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QRB, QFS, and the Probabilistic Powerdomain

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Abstract

We show that the first author's QRB-domains coincide with Li and Xu's QFS-domains, and also with Lawson-compact quasi-continuous dcpos, with stably-compact locally finitary compact spaces, with sober QFS-spaces, and with sober QRB-spaces. The first three coincidences were discovered independently by Lawson and Xi. The equivalence with sober QFS-spaces is then applied to give a novel, direct proof that the probabilistic powerdomain of a QRB-domain is a QRB-domain. This improves upon a previous, similar result, which was limited to pointed, second-countable QRB-domains.

Keywords: QRB-spaces, QFS-spaces, QRB-domains, QFS-domains, stably compact spaces, probabilistic powerdomain

1 Introduction

An outstanding problem in denotational semantics is whether there is a full subcategory of continuous dcpos that is both Cartesian-closed and closed under the action of the probabilistic powerdomain monad $\mathcal{V}$ [17]. Indeed, there are very few categories of dcpos that are known to be closed under $\mathcal{V}$: the category of all dcpos, that of all continuous dcpos [15], and that of all Lawson compact continuous dcpos [17]. To that list, one must add the pointed, second-countable QRB-domains [10].

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While QRB-domains are only quasi-continuous and not continuous domains, and do not form a Cartesian-closed category either, they have attracted considerable attention recently.

QRB-domains are defined by imitating RB-domains. Independently, Li and Xu used a similar process to define QFS-domains [23], imitating the construction of FS-domains [16]. Rather surprisingly, QRB and QFS-domains are the same thing (RB and FS-domains are not known to coincide), and are also exactly the Lawson-compact quasi-continuous domains. This was shown independently by the present authors and J. Lawson and X. Xi. We present our proof in Section 5 below; Lawson's and Xi's proof will appear as [21].

One of our characterizations of QRB is as so-called sober QFS-spaces, and this will turn out to be instrumental in proving that the category of all QRB-domains, and not just the second-countable ones, is closed under the action of the probabilistic powerdomain, as we shall see in Section 6. This improves upon [10], and relies on a rather different proof argument.

Outline. After some brief preliminaries (Section 2), we discuss the notion of functional approximation in Section 3. This is a central concept in domain theory, at the heart of RB-, FS-, QRB-, and QFS-domains. Another domain-theoretic leitmotiv is that one should always topologize (paraphrasing M. Stone), and we introduce QFS-*spaces* in Section 4 as the natural topological counterpart of QFS-domains. We give our proof that QRB-domains and QFS-domains are the same thing (and coincide with four other natural notions, including sober QFS-spaces) in Section 5. We apply this to the promised result that the probabilistic powerdomain of a QRB-domain is a QRB-domain in Section 6.

2 Preliminaries

We refer to the classic texts [6,1] for the required domain-theoretic background, and to [11] for topology.

We agree that a subset of a space is compact if and only if every open cover has a finite subcover, that is, we do not require separation. We take coherence to mean that the intersection of any two compact saturated subsets is compact. (A saturated subset is one that is equal to the intersection of its open neighborhoods.) A space is *stably compact* if it is sober, compact, locally compact and coherent. As is well-known, the patch topology of a stably compact space is compact Hausdorff, see [11, Section 9] or [6, Section VI-6] for more details.

Any sober space is *well-filtered*, meaning that if an open subset U contains a filtered intersection $\bigcap_{i\in I} Q_i$ of compact saturated subsets, then U contains Q_i for some $i \in I$. In a well-filtered space, every such filtered intersection is compact.

Given a T_0 topological space $(X;\tau)$, we will make heavy use of its *specialisation order* defined as $x \leq y$ if $x \in \overline{\{y\}}$. We write $\uparrow E$ for the upward closure (w.r.t. $\leq$) of a subset E. Subsets equal to their upward closure are exactly the saturated one. If E is finite, then $\uparrow E$ is compact, and we call such sets the *finitary compacts* of X.

The set of compact saturated subsets of a topological space $(X;\tau)$ may be equipped with an order by setting $A \leq B$ iff $A \supseteq B$, and we write $\mathcal{Q}(X)$ for the resulting poset. It may also be equipped with the *upper Vietoris* topology, which has a base of opens of the form $\Box U$, $U \in \tau$, where $\Box U$ denotes the collection of compact saturated subsets contained in U. This yields the *upper space* $\mathcal{Q}_{\mathsf{V}}(X)$ of X. Happily, the specialisation order of the upper space is precisely reverse inclusion. Analogously, We write $\mathrm{Fin}(X)$ for the collection of finitary compacts of X, and topologize it with the subspace topology, yielding a space that we write $\mathrm{Fin}_{\mathsf{V}}(X)$. When X is well-filtered (e.g., sober), $\mathcal{Q}(X)$ is a dcpo and directed suprema are computed as intersections.

For a finite subset E of a poset $(X, \leq)$ and $x \in X$, write $E \ll x$ iff every directed family $(x_i)_{i \in I}$ whose supremum $\sup_{i \in I} x_i$ is above x in X contains an element x_i that is above some element z of E. We also write $\uparrow E \ll x$ instead of $E \ll x$, stressing the fact that this is a property of the finitary compact $\uparrow E$, not just of the finite set E. The dcpo X is a *quasi-continuous domain* (see [7] or [6, Definition III-3.2]) if and only if for every $x \in X$, the collection of all $\uparrow E \in \mathrm{Fin}(X)$ that approximate x ($\uparrow E \ll x$) is directed (w.r.t. $\supseteq$) and their least upper bound in $\mathcal{Q}(X)$ is $\uparrow x$.

3 Functional approximation

We are concerned with spaces in which points are "systematically" approximated, by which we mean that we are given functions which produce approximants for each element. In domain theory, the idea goes back to Plotkin's characterization of SFP-domains, [24], as those dcpos X for which there is a chain of Scott-continuous functions $(\varphi_n)_{n \in \mathbb{N}}$ from X to X, satisfying the following properties

(i) for each $n \in \mathbb{N}$, $\varphi_n \leq \mathsf{id}_X$;

(ii) for each $n \in \mathbb{N}$, $\varphi_n \circ \varphi_n = \varphi_n$;

(iii) for each $n \in \mathbb{N}$, φ_n has finite image;

(iv) $\mathsf{id}_X = \bigvee^{\uparrow}_{n \in \mathbb{N}} \varphi_n$.

Plotkin also showed that the retracts of SFP-domains can be characterised similarly, by dropping the idempotency requirement (ii). If instead of a chain, only a directed family of such functions is present, then one obtains *RB-domains*. The concept was further generalized in the work of the second author, [16], where instead of requiring finite image, *finite separation* is stipulated: A function $\varphi \colon X \to X$ is *finitely separated* from id_X if there exists a finite set $M \subseteq X$ such that

$$\forall x \in X.\, \exists m \in M.\, \varphi(x) \leq m \leq x.$$

An FS-domain, then, is a dcpo which contains a directed family $(\varphi_i)_{i \in I}$ of continuous functions finitely separated from identity such that $\mathsf{id}_X = \bigvee^{\uparrow}_{i \in I} \varphi_i$.

In 2010 the first author, [9] realised that the concept of functional approximation could usefully be further generalized by allowing the approximating functions to produce compact neighborhoods rather than points, that is, the φ_i now take values in $\mathcal{Q}_{\mathsf{V}}(X)$ rather than X. With this generalization there are then two choices to be made about their *finiteness character*:

Choice 1 One can require the φ_i to have finite image in $\mathcal{Q}_{\mathsf{V}}(X)$ or not make any such restriction.

Choice 2 One can require the φ_i to only produce finitary compacts or allow general compact saturated sets.

Together this means that there are four variants that one might consider and it may come as a relief to the reader that they will in fact all turn out to lead to the same structures. Specifically, we will show that the most liberal notion, arbitrary image of general compact saturated sets, and the most restrictive one, finite image of finitary compacts, coincide. This will be true with and without assuming sobriety.

Definition 3.1 A continuous function $\varphi\colon X \to \mathcal{Q}_{\mathsf{V}}(X)$ is called a *quasi-deflation* if it is has finite image and for each $x \in X$, $x \in \varphi(x) \in \mathrm{Fin}\, X$. It is called *quasi-finitely separated* (or *qfs* for short) if there exists a finite set $M \subseteq X$ such that for every $x \in X$ there is $m \in M$ such that $x \in {\uparrow} m \subseteq \varphi(x)$. In this case, we say that φ is *separated by* M, or that M is a *separating set* for φ.

We shall agree to order continuous maps from X to $\mathcal{Q}_{\mathsf{V}}(X)$ in the pointwise extension of $\supseteq$. Accordingly, a family $(\varphi_i)_{i\in I}$ of continuous functions from X to $\mathcal{Q}_{\mathsf{V}}(X)$ is directed if and only if it is non-empty and for all $i, j \in I$, there is a $k \in I$ such that, for every $x \in X$, $\varphi_k(x) \subseteq \varphi_i(x), \varphi_j(x)$. We call it *approximating* if it is directed and furthermore, ${\uparrow} x = \bigcap_{i\in I} \varphi_i(x)$ holds for all $x \in X$.

We call a T_0 topological space $(X;\tau)$ a *QRB-space* if there is an approximating family of quasi-deflations for it. It is called a *QFS-space* if there is an approximating family of quasi-finitely separated maps.

A QFS- (or QRB-) space $(X;\tau;(\varphi_i)_{i\in I})$ is called *topological* if for all $U \in \tau$ and $x \in U$ there is $i \in I$ such that $\varphi_i(x) \subseteq U$.

Clearly, every quasi-deflation φ is also qfs because we can take the finitely many minimal elements of the finitely many possible images of φ as the separating set. Therefore, every QRB-space is also QFS. To explain the last part of the definition, we give an example to show that not every QFS-space is topological:

Example 3.2 Consider the poset P_1 in Figure 1 consisting of the natural numbers in their usual order plus an extra element a not related to any of the others. Equip this set with the Alexandroff topology (of all upper sets) and consider the map φ_m which maps each $n \in \mathbb{N}$ to ${\uparrow}\min\{m, n\}$ and a to ${\uparrow} m \cup \{a\}$. Clearly, each φ_m is a quasi-deflation. Furthermore, the family $(\varphi_m)_{m\in\mathbb{N}}$ is approximating and thus P_1 is a QRB-space. However, for no $m \in \mathbb{N}$ do we have that $\varphi_m(a) \subseteq {\uparrow} a = \{a\}$.

4 QFS-spaces

Proposition 4.1 *QFS-spaces are compact.*

Proof. Let X be a QFS-space and φ be any qfs map on X with separating set M. Then $X = {\uparrow} M$ since every $x \in X$ is above some $m \in M$ by definition. Since M is

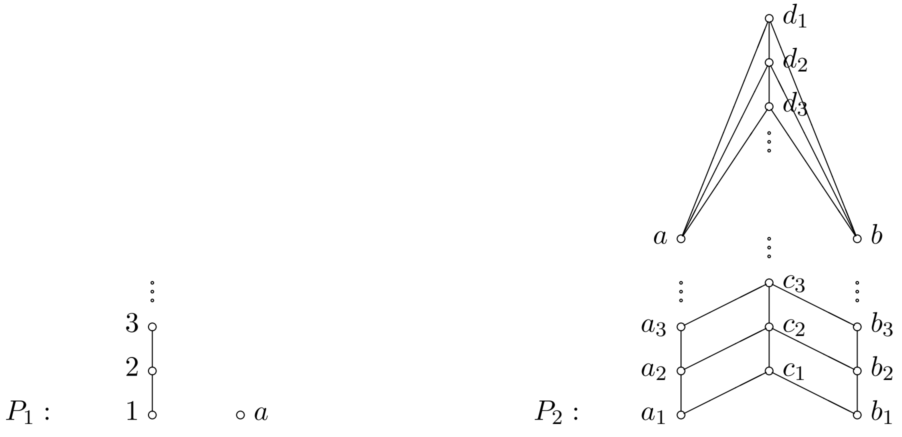

Fig. 1. Two example spaces

finite, we have compactness. □

For local compactness we start with a useful lemma:

Lemma 4.2 *Let φ be a qfs map on a topological space X, separated by the finite set M. Then for every $x \in X$, x is in the interior of $\uparrow(M \cap \varphi(x))$.*

Proof. Fix $x \in X$ and let $U = X \setminus \downarrow(M \setminus \varphi(x))$. Because of the finiteness of M, U is an open set and a neighborhood of $\varphi(x)$. Let $V = \varphi^{-1}(\Box U)$, which is an open set since φ is continuous. By construction, x is a member of V and, furthermore, we claim that $V \subseteq \uparrow(M \cap \varphi(x))$. Indeed, let $y \in V$. Then $\varphi(y) \subseteq U$ and hence the separating element $m \in M$ with $m \in \varphi(y)$ and $m \leq y$ also belongs to U. Hence $m \in M \cap \varphi(x)$ and $y \in \uparrow(M \cap \varphi(x))$ follows. □

A topological space X is *locally finitary compact* if every open neighborhood U of an arbitrary point x contains a locally finitary neighborhood $\uparrow E$ of x: $U \supseteq \uparrow E \supseteq \mathrm{int}(\uparrow E) \ni x$. The notion originates with Isbell [14], and the T_0 such spaces are called qc-spaces in [21]. Every quasi-continuous domain is locally finitary compact, since in this case $\mathrm{int}(\uparrow E) = \{x \in X \mid E \ll x\}$ [6, III-3.6(ii)]. The following is immediate from the definitions and the preceding lemma:

Lemma 4.3 *Every topological QFS-space is locally finitary compact.*

It would be nice if one could also show coherence for QFS-spaces but without further assumptions this is not possible, even for QRB-spaces:

Example 4.4 Consider the poset P_2 in Figure 1 together with the Scott topology (note that the only non-trivial directed suprema are $a = \bigvee^{\uparrow}_{n\in\mathbb{N}} a_n$ and $b = \bigvee^{\uparrow}_{n\in\mathbb{N}} b_n$). The QRB property is established by maps f_m, $m \in \mathbb{N}$, which map

$$\begin{array}{llll} a \mapsto a_m & b \mapsto b_m & c_n \mapsto c_{\min\{m,n\}} & d_n \mapsto c_m \text{ for } n > m \\ a_n \mapsto a_{\min\{m,n\}} & b_n \mapsto b_{\min\{m,n\}} & & d_n \mapsto d_n \text{ for } n \leq m \end{array}$$

and by setting $\varphi_m(x) = \uparrow f_m(x)$. The resulting QRB-space is topological because every Scott neighborhood of a (resp. b) must contain some final segment of a_n's (resp. b_n's). It is not coherent, though, because $\uparrow a \cap \uparrow b = \{d_n \mid n \in \mathbb{N}\}$ is not compact.

The situation is much nicer if we assume our spaces to be sober. First, since sobriety implies well-filteredness, we immediately have the following:

Lemma 4.5 *Sober QFS-spaces are topological.*

Combining the last two lemmas we get that sober QFS-spaces are locally finitary compact, and it is known from [3], or the equivalence between (6) and (11) in [22, Theorem 2], or [21, Corollary 3.6], or [11, Exercise 8.3.39], that the sober, locally finitary compact spaces are exactly the quasi-continuous dcpos in their Scott topology. Thus we have:

Proposition 4.6 *Sober QFS-spaces are quasi-continuous domains, and their given topology coincides with the Scott topology derived from the specialisation order.*

Thus it is appropriate to call sober QFS-spaces, *QFS-domains*, and similarly for sober QRB-spaces. A little amount of work should convince the reader that these QFS-domains are exactly the same of those defined by Li and Xu [23].

How far are (topological) QFS-spaces from QFS-domains? As it turns out, not very far as we will now show that sobrification leads from one to the other.

The sobrification $\hat{X}$ of a topological space $(X;\tau)$ can be described in a number of ways; the most convenient for our purposes is to realise it concretely as the set of closed irreducible[4] subsets of X, together with the topology $\hat{\tau}$ which consists of open sets $\hat{U} = \{A \in \hat{X} \mid A \cap U \neq \emptyset\}$, where U ranges over the open sets in τ. Note that X and $\hat{X}$ have isomorphic frames of opens.

Given a qfs map $\varphi\colon X \to \mathcal{Q}_{\mathsf{V}}(X)$ we replace $\varphi(x)$ with its set of open neighborhoods, defined as $\{U \in \tau \mid \varphi(x) \subseteq U\}$. This is always a Scott-open filter in the frame τ, and the Hofmann-Mislove Theorem tells us that, conversely, every Scott-open filter $\mathcal{F}$ of τ corresponds to a unique compact saturated set $Q_{\mathcal{F}}$ of the sobrification $\hat{X}$ of X. Indeed, $\mathcal{F}$ consists precisely of the opens U such that $\hat{U}$ is a neighborhood of $Q_{\mathcal{F}}$, that is, a closed irreducible set belongs to $Q_{\mathcal{F}}$ if and only if it meets every member of $\mathcal{F}$. For the upper Vietoris topology on $\mathcal{Q}_{\mathsf{V}}(\hat{X})$, the basic open set $\Box\hat{U}$ consists of those compacts $Q_{\mathcal{F}}$ where $\mathcal{F}$ ranges over the Scott-open filters which contain U. Using this setup, we define $\tilde{\varphi}\colon \hat{X} \to \mathcal{Q}_{\mathsf{V}}(\hat{X})$ by mapping $A \in \hat{X}$ to $Q_{\psi(A)}$ where $\psi(A) = \{U \in \tau \mid \exists a \in A.\ \varphi(a) \subseteq U\} = \{U \in \tau \mid A \cap \varphi^{-1}(\Box U) \neq \emptyset\}$.

Lemma 4.7 *For $\varphi\colon X \to \mathcal{Q}_{\mathsf{V}}(X)$ qfs, $\tilde{\varphi}$ is a qfs map.*

Proof. The function $\tilde{\varphi}$, equivalently ψ, is well-defined: if a directed union of opens belongs to $\psi(A)$ then it covers $\varphi(a)$ for some $a \in A$. Because $\varphi(a)$ is compact, one of them does so already. Filteredness follows from $\varphi^{-1}(\Box U \cap \Box V) = \varphi^{-1}(\Box(U \cap V)) = \varphi^{-1}(\Box U) \cap \varphi^{-1}(\Box V)$ and the assumption that A is irreducible.

[4] A set is irreducible if it meets every member of a finite family of open sets precisely if it meets their intersection (from which it follows that irreducible sets are non-empty).

For continuity, observe that $\tilde{\varphi}^{-1}(\Box\hat{U}) = \psi^{-1}(\{\mathcal{F} \mid U \in \mathcal{F}\}) = \{A \in \hat{X} \mid A \cap \varphi^{-1}(\Box U) \neq \emptyset\} = \widehat{\varphi^{-1}(\Box U)}$.

For finite separation, we assume that M is a separating set for φ. We show that the set $\hat{M} = \{\downarrow m \mid m \in M\}$ is separating for $\tilde{\varphi}$. Let A be a closed irreducible set. For every $U \in \psi(A)$ we have by definition that there is $a \in A$ such that $\varphi(a) \subseteq U$. It follows that $U \cap (M \cap A)$ is non-empty. Hence the family of these sets, indexed by $U \in \psi(A)$, is a proper filter on the finite set $M \cap A$ and so there is $m_A \in M$ belonging to all of them. We clearly have that $\downarrow m_A \subseteq A$ and because m_A is in every $U \in \psi(A)$, $\downarrow m_A$ meets every element of $\psi(A)$, whence $\downarrow m_A \in Q_{\psi(A)} = \tilde{\varphi}(A)$. □

The above construction has been chosen for its brevity but we may point out that the underlying idea relies on a natural transformation T (a "distributive law") from $\widehat{[-]} \circ \mathcal{Q}_{\mathsf{V}}$ to $\mathcal{Q}_{\mathsf{V}} \circ \widehat{[-]}$. Our map $\tilde{\varphi}$ is the composition $\hat{X} \xrightarrow{\hat{\varphi}} \widehat{\mathcal{Q}_{\mathsf{V}}(X)} \xrightarrow{T} \mathcal{Q}_{\mathsf{V}}(\hat{X})$. An even more explicit construction is also possible, and it demonstrates nicely the usefulness of the "Topological Rudin Lemma" presented in [13]: We invite the reader to use the latter to show that $T(C) = \{A \in \hat{X} \mid \forall Q \in C.\ Q \cap A \neq \emptyset\}$, and to also use it to reprove Lemma 4.7 with that definition.

Finally, we would like to show that the lifted family $(\tilde{\varphi}_i)_{i \in I}$ is approximating for $\hat{X}$. It is here where we need the condition that the original QFS space be topological, as without this condition this would not be the case. Consider again Example 3.2: The sobrification of the space P_1 consists of the sets $\downarrow x$, $x \in X$ plus one more, the chain $A = \mathbb{N}$. By definition, A belongs to each $\tilde{\varphi}_m(\downarrow a)$: check that, for every $a' \leq a$, A meets every open neighborhood U of $\varphi_m(a')$. Hence A is also in the intersection of all $\tilde{\varphi}_m(\downarrow a)$, but it does not belong to $\uparrow(\downarrow a) = \{\{a\}\}$.

We come to the main result of this section:

Theorem 4.8 *The sobrification of a topological QFS space is a QFS domain.*

Proof. All that remains is to show that the family $(\tilde{\varphi}_i)_{i \in I}$ is approximating for $\hat{X}$. Let $A \in \hat{X}$ be a closed irreducible subset of X and let B be another such, not above A. This means that B does not contain A (as subsets of X), and so let $a \in A \smallsetminus B$. By the definition of topological QFS spaces we obtain an index $i \in I$ such that $\varphi_i(a)$ is contained in the open set $X \smallsetminus B$. Writing $\psi_i(A)$ for $\{U \in \tau \mid \exists a \in A.\ \varphi_i(a) \subseteq U\}$, so that $\tilde{\varphi}_i(A) = Q_{\psi_i(A)}$, we obtain that $U \in \psi_i(A)$ for $U = X \smallsetminus B$. Since U does not meet B, B is not in $\tilde{\varphi}_i(A)$. □

5 QFS-domains

We have already seen that the addition of sobriety to the conditions for a QFS-space results in much nicer structures. The best is still to come, however. We begin by giving a short argument to show that QFS-domains are coherent, a result which appears as Corollary 3.9 in [21]. First a lemma, also from [21]:

Lemma 5.1 *If X is a QFS-domain then $\mathcal{Q}_{\mathsf{V}}(X)$ is an FS-domain.*

Proof. If φ is a qfs map on X separated by M, then $\Phi\colon \mathcal{Q}_{\mathsf{V}}(X) \to \mathcal{Q}_{\mathsf{V}}(X)$, defined by $\Phi(K) = \uparrow\varphi[K]$, is finitely separated: For the separating set consider all sets $\uparrow E$, $E \subseteq M$. $\square$

Proposition 5.2 *The topology of a QFS-domain is coherent.*

Proof. Let K, L be compact saturated sets of X. They are points in $\mathcal{Q}_{\mathsf{V}}(X)$ and generate principal upper, hence compact, sets $\uparrow_{\mathcal{Q}_{\mathsf{V}}(X)} K$ and $\uparrow_{\mathcal{Q}_{\mathsf{V}}(X)} L$. Since $\mathcal{Q}_{\mathsf{V}}(X)$ is an FS-domain, it is coherent, hence the set $\mathcal{K} = \uparrow_{\mathcal{Q}_{\mathsf{V}}(X)} K \cap \uparrow_{\mathcal{Q}_{\mathsf{V}}(X)} L$ is a compact saturated set. The claim follows from the observation that $K \cap L = \bigcup \mathcal{K}$ and the fact that $\bigcup$, as the multiplication of the upper powerspace monad ([25, Chapter 7]), is a continuous map from $\mathcal{Q}_{\mathsf{V}}(\mathcal{Q}_{\mathsf{V}}(X))$ to $\mathcal{Q}_{\mathsf{V}}(X)$. $\square$

For quasi-continuous domains, compactness plus coherence is the same as compactness in the Lawson topology. This follows, for example, from the fact that the Lawson and patch topologies coincide on quasi-continuous dcpos [6, Lemma V-5.15], and that every patch-compact space is coherent and compact [11, Proposition 9.1.27], while conversely quasi-continuous domains are locally compact and sober [11, Exercise 8.2.15]. We thus have the following refinement of Proposition 4.6:

Corollary 5.3 *QFS-domains are Lawson-compact quasi-continuous domains equipped with their Scott topology.*

We now work towards the converse of this:

Proposition 5.4 *Every compact, locally compact, coherent space X has an approximating family of maps $\varphi_{\mathcal{M}} \colon X \to \mathcal{Q}_{\mathsf{V}}(X)$ with finite image. Precisely, $\mathcal{M}$ ranges over the finite $\vee$-semi-lattices $\mathcal{M}$ of compact saturated sets of X, and $\varphi_{\mathcal{M}}$ maps each $x \in X$ to the smallest element of $\mathcal{M}$ whose interior contains x.*

Note that $\varphi_{\mathcal{M}}$ takes values in $\mathcal{Q}(X)$, not in $\mathrm{Fin}(X)$. Smallest is taken with respect to inclusion. A *$\vee$-semi-lattice of compact saturated sets* is a family of sets that is closed under finite intersections (in particular, contains X).

Proof. Define $\varphi_{\mathcal{M}}(x)$ as the intersection of all the elements Q of $\mathcal{M}$ that are neighborhoods of x. Using the fact that $\mathcal{M}$ is finite, $\varphi_{\mathcal{M}}(x)$ is the smallest neighborhood of x in $\mathcal{M}$, so $\varphi_{\mathcal{M}}(x)$ is well defined, and in $\mathcal{Q}(X)$ by coherence and compactness.

For continuity, let U be open and consider $x \in \varphi_{\mathcal{M}}^{-1}(\Box U)$. Let $Q = \varphi_{\mathcal{M}}(x)$. For every $y \in \mathrm{int}(Q)$, $\varphi_{\mathcal{M}}(y) \subseteq Q \subseteq U$, so y is in $\varphi_{\mathcal{M}}^{-1}(\Box U)$. Hence $\mathrm{int}(Q)$ is an open neighborhood of x included in $\varphi_{\mathcal{M}}^{-1}(\Box U)$, so $\varphi_{\mathcal{M}}^{-1}(\Box U)$ is open.

Clearly, if $\mathcal{M} \subseteq \mathcal{M}'$, then $\varphi_{\mathcal{M}}(x) \supseteq \varphi_{\mathcal{M}'}(x)$ for every $x \in X$. The family of all $\varphi_{\mathcal{M}}$ is directed: given $\mathcal{M}$ and $\mathcal{M}'$, there is a smallest semi-lattice $\mathcal{M} \sqcup \mathcal{M}'$ of compact saturated sets containing $\mathcal{M}$ and $\mathcal{M}'$, consisting of the intersections $Q \cap Q'$ with $Q \in \mathcal{M}$ and $Q' \in \mathcal{M}'$; coherence implies that each such $Q \cap Q'$ is compact saturated, and $\varphi_{\mathcal{M} \sqcup \mathcal{M}'}$ is above both $\varphi_{\mathcal{M}}$ and $\varphi_{\mathcal{M}'}$ (w.r.t. $\supseteq$).

All that remains to show is that the maps $\varphi_{\mathcal{M}}$ form an approximating family. Given $x \in X$, $\uparrow x \subseteq \bigcap_{\mathcal{M}} \varphi_{\mathcal{M}}(x)$ is by definition. For the reverse inclusion, we show that every open neighborhood U of x contains $\bigcap_{\mathcal{M}} \varphi_{\mathcal{M}}(x)$. By local compactness,

U contains a compact saturated neighborhood Q of x. $\mathcal{M} = \{Q, X\}$ qualifies as a semi-lattice of compact saturated sets, and we have $\varphi_{\mathcal{M}}(x) = Q \subseteq U$. $\square$

The following is standard:

Lemma 5.5 *Let X be a locally finitary compact space. For every compact saturated subset Q of X, and every open neighborhood U of Q, there is a further, finitary compact neighborhood $\uparrow E$ of Q contained in U.*

Proposition 5.6 *Every compact, locally finitary compact, coherent space X has an approximating family of quasi-deflations.*

Proof. Applying Proposition 5.4, we obtain an approximating family of maps $\varphi_{\mathcal{M}}$. We need to replace each compact saturated subset $Q \in \operatorname{im} \varphi_{\mathcal{M}}$ by a finitary compact.

Assume first that we are given an open neighborhood U_Q around each of them. Lemma 5.5 allows us to find finitary compact neighborhoods $\uparrow E_Q$ between Q and U_Q. We seek to find $\uparrow E_Q$ so that, additionally, $Q \subseteq Q'$ implies $\uparrow E_Q \subseteq \uparrow E_{Q'}$. To ensure this, we define $\uparrow E_Q$ step by step, always working on the largest $Q \in \mathcal{M}$ that is still to be considered (so we start with X itself, the largest element of $\mathcal{M}$). Given any $Q \in \mathcal{M}$ such that $\uparrow E_{Q'}$ is already defined for every strictly larger $Q' \in \mathcal{M}$, we apply Lemma 5.5 and define $\uparrow E_{Q'}$ as some finitary compact neighborhood of Q contained in $U_Q \cap \bigcap_{\substack{Q' \in \mathcal{M} \\ Q' \supsetneq Q}} \operatorname{int}(\uparrow E_Q)$.

We now replace each $Q \in \operatorname{im} \varphi_{\mathcal{M}}$ by the so chosen $\uparrow E_Q$, resulting in a function $\psi_{\mathcal{M},\mathcal{E},\mathcal{U}}$, where $\mathcal{U}$ is the collection of open neighborhoods U_Q we started with, and $\mathcal{E}$ is the collection of finitary compacts $\uparrow E_Q$. We need to check that $\psi_{\mathcal{M},\mathcal{E},\mathcal{U}}$ is continuous, and for that we check that $\psi^{-1}_{\mathcal{M},\mathcal{E},\mathcal{U}}(\Box U)$ is open for every open subset U of X. Let x be an element of $\psi^{-1}_{\mathcal{M},\mathcal{E},\mathcal{U}}(\Box U)$, and $Q = \varphi_{\mathcal{M}}(x)$; in particular, $\uparrow E_Q \subseteq U$. As in the proof of Proposition 5.4, every element y of $\operatorname{int}(Q)$ is such that $\varphi_{\mathcal{M}}(y) \subseteq Q$; for $Q' = \varphi_{\mathcal{M}}(y)$, $Q' \subseteq Q$ implies $\uparrow E_{Q'} \subseteq \uparrow E_Q$, so $\psi_{\mathcal{M},\mathcal{E},\mathcal{U}}(y) \subseteq \uparrow E_Q \subseteq U$. Therefore $\operatorname{int}(Q)$ is an open neighborhood of x included in $\psi^{-1}_{\mathcal{M},\mathcal{E},\mathcal{U}}(\Box U)$.

The family of all maps $\psi_{\mathcal{M},\mathcal{E},\mathcal{U}}$ (namely, with $\mathcal{E} = (\uparrow E_Q)_{Q \in \mathcal{M}}$ monotone, $\mathcal{U} = (U_Q)_{Q \in \mathcal{M}}$, and $Q \subseteq \operatorname{int}(\uparrow E_Q) \subseteq \uparrow E_Q \subseteq U_Q$ for each $Q \in \mathcal{M}$) is approximating, since we can choose the initial neighborhoods U_Q as close to each $Q \in \mathcal{M}$ as we like, and it remains to show that it is directed. It is non-empty: choose $\mathcal{M} = \{X\}$ and $\mathcal{U} = \mathcal{M}$, and define $\uparrow E_X$ as X itself, which is finitary compact as a consequence of Lemma 5.5 with $Q = U = X$. We find an upper bound of $\psi_{\mathcal{M},\mathcal{E},\mathcal{U}}$ and $\psi_{\mathcal{M}',\mathcal{E}',\mathcal{U}'}$ by defining $\mathcal{N} = \mathcal{M} \sqcup \mathcal{M}'$, and for the open neighborhood system $\mathcal{V}$ we let $V_N = \bigcap\{\operatorname{int}(\uparrow E_Q) \mid N \subseteq Q \in \mathcal{M}\} \cap \bigcap\{\operatorname{int}(\uparrow E'_{Q'}) \mid N \subseteq Q' \in \mathcal{M}'\}$ for each $N \in \mathcal{N}$. (We write $\mathcal{E} = (\uparrow E_Q)_{Q \in \mathcal{M}}$, $\mathcal{E}' = (\uparrow E'_{Q'})_{Q' \in \mathcal{M}'}$.) It is clear that $\psi_{\mathcal{N},\mathcal{F},\mathcal{V}}$ is above $\psi_{\mathcal{M},\mathcal{E},\mathcal{U}}$ and $\psi_{\mathcal{M}',\mathcal{E}',\mathcal{U}'}$. $\square$

Theorem 5.7 *Let X be a topological space. The following are equivalent:*

(i) *X is a stably compact, locally finitary compact space.*

(ii) *X is a sober QRB-space.*

(iii) *X is a sober QFS-space.*

(iv) *X is a QRB-domain with its Scott topology.*

(v) *X is a QFS-domain with its Scott topology.*

(vi) *X is a Lawson-compact quasi-continuous dcpo in its Scott topology.*

(vii) *X is a compact, coherent, quasi-continuous dcpo in its Scott topology.*

Proof. $(i) \Rightarrow (ii)$: X is a QRB-space by Proposition 5.6, and sober since stably-compact. $(ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (v)$ are obvious. $(iii) \Rightarrow (v)$ is Proposition 4.6, which also implies $(ii) \Rightarrow (iv)$ since QRB-spaces are instances of QFS-spaces.

$(v) \Rightarrow (vi)$. Every QFS-domain is quasi-continuous [23, Proposition 3.8], and Lawson-compact [23, Theorem 4.9].

$(vi) \Rightarrow (vii)$. For quasi-continuous domains, compactness plus coherence is the same as compactness in the Lawson topology.

$(vii) \Rightarrow (i)$. Every quasi-continuous dcpo is sober [11, Exercise 8.2.15] and locally finitary compact [11, Exercise 5.2.31]. With compactness and coherence, this implies that X is stably-compact. $\square$

Lawson and Xi's result mentioned in the introduction [21] is the equivalence $(iv) \Leftrightarrow (v) \Leftrightarrow (vi)$ above. Items (i)–(iii) offer other, purely topological characterizations of QRB-domains.

Returning to the topological beginnings of our paper, we note the following:

Theorem 5.8 *Topological QFS-spaces are QRB.*

Proof. Let $(X; \tau; (\varphi_i)_{i \in I})$ be a topological QFS-space. Then its sobrification $\hat{X}$ is a QFS-domain and so by the preceding theorem, a QRB-domain. Looking at the proof of Proposition 5.6 we find that we constructed the finitary compacts by invoking Lemma 5.5, so we should have a closer look at that in the case that we are dealing with a locally finitary compact space that is a sobrification. In that case, every element e of E is a closed irreducible set A that meets the open set $U \in \tau$. We may therefore replace e with the irreducible set $\downarrow a$ where a is an arbitrarily chosen element of $A \cap U$. In summary, then, we can make sure that the elements employed in the proof of Proposition 5.6 all stem from the image of the embedding $x \mapsto \downarrow x$ of X into its sobrification. $\square$

6 The Probabilistic Powerdomain over QRB-Domains

Let us turn to the *probabilistic powerdomain* $\mathcal{V}(X)$ over a space X. This was introduced by Jones in her PhD thesis [15] to give semantics to higher-order programs with probabilistic choice. Jones proved that $\mathcal{V}(X)$ was a continuous dcpo for every continuous dcpo X, but also that $\mathcal{V}(X)$ was not a continuous lattice, or even a bc-domain even for very simple continuous lattices or bc-domains X. We still do not know whether $\mathcal{V}(X)$ is an FS-domain, resp. an RB-domain whenever X is one, except in very specific cases [17]. However, the notion of functional approximation offered by QRB-domains is relaxed enough that the probabilistic powerdomain of a QRB-domain is again a QRB-domain. The first author proved this [10], for probability valuations (with total mass 1), and assuming second-countability.

Using Theorem 5.7, we shall see that the latter is an irrelevant assumption. We shall also prove it for spaces of continuous subprobability valuations, and of general, unbounded, continuous valuations. The nature of the proof is very different from [10]: we build an approximating family of qfs maps on $\mathcal{V}(X)$, directly [5].

The elements of $\mathcal{V}(X)$ are a slight variation on the idea of a measure, and are called continuous valuations. A *continuous valuation* ν on a space X is a Scott-continuous, strict, modular map from the complete lattice $\mathcal{O}(X)$ of open subsets of X to $\overline{\mathbb{R}^+} = \mathbb{R} \cup \{+\infty\}$. Strictness means that $\nu(\emptyset) = 0$, modularity states that $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$ for all $U, V \in \mathcal{O}(X)$.

Let $\mathcal{V}(X)$, the probabilistic powerdomain over X, denote the space of all continuous valuations on X, with the weak topology. We also write $\mathcal{V}^1(X)$ for the subspace of continuous probability valuations ($\nu(X) = 1$) and $\mathcal{V}^{\leq 1}(X)$ for the subspace of continuous subprobability valuations ($\nu(X) \leq 1$). We shall write $\mathcal{V}^\bullet(X)$ for $\mathcal{V}(X)$, $\mathcal{V}^1(X)$, or $\mathcal{V}^{\leq 1}(X)$. The *weak topology* on $\mathcal{V}^\bullet(X)$ has subbasic open sets of the form $[U > r]$, defined as $\{\nu \in \mathcal{V}^\bullet(X) \mid \nu(U) > r\}$ [19, Satz 8.5] (see also [12, Theorem 8.3]). Whatever $\bullet$ is, $\mathcal{V}^\bullet$ is a functor on the category of topological spaces, and its action $\mathcal{V}f$ on continuous maps $f\colon X \to Y$ is defined by $\mathcal{V}f(\nu)(V) = \nu(f^{-1}(V))$.

We again introduce a "distributivity law" θ, this time from $\mathcal{V}^\bullet\mathcal{Q}_\mathsf{V}$ to $\mathcal{Q}_\mathsf{V}\mathcal{V}^\bullet$. Given $\mu \in \mathcal{V}^\bullet\mathcal{Q}_\mathsf{V}(X)$, one may define $\theta(\mu)$ as the set of all $\nu \in \mathcal{V}^\bullet(X)$ such that $\nu(U) \geq \mu(\Box U)$ for every open U. It is not completely trivial that $\theta(\mu)$ is non-empty and compact, or that θ is continuous, but let us accept it for the moment. We may use θ to produce maps $\mathcal{V}^\bullet(X) \xrightarrow{\mathcal{V}\varphi_i} \mathcal{V}^\bullet\mathcal{Q}_\mathsf{V}(X) \xrightarrow{\theta} \mathcal{Q}_\mathsf{V}\mathcal{V}^\bullet(X)$ for an approximating family of quasi-deflations φ_i on X. It will be fairly easy to see that the resulting maps are approximating, but they certainly do not have finite image, and even the image of a single $\nu \in \mathcal{V}^\bullet(X)$ is in general not finitary. However, and up to a minor variation on the theme of θ (θ_f, see below), we will manage to show that these maps are qfs. Hence $\mathcal{V}^\bullet(X)$ will be a QFS space, and the equivalence between (iii) and (iv) of Theorem 5.7 will allow us to conclude.

Lemma 6.1 *Let $(X;\tau)$ be a stably compact space. If $\bullet$ is "1", assume X pointed, too. For every Scott-continuous map $f\colon \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ such that $f \leq \mathrm{id}_{\overline{\mathbb{R}^+}}$, the map θ_f defined by $\theta_f(\mu) = \{\nu \in \mathcal{V}^\bullet(X) \mid \forall U \in \tau.\ \nu(U) \geq f(\mu(\Box U))\}$ is a continuous and Scott-continuous map from $\mathcal{V}^\bullet(\mathcal{Q}_\mathsf{V}(X))$ to $\mathcal{Q}_\mathsf{V}(\mathcal{V}^\bullet(X))$.*

Proof. Let $[X \to \overline{\mathbb{R}^+}]$ denote the dcpo of Scott-continuous maps from X to $\overline{\mathbb{R}^+}$, in the pointwise ordering. For any monotonic set function ξ on the open subsets of X with values in $\overline{\mathbb{R}^+}$, and every $h \in [X \to \overline{\mathbb{R}^+}]$, one can define $\int_{x\in X} h(x)d\xi$ by the Choquet formula $\int_0^{+\infty} \xi(h^{-1}(t, +\infty])dt$, where the latter is a Riemann integral. For $\xi = \nu \in \mathcal{V}^\bullet(X)$, the functional $h \mapsto \int_{x\in X} h(x)d\nu$ is Scott-continuous and linear [26, Section 4]. Scott-continuity follows from the fact that Riemann integration of antitonic maps is itself Scott-continuous.

[5] This means proving the result using characterization 3 of QRB-domains given in Theorem 5.7. Characterization 1 may seem a better route, since $\mathcal{V}$ is already known to preserve stable compactness: only local finitary compactness remains to be proved. However, that seems quite a formidable effort by itself.

For $\xi(U) = \mu(\Box U)$, notice that $h_*(Q) = \min_{x \in Q} h(x)$ defines a continuous function of $Q \in \mathcal{Q}_{\mathsf{V}}(X)$, since $h_*^{-1}(t, +\infty] = \Box h^{-1}(t, +\infty]$ (by compactness, the inf of h is attained on Q), so that $\int_{x \in X} h(x) d\xi = \int_0^{+\infty} \mu(\Box h^{-1}(t, +\infty]) dt = \int_0^{+\infty} \mu(h_*^{-1}(t, +\infty]) dt = \int_{Q \in \mathcal{Q}_{\mathsf{V}}(X)} h_*(Q) d\mu$. Since $(h_1 + h_2)_* \geq h_{1*} + h_{2*}$, the functional $p\colon [X \to \overline{\mathbb{R}^+}] \to \overline{\mathbb{R}^+}$ that maps h to $\int_{Q \in \mathcal{Q}_{\mathsf{V}}(X)} h_*(Q) d\mu$ is superlinear, meaning that $p(h_1 + h_2) \geq p(h_1) + p(h_2)$ and $p(ah) = ap(h)$ for every $a \in \mathbb{R}^+$. Since $p(h) = \int_{x \in X} h(x) d\xi = \int_0^{+\infty} \xi(h^{-1}(t, +\infty]) dt$, p is also Scott-continuous in h.

$\theta_f(\mu)$ is non-empty. Define $q\colon [X \to \overline{\mathbb{R}^+}] \to \overline{\mathbb{R}^+}$ by: $q(h) = \sup_{x \in X} h(x)$ if $\bullet$ is "1" or "≤ 1", and $q(h) = +\infty . \sup_{x \in X} h(x)$ otherwise, agreeing that $+\infty . 0 = 0$. In each case, q is sublinear ($q(h_1 + h_2) \leq q(h_1) + q(h_2)$, and $q(ah) = aq(h)$ for every $a \in \mathbb{R}^+$), and $p \leq q$. The space $[X \to \overline{\mathbb{R}^+}]$ is a continuous dcpo, because X is locally compact hence core-compact, and using Proposition 2 of [5], for example. Together with the obvious, pointwise addition and scalar multiplication by non-negative reals, $[X \to \overline{\mathbb{R}^+}]$ is therefore a so-called continuous d-cone [27]. The Sandwich Theorem given there (Theorem 3.2) implies that there is a Scott-continuous linear map $\Lambda\colon [X \to \overline{\mathbb{R}^+}] \to \overline{\mathbb{R}^+}$ such that $p \leq \Lambda \leq q$. Defining $\nu(U)$ as $\Lambda(\chi_U)$, where χ_U is the characteristic map of U, yields a continuous valuation ν in $\mathcal{V}^\bullet(X)$ such that $\mu(\Box U) = p(\chi_U) \leq \Lambda(\chi_U) = \nu(U)$ for every open subset U of X.

Since $f \leq \mathrm{id}_{\overline{\mathbb{R}^+}}$, in particular $\theta_f(\mu)$ is non-empty.

$\theta_f(\mu)$ is compact saturated. To show this, we use the following results. Define $\nu^\dagger(Q)$, for $Q \in \mathcal{Q}(X)$, as $\inf\{\nu(U) \mid Q \subseteq U\}$, and $\langle Q \geq r \rangle_\bullet$ as $\{\nu \in \mathcal{V}^\bullet(X) \mid \nu^\dagger(Q) \geq r\}$. The latter sets are compact saturated subsets of $\mathcal{V}^\bullet(X)$: this is a consequence of [8, Lemma 6.6] if $\bullet$ is "≤ 1" or "1", and of [18, Theorem 6.5 (3)] otherwise.

We now notice that:

$$\theta_f(\mu) = \{\nu \in \mathcal{V}^\bullet(X) \mid \forall Q \in \mathcal{Q}(X) \cdot \nu^\dagger(Q) \geq a_Q^*\}, \tag{1}$$

where $a_Q^* = \inf_{U \text{ open} \supseteq Q} f(\mu(\Box U))$. Before we prove this, observe that $\theta_f(\mu)$ is therefore the intersection of the compact saturated subsets $\langle Q \geq a_Q^* \rangle$, $Q \in \mathcal{Q}(X)$, and is therefore itself compact, since $\mathcal{V}^\bullet(X)$ is stably compact. (The latter holds because X is stably compact, see [17,2]. Technically, this is proved there for $\mathcal{V}^1(X)$ and $\mathcal{V}^{\leq 1}(X)$, but the proof is similar for $\mathcal{V}(X)$.)

To prove (1), let $a_U = f(\mu(\Box U))$. Every $\nu \in \theta_f(\mu)$ trivially satisfies $\nu^\dagger(Q) \geq a_Q^*$. Conversely, assume the latter holds for every $Q \in \mathcal{Q}(X)$. For every open subset U of X, by local compactness U is the directed union of all $\mathrm{int}(Q)$, where Q ranges over the compact saturated subsets of U. Since $\Box$ commutes with directed unions, and μ and f are Scott-continuous, $a_U = \sup_{Q \subseteq U} f(\mu(\Box \mathrm{int}(Q)))$. Since $\mathrm{int}(Q) \subseteq U$ for every open neighborhood U of Q, this is less than or equal to $\sup_{Q \subseteq U} a_Q^*$, and the latter is less than or equal to a_U because $a_Q^* \leq a_U$ whenever $Q \subseteq U$. Therefore $a_U = \sup_{Q \subseteq U} a_Q^*$. A similar argument shows that $\nu(U) = \sup_{Q \subseteq U} \nu^\dagger(Q)$ (or see Tix [26, Satz 3.4 (1)]). It follows that $\nu(U) \geq a_U$. As U is arbitrary, ν is in $\theta_f(\mu)$.

θ_f is Scott-continuous. Monotonicity is clear, while for a directed family $(\mu_i)_{i \in I}$ in $\mathcal{V}^\bullet(X)$, $\theta_f(\sup_{i \in I} \mu_i) = \{\nu \in \mathcal{V}^\bullet(X) \mid \forall U \text{ open in } X \cdot \nu(U) \geq \sup_{i \in I} f(\mu_i(\Box U))\}$ (since f is Scott-continuous) $= \bigcap_{i \in I} \theta_f(\mu_i)$.

θ_f is continuous. Since X is T_0, well-filtered, and locally compact, $\mathcal{Q}(X)$ is a continuous dcpo, and the Scott and upper Vietoris topologies coincide [25, Section 7.3.4], i.e., $\mathcal{Q}(X) = \mathcal{Q}_{\mathsf{V}}(X)$. The Kirch-Tix Theorem states that given a continuous dcpo Y, the Scott and weak topologies coincide on $\mathcal{V}(Y)$ [26, Satz 4.10], and on $\mathcal{V}^{\leq 1}(Y)$ [19, Satz 8.6]; the same happens for $\mathcal{V}^1(Y)$ if additionally Y is pointed, by a trick due to Edalat [4, Section 3]: $Y' = Y \smallsetminus \{\bot\}$ is again a continuous dcpo, and $\mathcal{V}^1(Y)$ is isomorphic to $\mathcal{V}^{\leq 1}(Y')$. Taking $Y = \mathcal{Q}(X) = \mathcal{Q}_{\mathsf{V}}(X)$ (and noticing that this is pointed, as X is compact), we obtain that $\mathcal{V}^\bullet(\mathcal{Q}_{\mathsf{V}}(X))$ has the Scott topology of the pointwise ordering. To show that θ_f is continuous, it is therefore enough to show that $\theta_f^{-1}(\Box\mathcal{U})$ is open in the Scott topology for every open subset $\mathcal{U}$ of $\mathcal{V}^\bullet(X)$. Since $\Box\mathcal{U}$ is itself Scott-open by well-filteredness, this amounts to the Scott-continuity of θ_f. $\Box$

Theorem 6.2 *For every QRB-domain X, $\mathcal{V}(X)$, $\mathcal{V}^{\leq 1}(X)$, and also $\mathcal{V}^1(X)$ if X is pointed, are QRB-domains.*

Proof. Let X be a QRB-domain, and $(\varphi_i)_{i \in I}$ be an approximating family of quasi-deflations on X. By Theorem 5.7 (*iii*), we only need to show that $\mathcal{V}^\bullet(X)$ is a QFS-space. It is sober since stably compact, as we have noted earlier.

For $\epsilon \in (0,1]$, and $t \in \overline{\mathbb{R}^+}$, let $f_\epsilon(t) = \max(0, \min(t, 1/\epsilon) - \epsilon)$. This is a chain of Scott-continuous maps, as $\epsilon \geq \epsilon'$ implies $f_\epsilon \leq f_{\epsilon'}$. Also, $f_\epsilon \leq \mathrm{id}_{\overline{\mathbb{R}^+}}$. For short, write θ_ϵ for the map θ_{f_ϵ} given in Lemma 6.1, and define $\psi_{i\epsilon}$ as $\theta_\epsilon \circ \mathcal{V}\varphi_i \colon \mathcal{V}^\bullet(X) \to \mathcal{Q}_{\mathsf{V}}(\mathcal{V}^\bullet(X))$. The family $(\psi_{i\epsilon})_{i \in I, \epsilon \in (0,1]}$ is directed, and for every $\nu \in \mathcal{V}^\bullet(X)$, we claim that $\bigcap_{i \in I, \epsilon \in (0,1]} \psi_{i\epsilon}(\nu) = \uparrow \nu$.

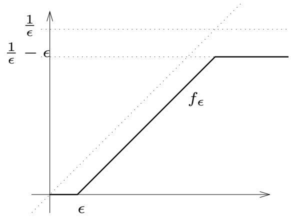

Fig. 2: The function f_ϵ

To this end, we notice that:

(*a*) $\bigcap_{\epsilon \in (0,1]} \theta_\epsilon(\mu) = \theta_{\mathrm{id}}(\mu)$. Indeed, $\bigcap_{\epsilon \in (0,1]} \theta_\epsilon(\mu) = \{\nu' \in \mathcal{V}^\bullet(X) \mid \forall U \text{ open in } X \cdot \nu'(U) \geq \sup_{\epsilon \in (0,1]} f_\epsilon(\mu(\Box U))\} = \{\nu' \in \mathcal{V}^\bullet(X) \mid \forall U \text{ open in } X \cdot \nu'(U) \geq \mu(\Box U)\} = \theta_{\mathrm{id}}(\mu)$.

(*b*) $\bigcap_{i \in I} \theta_{\mathrm{id}}(\mathcal{V}\varphi_i(\nu)) = \uparrow \nu$. This is proved as follows. For every open subset U of X, $\bigcup_{i \in I} \varphi_i^{-1}(\Box U) = U$: the elements x of U are those such that $\uparrow x \in \Box U$, and we obtain the desired equality by the defining property of quasi-deflations, plus well-filteredness. It follows that $\sup_{i \in I} \nu(\varphi_i^{-1}(\Box U)) = \nu(\bigcup_{i \in I} \varphi_i^{-1}(\Box U)) = \nu(U)$. The elements ν' of $\bigcap_{i \in I} \theta_{\mathrm{id}}(\mathcal{V}\varphi_i(\nu))$ are those elements of $\mathcal{V}^\bullet(X)$ such that, for every $i \in I$, for every open subset U of X, $\nu'(U) \geq \mathcal{V}\varphi_i(\nu)(\Box U)$; equivalently, such that $\nu'(U) \geq \sup_{i \in I} \mathcal{V}\varphi_i(\nu)(\Box U) = \nu(U)$, and we conclude.

Using these, $\bigcap_{i \in I, \epsilon \in (0,1]} \psi_{i\epsilon}(\nu) = \bigcap_{i \in I} \bigcap_{\epsilon \in (0,1]} \theta_\epsilon(\mathcal{V}\varphi_i(\nu)) = \bigcap_{i \in I} \theta_{\mathrm{id}}(\mathcal{V}\varphi_i(\nu)) = \uparrow \nu$, as announced.

It remains to show that $\psi_{i\epsilon}$ is qfs. Write δ_x for the Dirac mass at x, namely, the continuous valuation such that $\delta_x(U) = \chi_U(x)$ for every open U. Let E be the finite set of all elements that are minimal in some finitary compact in the image of φ_i, n be its cardinality, and let M be the finite set of continuous valuations of the form $\sum_{x \in E} a_x \delta_x$, where each a_x is an integer multiple of ϵ/n between 0 and $1/\epsilon$

(and with $\sum_{x\in E} a_x \leq 1$ if $\bullet$ is "≤ 1", $\sum_{x\in E} a_x = 1$ if $\bullet$ is "1"). This will be our separating set.

Fix $\nu \in \mathcal{V}^\bullet(X)$. We first simplify the expression of $\psi_{i\epsilon}(\nu)$. For $Q \in \mathcal{Q}(X)$, let $a^*_Q = \inf_{U \text{ open} \supseteq Q} f_\epsilon(\mathcal{V}\varphi_i(\nu)(\Box U)) = \inf_{U \text{ open}\supseteq Q} f_\epsilon(\nu(\varphi_i^{-1}(\Box U)))$. Let $Q_1, \cdots, Q_m$ be the finitely many finitary compacts in the image of φ_i. For $J \subseteq \{1, \cdots, m\}$, write Q_J for $\bigcup_{j\in J} Q_j$. We claim that $\psi_{i\epsilon}(\nu) = \bigcap_{J\subseteq\{1,\cdots,m\}} \langle Q_J \geq a^*_{Q_J}\rangle$. To this end, recall equality (1), which we have used in the course of proving Lemma 6.1: $\theta_f(\mu) = \{\nu' \in \mathcal{V}^\bullet(X) \mid \forall Q \in \mathcal{Q}(X) \cdot \nu'^\dagger(Q) \geq \inf_{U \text{ open}\supseteq Q} f(\mu(\Box U))\}$. So $\psi_{i\epsilon}(\nu) = \{\nu' \in \mathcal{V}^\bullet(X) \mid \forall Q \in \mathcal{Q}(X) \cdot \nu'^\dagger(Q) \geq a^*_Q\} = \bigcap_{Q\in\mathcal{Q}(X)} \langle Q \geq a^*_Q\rangle$. Looking back at the definition of a^*_Q, we see that, since φ_i has finite image, $\varphi_i^{-1}(\Box U)$ can only take finitely many values when U varies. The family of these values forms a (finite) filtered family of open sets, which therefore has a least element, which happens to be $\varphi_i^{-1}(\Box Q)$ (extending the $\Box$ notation in the obvious way). Hence $a^*_Q = f_\epsilon(\nu(\varphi_i^{-1}(\Box Q)))$. For every $\nu' \in \bigcap_{J\subseteq\{1,\cdots,m\}} \langle Q_J \geq a^*_{Q_J}\rangle$, and every $Q \in \mathcal{Q}(X)$, let J be the set of indices $j \in \{1, \cdots, m\}$ such that $Q_j \subseteq Q$. Since φ_i takes its values among $Q_1, \ldots, Q_m$, $\varphi_i^{-1}(\Box Q) = \varphi_i^{-1}(\Box Q_J)$, so $a^*_Q = a^*_{Q_J}$. It follows that $\nu'^\dagger(Q) \geq \nu^\dagger(Q_J) \geq a^*_{Q_J} = a^*_Q$, and as Q is arbitrary, $\nu' \in \bigcap_{Q\in\mathcal{Q}(X)} \langle Q \geq a^*_Q\rangle = \psi_{i\epsilon}(\nu)$. The converse inclusion $\psi_{i\epsilon}(\nu) \subseteq \bigcap_{J\subseteq\{1,\cdots,m\}} \langle Q_J \geq a^*_{Q_J}\rangle$ is obvious.

To show that $\psi_{i\epsilon}$ is qfs, it will therefore be enough to find an element $\sum_{x\in E} a_x\delta_x$ of M below ν and in $\psi_{i\epsilon}(\nu) = \bigcap_{J\subseteq\{1,\cdots,m\}} \langle Q_J \geq a^*_{Q_J}\rangle$.

Let L be the finite lattice of all intersections of sets of the form $\uparrow A$, $A \subseteq E$. Tix observed that $\nu^\dagger$ defined a valuation on the compact saturated subsets of X [26, Satz 3.4 (2–4)]. In particular $\nu^\dagger$ restricts to a valuation on L. Using the Smiley-Horn-Tarski Theorem (see, e.g., [20, Theorem 3.4]), $\nu^\dagger$ extends to an additive measure on the algebra ρL of subsets generated by L. The algebra ρL is the smallest collection of subsets containing L and closed under unions, intersections, and complements. Its elements are the finite disjoint unions of sets of the form $C_A = \bigcap_{x\in A} \uparrow x \smallsetminus \bigcup_{x\in E\smallsetminus A} \uparrow x$, $A \subseteq E$.

For each $x \in E$, let $b_x = \nu^\dagger(C_{A_x})$ where A_x is the unique subset of E such that C_{A_x} contains x (that is, $A_x = \downarrow x \cap E$). This definition ensures that $\nu^\dagger(B) = \sum_{x\in B\cap E} b_x = (\sum_{x\in E} b_x\delta_x)(B)$ for every $B \in L$. For every open subset U of X, and every $x \in E \cap U$, $C_{A_x} \subseteq \uparrow x \subseteq U$, so $(\sum_{x\in E} b_x\delta_x)(U) = \sum_{x\in E\cap U} \nu^\dagger(C_{A_x}) = \nu^\dagger(\bigcup_{x\in E\cap U} C_{A_x})$ (since the sum is disjoint) $\leq \nu^\dagger(\uparrow(E \cap U)) \leq \nu(U)$.

When $\bullet$ is "≤ 1", we define the desired element $\sum_{x\in E} a_x\delta_x$ of M by letting a_x be the nearest multiple of ϵ/n below b_x, namely $\frac{\epsilon}{n}\lfloor\frac{n}{\epsilon}b_x\rfloor$. Clearly, $\sum_{x\in E} a_x\delta_x \leq \sum_{x\in E} b_x\delta_x \leq \nu$. Moreover, for every $J \subseteq \{1, \cdots, m\}$, $(\sum_{x\in E} a_x\delta_x)^\dagger(Q_J) + \epsilon = \sum_{x\in E\cap Q_J} a_x + \epsilon \geq \sum_{x\in E\cap Q_J}(a_x + \frac{\epsilon}{n}) \geq \sum_{x\in E\cap Q_J} b_x = \nu^\dagger(Q_J)$, since Q_J belongs to ρL. Since $\varphi_i(x) \supseteq \uparrow x$ for every $x \in X$, $\varphi^{-1}(\Box Q_J) \subseteq Q_J$. For every $Q \in \mathcal{Q}(X)$, recall that $\varphi_i^{-1}(\Box Q)$ is the least element of some finite filtered family of open sets, hence is open. It follows that the notation $\nu(\varphi_i^{-1}(\Box Q_J))$ makes sense. From $\varphi_i^{-1}(\Box Q_J) \subseteq Q_J$, we obtain $\nu^\dagger(Q_J) \geq \nu(\varphi_i^{-1}(\Box Q_J))$. We have just shown that $(\sum_{x\in E} a_x\delta_x)^\dagger(Q_J) + \epsilon \geq \nu(\varphi_i^{-1}(\Box Q_J))$, whence $(\sum_{x\in E} a_x\delta_x)^\dagger(Q_J) \geq \max(0, \nu(\varphi_i^{-1}(\Box Q_J)) - \epsilon) = f_\epsilon(\nu(\varphi_i^{-1}(\Box Q_J))) = a^*_{Q_J}$. (We are silently using the fact that $f_\epsilon(t) = \max(0, t - \epsilon)$ for every $t \in [0, 1]$.) Therefore $\sum_{x\in E} a_x\delta_x$ is in

$\langle Q_J \geq a^*_{Q_J}\rangle$, and as J is arbitrary, it is in $\psi_{i\epsilon}(\nu)$.

When $\bullet$ is "1" instead, we use the standard trick of putting all the missing mass on the bottom element $\perp$. In other words, we define a_x as above for $x \neq \perp$, and as $1 - \sum_{x\in E, x\neq\perp} a_x$ otherwise. (Note that E contains $\perp$. Indeed, it appears as the minimal element of $\varphi_i(\perp) = X$.) We check that $(\sum_{x\in E} a_x\delta_x)(U) \leq \nu(U)$ as above when U does not contain $\perp$, while the same inequality reduces to the trivial $1 \leq 1$ when U contains $\perp$, namely when $U = X$. Since we are using larger coefficients a_x than in the "≤ 1" case, the fact that $(\sum_{x\in E} a_x\delta_x)^\dagger(Q_J) \geq a^*_{Q_J}$ follows by the same arguments. It follows, again, that $\sum_{x\in E} a_x\delta_x$ is in $\bigcap_{J\subseteq\{1,\cdots,m\}}\langle Q_J \geq a^*_{Q_J}\rangle = \psi_{i\epsilon}(\nu)$.

Finally, when $\bullet$ is neither "≤ 1" not "1", we argue as in the "≤ 1" case, except we now define a_x as $\frac{\epsilon}{n}\lfloor\frac{n}{\epsilon}\min(\frac{1}{\epsilon}, b_x)\rfloor$. To check that $(\sum_{x\in E} a_x\delta_x)^\dagger(Q_J) \geq a^*_{Q_J}$, we compute $(\sum_{x\in E} a_x\delta_x)^\dagger(Q_J) + \epsilon = \sum_{x\in E\cap Q_J} a_x + \epsilon \geq \sum_{x\in E\cap Q_J}(a_x + \frac{\epsilon}{n}) \geq \sum_{x\in E\cap Q_J}\min(\frac{1}{\epsilon}, b_x)$. If every b_x is less than or equal to $\frac{1}{\epsilon}$, the latter is equal to $\nu^\dagger(Q_J)$, hence greater than or equal to $\min(\frac{1}{\epsilon}, \nu^\dagger(Q_J))$. If $b_x > \frac{1}{\epsilon}$ for some $x \in E\cap Q_J$, then $\nu^\dagger(Q_J) \geq \nu^\dagger(C_{A_x}) = b_x > \frac{1}{\epsilon}$, so $\sum_{x\in E\cap Q_J}\min(\frac{1}{\epsilon}, b_x) \geq \frac{1}{\epsilon} = \min(\frac{1}{\epsilon}, \nu^\dagger(Q_J))$. In any case, $(\sum_{x\in E} a_x\delta_x)^\dagger(Q_J) + \epsilon \geq \min(\frac{1}{\epsilon}, \nu^\dagger(Q_J))$. As in the "$\leq 1$" case, this implies $(\sum_{x\in E} a_x\delta_x)^\dagger(Q_J) + \epsilon \geq \min(\frac{1}{\epsilon}, \nu(\varphi_i^{-1}(\Box Q_J)))$, so $(\sum_{x\in E} a_x\delta_x)^\dagger(Q_J) \geq f_\epsilon(\nu(\varphi_i^{-1}(\Box Q_J))) = a^*_{Q_J}$. $\square$

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