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Locally finite groups in which every non-cyclic subgroup is self-centralizing

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Abstract

Locally finite groups having the property that every non-cyclic subgroup contains its centralizer are classified.

Keywords: Self-centralizing subgroup, Frobenius group, locally finite group

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1. Introduction

A subgroup H of a group G is *self-centralizing* if the centralizer $C_G(H)$ is contained in H . In [1] it has been remarked that a locally graded group in which all non-trivial subgroups are self-centralizing has to be finite; therefore it has to
5 be either cyclic of prime order or non-abelian of order being the product of two different primes.

In this article, we consider the more extensive class \mathfrak{X} of all groups in which every non-cyclic subgroup is self-centralizing. In what follows we use the term \mathfrak{X} -groups in order to denote groups in the class \mathfrak{X} . The study of properties of
10 \mathfrak{X} -groups was initiated in [1]. In particular, the first four authors determined the structure of finite \mathfrak{X} -groups which are either nilpotent, supersoluble or simple.

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In this paper, Theorem 2.1 gives a complete classification of finite \mathfrak{X} -groups. We remark that this result does not depend on classification of the finite simple groups rather only on the classification of groups with dihedral or semidihedral
15 Sylow 2-subgroups. We also determine the infinite soluble \mathfrak{X} -groups, and the locally finite \mathfrak{X} -groups the results being presented in Theorems 3.6 and 3.7. It turns out that these latter groups are suitable finite extensions either of the infinite cyclic group or of a Prüfer p -group, \mathbb{Z}_{p^∞} , for some prime p .

We follow [2] for basic group theoretical notation. In particular, we note that
20 $F^*(G)$ denotes the generalized Fitting subgroup of G , that is the subgroup of G generated by all subnormal nilpotent or quasisimple subgroups of G . The latter subgroups are the components of G . We see from [2, Section 31] that distinct components commute. The fundamental property of the generalized Fitting subgroup that we shall use is that it contains its centralizer in G [2, (31.13)].
25 We denote the alternating group and symmetric group of degree n by $\text{Alt}(n)$ and $\text{Sym}(n)$ respectively. We use standard notation for the classical groups. The notation $\text{Dih}(n)$ denotes the dihedral group of order n and Q_8 is the quaternion group of order 8. The term quaternion group will cover groups which are often called generalized quaternion groups. The cyclic group of order n is represented
30 simply by n , so for example $\text{Dih}(12) \cong 2 \times \text{Dih}(6) \cong 2 \times \text{Sym}(3)$. Finally $\text{Mat}(10)$ denotes the Mathieu group of degree 10. The Atlas [3] conventions are used for group extensions. Thus, for example, $p^2:\text{SL}_2(p)$ denotes the split extension of an elementary abelian group of order p^2 by $\text{SL}_2(p)$.

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35 We would like to thank Professor Hermann Heineken for pointing out an oversight in the statement of (2.1.3) of Theorem 2.1. We also thank our referee for valuable suggestions.

2. Finite \mathfrak{X} -groups

In this section we determine all the finite groups belonging to the class \mathfrak{X} .

40 The main result is the following.

Theorem 2.1. *Let G be a finite \mathfrak{X} -group. Then one of the following holds:*

(1) *If G is nilpotent, then either*

(1.1) *G is cyclic;*

(1.2) *G is elementary abelian of order p^2 for some prime p ;*

45 (1.3) *G is an extraspecial p -group of order p^3 for some odd prime p ; or*

(1.4) *G is a dihedral, semidihedral or quaternion 2-group.*

(2) *If G is supersoluble but not nilpotent, then, letting p denote the largest prime divisor of $|G|$ and $P \in \text{Syl}_p(G)$, we have that P is a normal subgroup of G and one of the following holds:*

50 (2.1) *P is cyclic and either*

(2.1.1) *$G \cong D \rtimes C$, where C is cyclic, D is cyclic and *every non-trivial element of D acts fixed point freely on C (so G is a Frobenius group);**

55 (2.1.2) *$G = D \rtimes C$, where C is a cyclic group of odd order, D is a quaternion group, and $C_G(C) = C \times D_0$ where D_0 is a cyclic subgroup of index 2 in D with G/D_0 a dihedral group; or*

(2.1.3) *$G = D \rtimes C$, where D is a cyclic q -group, C is a cyclic q' -group (here q denotes the smallest prime dividing the order of G), $1 < Z(G) < D$ and $G/Z(G)$ is a Frobenius group;*

60 (2.2) *P is extraspecial and G is a Frobenius group with cyclic Frobenius complement of odd order dividing $p - 1$.*

(3) *If G is not supersoluble and $F^*(G)$ is nilpotent, then either (3.1) or (3.2) below holds.*

(3.1) *$F^*(G)$ is elementary abelian of order p^2 , $F^*(G)$ is a minimal normal subgroup of G and one of the following holds:*

65

(3.1.1) *$p = 2$ and $G \cong \text{Sym}(4)$ or $G \cong \text{Alt}(4)$; or*

(3.1.2) p is odd and $G = G_0 \rtimes N$ is a Frobenius group with Frobenius kernel N and Frobenius complement G_0 which is itself an \mathfrak{X} -group. Furthermore, either

- 70 (3.1.2.1) G_0 is cyclic of order dividing $p^2 - 1$ but not dividing $p - 1$;
 (3.1.2.2) G_0 is quaternion;
 (3.1.2.3) G_0 is supersoluble as in (2.1.2) with $|C|$ dividing $p - \epsilon$ where $p \equiv \epsilon \pmod{4}$;
 (3.1.2.4) G_0 is supersoluble as in (2.1.3) with D a 2-group, $C_D(C)$ a
 75 non-trivial maximal subgroup of D and $|C|$ odd dividing $p - 1$ or $p + 1$;
 (3.1.2.5) $G_0 \cong \mathrm{SL}_2(3)$;
 (3.1.2.6) $G_0 \cong \mathrm{SL}_2(3) \cdot 2$ and $p \equiv \pm 1 \pmod{8}$; or
 (3.1.2.7) $G_0 \cong \mathrm{SL}_2(5)$ and 60 divides $p^2 - 1$.

80 (3.2) $F^*(G)$ is extraspecial of order p^3 and one of the following holds:

- (3.2.1) $G \cong \mathrm{SL}_2(3)$ or $G \cong \mathrm{SL}_2(3) \cdot 2$ (with quaternion Sylow 2-subgroups of order 16); or
 (3.2.2) $G = K \rtimes N$ where N is extraspecial of order p^3 and exponent p with p an odd prime, K centralizes $Z(N)$ and is cyclic of odd
 85 order dividing $p + 1$. Furthermore, $G/Z(N)$ is a Frobenius group.

(4) If $F^*(G)$ is not nilpotent, then either

- (4.1) $F^*(G) \cong \mathrm{SL}_2(p)$ where p is a Fermat prime, $|G/F^*(G)| \leq 2$ and G has quaternion Sylow 2-subgroups; or
 (4.2) $G \cong \mathrm{PSL}_2(9)$, $\mathrm{Mat}(10)$ or $\mathrm{PSL}_2(p)$ where p is a Fermat or Mersenne
 90 prime.

Furthermore, all the groups listed above are \mathfrak{X} -groups.

We make a brief remark about the group $\mathrm{SL}_2(3) \cdot 2$ and the groups appearing in part (4.1) of Theorem 2.1 in the case $G > F^*(G)$. To obtain such groups, take $F = \mathrm{SL}_2(p^2)$, then the groups in question are isomorphic to the normalizer
 95 in F of the subgroup isomorphic to $\mathrm{SL}_2(p)$. We denote these groups by $\mathrm{SL}_2(p) \cdot 2$

to indicate that the extension is not split (there are no elements of order 2 in the outer half of the group).

We shall repeatedly use the fact that if L is a subgroup of an \mathfrak{X} -group X , then L is an \mathfrak{X} -group. Indeed, if $H \leq L$ is non-cyclic, then $C_L(H) \leq C_X(H) \leq H$.

100 The following elementary facts will facilitate our proof that the examples listed are indeed \mathfrak{X} -groups.

Lemma 2.2. *The finite group X is an \mathfrak{X} -group if and only if $C_X(x)$ is an \mathfrak{X} -group for all $x \in X$ of prime order.*

Proof. If X is an \mathfrak{X} -group, then, as \mathfrak{X} is subgroup closed, $C_X(x)$ is an \mathfrak{X} -group
 105 for all $x \in X$ of prime order. Conversely, assume that $C_X(x)$ is an \mathfrak{X} -group for all $x \in X$ of prime order (and hence of any order). Let $H \leq X$ be non-cyclic. We shall show $C_X(H) \leq H$. If $C_X(H) = 1$, then $C_X(H) \leq H$ and we are done. So assume $x \in C_X(H)$ and $x \neq 1$. Then $H \leq C_X(x)$ which is an \mathfrak{X} -group. Hence $x \in C_{C_X(x)}(H) \leq H$. Therefore $C_X(H) \leq H$, and X is an \mathfrak{X} -group. \square

110 **Lemma 2.3.** *Suppose that X is a Frobenius group with kernel K and complement L . If K and L are \mathfrak{X} -groups, then X is an \mathfrak{X} -group.*

Proof. Let $x \in X$ have prime order. Then, as K and L have coprime orders, $x \in K$ or x is conjugate to an element of L . But then, since X is a Frobenius group, either $C_X(x) \leq K$ or $C_X(x)$ is conjugate to a subgroup of L . Since
 115 K and L are \mathfrak{X} -groups, $C_X(x)$ is an \mathfrak{X} -group. Hence X is an \mathfrak{X} -group by Lemma 2.2. \square

The rest of this section is dedicated to the proof of Theorem 2.1; therefore G always denotes a finite \mathfrak{X} -group. Parts (1) and (2) of Theorem 2.1 are already proved in [1, Theorems 2.2, 2.4, 3.2 and 3.4]. However, our statement in (2.1.3)
 120 adds further detail which we now explain. So, for a moment, assume that G is supersoluble, q is the smallest prime dividing $|G|$, D is a cyclic q -group and C is a cyclic q' -group. In addition, $1 \neq Z(G) = C_D(C)$. Assume that $d \in D \setminus Z(G)$. Then, as $d \notin Z(G)$, C is not centralized by d . By coprime action, $C = [C, d] \times C_C(d)$ and so $Y = [C, d]\langle d \rangle$ is centralized by $C_C(d)$. As Y

125 is non-abelian and $C_C(d) \cap Y = 1$, we deduce that $C_C(d) = 1$. Hence $G/Z(G)$ is a Frobenius group. This means that we can assume that (1) and (2) hold and, in particular, we assume that G is not supersoluble.

The following lemma provides the basic case subdivision of our proof.

Lemma 2.4. *One of the following holds:*

- 130 (i) $F^*(G)$ is elementary abelian of order p^2 for some prime p .
(ii) $F^*(G)$ is extraspecial of order p^3 for some prime p .
(iii) $F^*(G)$ is quasisimple.

Proof. Suppose first that $F^*(G)$ is nilpotent. Then its structure is given in part (1) of Theorem 2.1. Suppose that $F^*(G)$ is cyclic. Since $C_G(F^*(G)) = F^*(G)$, 135 we have $G/F^*(G)$ is isomorphic to a subgroup of $\text{Aut}(F^*(G))$. Because the automorphism group of a cyclic group is abelian, we have that G is supersoluble. Therefore, by our assumption concerning G , $F^*(G)$ is not cyclic. Hence $F^*(G)$ is either elementary abelian of order p^2 for some prime p , is extraspecial of order p^3 for some odd prime p or $F^*(G)$ is a dihedral, semidihedral or quaternion 2- 140 group. Since the automorphism groups of dihedral, semidihedral and quaternion groups of order at least 16 are 2-groups, we deduce that when $p = 2$ and $F^*(G)$ is non-abelian, $F^*(G)$ is extraspecial. This proves the lemma when $F^*(G)$ is nilpotent.

If $F^*(G)$ is not nilpotent, then there exists a component $K \leq F^*(G)$. As 145 $F^*(G) = C_{F^*(G)}(K)K$ and K is non-abelian, we have $F^*(G) = K$ and this is case (iii). \square

Lemma 2.5. *Suppose that p is a prime and $F^*(G)$ is extraspecial of order p^3 . Then one of the following holds:*

- (i) $G \cong \text{SL}_2(3)$, $G \cong \text{SL}_2(3) \cdot 2$ (with quaternion Sylow 2-subgroups of order 16); or 150
(ii) $G = NK$ where N is extraspecial of order p^3 of exponent p with p an odd prime, K centralizes $Z(N)$ and is cyclic of odd order dividing $p + 1$. Furthermore, $G/Z(N)$ is a Frobenius group.

Proof. Let $N = F^*(G)$. We have that N is extraspecial of order p^3 by assumption.
155 Suppose first that $p = 2$, then we have $N \cong Q_8$ as the dihedral group of order 8 has no odd order automorphisms and G is not a 2-group. Since $\text{Aut}(Q_8) \cong \text{Sym}(4)$, $G/Z(N)$ is isomorphic to a subgroup of $\text{Sym}(4)$ containing $\text{Alt}(4)$. If $G/Z(N) \cong \text{Alt}(4)$, then $G = NT \cong \text{SL}_2(3)$ where T is a cyclic subgroup of order 3. When $G/Z(N) \cong \text{Sym}(4)$, taking $T \in \text{Syl}_3(G)$, we have
160 $NT \cong \text{SL}_2(3)$, $N_G(T)$ has order 12 and $N_G(T)/Z(N) \cong \text{Sym}(3)$. Since $N_G(T)$ is an \mathfrak{X} -group and $N_G(T)$ is supersoluble, we see that $N_G(T)$ is a product DT where D is cyclic of order 4 by (2.1.3). Because the Sylow 2-subgroups of G are either dihedral, semidihedral or quaternion and $D \not\leq N$, we see that ND is quaternion. Thus $G \cong \text{SL}_2(3) \cdot 2$ as claimed in (i).

165 Assume that p is odd. We know that the outer automorphism group of N is isomorphic to a subgroup of $\text{GL}_2(p)$ and $C_{\text{Aut}(N)}(Z(N))/\text{Inn}(N)$ is isomorphic to a subgroup of $\text{SL}_2(p)$. Since p is odd and the Sylow p -subgroups of G are \mathfrak{X} -groups, we have $N \in \text{Syl}_p(G)$ and G/N is a p' -group by part (1) of Theorem 2.1. Set $Z = Z(N)$. Since G/N and N have coprime orders, the
170 Schur Zassenhaus Theorem says that G contains a complement K to N . Set $K_1 = C_K(Z)$. Then K_1 commutes with Z and so K_1 is cyclic. If $K_1 = 1$, then $|K|$ divides $p - 1$ and we find that G is supersoluble, which is a contradiction. Hence $K_1 \neq 1$. Let $x \in K_1$. Then $[N, x]$ and $C_N(x)$ commute by the Three Subgroups Lemma. Hence $C_N(x)$ centralizes $[N, x]\langle x \rangle$ which is non-abelian. It
175 follows that $[N, x] = N$ and $C_N(x) = Z$. If $\langle x \rangle$ does not act irreducibly on N/Z , then there exists $Z < N_1 < N$ which is $\langle x \rangle$ -invariant. If N_1 is cyclic, then, as $\langle x \rangle$ centralizes $\Omega_1(N_1) = Z$, $\langle x \rangle$ centralizes $N_1 > Z$, a contradiction. If N_1 is elementary abelian, then, as $\langle x \rangle$ centralizes Z , $[N_1, \langle x \rangle]$ has order at most p by Maschke's Theorem. If $[N_1, \langle x \rangle] \neq 1$, then $[N_1, \langle x \rangle]\langle x \rangle$ is non-abelian and
180 Z centralizes $[N_1, \langle x \rangle]\langle x \rangle$, a contradiction. Hence $\langle x \rangle$ centralizes N_1 contrary to $C_N(\langle x \rangle) = Z$. We conclude that every element of K_1 acts irreducibly on $N/Z(N)$. In particular, since K_1 is isomorphic to a subgroup of $\text{SL}_2(p)$, we have that K_1 is cyclic of odd order dividing $p + 1$. Furthermore, as K_1 acts irreducibly on $N/Z(N)$, N has exponent p .

185 By the definition of K_1 , $|K/K_1|$ divides $|\text{Aut}(Z)| = p - 1$. Assume that
 $K \neq K_1$ and let $y \in K \setminus K_1$ have prime order r . Then r does not divide $|K_1|$
and $Z\langle y \rangle$ is non-abelian. Since K_1 centralizes Z , we have $C_{K_1}(y) = 1$. Let
 $w \in K_1$ have prime order q . Then $\langle y \rangle \langle w \rangle$ is non-abelian and acts faithfully on
 $V = N/Z$. Therefore [2, 27.18] implies that $C_N(y) \neq 1$. As $C_N(y) \cap Z = 1$ and
190 $C_N(y)$ centralizes $Z\langle y \rangle$, we have a contradiction. Hence $K = K_1$. Finally, we
note that $NK/Z(N)$ is a Frobenius group.

It remains to show that the groups listed are \mathfrak{X} -groups. We consider the
groups listed in (ii) and leave the groups in (i) to the reader. Assume that
 $H \leq G$ is non-cyclic. We shall show that $C_G(H) \leq H$. If $H \geq N$, then
195 $C_G(H) \leq C_G(N) \leq N \leq H$ and we are done. Suppose that $H \leq N$. Then,
as N is extraspecial of exponent p , H is elementary abelian of order p^2 and
 $C_N(H) = H$. Since G/N is cyclic of odd order dividing $p + 1$, we see that
 $N_G(H) = N$ and so $C_G(H) = C_N(H) = H$ and we are done in this case.
Suppose that $H \not\leq N$ and $N \not\leq H$. Let $h \in H \setminus N$. Then, as $|G/N|$ divides
200 $p + 1$ and is odd, we either have $H \cap N = N$ or $H \cap N = Z$. So we must have
 $H \cap N = Z = Z(G)$. Now $H/Z \cong G/N$ is cyclic of order dividing $p + 1$ and so
we get that H is cyclic, a contradiction. Thus G is an \mathfrak{X} -group. \square

Lemma 2.6. *Suppose that $N = F^*(G)$ is elementary abelian of order p^2 . Then
one of the following holds:*

- 205 (i) $p = 2$, $G \cong \text{Sym}(4)$ or $\text{Alt}(4)$; or
- (ii) p is odd and $G = NG_0$ is a Frobenius group with Frobenius kernel N and
Frobenius complement G_0 which is itself an \mathfrak{X} -group. Furthermore, either
 - (a) G_0 is cyclic of order dividing $p^2 - 1$ but not dividing $p - 1$;
 - (b) G_0 is quaternion;
 - 210 (c) G_0 is supersoluble as in part (2.1.2) of Theorem 2.1 with $|C|$ dividing
 $p - \epsilon$ where $p \equiv \epsilon \pmod{4}$;
 - (d) G_0 is supersoluble as in part (2.1.3) of Theorem 2.1 with D a 2-group,
 $C_D(C)$ a non-trivial maximal subgroup of D and $|C|$ odd dividing
 $p - 1$ or $p + 1$;

- 215 (e) $G_0 \cong \text{SL}_2(3)$;
 (f) $\text{SL}_2(3) \cdot 2$ and $p \equiv \pm 1 \pmod{8}$; or
 (g) $G_0 \cong \text{SL}_2(5)$ and 60 divides $p^2 - 1$.

Furthermore, all the groups listed are \mathfrak{X} -groups.

220 *Proof.* We have N has order p^2 , is elementary abelian and G/N is isomorphic to a subgroup of $\text{GL}_2(p)$. If $p = 2$, then we quickly obtain part (i). So assume that p is odd.

Suppose that p divides the order of G/N . Let $P \in \text{Syl}_p(G)$. Then P is extraspecial of order p^3 and P is not normal in G . Hence by [4, Theorem 2.8.4] there exists $g \in G$ such that $G \geq K = \langle P, P^g \rangle \cong p^2:\text{SL}_2(p)$. Let $Z = Z(P)$, t be an involution in K , $K_0 = C_K(t)$ and $P_0 = P \cap K_0$. Then, as t inverts N , 225 $K_0 \cong \text{SL}_2(p)$, P_0 has order p and centralizes $Z\langle t \rangle$, which is a contradiction as $Z\langle t \rangle \cong \text{Dih}(2p)$. Hence G/N is a p' -group.

Suppose that $x \in G \setminus N$. If $C_N(x) \neq 1$, then $C_N(x)$ centralizes $[N, x]\langle x \rangle$ which is non-abelian, a contradiction. Thus $C_N(x) = 1$ for all $x \in G \setminus N$. It 230 follows that G is a Frobenius group with Frobenius kernel N . Let G_0 be a Frobenius complement to N . As $G_0 \leq G$, G_0 is an \mathfrak{X} -group. Recall that the Sylow 2-subgroups of G_0 are either cyclic or quaternion and that the odd order Sylow subgroups of G_0 are all cyclic [5, V.8.7].

Assume that N is not a minimal normal subgroup of G . Then G/N is 235 conjugate in $\text{GL}_2(p)$ to a subgroup of the diagonal subgroup. Therefore G is supersoluble, which is a contradiction. Hence N is a minimal normal subgroup of G and G_0 is isomorphic to an irreducible subgroup of $\text{GL}_2(p)$. This completes the general description of the structure of G . It remains to determine the structure of G_0 .

240 If G_0 is nilpotent, then Theorem 2.1 (1) applies to give G_0 is either quaternion or cyclic. In the latter case, as G_0 acts irreducibly on N it is isomorphic to a subgroup of the multiplicative group of $\text{GF}(p^2)$ and is not of order dividing $p - 1$. This gives the structures in (ii) (a) and (b).

If G_0 is supersoluble, then the structure of G_0 is described in part (2.1) of

245 Theorem 2.1, as $\mathrm{GL}_2(p)$ contains no extraspecial subgroups of odd order. We
adopt the notation from (2.1). By [5, V.8.18 c)], $Z(G_0) \neq 1$. Hence (2.1.1)
cannot occur. Case (2.1.2) can occur and, as C commutes with a cyclic subgroup
of order at least 4 and G_0 is isomorphic to a subgroup of $\mathrm{GL}_2(p)$, $|C|$ divides
 $p - 1$ if $p \equiv 1 \pmod{4}$ and $|C|$ divides $p + 1$ if $p \equiv 3 \pmod{4}$. In the situation
250 described in part (2.1.3) of Theorem 2.1, the groups have no 2-dimensional
faithful representations unless $q = 2$ and $C_D(C)$ has index 2. In this case $|C|$ is
an odd divisor of $p - 1$ or $p + 1$.

Suppose that G_0 is not supersoluble. Refereing to Lemma 2.4 and using the
fact that the Sylow subgroups of G_0 are either cyclic or quaternion, we have
255 that $F^*(G_0)$ is either quaternion of order 8 or $F^*(G_0)$ is quasisimple. In the first
case we obtain the structures described in parts (b), (e) and (f) from Lemma 2.5
where for part (f) we note that we require $\mathrm{SL}_2(p)$ to have order divisible by 16.

If $F^*(G_0)$ is quasisimple, then Zassenhaus's Theorem [6, Theorem 18.6, p.
204] gives $G_0 = WM$ where $W \cong \mathrm{SL}_2(5)$ and M is metacyclic. Since G_0 is an
260 \mathfrak{X} -group, this means that $M \leq W$ and $G_0 \cong \mathrm{SL}_2(5)$. Since $\mathrm{SL}_2(5)$ is isomorphic
to a subgroup of $\mathrm{GL}_2(p)$ only when $p = 5$ or 60 divides $p^2 - 1$ and $p \neq 5$ part
(g) holds.

That $\mathrm{Sym}(4)$ and $\mathrm{Alt}(4)$ are \mathfrak{X} -groups is easy to check. The groups listed in
(ii) are \mathfrak{X} -groups by Lemma 2.3. □

265 The finite simple \mathfrak{X} -groups are determined in [1]. We have to extend the
arguments to the cases where $F^*(G)$ is simple or quasisimple. This is relatively
elementary.

Lemma 2.7. *Suppose that $F^*(G)$ is simple. Then $G \cong \mathrm{SL}_2(4)$, $\mathrm{PSL}_2(9)$,
 $\mathrm{Mat}(10)$ or $\mathrm{PSL}_2(p)$ where p is a Fermat or Mersenne prime.*

270 *Proof.* Set $H = F^*(G)$. As \mathfrak{X} is subgroup closed, H is an \mathfrak{X} -group and so H
is one of the groups listed in the statement by Theorem 3.7 of [1]. Hence we
obtain $H \cong \mathrm{SL}_2(4)$, $\mathrm{PSL}_2(9)$ or $\mathrm{PSL}_2(p)$ for p a Fermat or Mersenne prime.

Suppose that $G > H$. If $H \cong \mathrm{SL}_2(4)$, then $G \cong \mathrm{Sym}(5)$ and the subgroup
 $2 \times \mathrm{Sym}(3)$ witnesses the fact that $\mathrm{Sym}(5)$ is not an \mathfrak{X} -group. Suppose $H \cong$

275 $\mathrm{PSL}_2(9) \cong \mathrm{Alt}(6)$. If $G \geq K \cong \mathrm{Sym}(6)$, then G contains $\mathrm{Sym}(5)$ which is impossible. Therefore $G \cong \mathrm{PGL}_2(9)$ or $G \cong \mathrm{Mat}(10)$. In the first case, G contains a subgroup $\mathrm{Dih}(20) \cong 2 \times \mathrm{Dih}(10)$ which is impossible. Thus $G \cong \mathrm{Mat}(10)$ and this group is easily shown to satisfy the hypothesis as all the centralizer of elements of prime order are \mathfrak{X} -groups.

280 If $H \cong \mathrm{PSL}_2(p)$, p a Fermat or Mersenne prime, then $G \cong \mathrm{PGL}_2(p)$ and contains a dihedral group of order $2(p+1)$ and one of order $2(p-1)$. One of these is not a 2-group and this contradicts G being an \mathfrak{X} -group. \square

Lemma 2.8. *Suppose that $F^*(G)$ is quasisimple but not simple. Then $F^*(G) \cong \mathrm{SL}_2(p)$ where p is a Fermat prime, $|G/H| \leq 2$ and G has quaternion Sylow*
 285 *2-subgroups.*

Proof. Let $H = F^*(G)$ and $Z = Z(H)$. Since H centralizes Z , we have Z is cyclic. Let $S \in \mathrm{Syl}_2(H)$. If $Z \not\leq S$, then S must be cyclic. Since groups with a cyclic Sylow 2-subgroup have a normal 2-complement [2, 39.2], this is impossible. Hence $Z \leq S$. In particular, $Z(G) \neq 1$ as the central involution of H is central
 290 in G . It follows also that all the odd order Sylow subgroups of G are cyclic. By part (1) of Theorem 2.1, S is either abelian, dihedral, semidihedral or quaternion. If S is abelian, then S/Z is cyclic and again we have a contradiction. So S is non-abelian. Thus S/Z is dihedral (including elementary abelian of order 4). Hence $H/Z \cong \mathrm{Alt}(7)$ or $\mathrm{PSL}_2(q)$ for some odd prime power q [4, Theorem 16.3].
 295 Since the odd order Sylow subgroups of G are cyclic, we deduce that $H \cong \mathrm{SL}_2(p)$ for some odd prime p . If $p-1$ is not a power of 2, then H has a non-abelian subgroup of order pr where r is an odd prime divisor of $p-1$ which is centralized by Z . Hence p is a Fermat prime.

Suppose that $G > H$ with $H \cong \mathrm{SL}_2(p)$, p a Fermat prime. Note G/H has
 300 order 2. Let $S \in \mathrm{Syl}_2(G)$. Then $S \cap H$ is a quaternion group. Suppose that S is not quaternion Then there is an involution $t \in S \setminus H$. By the Baer-Suzuki Theorem, there exists a dihedral group D of order $2r$ for some odd prime r which contains t . Since D and Z commute, this is impossible. Hence S is quaternion. This gives the structure described in the lemma.

305 It remains to demonstrate that the groups $\mathrm{SL}_2(p)$ and $\mathrm{SL}_2(p)\cdot 2$ with p a Fermat prime are indeed \mathfrak{X} -groups. Let G denote one of these group, $H = F^*(G) \cong \mathrm{SL}_2(p)$. Recall from the comments just after the statement of Theorem 2.1 that G is isomorphic to a subgroup of $X = \mathrm{SL}_2(p^2)$. Let V be the natural $\mathrm{GF}(p^2)$ representation of X and thereby a representation of G . Assume that $L \leq G$ is non-cyclic. Since H has no abelian subgroups which are not cyclic, L is non-abelian and L acts irreducibly on V . Schur's Lemma implies that $C_X(L)$ consists of scalar matrices and so has order at most 2. If L has even order, then as G has quaternion Sylow 2-subgroups, $L \geq C_G(L)$. So suppose that L has odd order. Then using Dickson's Theorem [7, 260, page 285], as p is a Fermat prime, we find that L is cyclic, a contradiction. Thus G is an \mathfrak{X} -group. \square

Proof of Theorem 2.1. This follows from the combination of the lemmas in this section. \square

3. Locally finite \mathfrak{X} -groups

It has been proved in [1, Theorem 2.2] that an infinite abelian group is in the class \mathfrak{X} if and only if it is either cyclic or isomorphic to \mathbb{Z}_{p^∞} (the Prüfer p -group) for some prime p . Moreover, Theorem 2.3 and Theorem 2.5 of [1] imply that every infinite nilpotent \mathfrak{X} -group is abelian. We start this section by showing that some extensions of infinite abelian \mathfrak{X} -groups provide further examples of infinite \mathfrak{X} -groups.

325 **Lemma 3.1.** *The infinite dihedral group belongs to the class \mathfrak{X} .*

Proof. Write $G = \langle a, y \mid y^2 = 1, a^y = a^{-1} \rangle$. Then for every non-cyclic subgroup H of G there exist non-zero integers n and m such that $a^n, a^m y \in H$. It easily follows that $C_G(H) = 1$. \square

Lemma 3.2. *Let $G = A\langle y \rangle$ where $A \cong \mathbb{Z}_{2^\infty}$ and $\langle y \rangle$ has order 2 or 4, with $y^2 \in A$ and $a^y = a^{-1}$, for all $a \in A$. Then G belongs to the class \mathfrak{X} .*

Proof. It is clear that G/A has order 2, and A is the Fitting subgroup of G . Also $C_G(A) = A$ and $Z(G)$ is the subgroup of order 2 of A . Let H be a non-cyclic subgroup of G with $H \neq A$. Then $H \not\leq A$ as every proper subgroup of A is cyclic. Pick any element $h \in H \setminus A$. Then $G = A\langle h \rangle$ since $|G : A| = 2$. Therefore
335 by the Dedekind modular law we get $H = C\langle h \rangle$, where $C = A \cap H > 1$ is finite.

Since $h = bv$ with $b \in A$ and $v \in \langle y \rangle \setminus A$, we get $a^h = a^{-1}$ for all $a \in A$. In particular, $C_A(h)$ has order 2 and $C_G(h)$ has order 4. Since C has a unique involution and $h \in C_G(H)$, we conclude that $C_G(H) \leq H$ and so G is an \mathfrak{X} -group. \square

340 When $\langle y \rangle$ has order 2, the group $G = A \rtimes \langle y \rangle$ of Lemma 3.2 is a generalized dihedral group.

Let p denote any odd prime. Then, by Hensel's Theorem (see for instance [8, Theorem 127.5]), the group \mathbb{Z}_{p^∞} has an automorphism of order $p - 1$, say ϕ .

Lemma 3.3. *The groups $G = \mathbb{Z}_{p^\infty} \rtimes \langle \phi^j \rangle$ for $1 \leq j \leq p - 1$ are \mathfrak{X} -groups.*

345 *Proof.* As \mathfrak{X} is subgroup closed, it suffices to show that $G = \mathbb{Z}_{p^\infty} \rtimes \langle \phi \rangle$ is an \mathfrak{X} -group. Write the elements of G in the form ay with $a \in A \cong \mathbb{Z}_{p^\infty}$ and $y \in \langle \phi \rangle$. Suppose there exist non-trivial elements $a \in A$ and $y \in \langle \phi \rangle$ such that $a^y = a$. For a suitable non-negative integer n , the element a^{p^n} has order p and it is fixed by y . Then y centralizes all elements of order p in A , and therefore $y = 1$ by a
350 result due to Baer (see, for instance, [9, Lemma 3.28]). This contradiction shows that $\langle \phi \rangle$ acts fixed point freely on A .

Let H be any non-cyclic subgroup of G . Then, as G/A is cyclic, $A \cap H \neq 1$. If $H = A$ then of course $C_G(H) = H$. Thus we can assume that there exist non-trivial elements $a, b \in A$ and $y \in \langle \phi \rangle$ such that $a, by \in H$. Let $g \in C_G(H)$.
355 If $g \in A$ then $1 = [g, by] = [g, y]$, so $g = 1$. Now let $g = cz$ with $c \in A$ and $1 \neq z \in \langle \phi \rangle$. Thus $1 = [cz, a] = [z, a]$, and $a = 1$, a contradiction. Therefore $C_G(H) \leq H$ for all non-cyclic subgroups H of G , so G is an \mathfrak{X} -group. \square

Lemma 3.4. *An infinite polycyclic group belongs to the class \mathfrak{X} if and only if it is either cyclic or dihedral.*

360 *Proof.* Arguing as in the proof of Theorem 3.1 of [1], one can easily prove that every infinite polycyclic \mathfrak{X} -group is either cyclic or dihedral. On the other hand, the infinite dihedral group belongs to the class \mathfrak{X} by Lemma 3.1. \square

Proposition 3.5. *A torsion-free soluble group belongs to the class \mathfrak{X} if and only if it is cyclic.*

365 *Proof.* Let G be a torsion-free soluble \mathfrak{X} -group. Then every abelian subgroup of G is cyclic, so G satisfies the maximal condition on subgroups by a result due to Mal'cev (see, for instance, [10, 15.2.1]). Thus G is polycyclic by [10, 5.4.12]. Therefore G has to be cyclic. \square

In next theorem we determine all infinite soluble \mathfrak{X} -groups.

370 **Theorem 3.6.** *Let G be an infinite soluble group. Then G is an \mathfrak{X} -group if and only if one of the following holds:*

- (i) G is cyclic;
- (ii) $G \cong \mathbb{Z}_{p^\infty}$ for some prime p ;
- (iii) G is dihedral;
- 375 (iv) $G = A\langle y \rangle$ where $A \cong \mathbb{Z}_{2^\infty}$ and $\langle y \rangle$ has order 2 or 4, with $y^2 \in A$ and $a^y = a^{-1}$, for all $a \in A$;
- (v) $G \cong A \rtimes D$, where $A \cong \mathbb{Z}_{p^\infty}$ and $1 \neq D \leq C_{p-1}$ for some odd prime p .

Proof. First let G be an \mathfrak{X} -group. If G is abelian then (i) or (ii) holds by [1, Theorem 2.2]. Assume G is non-abelian, and let A be the Fitting subgroup of G . Then $A \neq 1$ and $C_G(A) \leq A$ as G is soluble. 380 Let N be a nilpotent normal subgroup of G . Then N is finite, as, otherwise, using N is self-centralizing and $G/Z(N)$ is a subgroup of $\text{Aut}(N)$, we obtain G is finite, which is a contradiction. Thus [1, Theorems 2.3 and 2.5] imply that N is abelian. In particular, as the product of any two normal nilpotent subgroups of G is again a normal nilpotent 385 subgroup by Fitting's Theorem, we see that the generators of $A = F(G)$ commute. Hence A is abelian. As A is infinite and abelian, $A = C_G(A)$ is either infinite cyclic or isomorphic to \mathbb{Z}_{p^∞} for some prime p . In the former case clearly $G' \leq A$.

In the latter case, let C be any proper subgroup of A . Thus C is finite cyclic. Moreover C is characteristic in A , so it is normal in G , and $G/C_G(C)$ is abelian since it is isomorphic to a subgroup of $\text{Aut}(C)$. It follows that $G' \leq C_G(C)$, and again $G' \leq C_G(A) = A$. Therefore G/A is abelian.

If A is infinite cyclic, then the argument in the proof of Theorem 3.1 of [1] shows that G is dihedral. Thus (iii) holds.

Let $A \cong \mathbb{Z}_{p^\infty}$ for some prime p , and suppose there exists an element $x \in G$ of infinite order. Then $x \in G \setminus A$, and so there exists an element $y \in A$ such that $[x, y] \neq 1$. Then $\langle y \rangle$ is a finite normal subgroup of G , so conjugation by x induces a non-trivial automorphism of $\langle y \rangle$. Since $\text{Aut}(\langle y \rangle)$ is finite, it follows that there is an integer n such that $[x^n, y] = 1$. Now y is a torsion element and x^n has infinite order and so $\langle x^n, y \rangle$ is neither periodic nor torsion free and this contradicts [1, Theorems 2.2]. Therefore G is periodic, and G/A is isomorphic to a periodic subgroup of automorphisms of \mathbb{Z}_{p^∞} .

It is well-known that $\text{Aut}(\mathbb{Z}_{p^\infty})$ is isomorphic to the multiplicative group of all p -adic units. It follows that periodic automorphisms of \mathbb{Z}_{p^∞} form a cyclic group having order 2 if $p = 2$, and order $p - 1$ if p is odd (see, for instance, [11] for details). In the latter case (v) holds. Finally, let $p = 2$. Then $G/A = \langle yA \rangle$ has order 2, and $G = A\langle y \rangle$ with $y \notin A$ and $y^2 \in A$. Moreover $a^y = a^{-1}$, for all $a \in A$. If y has order 2 then $G = A \rtimes \langle y \rangle$. Otherwise from $y^2 \in A$ it follows $y^2 = (y^2)^y = y^{-2}$, so y has order 4. Therefore G has the structure described in (iv).

On the other hand, Lemmas 3.1 – 3.3 show that the groups listed in (i) – (v) are \mathfrak{X} -groups. □

Finally, we determine all infinite locally finite \mathfrak{X} -groups.

Theorem 3.7. *Let G be an infinite locally finite group. Then G is an \mathfrak{X} -group if and only if one of the following holds:*

- (i) $G \cong \mathbb{Z}_{p^\infty}$ for some prime p ;
- (ii) $G = A\langle y \rangle$ where $A \cong \mathbb{Z}_{2^\infty}$ and $\langle y \rangle$ has order 2 or 4, with $y^2 \in A$ and $a^y = a^{-1}$, for all $a \in A$;

(iii) $G \cong A \rtimes D$, where $A \cong \mathbb{Z}_{p^\infty}$ and $1 \neq D \leq C_{p-1}$ for some odd prime p .

Proof. Any abelian subgroup of G is either finite or isomorphic to \mathbb{Z}_{p^∞} for some
420 prime p , so it satisfies the minimal condition on subgroups. Thus G is a Černikov
group by a result due to Šunkov (see, for instance [10, page 436, I]). Hence there
exists an abelian normal subgroup A of G such that $A \cong \mathbb{Z}_{p^\infty}$ for some prime
 p , and G/A is finite. It follows that G is metabelian, arguing as in the proof of
Theorem 3.6. Therefore the result follows from Theorem 3.6. \square

425 **Corollary 3.8.** *Let G be an infinite locally nilpotent group. Then G is an
 \mathfrak{X} -group if and only if one of the following holds:*

- (i) G is cyclic;
- (ii) $G \cong \mathbb{Z}_{p^\infty}$ for some prime p ;
- (iii) $G = A\langle y \rangle$ where $A \cong \mathbb{Z}_{2^\infty}$ and $\langle y \rangle$ has order 2 or 4, with $y^2 \in A$ and
430 $a^y = a^{-1}$, for all $a \in A$.

Proof. Suppose G is not abelian. Every finitely generated subgroup of G is
nilpotent, so it is either abelian or finite. It easily follows that all torsion-free
elements of G are central. Thus G is periodic (see [12, Proposition 1]). Therefore
 G is direct product of p -groups (see, for instance, [10, Proposition 12.1.1]).
435 Actually only one prime can occur since G is an \mathfrak{X} -group, so G is a locally finite
 p -group. Thus the result follows by Theorem 3.7. \square

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