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A note on coherence of dcpos

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Abstract

In this note, we prove that a well-filtered dcpo \( L \) is coherent in its Scott topology if and only if for every \( x, y \in L \), \( \uparrow x \cap \uparrow y \) is compact in the Scott topology. We use this result to prove that a well-filtered dcpo \( L \) is Lawson-compact if and only if it is patch-compact if and only if \( L \) is finitely generated and \( \uparrow x \cap \uparrow y \) is compact in the Scott topology for every \( x, y \in L \).

Keywords: coherence, well-filtered dcpo, Lawson compactness, patch topology

2000 MSC: 54B20, 06B35, 06F30

1. Introduction

In this paper, we investigate the coherence with respect to the Scott topology on directed-complete partial ordered sets (dcpo’s for short). Coherence, which states that the intersection of any two compact saturated sets is again compact, is an important property in domain theory \cite{1, 3}. For instance, coherence is equivalent to Lawson compactness on pointed continuous domains \cite{2}. This equivalence enabled the second author to characterise the Lawson compactness of continuous domains by the so-called “property M”, and use this element-level characterization to classify the category of continuous domains with respect to the cartesian closedness \cite{5, 6}.

In \cite{9, 8}, the equivalence between coherence and Lawson compactness was generalised to quasicontinuous domains. In Chapter 3 of \cite{3}, one even sees that on finitely generated quasicontinuous domains the compactness of \( \uparrow x \cap \uparrow y \) for any \( x, y \in L \), which seems much weaker than what coherence requires, already implies the Lawson compactness of \( L \). In this note, we greatly generalize this result to well-filtered dcpos. Indeed, since every quasicontinuous domain is locally finitary compact and sober (see for example, \cite{3}), our proof drops the locally finitary compact property and only uses well-filteredness, which is even strictly weaker than sobriety \cite{7}.

2. Preliminaries

We refer to \cite{1, 3} for the standard definitions and notations of order theory and domain theory, and to \cite{4} for topology.
A topological space is called well-filtered if, whenever an open set \( U \) contains a filtered intersection \( \bigcap_{i \in I} Q_i \) of compact saturated subsets, then \( U \) contains \( Q_i \) for some \( i \in I \). Any sober space is well-filtered (see [3, Theorem II-1.21]). We take coherence of a topological space to mean that the intersection of any two compact saturated subsets is compact. A stably compact space is a topological space which is compact, locally compact, well-filtered and coherent. We call a dcpo \( L \) well-filtered (respectively, compact, sober, coherent, locally compact, stably compact) if \( L \) with its Scott topology \( \sigma(L) \) is a well-filtered (respectively, compact, sober, coherent, locally compact, stably compact) space. Without further reference, we always equip \( L \) with the Scott topology \( \sigma(L) \). Finally, a dcpo \( L \) is said to be core-compact if its Scott topology \( \sigma(L) \) is a continuous lattice in the inclusion order.

For a topological space \( X \), we denote the set of all compact saturated sets of \( X \) by \( Q(X) \). We consider the upper Vietoris topology \( v \) on \( Q(X) \), generated by the sets

\[
\square U = \{ K \in Q(X) \mid K \subseteq U \},
\]

where \( U \) ranges over the open subsets of \( X \). We use \( Q_v(X) \) to denote the resulting topological space. For a dcpo \( L \), we use \( Q_v(L) \) to denote \( Q_v((L, \sigma(L))) \).

3. Main results

Lemma 3.1. Let \( L \) be a well-filtered dcpo. Then \( L \) is coherent if and only if \( \uparrow x \cap \uparrow y \) is compact for all \( x, y \in L \).

Proof. If \( L \) is coherent, it is obvious that \( \uparrow x \cap \uparrow y \) is compact for all \( x, y \in L \), since \( \uparrow x, \uparrow y \) are compact saturated.

For the reverse, suppose \( \uparrow x \cap \uparrow y \) is compact for all \( x, y \in L \). We proceed to prove that for any compact saturated sets \( A, B \subseteq L \), \( A \cap B \) is compact in \( L \). To this end, fix some element \( a \in L \); we define a function \( f \) from \( L \) to \( Q_v(L) \) by sending an element \( x \) to the compact saturated set \( \uparrow x \cap \uparrow a \). We claim that \( f \) is continuous. Indeed, for every Scott open subset \( U \subseteq L \), \( f^{-1}(\square U) = \{ x \mid \uparrow x \cap \uparrow a \subseteq U \} \) is obviously an upper set. Let \( D \subseteq L \) be a directed subset with \( \sup D \in f^{-1}(\square U) \), then one has \( \uparrow (\sup D) \cap \uparrow a \subseteq U \), that is, \( \bigcap_{d \in D} (\uparrow d \cap \uparrow a) \subseteq U \). Note that \( L \) is well-filtered and \( \{ \uparrow d \cap \uparrow a \mid d \in D \} \) is a filtered family of compact saturated sets by assumption, so we have some \( d \in D \) such that \( \uparrow d \cap \uparrow a \subseteq U \), i.e., \( d \in f^{-1}(\square U) \). Hence \( f \) is continuous.

Since \( f \) is continuous, for the given compact saturated subset \( A \subseteq L \), \( f(A) = \{ \uparrow x \cap \uparrow a \mid x \in A \} \) is a compact subset of \( Q_v(L) \). We now claim that the union of \( f(A) \), which is just \( A \cap \uparrow a \), is compact in \( L \). Indeed, for any compact subset \( C \) of \( Q_v(L) \), let \( \{ U_\alpha \} \) be a directed family of open sets of \( L \) covering \( \bigcup C \). By compactness, every element \( K \) of \( C \) is already covered by one \( U_\alpha \); in other words, \( K \subseteq \bigcup U_\alpha \). It follows that \( \{ \square U_\alpha \} \) is a directed family covering \( C \), and now the compactness of \( C \) tells us that \( C \subseteq \bigcup U_\alpha \) for some \( \alpha \). Hence \( \bigcup C \subseteq U_\alpha \) for this \( \alpha \). (This argument is similar to the one employed by Andrea Schalk in [13, Chapter 7] for showing that \( \bigcup Q_v(Q_v(X)) \rightarrow Q_v(X) \) is well-defined.)

Now for such \( A \) the above argument enables us to define another function \( g \) from \( L \) to \( Q_v(L) \) as: \( g(x) = \uparrow x \cap A \) for every \( x \in L \). A similar deduction shows that \( g \) is continuous. So for the compact saturated subset \( B \) of \( L \), \( g(B) \) is compact in \( Q_v(L) \), and again the union of \( g(B) \), which is \( A \cap B \), is compact in \( L \). So \( L \) is coherent. \( \square \)
Corollary 3.2. Every well-filtered complete lattice $L$ is coherent.

Proof. For every $x, y \in L$ the intersection of $\uparrow x$ and $\uparrow y$, which is $\uparrow (x \lor y)$, is always compact, so the statement follows from Lemma 3.1. 

The following fact about core-compact complete lattices is essentially due to G. Gierz and K.H. Hofmann [2]; we collect it here as a corollary to the previous result.

Corollary 3.3. For a complete lattice $L$, the following statements are equivalent:

1. $L$ is core-compact, i.e., $\sigma(L)$ is a continuous lattice;
2. $(L, \sigma(L))$ is stably compact.

Proof. The only interesting part is that 1 implies 2. Suppose $L$ is a complete lattice and $\sigma(L)$ is continuous, then $(L, \sigma(L))$ is a locally compact sober space by [3, Proposition VII-4.1]. Since sober spaces are well-filtered, $L$ is coherent by Corollary 3.2. Finally, $L$ is obviously compact in its Scott topology since it has a least element.

We now come to a characterization of the compactness of Lawson and patch topologies on $L$. Recall that the patch topology on $L$ arises by taking all Scott closed sets together with all compact saturated sets as a subbasis for the closed sets; whereas the (coarser) Lawson topology is generated by the Scott closed subsets and principal upper sets $\uparrow x$. The following theorem is a generalization of [3, Theorem III-5.8] which is stated for quasicontinuous domains.

Theorem 3.4. Let $L$ be a well-filtered dcpo. Then the following statements are equivalent:

1. $L$ is patch-compact, i.e., $L$ is compact in the patch topology;
2. $L$ is Lawson-compact;
3. $L$ is compact and $\uparrow x \cap \uparrow y$ is compact for every $x, y \in L$;
4. $L$ is finitely generated and $\uparrow x \cap \uparrow y$ is compact for every $x, y \in L$;
5. $L$ is finitely generated and coherent.

Proof. (1$\Rightarrow$2): That 1 implies 2 is true for all dcpos since the patch topology is finer than the Lawson topology.

(2$\Rightarrow$3): It is obvious that $L$ is compact since the Lawson topology is finer than the Scott topology. For every $x, y \in L$, $\uparrow x \cap \uparrow y$ is Lawson closed; therefore it is Lawson-compact, thus Scott compact.

(3$\Rightarrow$4): Since $L$ is compact, by the Hausdorff Maximal principle, every element is above some minimal element of $L$. Denote the set of all minimal elements of $L$ by $M$. The set of all minimal elements of $L$ by $M$; $M$ must be finite. Otherwise, the family $\{M \setminus F \mid F \subseteq \text{fin} M\}$ is a filter of non-empty Scott closed sets with an empty intersection, which contradicts compactness.

(4$\Rightarrow$5): This is from Lemma 3.1.

(5$\Rightarrow$1): This is a straightforward consequence of [3, Lemma VI-6.5].

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\footnote{This lemma works for sober dcpos in [3]. However, one can find that its proof only uses well-filteredness.}
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