w$-store associates to each cell $l$ in $w$ a value $w \vdash v_{\text{sort}(l)} : D_{\text{sort}(l)}$. The semantic domain for these is

$$S_w = \prod_{l \in w} [D_{\text{sort}(l)}]_w$$

Although for ground references $S$ is a functor $\text{Inj}^{op} \to \text{Set}$, that is not so for full ground references.

Computation types have more subtle semantics. To understand it, say that a $w$-store $s$ associates to each cell $l$ in $w$ a value $w \vdash v_{\text{sort}(l)} : D_{\text{sort}(l)}$. An SC-configuration $\Gamma \vdash_{sc} x, s, M : B$ consists of

- a world $x$—we think of cells in $x$ as local/private, whereas those in $w$ are global/public
- a $w + x$-store $s$
- and a computation $w + x, \Gamma \vdash M : B$.

These arise in the operational semantics of a language that has both $w$-many global cells and generation of local cells.

We want $[B]_w$ to be a semantic domain for closed SC-configurations in world $w$. What category should $[B]$ be a functor from?

Let’s start with the ground ref setting. An initialisation $(i, p) : w \to w'$ consists of

- an injection $i : w \to w'$—we write $\text{new}(i)$ for the cells in $w'$ not in the range of $i$
- and an element $p \in \prod_{l \in \text{new}(i)} [D_{\text{sort}(l)}]$

These form a category $\text{Init}$. A partial initialisation is defined similarly, except that $i$ is a partial injection. The latter form a category $\text{PInit}$ that contains both $\text{Inj}^{op}$ and $\text{Init}$, and indeed is freely generated by these subcategories modulo two equations.

A partial initialisation $i : w \to w'$ converts an SC-configuration in world $w$ to one in world $w'$, by
• hiding the cells in \( w \) that are not in the domain of \( i \)
• renaming each cell in the domain of \( i \) to one in the range
• creating each cell \( l \in \text{new}(i) \) with value \( p_l \).

So a computation type \( B \) should denote a functor \( \text{PInit} \to \text{Set} \).

Turning to full ground references, an initialization \( (i, p) : w \to w' \) consists of
• an injection \( i : w \to w' \)
• and an element \( p \in \prod_{l \in \text{new}(i)} \| D_{\text{sort}(i)} \| (w + \text{new}(i)) \)

These form a category \( \text{Init} \), and \( S \) is a functor \( \text{Init} \to \text{Set} \). But partial initializations are more subtle. Let’s first say that a stateful value \( \Gamma \vdash^v x, s, V : A \) consists of
• a world \( x \)—again, cells in \( x \) are local/private, whilst those in \( x \) are global/public
• for each local cell \( l \) in \( x \), a value \( w + x \vdash^v s_l : D_{\text{sort}(l)} \)
• and a value \( w + x \vdash^v V : A \).

For any functor \( A : \text{Inj} \to \text{Set} \), define
\[
(\Psi A) w \overset{\text{def}}{=} \int_{x \in \text{Init}} \prod_{l \in x} \| D_{\text{sort}(l)} \| (w + x) \times [A](w + x)
\]
so that \( (\Psi[A]) w \) is a semantic domain for closed stateful values of type \( A \) in world \( w \). If \( A \) is a constant functor, then \( (\Psi A) w \cong Aw \). A partial initialization \( (i, p) : w \to w' \) consists of
• a partial injection \( i : w \to w' \)
• and an element \( p \in (\Psi \prod_{l \in \text{new}(i)} [D_{\text{sort}(i)}])((w + \text{new}(i)) \).

These form a category \( \text{PInit}^{\text{op}} \) that contains both \( \text{Inj}^{\text{op}} \) and \( \text{Init} \) and indeed is freely generated by these subcategories modulo two equations. A computation type \( B \) should denote a functor \( \text{PInit} \to \text{Set} \).

Also, if \( A : \text{Inj} \to \text{Set} \) then \( \Psi A : \text{PInit}^{\text{op}} \to \text{Set} \).

Semantics of judgements:
• A value \( w, \Gamma \vdash^v V : A \) denotes a family of functions \( [\Gamma](w + x) \to \| A \| (w + x) \), natural in \( x \in \text{Inj} \).
• A computation \( w, \Gamma \vdash^v M : B \) denotes a family of functions \( S(w + x) \times [\Gamma](w + x) \to \| B \| (w + x) \), natural in \( x \in \text{Init} \). Here overline represents the forgetful functor \( \text{Init} \to \text{Inj} \).

Semantics of types:
\[
\| FA \| w \overset{\text{def}}{=} \int_{x \in \text{Init}} S(w + x) \times [A](w + x)
\]
\[
\| UB \| w \overset{\text{def}}{=} \int_{x \in \text{Init}} S(w + x) \to \| B \| (w + x)
\]
\[
\prod_{i \in I} \| B_i \| w \overset{\text{def}}{=} \prod_{i \in I} \| B_i \| w
\]
\[
\| A \to B \| w \overset{\text{def}}{=} \int_{x \in \text{Init}} [A](w + x) \to \| B \| (w + x)
\]
\[
\| U \prod_{i \in I} (A_i \to B_i) \| w \overset{\text{def}}{=} \int_{x \in \text{Init}} S(w + x) \to \prod_{i \in I} ([A_i](w + x) \to \| B_i \| (w + x))
\]

References