A note on coherence of dcpos

Xiaodong Jia\textsuperscript{a,*}, Achim Jung\textsuperscript{a}, Qingguo Li\textsuperscript{b}

\textsuperscript{a}School of Computer Science, University of Birmingham, Birmingham, B15 2TT, United Kingdom
\textsuperscript{b}College of Mathematics and Econometrics, Hunan University, Changsha, Hunan, 410082, China

Abstract
In this note, we prove that a well-filtered dcpo $L$ is coherent in its Scott topology if and only if for every $x, y \in L$, $\uparrow x \cap \uparrow y$ is compact in the Scott topology. We use this result to prove that a well-filtered dcpo $L$ is Lawson-compact if and only if it is patch-compact if and only if $L$ is finitely generated and $\uparrow x \cap \uparrow y$ is compact in the Scott topology for every $x, y \in L$.

Keywords: coherence, well-filtered dcpo, Lawson compactness, patch topology

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1. Introduction

In this paper, we investigate the coherence with respect to the Scott topology on directed-complete partial ordered sets (dcpo’s for short). Coherence, which states that the intersection of any two compact saturated sets is again compact, is an important property in domain theory \cite{1, 3}. For instance, coherence is equivalent to Lawson compactness on pointed continuous domains \cite{5}. This equivalence enabled the second author to characterise the Lawson compactness of continuous domains by the so-called “property M”, and use this element-level characterization to classify the category of continuous domains with respect to the cartesian closedness \cite{5, 6}.

In \cite{9, 8}, the equivalence between coherence and Lawson compactness was generalised to quasicontinuous domains. In Chapter 3 of \cite{3}, one even sees that on finitely generated quasicontinuous domains the compactness of $\uparrow x \cap \uparrow y$ for any $x, y \in L$, which seems much weaker than what coherence requires, already implies the Lawson compactness of $L$. In this note, we greatly generalize this result to well-filtered dcpos. Indeed, since every quasicontinuous domain is locally finitary compact and sober (see for example, \cite{4}), our proof drops the locally finitary compact property and only uses well-filteredness, which is even strictly weaker than sobriety \cite{7}.

2. Preliminaries

We refer to \cite{1, 5} for the standard definitions and notations of order theory and domain theory, and to \cite{4} for topology.
A topological space is called well-filtered if, whenever an open set $U$ contains a filtered intersection $\bigcap_{i \in I} Q_i$ of compact saturated subsets, then $U$ contains $Q_i$ for some $i \in I$. Any sober space is well-filtered (see [3, Theorem II-1.21]). We take coherence of a topological space to mean that the intersection of any two compact saturated subsets is compact. A stably compact space is a topological space which is compact, locally compact, well-filtered and coherent. We call a dcpo $L$ well-filtered (respectively, compact, sober, coherent, locally compact, stably compact) if $L$ with its Scott topology $\sigma(L)$ is a well-filtered (respectively, compact, sober, coherent, locally compact, stably compact) space. Without further reference, we always equip $L$ with its Scott topology $\sigma(L)$. Finally, a dcpo $L$ is said to be core-compact if its Scott topology $\sigma(L)$ is a continuous lattice in the inclusion order.

For a topological space $X$, we denote the set of all compact saturated sets of $X$ by $Q(X)$. We consider the upper Vietoris topology $v$ on $Q(X)$, generated by the sets

$$\Box U = \{K \in Q(X) \mid K \subseteq U\},$$

where $U$ ranges over the open subsets of $X$. We use $Q_v(X)$ to denote the resulting topological space. For a dcpo $L$, we use $Q_v(L)$ to denote $Q_v((L, \sigma(L)))$.

3. Main results

**Lemma 3.1.** Let $L$ be a well-filtered dcpo. Then $L$ is coherent if and only if $\uparrow x \cap \uparrow y$ is compact for all $x, y \in L$.

**Proof.** If $L$ is coherent, it is obvious that $\uparrow x \cap \uparrow y$ is compact for all $x, y \in L$, since $\uparrow x, \uparrow y$ are compact saturated.

For the reverse, suppose $\uparrow x \cap \uparrow y$ is compact for all $x, y \in L$. We proceed to prove that for any compact saturated sets $A, B \subseteq L$, $A \cap B$ is compact in $L$. To this end, fix some element $a \in L$; we define a function $f$ from $L$ to $Q_v(L)$ by sending an element $x$ to the compact saturated set $\uparrow x \cap \uparrow a$. We claim that $f$ is continuous. Indeed, for every Scott open subset $U \subseteq L$, $f^{-1}(\Box U) = \{x \mid \uparrow x \cap \uparrow a \subseteq U\}$ is obviously an upper set. Let $D \subseteq L$ be a directed subset with $\sup D \in f^{-1}(\Box U)$, then one has $\uparrow(\sup D) \cap \uparrow a \subseteq U$, that is, $\bigcap_{d \in D}(\uparrow d \cap \uparrow a) \subseteq U$. Note that $L$ is well-filtered and $\{\uparrow d \cap \uparrow a \mid d \in D\}$ is a filtered family of compact saturated sets by assumption, so we have some $d \in D$ such that $\uparrow d \cap \uparrow a \subseteq U$, i.e., $d \in f^{-1}(\Box U)$. Hence $f$ is continuous.

Since $f$ is continuous, for the given compact saturated subset $A \subseteq L$, $f(A) = \{\uparrow x \cap \uparrow a \mid x \in A\}$ is a compact subset of $Q_v(L)$. We now claim that the union of $f(A)$, which is just $A \cap \uparrow a$, is compact in $L$. Indeed, for any compact subset $C$ of $Q_v(L)$, let $\{U_\alpha\}$ be a directed family of open sets of $L$ covering $\bigcup C$. By compactness, every element $K$ of $C$ is already covered by one $U_\alpha$; in other words, $K \subseteq \Box U_\alpha$. It follows that $\{\Box U_\alpha\}$ is a directed family covering $C$, and now the compactness of $C$ tells us that $C \subseteq \Box U_\alpha$ for some $\alpha$. Hence $\bigcup C \subseteq U_\alpha$ for this $\alpha$. (This argument is similar to the one employed by Andrea Schalk in [4, Chapter 7] for showing that $\bigcup : Q_v(Q_v(X)) \rightarrow Q_v(X)$ is well-defined.)

Now for such $A$ the above argument enables us to define another function $g$ from $L$ to $Q_v(L)$ as: $g(x) = \uparrow x \cap A$ for every $x \in L$. A similar deduction shows that $g$ is continuous. So for the compact saturated subset $B$ of $L$, $g(B)$ is compact in $Q_v(L)$, and again the union of $g(B)$, which is $A \cap B$, is compact in $L$. So $L$ is coherent. □
Corollary 3.2. Every well-filtered complete lattice $L$ is coherent.

Proof. For every $x, y \in L$ the intersection of $\uparrow x$ and $\uparrow y$, which is $\uparrow (x \lor y)$, is always compact, so the statement follows from Lemma 3.1.

The following fact about core-compact complete lattices is essentially due to G. Gierz and K.H. Hofmann [2]; we collect it here as a corollary to the previous result.

Corollary 3.3. For a complete lattice $L$, the following statements are equivalent:

1. $L$ is core-compact, i.e., $\sigma(L)$ is a continuous lattice;
2. $(L, \sigma(L))$ is stably compact.

Proof. The only interesting part is that 1 implies 2. Suppose $L$ is a complete lattice and $\sigma(L)$ is continuous, then $(L, \sigma(L))$ is a locally compact sober space by [3, Proposition VII-4.1]. Since sober spaces are well-filtered, $L$ is coherent by Corollary 3.2. Finally, $L$ is obviously compact in its Scott topology since it has a least element.

We now come to a characterization of the compactness of Lawson and patch topologies on $L$. Recall that the patch topology on $L$ arises by taking all Scott closed sets together with all compact saturated sets as a subsbasis for the closed sets; whereas the (coarser) Lawson topology is generated by the Scott closed subsets and principal upper sets $\uparrow x$. The following theorem is a generalization of [3, Theorem III-5.8] which is stated for quasicontinuous domains.

Theorem 3.4. Let $L$ be a well-filtered dcpo. Then the following statements are equivalent:

1. $L$ is patch-compact, i.e., $L$ is compact in the patch topology;
2. $L$ is Lawson-compact;
3. $L$ is compact and $\uparrow x \cap \uparrow y$ is compact for every $x, y \in L$;
4. $L$ is finitely generated and $\uparrow x \cap \uparrow y$ is compact for every $x, y \in L$;
5. $L$ is finitely generated and coherent.

Proof. (1$\Rightarrow$2): That 1 implies 2 is true for all dcpos since the patch topology is finer than the Lawson topology.

(2$\Rightarrow$3): It is obvious that $L$ is compact since the Lawson topology is finer than the Scott topology. For every $x, y \in L$, $\uparrow x \cap \uparrow y$ is Lawson closed; therefore it is Lawson-compact, thus Scott compact.

(3$\Rightarrow$4): Since $L$ is compact, by the Hausdorff Maximality Principle, every element is above some minimal element of $L$. Denote the set of all minimal elements of $L$ by $M$. The set of all minimal elements of $L$ by $M$; $M$ must be finite. Otherwise, the family $\{M \setminus F \mid F \subseteq M\}$ is a filtered set of non-empty Scott closed sets with an empty intersection, which contradicts compactness.

(4$\Rightarrow$5): This is from Lemma 3.1.

(5$\Rightarrow$1): This is a straightforward consequence of [3, Lemma VI-6.5].

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\footnote{This lemma works for sober dcpos in [3]. However, one can find that its proof only uses well-filteredness.}
References


