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Fast Recoloring of Sparse Graphs

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Abstract

In this paper, we show that for every graph of maximum average degree bounded away from d and any two $(d+1)$-colorings of it, one can transform one coloring into the other one within a polynomial number of vertex recolorings so that, at each step, the current coloring is proper. In particular, it implies that we can transform any 8-coloring of a planar graph into any other 8-coloring with a polynomial number of recolorings. These results give some evidence on a conjecture of Cereceda et al [8] which asserts that any $(d+2)$ coloring of a d-degenerate graph can be transformed into any other one using a polynomial number of recolorings.

We also show that any $(2d+2)$-coloring of a d-degenerate graph can be transformed into any other one with a linear number of recolorings.

1 Introduction

Reconfiguration problems consist in finding step-by-step transformations between two feasible solutions such that all intermediate states are also feasible. Such problems model dynamic situations where a given solution is in place and has to be modified, but no property disruption can be afforded. Recently, reconfigurations problems have raised a lot of interest in the context of constraint satisfaction problems [6, 12] and of graph invariants like independent sets [13], dominating sets [3, 15] or vertex colorings [4, 5].

In this paper $G = (V, E)$ is a graph where n denotes the size of V and k is an integer. For standard definitions and notations on graphs, we refer the reader to [10]. A *"proper" k-coloring* of G is a function $f : V(G) \to \{1, \dots, k\}$ such that, for every $xy \in E$, $f(x) \neq f(y)$. Throughout the paper we will only consider proper colorings. In the following, we will omit the proper for brevity. The *chromatic number* $\chi(G)$ of a graph G is the smallest k such that G admits a k-coloring. Two k-colorings are *adjacent* if they differ on exactly one vertex. The *k-recoloring graph of G*, denoted by $\mathcal{C}_k(G)$ and defined for any $k \geq \chi(G)$, is the graph whose vertices are k-colorings of G, with the adjacency condition defined above. Note that two colorings equivalent up to color permutation are distinct vertices in the recoloring graph. The graph G is *k-mixing* if $\mathcal{C}_k(G)$ is connected. Cereceda, van den Heuvel and Johnson provided an algorithm to decide whether, given two 3-colorings of a graph, one can transform the one into the other in polynomial time [8, 9]. In particular, their result characterizes

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3-mixing graphs. The easiest way to prove that a graph G is not k-mixing is to exhibit a *frozen* k-coloring of G, *i.e.* a coloring in which every vertex is adjacent to vertices of all other colors. Such a coloring is an isolated vertex in $\mathcal{C}_k(G)$.

Given any two colorings of a graph, to decide whether one can be transformed into the other, is **PSPACE**-complete for $k \geq 4$ [5]. The *k-recoloring diameter* of a k-mixing graph is the diameter of $\mathcal{C}_k(G)$. In other words, it is the minimum D for which any k-coloring can be transformed into any other one through a sequence of at most D adjacent k-colorings. Bonsma and Cereceda [5] proved that there exists a family of graphs and an integer k such that, for every graph G in the family there exist two k-colorings whose distance in the k-recoloring graph is finite and super-polynomial in n. However, the diameter of the k-recoloring may be polynomial when we restrict to a well-structured class of graphs and k is large enough. Graphs with bounded degeneracy are natural candidates.

The diameter of the k-recoloring graphs has been already studied in terms of the degeneracy of a graph. It was shown independently by Dyer et al [11] and by Cereceda et al. [8] that for any $(d-1)$-degenerate graph G and every $k \geq d+1$, $\mathcal{C}_k(G)$ is connected ($\mathrm{diam}(\mathcal{C}_k(G)) < \infty$). Moreover, Cereceda [7] also showed that for any $(d-1)$-degenerate graph G and every $k \geq 2d-1$, we have $\mathrm{diam}(\mathcal{C}_k(G)) = O(n^2)$.

Cereceda et al. conjectured in [8] that, for any $(d-1)$-degenerate graph G and every $k \geq d+1$, we have $\mathrm{diam}(\mathcal{C}_k(G)) = O(n^2)$. No general result is known so far on this conjecture, but several particular cases have been treated in the last few years. Bonamy et al. [4] showed that for every $(d-1)$-degenerate chordal graph and every $k \geq d+1$, $\mathrm{diam}(\mathcal{C}_k(G)) = O(n^2)$, improving the results of [8, 11]. This result was then extended to graphs of bounded treewidth by Bonamy and Bousquet in [1]. Unfortunately, all these results are based on the existence of an underlying tree structure. This leads to nice proofs but new ideas are required to extend these results to other classes of graphs.

Our results. In Section 2, we show that Cereceda's quadratic bound on the recoloring diameter can be improved into a linear bound if one more color is available. More precisely we show that for every $(d-1)$-degenerate graph G and every $k \geq 2d$, the recoloring diameter of G is at most dn.

In Section 3, we study the k-recoloring diameter from another invariant of graphs related to degeneracy: the maximum average degree. The *maximum average degree* of G, denoted by $\mathrm{mad}(G)$, is the maximum average degree of a (non-empty) induced subgraph H of G. We prove that for every integer $d \geq 1$ and for every $\varepsilon > 0$, there exists $c = c(d, \varepsilon) \geq 1$ such that for every graph G satisfying $\mathrm{mad}(G) \leq d - \varepsilon$ and for every $k \geq d+1$, $\mathrm{diam}(\mathcal{C}_k(G)) = O(n^c)$. The proof goes as follows. We first show that the vertex set can be partitioned into a logarithmic number of sparse sets. Using this partition, we show that one color can be eliminated after a polynomial number of recolorings and then we finally conclude by an iterative argument.

Since every planar graph G satisfies $\mathrm{mad}(G) \leq 6$, our result implies that for every $k \geq 8$ the diameter of the k-recoloring graph of G is polynomial in n. Bousquet and Bonamy observed in [2] that $k \geq 7$ is needed to obtain such a conclusion and conjectured that $k = 7$ is enough (this is the planar graph version of the conjecture raised by Cereceda et al. [8] for degenerated graphs). We also discuss the limitations of our approach by showing that it cannot provide a polynomial bound on the diameter of the 7-recoloring graph of a planar graph. Finally, we also mention other consequences of our result to triangle-free planar graphs.

The degeneracy is closely related to the maximum average degree: a graph G satisfying $\mathrm{mad}(G) \leq d$ is d-degenerate and every d-degenerate graph has maximum average degree at most $2d$ (see e.g. Proposition 3.1 of [14]). Using the latter inequality, one can deduce from our result that if G has

degeneracy $d-1$, the diameter of the $2d$-recoloring graph of G is polynomial in n. However, as the first part of our paper shows, better results can be attained in such case.

2 Linear diameter with $2d$ colors

Let us first set some basic notations. Let X be a subset of V. The *size* $|X|$ of X is its number of elements. Let $G = (V, E)$ be a graph. For any coloring α of G, we denote by $\alpha(H)$ the set of colors used by α on the subgraph H of G. The *neighborhood* of a vertex x in G, denoted by $N_G(x)$, is the subset of vertices $y \in V(G)$ such that $xy \in E(G)$. If the graph G is clear from the context, we will denote $N_G(x)$ by $N(x)$. The *length* of a path P is its number of edges and its size, denoted by $|P|$, is its number of vertices. The *distance* between two vertices x and y, denoted by $d(x, y)$, is the minimum length of a path between these two vertices. When there is no such path, $d(x, y)$ is set to infinite. The *distance* between two k-colorings of G is implicitly the distance between them in the k-recoloring graph $\mathcal{C}_k(G)$. The *diameter* of G is the maximum over all the pairs $u, v \in V(G)$ of the distance between u and v.

Theorem 1. *For every $(d-1)$-degenerate graph G on n vertices and every $k \geq 2d$, diam$(\mathcal{C}_k(G)) \leq dn$. Even more, for any two k-colorings there exists a recoloring procedure that transforms one into the other and where every vertex is recolored at most d times.*

Proof. Let α and β be two k-colorings. We will show by induction on the number of vertices that there exists a recoloring procedure that transforms α into β and where every vertex is recolored at most d times. If $n = 1$ the result is obviously true. Let G be a $(d-1)$-degenerate graph on $(n+1)$ vertices and let u be a vertex of degree at most $d-1$. Consider $\tilde{G}$ to be the graph induced by $V \setminus \{u\}$. Let us denote by $\tilde{\alpha}$ and $\tilde{\beta}$ the restrictions of α and β to $\tilde{G}$. By induction, the coloring α' can be transformed into $\tilde{\beta}$ so that every vertex is recolored at most d times and at every step, the k-coloring is proper in $\tilde{G}$.

Since u has at most $d-1$ neighbors and since each vertex in $\tilde{G}$ is recolored at most d times, the neighbors of u are recolored $\ell \leq d(d-1)$ times in this recoloring sequence. Let $t_1, \ldots, t_\ell$ be the times in the recoloring sequence when a neighbor of u changes its color. For any time t in the sequence, let c_t be the new color assigned at this time.

Consider again the initial graph G. Now, let us try to add some recolorings of the vertex u in the sequence of recolorings obtained for $\tilde{G}$ to guarantee that the k-colorings are proper in G. We claim that the vertex u can be inserted in the recoloring sequence of $\tilde{G}$ with the addition of at most d new recoloring steps that change the color of u. Consider the following recoloring algorithm: at each step of the recoloring process, some vertex v is recolored from color a to color b. If v is not a neighbor of u or if the current color of u is not b, the obtained coloring is still proper in G and we do not perform any recoloring of u. Assume now that $v \in N(u)$ and that the color of u is b. This happens at some time t_i, with $i \leq \ell$. In this case, we add a new recoloring step in our sequence right before the recoloring of v at time t_i, in which we change the color of u. In order to maintain the proper coloring, we want to assign to u a color distinct from the colors in $N(u)$ (there are at most $d-1$ different colors there). So there remain at least $k-(d-1) \geq d+1$ choices of colors for u that do not create monochromatic edges. Thus, we assign to u a color distinct from $c_{t_i}, \ldots, c_{t_{i+d-1}}$. By choosing this color, we make sure that u will require no recoloring before time t_{i+d} in the sequence.

Let s be the number of recolorings of u and let $t_{i_1}, \ldots, t_{i_s}$ the corresponding recoloring times in the original sequence. By the construction of the new sequence, observe that $i_{j+1} - i_j \geq d$ for every $j < s$. Since $\ell \leq d(d-1)$ and $i_s \leq \ell$, we have that $s \leq d-1$. At the end of the procedure we may have

to change the color of u to $\beta(u)$ if it is not its current color. Hence, the recoloring of $V \setminus \{u\}$ can be extended to V by recoloring the vertex u at most d times, which concludes the proof. $\square$

3 Recoloring sparse graphs

The *maximum average degree* of a graph G is defined as

$$\text{mad}(G) = \max_{\emptyset \neq H \subseteq G} \frac{2|E(H)|}{|V(H)|} .$$

We will prove the following theorem that relates the maximum average of the graph with the diameter of its recoloring graph.

Theorem 2. *For every integer $d \geq 0$ and for every $\varepsilon > 0$, there exists $c = c(d, \varepsilon) \geq 1$ such that for every graph G on n vertices satisfying mad$(G) \leq d - \varepsilon$ and for every $k \geq d + 1$, we have diam$(\mathcal{C}_k(G)) = O(n^c)$.*

For every graph G, every t-partition $\{V_1, \ldots, V_t\}$ of the vertex set of G and every $i \leq t$, we consider the following induced subgraph,

$$G_i = G\left[\cup_{j \geq i} V_j\right] .$$

The *level function* of a t-partition, denoted by $L : V(G) \longrightarrow \{1, \ldots, t\}$, labels each vertex with its corresponding part of the partition, that is $L(v) = i$ for every $v \in V_i$. A *t-partition of degree ℓ of G* is a t-partition $\{V_1, \ldots, V_t\}$ of the vertex set of G such that every vertex $v \in V_i$ has degree at most ℓ in $V(G_i)$.

The existence of a t-partition of degree $(d - 1)$ is crucial in the proof of Theorem 2. Let us briefly explain why. If $k \geq d + 1$, then for any k-coloring α of G and for every vertex $v \in V$ there exists at least one color, say a, such that $a \neq \alpha(v)$ and it does not appear in $N_{G_{L(v)}}(v)$. Indeed the vertex v has at most $(d - 1)$ neighbors in $G_{L(v)}$ and there are $k \geq d + 1$ colors. Thus, we can always change the color $\alpha(v)$ by a without creating any monochromatic edge in G_i. Nevertheless, this recoloring step may create monochromatic edges in G. In order to prove Theorem 2, we will provide a recursive recoloring algorithm that, given a coloring α and a vertex v, obtains a new coloring α' with $\alpha'(v) = a$ by only performing a polynomial number of valid recolorings.

Consider a total order $\prec$ on the set of vertices such that if $u \prec w$ then $L(u) \leq L(w)$. We proceed to describe Procedure 1 which has as an input a tuple (γ, P), where γ is a coloring of G and P is a list of vertices that forms a path in G. The recoloring algorithm will consist in the call of Procedure 1 with input $(\alpha, \{v\})$.

A procedure call C *generates (or calls) a call* D if D is started during the call of C. In this case we also say that D is a *recursive call of* C. Similarly, a procedure call D is *generated by* C if there exists a sequence of calls $C = C_1, C_2, \ldots, C_t = D$ such that C_i generates C_{i+1} for every $i < t$. The vertex u in the last position of the list P in a particular call of Procedure 1, will be called the *current vertex of the call*. Recursive calls made in a call where u is the current vertex are called *recursive calls of* u. We say that a procedure call C is *generated by* u if a call of a procedure with current vertex u generates C.

Let us first state a few immediate remarks concerning this procedure. Observe that in each recursive call, we add one vertex in the list P. By construction, the vertex added in P in the recursive call is a neighbor of the current vertex u and has level strictly smaller than u. Since in any call C of the procedure the unique recolored vertex is the current vertex u, an immediate induction argument ensures that any recolored vertex in calls generated by C has level strictly smaller than $L(u)$. So we have the following:

Procedure 1 Recoloring Algorithm

Input: A coloring γ of G, a list P of vertices.
Output: A coloring γ' of G which agrees with γ on $V(G_{L(u)}) \setminus \{u\}$ where u is the last element of P. Moreover $\gamma(u) \neq \gamma'(u)$.

Let u be the last element of P. — u is the current vertex of the procedure.
Let a be a color not in $\gamma\left(\{u\} \cup N_{G_{L(u)}}(u)\right)$. — Such a color exists and is the *target color* for u.
Let $\gamma' = \gamma$. — γ' is the current coloring.
Let $X = \{v_1 \succ \ldots \succ v_s\}$ be the set of neighbors of u in $\cup_{j<L(u)} V_j$.
for $v_i \in X$ with i increasing **do**
  if v_i is colored with a **then**
    Add v_i at the end of P.
    $\gamma' \leftarrow$ Procedure 1 with input (γ', P). — The color of v_i is now different from a.
    Delete v_i from the end of P.
  end if
end for
Change the color of u to a in γ'.
Output γ'

Observation 1. *If the procedure call C with input (γ, P) makes some recursive calls, then the size of P increases in these calls. Moreover, the level of the vertex v_i added at the end of P during C, is strictly smaller than the level of the current vertex u and both vertices are adjacent.*
This implies that for every vertex w recolored in a call generated by C we have $L(w) < L(u)$, i.e. *the coloring output by a call with current vertex u agrees with γ on $V(G_{L(u)}) \setminus \{u\}$.*

Recall that the vertices of G are equipped with a total order $\prec$. A path P_1 is *lexicographically smaller* than P_2, denoted by $P_1 \prec_l P_2$ if:

- P_2 is empty and P_1 is not.
- The first vertex of P_1 is smaller than the first vertex of P_2.
- The first vertices of both paths are the same and the path P_1 without its first vertex is lexicographically smaller than the path P_2 without its first vertex.

Informally, we compare the first vertex of each path (which in our case will be the largest) and if they are not equal, the largest path is the one with the largest vertex; otherwise we compare the remaining paths. Notice that if P_2 is contained in the first positions of a path P_1, then $P_1 \prec_l P_2$. In particular, with this definition, the empty path is the largest one.

The *path of the procedure call C*, denoted by P_C, is the path P given as an input of C. We will show that the sequence of paths used as an input of successive calls of Procedure 1, is lexicographically strictly decreasing (in particular two calls cannot have the same path P).

Claim A. *If a procedure call D is initiated after a call C, then $P_D \prec_l P_C$.*

Proof. First note that if D is called by C then $P_D \prec_l P_C$. Indeed, the path P_C is contained in the first positions of P_D. Consider now two calls C and D of Procedure 1 such that D is generated by C. An immediate induction argument using the previous observation ensures that $P_D \prec_l P_C$.

So we may assume that D is not generated by C. Let us denote by I the initial call of Procedure 1. Recall that all the calls are generated by I and that they are organized in a tree structure. So there exists a unique sequence $\mathcal{S}_1 : I = C_1, C_2, \ldots, C_{t_1} = C$ such that C_j calls C_{j+1} for every $j < t_1$ and a unique sequence $\mathcal{S}_2 : I = D_1, D_2, \ldots, D_{t_2} = D$ such that D_j calls D_{j+1} for every $j < t_2$. Let us denote by B the last common procedure call in $\mathcal{S}_1$ and $\mathcal{S}_2$. Since D is not generated by C, $\mathcal{S}_1$ is not included in $\mathcal{S}_2$ and then B is not the last element of $\mathcal{S}_1$ or $\mathcal{S}_2$. Let us denote by B_C the procedure called by B in $\mathcal{S}_1$ and by B_D the procedure called by B in $\mathcal{S}_2$. We have:

- B_C and B_D are called by B in this order (otherwise D would have been initiated before C),
- either $B_C = C$ or B_C generates C, and
- either $B_D = D$ or B_D generates D.

The previous observations ensure that $P_D \preceq_l P_{B_D}$. Thus, it suffices to show that $P_{B_D} \prec_l P_C$. Since B_C and B_D are called by B, the corresponding paths P_{B_C} and P_{B_D} are both P_B plus a last additional vertex, denoted respectively by v_{B_C} and v_{B_D}. Since B_C is called before B_D, by construction of Procedure 1 we have $v_{B_D} \prec v_{B_C}$. Notice that C is generated by B_C, which implies that P_{B_C} is contained in the first $|P_{B_C}|$ positions of P_C. So the path P_C is lexicographically larger than P_{B_D}: they coincide in the first $|P_B|$ positions and at the first position where they differ we have $v_{B_D} \prec v_{B_C}$. □

A path $P = (u_1, \ldots, u_s)$ is *level-decreasing* if $L(u_i) > L(u_{i+1})$ for every $i < s$. Observation 1 ensures that P_C is a level-decreasing path for any call C. In the follow claim we will show that if the t-partition used in Procedure 1 has small degree, then the number of paths passing through any vertex is bounded.

Claim B. *Given that the t-partition $\{V_1, \ldots, V_t\}$ of G has degree at most ℓ, the number of level-decreasing paths between two vertices u and w in different levels is at most ℓ^{i-1} where $i = |L(u) - L(w)|$.*

Proof. Without loss of generality, we may assume $L(w) < L(u)$. Let us prove the claim by induction on i. If $i = 1$, then there is at most one level-decreasing path between u and w which is the edge uw if it exists. Assume now that $L(u) - L(w) = i$. By the definition of a t-partition of degree ℓ, the vertex w has at most ℓ neighbors in $G_{L(w)}$, and, in particular, $s \leq \ell$ neighbors in $\cup_{j=L(w)+1}^{L(u)-1} V_j$. Let us denote by $w_1, \ldots, w_s$ these neighbors of w. Notice that $1 \leq L(u) - L(w_j) \leq i - 1$ for every w_j. Notice that for any level-decreasing path P from u to w, the before last element of P should be in $\{w_1, \ldots, w_s\}$. By induction, for every w_j, there are at most ℓ^{i-2} level-decreasing paths from u to w_j. Therefore, the number of level-decreasing paths from u to w is at most

$$\sum_{j=1}^{s} \ell^{i-2} \leq \ell^{i-1} ,$$

which concludes the proof of the claim. □

We say that two colorings α and β *agree* on some subset X if $\alpha(x) = \beta(x)$ for every $x \in X$.

Lemma 3. *Suppose that G admits a t-partition of degree ℓ. For every $v \in V$ and every $(\ell + 2)$-coloring α, there exists an $(\ell + 2)$-coloring α' with $\alpha(v) \neq \alpha'(v)$, such that α' can be obtained from α by recoloring each vertex at most $\ell^{L(v)}$ times. Moreover, α' agrees with α in $V(G_{L(v)}) \setminus \{v\}$ at any recoloring step.*

Proof. Let us now prove that the recoloring algorithm ends, that it makes the right amount of recolorings and that it is correct.

Termination and number of recolorings in the recoloring algorithm. Each call of Procedure 1 creates at most n recursive calls (we have a priori no good upper bound on the number of neighbors of u in $\cup_{j<L(u)} V_j$). Since the level of the current vertex u decreases at every recursive call, the depth of the recursion is at most $L(u) \leq t$. This implies that the recoloring algorithm will terminate in at most n^t iterations. We need an additional argument to show that the number of recolorings is at most ℓ^t as stated in the lemma.

Notice that the number of recolorings is exactly the number of calls of Procedure 1; every procedure call C only recolors one vertex once, the current one in C. Thus, if we can bound the number of calls where v is the current vertex, then we can bound the number of times we recolored v.

Let I be the initial call of Procedure 1. Recall that $P_I = \{v\}$. Since each call C is generated by I, the first vertex in the path P_C is v. By Claim A, the sequence of paths used as an input of successive calls of Procedure 1 is lexicographically strictly decreasing. By Claim B, the number of level-decreasing paths from the vertex v to any given w is at most ℓ^{i-1}, where $i = L(v) - L(w) \leq L(v)$. Since the unique recolored vertex in each procedure is the current vertex u, we obtain that for every $v \in V$ and every $(\ell+2)$-coloring α we can change the color of v by recoloring each vertex at most $\ell^{L(v)}$ times.

Correctness of the recoloring algorithm. Let us now prove that if the initial coloring is proper, then at any step of the algorithm, the current coloring is also proper. We have already seen that in each call of Procedure 1 the unique recolored vertex is the current vertex u (the last vertex in P) and in any recursive call of u, the current vertex w satisfies $L(w) < L(u)$.

It suffices to show that when u is recolored in a procedure call C with color a, no neighbor of u has color a. Color a is chosen in Procedure 1 such that no neighbor of u in $V(G_{L(u)})$ is colored with a. Since, by Observation 1 the vertices of $V(G_{L(u)})$ are not recolored by any of the calls generated by C, recoloring u with a does not create monochromatic edges in $V(G_{L(u)})$.

Let $v_1, \ldots, v_s$ be the neighbors of u in $\cup_{j<L(u)} V_j$ in decreasing order with respect to $\prec$. Let $\gamma'_0 = \gamma$ be the coloring used as an input of the call C and, for every $i \leq s$, let γ'_i be the coloring γ' output by the procedure called by C whose current vertex is v_i. Recall that when the recoloring of u is performed, the current coloring is γ'_s. We will show that $\gamma'_s(v_i) \neq a$ for every $i \leq s$.

If $\gamma'_{i-1}(v_i) \neq a$, then we do not call any procedure to change the color of v_i and $\gamma'_i = \gamma'_{i-1}$. If $\gamma'_{i-1}(v_i) = a$, then γ'_i is the output of Procedure 1 with input parameters $\gamma = \gamma'_{i-1}$ and $P = (P_C, v_i)$. Since v_i is now the last vertex of P, by construction of the procedure, the coloring γ'_i satisfies that $\gamma'_i(v_i) \neq a$.

It remains to show that the color of v_i is not modified between γ'_i and the final coloring γ'_s. For the sake of contradiction assume that $j_* \in \{i+1, \ldots, s\}$ is the smallest integer j such that $\gamma'_j(v_i) \neq \gamma'_i(v_i)$. This implies that v_i is the current vertex of a call D generated by the procedure call corresponding to v_{j_*}. Hence, the vertex v_{j_*} appears before than v_i in P_D. On the one hand, since $i \leq j_*$, by the order given on the neighbors of u, we have $L(v_i) \geq L(v_{j_*})$. On the other hand, since the path P_D is level-decreasing, $L(v_{j_*}) > L(v_i)$, leading a contradiction. So the recoloring algorithm is correct.

Let us finally prove Lemma 3. The recoloring algorithm calls Procedure 1 with the initial coloring α and the list $P = \{v\}$ and outputs an $(\ell+2)$-coloring α' with $\alpha'(v) \neq \alpha(v)$. Moreover, it provides a sequence of proper colorings where, apart from v, no other vertex with level at least $L(v)$ has been recolored. This concludes the proof of Lemma 3. □

The next lemma is a natural consequence of Lemma 3.

Lemma 4. *Suppose that a graph G on n vertices admits a t-partition of degree ℓ. Then, for any $(\ell+2)$-coloring α there exists a $(\ell+1)$-coloring β (that is, $\beta(v) \neq \ell+2$ for every $v \in V(G)$) such that $d(\alpha, \beta) \leq \ell^t n^2$.*

Proof. Let us fix a t-partition of degree ℓ of G and denote by $V_1, \dots, V_t$ its parts. By Lemma 3, we can change the color of every vertex in $v \in V_i$ colored with color $\ell + 2$ by performing at most ℓ^i recolorings for each vertex in $\cup_{j<i} V_j$. Thus, first remove color $\ell + 2$ from V_t by recoloring each vertex in G at most $\ell^t |V_t|$ times, then remove it from V_{t-1} by recoloring each vertex at most $\ell^{t-1}|V_{t-1}|$, and so on. By Claim 3, while removing color $\ell + 2$ from V_i, we do not recolor any of the vertices in G_i (apart from the ones with color $\ell + 2$). Therefore, while recoloring V_i we never create new vertices in color $\ell + 2$ in G_i. After removing color $\ell + 2$ from V_1 we have a proper coloring β of G that does not use the color $\ell + 2$. Moreover, we have recolored each vertex at most

$$\ell|V_1| + \ell^2|V_2| + \cdots + \ell^t|V_t| \leq \ell^t n \ ,$$

times. Thus the total number of recolorings is at most $\ell^t n^2$ concluding the first part of the lemma. □

The following lemma shows that we can select a canonical stable set S of G, such that $G \setminus S$ has a partition with smaller degree.

Lemma 5. *Let G be a graph that admits a t-partition of degree ℓ. Then there exists a stable set S such that $G \setminus S$ admits a t-partition of degree $\ell - 1$.*

Proof. Fix a t-partition of degree ℓ of G and denote by $V_1, \dots, V_t$ its parts. Let S_t be a maximal (by inclusion) stable set in G_t. Define recursively S_i to be a maximal (by inclusion) stable set in $G_i \setminus T_i$, where $T_i = \bigcup_{j \geq i} (S_j \cup N_{G_i}(S_j))$ (recall that $N_G(X)$ is the set of vertices in $V(G) \setminus X$ at distance one from some vertex in X) and let $S = S_1 \cup \cdots \cup S_t$. By construction of T_i, any vertex in S_i is not in the neighborhood of S_j for any $j > i$, thus S is a stable set.

We claim that $\{V_1 \setminus S_1, \dots, V_t \setminus S_t\}$ is a t-partition of degree $\ell - 1$ of $G \setminus S$. We just need to show that every $v \in V_i \setminus S_i$ has degree at most $\ell - 1$ in $G'_i = G_i \setminus S$. By the maximality condition of the selected stable sets, any such v has at least one neighbor in S. In particular, by the order of the construction (from V_t to V_1), it has at least one neighbor in $\cup_{j \geq i} S_j$ (otherwise v could be included in S_i, contradicting the maximality of it). Since $\{V_1, \dots, V_t\}$ is a t-partition of degree ℓ, any $v \in V_i$ has at most ℓ neighbors in G_i. Therefore the degree of v in G'_i is at most $\ell - 1$ and $G \setminus S$ admits a t-partition of degree $\ell - 1$. □

The following result shows that a bounded average degree implies the existence of a good partition.

Proposition 6. *For every $d \geq 1$ and every $\varepsilon > 0$, there exists a $C = C(d, \varepsilon) > 0$ such that for every graph G on n vertices that satisfies mad$(G) \leq d - \varepsilon$, G admits a $(C \log_d n)$-partition of degree $d - 1$.*

Proof. Set $C = (\log_d(d/(d-\varepsilon)))^{-1}$. By the definition of the maximum average degree of a graph, every nonempty subgraph of G has density at most $d - \varepsilon$. Partition the set $V = U_{<d} \cup U_{\geq d}$ in two parts where $v \in U_{<d}$ if the degree of v is at most $d - 1$ and $v \in U_{\geq d}$ otherwise. We have,

$$(d - \varepsilon) \cdot n \geq 2|E(G)| = \sum_{v \in V} \deg(v) \geq \sum_{v \in U_{\geq d}} \deg(v) \geq d|U_{\geq d}| \ .$$

This directly implies that $|U_{\geq d}| \leq \frac{d-\varepsilon}{d} \cdot n$. Set the first part of the t-partition as $V_1 = U_{<d}$. Notice that $|V_1| \geq \frac{\epsilon}{d} \cdot n$. Since the graph $G^{(2)} = G \setminus V_1$ is a subgraph of G, its maximum average degree is at most $d - \varepsilon$ and thus we can repeat the same procedure on it. Moreover, $|V(G^{(2)})| \leq \frac{d-\varepsilon}{d} \cdot n$. After m iterations of this procedure, we have $|V(G^{(m)})| \leq \left(\frac{d-\epsilon}{d}\right)^m n$, and thus, we have to repeat this procedure at most $t = \log_{\frac{d}{d-\varepsilon}} n = \frac{\log_d n}{\log_d(d/(d-\varepsilon))} = C \log_d n$ times before we finish the construction of the partition of degree $d - 1$. □

Set

$$c(d,\varepsilon) = \frac{1}{\log_d(d/(d-\varepsilon))} + 2 \ .$$

Recall that G admits a $((c(d,\varepsilon)-2)\cdot \log_d n)$-partition of degree at most $d-1$.

Now we show that the recoloring graph of a graph with a low degree partition has small diameter.

Proposition 7. *Let G be a graph on n vertices that admits a t-partition of degree ℓ. Then for every $k \geq \ell+2$ we have*

$$\mathit{diam}(\mathcal{C}_k(G)) \leq 4\ell^{t+1}n^2 \ .$$

Proof. We will show that any k-coloring α can be reduced to a canonical k-coloring γ^* using at most $2\ell^{t+1}n^2$ recoloring steps. This canonical coloring γ^* only depends on structural properties of G and not on the coloring α (the precise definition of γ^* will be detailed below). The previous claim implies the statement of the theorem: between any pair of colorings α_1 and α_2 there exists a path in the k-recoloring graph of length at most $4\ell^{t+1}n^2$ (which in particular goes through γ^*).

Let us show how to transform α into the canonical coloring γ^*. Let $G_\ell = G$ and $\alpha_\ell = \alpha$. For every j from ℓ to 1, we do the following recoloring procedure:

1. Use Lemma 4 on G_j in order to transform the $(k-\ell+j)$-coloring α_j into a $(k-\ell+j-1)$-coloring β using at most $j^t|V(G_j)|^2$ many recoloring steps.
2. Let S_j be the stable set of G_j provided by Lemma 5. Observe that S_j does not depend on the coloring α_j. Construct the $(k-\ell+j)$-coloring β' from β by recoloring the vertices in S_j with color $(k-\ell+j)$.
3. Consider the graph $G_{j-1} = G_j \setminus S_j$ and let α_{j-1} be the $(k-\ell+j-1)$-coloring obtained by restricting β' into G_{j-1}. Notice that, by Lemma 5, there exists G_{j-1} admits a t-partition of degree $j-1$.

By Lemma 4, at Step 1 of every iteration we perform at most $j^t \cdot |V(G_j)|^2 \leq \ell^t n^2$ many recolorings. At Step 2 of each iteration we perform at most $|S_j| \leq n$ many recolorings. Recall that the number of iterations is ℓ. Thus, the number of recolorings during the recoloring procedure is at most $\ell(\ell^t n^2 + n) \leq 2\ell^{t+1}n^2$.

Let α_0 be the k-coloring obtained at the end of the procedure. Since the set S_j obtained at Step 2 only depends on the graph G_j and the selected t-partition of degree ℓ of the graph G but not on the coloring α_j, the coloring α_0 restricted to $G \setminus G_0$, does not depend on α. Indeed, all the vertices of S_j are colored with color $(k-\ell+j)$ for every j between 1 and ℓ. Moreover $G_0 = G \setminus (S_1 \cup \cdots \cup S_\ell)$, has a t-partition $\{V_1,\ldots,V_t\}$ of degree 0, or, in other words, G_0 is the edgeless subgraph. Hence, α_0 can be transformed into γ^* by recoloring all the vertices in G_0 with color 1 (in fact, only $\ell+1$ colors are used in γ^*). This can be done in at most n recoloring steps.

Thus, we can transform any k-coloring α into a canonical k-coloring γ^* (*i.e.* a coloring that does not depend on α) using at most $2\ell^{t+1}n^2$ many recolorings. This implies that for any two k-colorings α_1 and α_2, we have $d(\alpha_1,\alpha_2) \leq 2\ell^{t+1}n^2$. Indeed, α_1 can be transformed into γ^* with at most $2\ell^{t+1}n^2$ recolorings and α_2 can be transformed into γ^* with at most $2\ell^{t+1}n^2$ recolorings. Therefore,

$$\mathrm{diam}(\mathcal{C}_k(G)) \leq 4\ell^{t+1}n^2 \ ,$$

concluding the proof of the proposition. □

The proof of the main theorem of this section, follows as a corollary of the two previous propositions.

Proof of Theorem 2. Recall that G satisfies $\text{mad}(G) \leq d - \varepsilon$. Thus, by Proposition 6, there exists a $C > 0$ such that G admits a $C \cdot \log_d n$ partition of degree $d - 1$. Finally, by Proposition 7, for every $k \geq \ell + 2 = d + 1$, G satisfies

$$\text{diam}(\mathcal{C}_k(G)) \leq 4(d-1)^{c \log_d n + 1} n^2 = O(n^c) \ ,$$

for some $c > 0$. □

We did not make any attempt to improve the constant c obtained in Theorem 2. However, this constant can be decreased if we are more careful. For instance, the n^2 factor obtained in Lemma 4 can be replaced by n, since Claim B actually bounds the number of decreasing paths between w and vertices at the same level as u (if we assume that $L(w) < L(u)$).

Note that the proof also provides an algorithm which runs in polynomial time. Indeed Procedure 1 runs in polynomial time. Moreover the partition of Proposition 6 can be found in polynomial time as well as the stable set provided by Lemma 5. So the proof provides an algorithm such that given any two k-colorings transforms one into the other in polynomial time, provided that $\text{mad}(G) \leq d - \varepsilon$ for some $\varepsilon > 0$, and that $k \geq d + 1$.

4 Recoloring planar graphs and related classes

As observed in [2], there is a planar graph G (the graph of the icosahedron, see Figure 1) such that $\mathcal{C}_6(G)$ is not even connected ($\text{diam}(\mathcal{C}_6(G)) = \infty$) . There also exists a planar graph G such that $\mathcal{C}_5(G)$ is not connected ($\text{diam}(\mathcal{C}_5(G)) = \infty$) (for instance consider the graph of Figure 1 where vertices colored with 6 were deleted). In both cases the reason is the same: the colorings are frozen and then no vertex can be recolored, or, otherwise stated, the coloring is an isolated vertex in the recoloring graph.

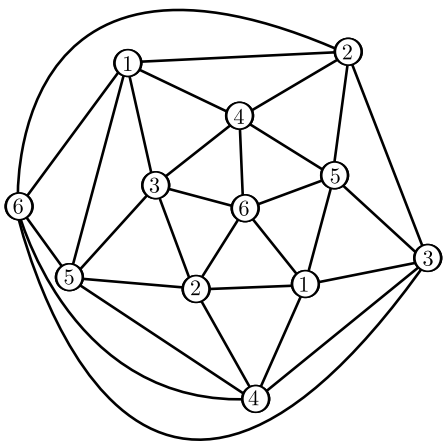

Figure 1: A 6-coloring corresponding to an isolated vertex in $\mathcal{C}_6(G)$.

Recall that any planar graph G is 5-degenerate. The result of Cereceda [7] on the degeneracy of implies that for any planar graph G, $\text{diam}(\mathcal{C}_{11}(G)) = O(n^2)$. The result of Dyer et al [11] show that $\mathcal{C}_k(G)$ is connected for every $k \geq 7$. The best known upper bound for the diameter in the cases $k = 7, 8, 9, 10$ is the trivial one due to Dyer et al. [11], *i.e.* $\text{diam}(\mathcal{C}_k(G)) \leq k^n$.

As a corollary of Theorem 2, we obtain that $\mathcal{C}_8(G)$ has polynomial diameter.

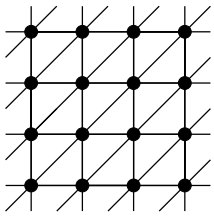

Figure 2: The structure of a planar graph with no $\frac{\sqrt{n}}{2}$-partition of degree 5.

Corollary 8. *For any planar graph G on n vertices and any $k \geq 8$,*

$$diam(\mathcal{C}_k(G)) = Poly(n) \ .$$

Proof. Euler formula ensures that for every planar graph H, $|E(H)| \leq 3|V(H)| - 6$. Since every subgraph of a planar graph is also planar, we have $\text{mad}(G) < 6$. So we just have to apply Theorem 2 with $d = 7$ and $\varepsilon = 1$ to conclude. □

It would be interesting to determine whether the diameter of $\mathcal{C}_7(G)$ is polynomial or not. Observe that while Theorem 2 is able to prove such statement for a graph G with $\text{mad}(G) = 5.99$, it is not enough to prove it for a planar graphs because their maximum average degree is not bounded away from 6. Unfortunately, the same partition argument that we used for the proof of Theorem 2 will not be able to show that the diameter of the 7-recoloring graph is small. Here we briefly sketch the argument

Proposition 9. *There exists a planar graph G on n vertices that does not admit a $\frac{\sqrt{n}}{2}$-partition of degree 5.*

Proof. Suppose that $n = 4m^2$ and let G be the graph with vertex set $V(G) = \{(i,j) : 1 \leq i,j \leq 2m\}$ and edge set $E(G) = \{(i_1,j_1)(i_2,j_2) : |i_1 - i_2| + |j_1 - j_2| = 1\} \cup \{(i_1,j_1)(i_2,j_2) : i_1 = i_2 + 1,\ j_1 = j_2 + 1\}$. This can be seen as a triangulated grid where every *inner vertex* (*i.e.* a vertex with both coordinates in $\{2, \ldots, 2m-1\}$) has degree 6 (see Figure 2).

We claim that the vertex $v = (m,m)$ does not belong to V_i, for every partition of degree 5 and for every $i < m$. We show it by induction on m. If $m = 1$ there is nothing to prove. Since any inner vertex has degree 6, for any partition of degree 5 the set V_1 does not contain inner vertices. We can assume that V_1 is composed by all the vertices of degree at most 5 in G, that is the ones lying on the boundary of the grid. Now, $G^{(2)} = G \setminus V_1$ is a $2(m-1) \times 2(m-1)$ triangulated grid. Thus, by induction hypothesis, the vertex $v = (m-1, m-1)$ does not belong to V_i for every partition of degree 5 of $G^{(2)}$ and for every $i < m-1$. This proves the claim. □

Closing the gap between 7 and 8 on planar graphs is an interesting open problem which may give new methods for tackling Cereceda et al.'s degeneracy conjecture. Moreover note that since the graph presented in Proposition 9 is 3-colorable, the method introduced for Theorem 2 is not useful to prove that the diameter of $\mathcal{C}_7(G)$ is polynomial even if G is a 3-colorable planar graph.

However, an interesting result can be obtained for triangle-free planar graphs (recall that triangle-free planar graphs are 3-colorable by Grötzsch's theorem).

Corollary 10. *For any triangle-free planar graph G on n vertices and any $k \geq 6$ we have*

$$diam(\mathcal{C}_k(G)) = Poly(n) \ .$$

Besides, there exists a triangle-free planar graph G on n vertices that does not admit a $\frac{\sqrt{n}}{2}$-partition of degree 3.

Proof. Again, a slight variant of the Euler formula ensures that for every triangle-free planar graph H, $|E(H)| \leq 2|V(H)| - 4$. Since every subgraph of a triangle-free planar graph is also triangle-free and planar, we have $\text{mad}(G) < 4$. So we just have to apply Theorem 2 with $d = 5$ and $\varepsilon = 1$ to conclude.

For the second part of the statement, suppose that $n = 4m^2$ and let G be the graph with vertex set $V(G) = \{(i, j) : 1 \leq i, j \leq 2m\}$ and edge set $E(G) = \{(i_1, j_1)(i_2, j_2) : |i_1 - i_2| + |j_1 - j_2| = 1\}$, that is G is a $2m \times 2m$ grid. We claim that $v = (m, m) \notin V_i$, for any partition of degree 3 and $i < m$, which can be proved as in Proposition 9. So the argument cannot be extended to 5-colorings of triangle-free planar graphs. □

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