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Well-posedness and qualitative behaviour of a semi-linear parabolic Cauchy problem arising from a generic model for fractional-order autocatalysis

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In this paper, we examine a semi-linear parabolic Cauchy problem with non-Lipschitz nonlinearity which arises as a generic form in a significant number of applications. Specifically, we obtain a well-posedness result and examine the qualitative structure of the solution in detail. The standard classical approach to establishing well-posedness is precluded owing to the lack of Lipschitz continuity for the nonlinearity. Here, existence and uniqueness of solutions is established via the recently developed generic approach to this class of problem (Meyer & Needham 2015 *The Cauchy problem for non-Lipschitz semi-linear parabolic partial differential equations*. London Mathematical Society Lecture Note Series, vol. 419) which examines the difference of the maximal and minimal solutions to the problem. From this uniqueness result, the approach of Meyer & Needham allows for development of a comparison result which is then used to exhibit global continuous dependence of solutions to the problem on a suitable initial dataset. The comparison and continuous dependence results obtained here are novel to this class of problem. This class of problem arises specifically in the study of a one-step autocatalytic reaction, which is schematically given by $A \to B$ at rate $a^p b^q$ (where a and b are the concentrations of A and B, respectively, with $0 < p, q < 1$) and well-posedness for this problem has been lacking up to the present.

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1. Introduction and motivation

In this paper, we consider an initial-boundary value problem arising as a generic model for a one-step autocatalytic reaction. The initial-boundary value problem is of semi-linear parabolic type, and in dimensionless form is given by

$$u_t = u_{xx} + f(u) \quad \forall (x,t) \in \mathbb{R} \times \mathbb{R}^+ \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \forall x \in \mathbb{R} \tag{1.2}$$

and

$$u(x,t) \text{ is uniformly bounded as } |x| \to \infty \quad \text{for } t \in [0,T]. \tag{1.3}$$

The nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(u) = \begin{cases} u^p(1-u)^q; & u \in [0,1] \\ 0; & u \in (-\infty,0) \cup (1,\infty), \end{cases} \tag{1.4}$$

and the initial data $u_0 : \mathbb{R} \to \mathbb{R}$ (which will be from a sufficiently smooth class of bounded functions) is such that

$$u_0(x) \geq 0 \quad \forall x \in \mathbb{R}. \tag{1.5}$$

The indices $p, q > 0$ represent the reaction order. The chemical background of the model is reviewed in detail in [1]. Particular reactions which have been established as autocatalytic include the iodate–arsenic reaction, the acidic nitrate–ferroin reaction and the hydroxylamine–nitrate reaction. Autocatalytic rate laws also arise in enzyme reactions (such as glycolysis) and in the calcium deposition in bone formation. Details on the occurrence of autocatalytic steps in biochemical reactions may be found in Murray [2, chs 5–7]. In a number of the above applications, it is possible that both $0 < p, q < 1$. When $p, q \geq 1$, the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous on every closed bounded interval. In this case, the initial-boundary value problem (1.1)–(1.4) has been studied extensively. In particular, classical Hadamard well-posedness has been established, along with considerable qualitative information regarding the structure of the solution to (1.1)–(1.3). Specific attention has been focused on the convergence to the equilibrium state $u = 1$ via the evolution of travelling wave structures in the solution to (1.1)–(1.5) when the initial data is non-trivial, as $t \to \infty$, their propagation speed, shape and form [2–16]. The cases when $0 < p < 1$ and/or $0 < q < 1$ have received much less attention, primarily because the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ lacks Lipschitz continuity in these cases owing to the behaviour at $u = 0$ and/or $u = 1$ and the classical comparison theorems and continuous dependence theorems fail to apply. However, the case when $0 < p < 1$ and $q = 1$ has been considered in some detail in [17–19] in the context of global existence and uniqueness, although full Hadamard well-posedness was not established. The qualitative features concerning the solution to (1.1)–(1.5) with non-trivial initial data, in this case, do not exhibit travelling wave structure, but uniform convergence over $x \in \mathbb{R}$, to the equilibrium state $u = 1$, as $t \to \infty$. This represents a significant bifurcation in the structure of the solution to (1.1)–(1.5) for $p \geq 1$ and $0 < p < 1$, respectively. It is the purpose of this paper to address the initial-boundary value problem (1.1)–(1.5) when $0 < p < 1$ and $0 < q < 1$. We establish, via novel comparison and continuous dependence theorems, global well-posedness and, under mild restrictions, uniform global well-posedness, together with detailed qualitative features relating to the solution to (1.1)–(1.5). The approach to achieving these results is based on the recent generic theory developed in [20,21] and relies heavily on these results. Qualitatively, we find that the global solution to (1.1)–(1.5) does not lead to the development of travelling wave structures. In the physical context in which the model (1.1)–(1.5) arises, this anomalous behaviour arises through the interaction of Fick's law with a singular behaviour in the reaction rate ($f'(u) \to \infty$ as $u \to 0^+$). This has been discussed in detail in [22–26] and references therein, where it has been proposed that a relaxation of this behaviour requires a suitable relaxation term to be included in a modified Fick's law.

The paper is structured as follows. In §2, we introduce the notation used within the framework of the theoretical study of semi-linear parabolic partial differential equations, as found in [21,27], and establish some elementary results, which will be of use in later sections. In §3, we establish that, for any initial data in the set of interest, there exists a global minimal solution and a global maximal solution to the initial-boundary value problem, via results contained in [20,21]. In §4, we obtain a uniqueness result via adapting methods and results contained in [21,28]. In §5, we obtain a continuous dependence result on the initial data for solutions to the initial-boundary value problem. In §6, we bring together these results to establish a statement about well-posedness, and address qualitative features of the solution to (1.1)–(1.5).

2. Problem statement and preliminaries

Here, we formally introduce the problem which this paper addresses together with notation and definitions which will be used throughout the paper. To begin, it is convenient to introduce the following sets:

$$D_T = (-\infty, \infty) \times (0, T], \quad \bar{D}_T = (-\infty, \infty) \times [0, T], \quad \partial D = (-\infty, \infty) \times \{0\},$$

with $T > 0$. We also introduce the set of initial data $\mathcal{U}_0$ as the set of all functions $u_0 : \mathbb{R} \to \mathbb{R}$ such that u_0 is bounded, continuous with bounded and continuous derivative and bounded and piecewise continuous second derivative. Additionally, we introduce the subset $\mathcal{U}_{0+} \subset \mathcal{U}_0$ as

$$\mathcal{U}_{0+} = \{u_0 \in \mathcal{U}_0 : u_0(x) \geq 0 \ \forall x \in \mathbb{R} \text{ and } \exists x^* \in \mathbb{R} \text{ s.t. } u_0(x^*) > 0\}.$$

Throughout the paper, we consider classical solutions $u : \bar{D}_T \to \mathbb{R}$ to the following semi-linear parabolic Cauchy problem:

$$\begin{gathered} u_t = u_{xx} + f(u) \quad \forall (x,t) \in D_T, \\ u(x,0) = u_0(x) \quad \forall x \in \mathbb{R}, \\ u(x,t) \text{ is uniformly bounded as } |x| \to \infty \quad \text{for } t \in [0,T], \end{gathered}$$

where $u_0 \in \mathcal{U}_{0+}$, $u \in C(\bar{D}_T) \cap C^{2,1}(D_T)$ and $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(u) = \begin{cases} u^p(1-u)^q; & u \in [0,1] \\ 0; & u \in (-\infty, 0) \cup (1, \infty) \end{cases} \tag{2.1}$$

with $p, q \in (0,1)$. For the initial data $u_0 \in \mathcal{U}_{0+}$, we define

$$\sup_{x \in \mathbb{R}} \{u_0(x)\} = M_0 > 0, \quad \inf_{x \in \mathbb{R}} \{u_0(x)\} = m_0 \geq 0. \tag{2.2}$$

We refer to this initial-boundary value problem as (S) throughout the rest of the paper. For convenience, we introduce $\gamma = p/(p+q)$ and observe that

$$\sup_{u \in \mathbb{R}} f(u) = f(\gamma) = (\gamma)^p (1-\gamma)^q, \tag{2.3}$$

and that $f : (-\infty, \gamma] \to \mathbb{R}$ is non-decreasing and $f : [\gamma, \infty) \to \mathbb{R}$ is non-increasing.

In what follows, we denote by H_α the set of functions $g : \mathbb{R} \to \mathbb{R}$ which are Hölder continuous of degree $\alpha \in (0,1]$ on every closed bounded interval. In addition, a function $g : \mathbb{R} \to \mathbb{R}$ is said to be upper Lipschitz continuous when g is continuous, and for any closed bounded interval $E \subset \mathbb{R}$,

there exists a constant $k_E > 0$ such that for all $x, y \in E$, with $y \geq x$,

$$g(y) - g(x) \leq k_E(y - x).$$

This set of functions is denoted by L_u. It is straightforward to establish that $f : \mathbb{R} \to \mathbb{R}$ as given by (2.1) satisfies $f \in H_{\alpha'}$ with $\alpha' = \min\{p, q\}$ and Hölder constant $k_{\mathrm{H}} = 1$ on every closed bounded interval $E \subset \mathbb{R}$. In addition, we have

$$f(y) - f(x) \leq (y - x)^p \quad \forall y \geq x, \tag{2.4}$$

while it follows from the mean value theorem that

$$f(y) - f(x) \leq p\theta^{p-1}(y - x) \quad \forall y > x \geq 0 \tag{2.5}$$

with $\theta \in (x, y)$.

We are now in a position to address well-posedness of the problem (S) on $\mathcal{U}_{0+}$. Here, we adopt the definition of Hadamard, given by

(P1) (Existence) For each $u_0 \in \mathcal{U}_{0+}$, there exists a solution $u : \bar{D}_T \to \mathbb{R}$ to (S) on $\bar{D}_T$ for each $T > 0$.

(P2) (Uniqueness) Whenever $u : \bar{D}_T \to \mathbb{R}$ and $v : \bar{D}_T \to \mathbb{R}$ are solutions to (S) on $\bar{D}_T$ for the same $u_0 \in \mathcal{U}_{0+}$, then $u = v$ on $\bar{D}_T$ for each $T > 0$.

(P3) (Continuous dependence) Given that (P1) and (P2) are satisfied for (S), then given any $u_0 \in \mathcal{U}_{0+}$ and $\epsilon > 0$, there exists a $\delta > 0$ (which may depend on u_0, T and ϵ) such that, for all $v_0 \in \mathcal{U}_{0+}$, then

$$\sup_{x \in \mathbb{R}} |v_0(x) - u_0(x)| < \delta \implies \sup_{(x,t) \in \bar{D}_T} |v(x, t) - u(x, t)| < \epsilon,$$

where $v : \bar{D}_T \to \mathbb{R}$ and $u : \bar{D}_T \to \mathbb{R}$ are the unique solutions to (S) corresponding, respectively, to $v_0, u_0 \in \mathcal{U}_{0+}$. This must hold for each $T > 0$.

When the above three properties (P1)–(P3) are satisfied, then (S) is said to be *globally well-posed* on $\mathcal{U}_{0+}$. Moreover, when (P1)–(P3) are satisfied by (S) and the constant δ in (P3) depends only on u_0 and ϵ (i.e. being independent of T), then (S) is said to be *uniformly globally well-posed* on $\mathcal{U}_{0+}$. In what follows, it is also convenient to label as ($\hat{\mathrm{S}}$) that semi-linear parabolic Cauchy problem obtained from (S) by exchanging $f : \mathbb{R} \to \mathbb{R}$ as given by (2.1), with $\hat{f} : \mathbb{R} \to \mathbb{R}$. A regular subsolution and a regular supersolution to (S) or ($\hat{\mathrm{S}}$) will be as defined in [20, definition 4.1]. We first address the question of existence for the problem (S).

3. Existence

We now establish an existence result for (S). We first introduce the function $f_\eta : \mathbb{R} \to \mathbb{R}$, for any $\eta \in (0, \gamma]$, such that

$$f_\eta(u) = \begin{cases} f(\eta); & u < \eta \\ f(u); & u \geq \eta, \end{cases} \tag{3.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is given by (2.1). It follows from (2.5) that $f_\eta \in L_u$, and

$$f_\eta(u) \geq f(u) \quad \forall u \in \mathbb{R}. \tag{3.2}$$

We next establish that (S), with any $u_0 \in \mathcal{U}_{0+}$, is *a priori* bounded on $\bar{D}_T$ for any $T > 0$. We have the following.

Proposition 3.1. *Let $u : \bar{D}_T \to \mathbb{R}$ be any solution to* (S) *with initial data $u_0 \in \mathcal{U}_{0+}$. Then,*

$$m_0 \le u(x,t) \le \max\{M_0, 1\} \quad \forall (x,t) \in \bar{D}_T.$$

Proof. Introduce $v : \bar{D}_T \to \mathbb{R}$ such that

$$v(x,t) = m_0 - u(x,t) \quad \forall (x,t) \in \bar{D}_T.$$

Then,

$$v_t - v_{xx} = -f(m_0 - v) \le 0 \quad \forall (x,t) \in D_T$$
$$v(x,0) \le 0 \quad \forall x \in \mathbb{R},$$

after which an application of the extended maximum principle in [20, theorem 3.4] establishes that $v \le 0$ on $\bar{D}_T$, and so

$$u(x,t) \ge m_0 \quad \forall (x,t) \in \bar{D}_T.$$

Next, introduce $\bar{u}, \underline{u} : \bar{D}_T \to \mathbb{R}$ as

$$\bar{u}(x,t) = \max\{M_0, 1\}, \quad \underline{u}(x,t) = u(x,t) \ \ \forall (x,t) \in \bar{D}_T. \tag{3.3}$$

It follows from (3.1) and (3.2) that

$$\left.\begin{aligned} \bar{u}_t - \bar{u}_{xx} - f_\eta(\bar{u}) &\ge 0 \\ \underline{u}_t - \underline{u}_{xx} - f_\eta(\underline{u}) = f(\underline{u}) - f_\eta(\underline{u}) &\le 0 \end{aligned}\right\} \quad \forall (x,t) \in D_T \tag{3.4}$$

and

$$\underline{u}(x,0) \le u_0(x) \le \bar{u}(x,0) \quad \forall x \in \mathbb{R}. \tag{3.5}$$

Because $f_\eta \in L_u$, a direct application of the comparison theorem in [20, theorem 4.4] establishes that $\underline{u} \le \bar{u}$ on $\bar{D}_T$, and so

$$u(x,t) \le \max\{M_0, 1\} \quad \forall (x,t) \in \bar{D}_T,$$

as required. ■

Before stating the main existence result, we refer to [21, remark 8.4] for the definitions of a constructed maximal solution and a constructed minimal solution to (S) on $\bar{D}_T$. We now have the following.

Theorem 3.2 (Existence). *The problem* (S) *with $u_0 \in \mathcal{U}_{0+}$ has a global constructed maximal solution $\bar{u}^c : \bar{D}_\infty \to \mathbb{R}$ and a global constructed minimal solution $\underline{u}^c : \bar{D}_\infty \to \mathbb{R}$. Moreover, any solution $u : \bar{D}_\infty \to \mathbb{R}$ to* (S)*, with $u_0 \in \mathcal{U}_{0+}$, satisfies*

$$m_0 \le \underline{u}^c(x,t) \le u(x,t) \le \bar{u}^c(x,t) \le \max\{M_0, 1\} \quad \forall (x,t) \in \bar{D}_\infty.$$

Proof. It follows from proposition 3.1 that (S), with any $u_0 \in \mathcal{U}_{0+}$, is *a priori* bounded on $\bar{D}_T$ uniformly for $T > 0$, and hence is *a priori* bounded on $\bar{D}_\infty$. Existence of the global constructed maximal/minimal solution then follows directly from [21, theorem 8.25], because $f \in H_{\alpha'}$. The bounds follow from proposition 3.1. ■

It follows from theorem 3.2 that (P1) is satisfied. We now turn to the question of uniqueness for the problem (S).

4. Uniqueness

It is first instructive to consider the problem (S) when $u_0 : \mathbb{R} \to \mathbb{R}$ is given by $u_0(x) = 0$ for all $x \in \mathbb{R}$ (and so $u_0 \notin \mathcal{U}_{0+}$). It is then straightforward to observe that $u_1 : \bar{D}_\infty \to \mathbb{R}$, given by

$$u_1(x,t) = 0 \quad \forall (x,t) \in \bar{D}_\infty,$$

is a global solution to (S) in this case. However, now consider $u_2 : \bar{D}_\infty \to \mathbb{R}$, given by

$$u_2(x,t) = \begin{cases} \phi(t); & (x,t) \in \bar{D}_{t^*} \\ 1; & (x,t) \in \bar{D}_\infty \backslash \bar{D}_{t^*}, \end{cases}$$

where $\phi : [0, t^*] \to \mathbb{R}$ is given implicitly by

$$\int_0^{\phi(t)} \frac{\mathrm{d}s}{s^p(1-s)^q} = t \quad \forall t \in [0, t^*]$$

and

$$t^* = \int_0^1 \frac{\mathrm{d}s}{s^p(1-s)^q}.$$

It is readily verified that $u_2 : \bar{D}_\infty \to \mathbb{R}$ is also a global solution to (S) in this case. It follows that, in this case, (S) exhibits non-uniqueness. However, in what follows, with $u_0 \in \mathcal{U}_{0+}$, we establish uniqueness for (S).

It is convenient at this stage to introduce the following sup norms for the continuous and bounded functions $v : \bar{D}_T \to \mathbb{R}$ and $w : \mathbb{R} \to \mathbb{R}$ as follows:

$$\|v\|_A = \sup_{(x,t) \in \bar{D}_T} \{|v(x,t)|\}, \quad \|w\|_B = \sup_{x \in \mathbb{R}} \{|w(x)|\}.$$

Before we can establish a uniqueness argument, we first require an improved lower bound for solutions to (S). We have the following.

Theorem 4.1. *For $k \in (0,1)$, the constructed minimal solution $\underline{u}^c : \bar{D}_\infty \to \mathbb{R}$ to (S) with $u_0 \in \mathcal{U}_{0+}$ satisfies*

$$\underline{u}^c(x,t) \geq ((1-p)kt)^{1/(1-p)} \quad \forall (x,t) \in \bar{D}_{T_k},$$

where

$$T_k = \frac{(1-(1/2)^{(1-p)})(1-k^{1/q})^{(1-p)}}{(1-p)}. \tag{4.1}$$

Proof. To begin, fix $k \in (0,1)$ and let $\bar{u}^c, \underline{u}^c : \bar{D}_\infty \to \mathbb{R}$ be the constructed maximal solution and constructed minimal solution to (S) with initial data $u_0 \in \mathcal{U}_{0+}$, respectively, as in theorem 3.2. Now, consider the problem ($\hat{\text{S}}$) with $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by

$$\hat{f}(u) = \begin{cases} ku^p; & u \in [0, u_k] \\ f(u); & u \notin [0, u_k] \end{cases} \leq f(u) \quad \forall u \in \mathbb{R}, \tag{4.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is given by (2.1), $u_k = (1-k^{1/q}) \in (0,1)$, and initial data $\hat{u}_0 \in \mathcal{U}_{0+}$ is given by

$$\hat{u}_0(x) = \frac{u_k u_0(x)}{2\max\{1, M_0\}} \leq \min\left\{\frac{1}{2}u_k, u_0(x)\right\} \quad \forall x \in \mathbb{R}. \tag{4.3}$$

It follows from (4.2) that $\hat{f} \in H_{\alpha'}$. Now, let $u : \bar{D}_T \to \mathbb{R}$ be any solution to ($\hat{\text{S}}$) above, then because $\hat{f} : \mathbb{R} \to \mathbb{R}$ is non-negative and $\hat{u}_0 \in \mathcal{U}_{0+}$, it follows from an application of the extended maximum

principle in [20, theorem 3.4] that

$$u(x,t) \geq 0 \quad \forall (x,t) \in \bar{D}_T. \tag{4.4}$$

Moreover, because $u : \bar{D}_T \to \mathbb{R}$ is a solution to $(\hat{\mathrm{S}})$ above, it follows that

$$u_t - u_{xx} - f(u) = \hat{f}(u) - f(u) \leq 0 \quad \forall (x,t) \in D_T. \tag{4.5}$$

It then follows from (4.3) and (4.5) that $u : \bar{D}_T \to \mathbb{R}$ is a regular subsolution to (S) with initial data $u_0 \in \mathcal{U}_{0+}$. Therefore, an application of the comparison result in [21, proposition 8.26] gives

$$u(x,t) \leq \bar{u}^{\mathrm{c}}(x,t) \quad \forall (x,t) \in \bar{D}_T. \tag{4.6}$$

Thus, via (4.6) and theorem 3.2, we have

$$u(x,t) \leq \max\{1, M_0\} \quad \forall (x,t) \in \bar{D}_T, \tag{4.7}$$

and so, from (4.4) and (4.7), we conclude that $(\hat{\mathrm{S}})$ above is *a priori* bounded on $\bar{D}_T$ uniformly in $T > 0$. Thus, it follows from [21, theorem 8.25] that because $\hat{f} \in H_{\alpha'}$ there exists a constructed minimal solution $\underline{\hat{u}} : \bar{D}_\infty \to \mathbb{R}$ to $(\hat{\mathrm{S}})$ above. Now, because $\hat{f} \in H_{\alpha'}$, while $\underline{\hat{u}} : \bar{D}_\infty \to \mathbb{R}$ is the constructed minimal solution to $(\hat{\mathrm{S}})$ above and $\underline{u}^{\mathrm{c}} : \bar{D}_\infty \to \mathbb{R}$ is a regular supersolution to $(\hat{\mathrm{S}})$ above, then an application of the comparison result in [21, proposition 8.26] together with (4.4) gives

$$0 \leq \underline{\hat{u}}(x,t) \leq \underline{u}^{\mathrm{c}}(x,t) \quad \forall (x,t) \in \bar{D}_\infty. \tag{4.8}$$

Next, because $\underline{\hat{u}} : \bar{D}_T \to \mathbb{R}$ is a solution to $(\hat{\mathrm{S}})$ above on $\bar{D}_T$, then, via (4.2), we have

$$\underline{\hat{u}}_t - \underline{\hat{u}}_{xx} - \underline{\hat{u}}^p = \hat{f}(\underline{\hat{u}}) - \underline{\hat{u}}^p \leq 0 \quad \forall (x,t) \in D_T. \tag{4.9}$$

It follows from (4.9) and (4.3) that $\underline{\hat{u}} : \bar{D}_T \to \mathbb{R}$ is a regular subsolution to $(\hat{\mathrm{S}})$ with now $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by

$$\hat{f}(u) = \begin{cases} u^p; & u \geq 0 \\ 0; & u < 0 \end{cases} \tag{4.10}$$

and with initial data $\hat{u}_0 \in \mathcal{U}_{0+}$. Now, we define $\tilde{u} : \bar{D}_T \to \mathbb{R}$ to be

$$\tilde{u}(x,t) = \left((1-p)t + \left(\frac{u_k}{2} \right)^{(1-p)} \right)^{1/(1-p)} \quad \forall (x,t) \in \bar{D}_T. \tag{4.11}$$

It follows from (4.11) and (4.3) that, for $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by (4.10),

$$\tilde{u}_t - \tilde{u}_{xx} - \hat{f}(\tilde{u}) = 0 \geq 0 \quad \forall (x,t) \in D_T \tag{4.12}$$

and

$$\tilde{u}(x,0) = \tfrac{1}{2} u_k \geq \hat{u}_0(x) \quad \forall x \in \mathbb{R}. \tag{4.13}$$

It follows that $\tilde{u} : \bar{D}_T \to \mathbb{R}$ is a regular supersolution to $(\hat{\mathrm{S}})$ with $\hat{f}$ given by (4.10) and initial data $\hat{u}_0 \in \mathcal{U}_{0+}$. Thus, an application of the comparison result given in [28, remark 2.17], with (4.8), gives

$$0 \leq \underline{\hat{u}}(x,t) \leq \tilde{u}(x,t) \quad \forall (x,t) \in \bar{D}_T.$$

It then follows that

$$0 \leq \underline{\hat{u}}(x,t) \leq u_k \quad \forall (x,t) \in \bar{D}_{T_k}, \tag{4.14}$$

where T_k is given by (4.1). Consequently, from (4.2), (4.3) and (4.14), we have that $\underline{\hat{u}} : \bar{D}_{T_k} \to \mathbb{R}$ is a solution to $(\hat{\mathrm{S}})$ with $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by

$$\hat{f}(u) = \begin{cases} ku^p; & u > 0 \\ 0; & u \leq 0 \end{cases} \quad \forall u \in \mathbb{R}, \tag{4.15}$$

and initial data $\hat{u}_0 \in \mathcal{U}_{0+}$. Next, define the function $z : \bar{D}_{kT_k} \to \mathbb{R}$ to be

$$z(\tilde{x}, \tilde{t}) = \underline{\hat{u}}(x,t) \quad \forall (\tilde{x}, \tilde{t}) \in \bar{D}_{kT_k}, \tag{4.16}$$

where $\tilde{x} = k^{1/2}x$ and $\tilde{t} = kt$. We observe from (4.15) and (4.16) that

$$z_{\tilde{t}} - z_{\tilde{x}\tilde{x}} - z^p = 0 \geq 0 \quad \forall(\tilde{x}, \tilde{t}) \in D_{kT_k}, \tag{4.17}$$

with initial data $z(\cdot, 0) \in \mathcal{U}_{0+}$. Now, define $\underline{z} : \bar{D}_{kT_k} \to \mathbb{R}$ to be

$$\underline{z}(\tilde{x}, \tilde{t}) = ((1-p)\tilde{t})^{1/(1-p)} \quad \forall(\tilde{x}, \tilde{t}) \in \bar{D}_{kT_k}. \tag{4.18}$$

It follows that

$$\underline{z}_t - \underline{z}_{xx} - \underline{z}^p = 0 \leq 0 \quad \forall(\tilde{x}, \tilde{t}) \in D_{kT_k},$$
$$\underline{z}(\tilde{x}, 0) = 0 \leq z(\tilde{x}, 0) \quad \forall \tilde{x} \in \mathbb{R}.$$

Therefore, $z : \bar{D}_{kT_k} \to \mathbb{R}$ and $\underline{z} : \bar{D}_{kT_k} \to \mathbb{R}$ are a non-negative regular supersolution and a regular subsolution to the problem $(\hat{\mathrm{S}})$, with $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by (4.10) and initial data $z(\cdot, 0) \in \mathcal{U}_{0+}$, respectively, and, hence, an application of the comparison result given in [28, remark 2.17] gives

$$z(\tilde{x}, \tilde{t}) \geq ((1-p)\tilde{t})^{1/(1-p)} \quad \forall(\tilde{x}, \tilde{t}) \in \bar{D}_{kT_k},$$

from which, via (4.16), it follows that

$$\underline{\hat{u}}(x, t) \geq ((1-p)kt)^{1/(1-p)} \quad \forall(x, t) \in \bar{D}_{T_k}. \tag{4.19}$$

The result follows from (4.19) and (4.8), as required. ■

We can now establish a uniqueness result for (S). The proof follows a similar approach to that of Aguirre & Escobedo [28], with theorem 4.1 and the existence of the constructed minimal solution in theorem 3.2 playing a crucial role.

Theorem 4.2 (Uniqueness). *The constructed minimal solution* $\underline{u}^{\mathrm{c}} : \bar{D}_\infty \to \mathbb{R}$ *to* (S) *with* $u_0 \in \mathcal{U}_{0+}$ *is the unique solution to* (S).

Proof. We must establish that $\underline{u}^{\mathrm{c}} = \bar{u}^{\mathrm{c}}$ on $\bar{D}_\infty$. For $u_0 \in \mathcal{U}_{0+}$ with $m_0 > 0$ in (2.2), via theorem 3.2 $\underline{u}^{\mathrm{c}}, \bar{u}^{\mathrm{c}} : \bar{D}_\infty \to \mathbb{R}$ are both solutions to $(\hat{\mathrm{S}})$ with $\hat{f} = f_\eta : \mathbb{R} \to \mathbb{R}$ given by (3.1), where $\eta = \min\{m_0, \gamma\}$ and $u_0 \in \mathcal{U}_{0+}$. Because $f_\eta \in L_u$, an application of the uniqueness result in [20, theorem 4.5] gives $\underline{u}^{\mathrm{c}} = \bar{u}^{\mathrm{c}}$ on $\bar{D}_\infty$, as required.

Now, consider $u_0 \in \mathcal{U}_{0+}$ with $m_0 = 0$ in (2.2). Then, via (2.4) and the Hölder equivalence lemma in [21, lemma 5.10], we have

$$(\bar{u}^{\mathrm{c}} - \underline{u}^{\mathrm{c}})(x, t) = \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} (f(\bar{u}^{\mathrm{c}}) - f(\underline{u}^{\mathrm{c}}))(x + 2\sqrt{t-\tau}\lambda, \tau)\, \mathrm{e}^{-\lambda^2}\, \mathrm{d}\lambda\, \mathrm{d}\tau \tag{4.20}$$
$$\leq \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} (\bar{u}^{\mathrm{c}} - \underline{u}^{\mathrm{c}})^p(x + 2\sqrt{t-\tau}\lambda, \tau)\, \mathrm{e}^{-\lambda^2}\, \mathrm{d}\lambda\, \mathrm{d}\tau$$
$$\leq \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} \|(\bar{u}^{\mathrm{c}} - \underline{u}^{\mathrm{c}})(\cdot, \tau)\|_B^p\, \mathrm{e}^{-\lambda^2}\, \mathrm{d}\lambda\, \mathrm{d}\tau$$
$$\leq \int_0^t \|(\bar{u}^{\mathrm{c}} - \underline{u}^{\mathrm{c}})(\cdot, \tau)\|_B^p\, \mathrm{d}\tau \tag{4.21}$$

for all $(x, t) \in \bar{D}_T$ and any $T > 0$, on noting, via [21, corollary 5.16], that $\underline{u}^{\mathrm{c}}, \bar{u}^{\mathrm{c}} : \bar{D}_\infty \to \mathbb{R}$ are uniformly continuous on $\bar{D}_T$, and so $\|(\bar{u}^{\mathrm{c}} - \underline{u}^{\mathrm{c}})(\cdot, t)\|_B$ is continuous for $t \in [0, T]$. Moreover, the right-hand side of (4.21) is independent of x, from which we obtain

$$\|(\bar{u}^{\mathrm{c}} - \underline{u}^{\mathrm{c}})(\cdot, t)\|_B \leq \int_0^t \|(\bar{u}^{\mathrm{c}} - \underline{u}^{\mathrm{c}})(\cdot, \tau)\|_B^p\, \mathrm{d}\tau \quad \forall t \in [0, T],$$

which gives, after an integration,

$$\|(\bar{u}^{\mathrm{c}} - \underline{u}^{\mathrm{c}})(\cdot, t)\|_B \leq ((1-p)t)^{1/(1-p)} \quad \forall t \in [0, T]. \tag{4.22}$$

Now, via (2.5) and theorem 4.1, for any $(s,\tau)\in D_{T_k}$, there exists $\theta\in[\underline{u}^{\mathrm{c}}(s,\tau),\bar{u}^{\mathrm{c}}(s,\tau)]$ such that

$$f(\bar{u}^{\mathrm{c}}(s,\tau))-f(\underline{u}^{\mathrm{c}}(s,\tau))\le p\theta^{p-1}(\bar{u}^{\mathrm{c}}(s,\tau)-\underline{u}^{\mathrm{c}}(s,\tau)) \tag{4.23}$$

$$\le p((1-p)k\tau)^{-1}(\bar{u}^{\mathrm{c}}(s,\tau)-\underline{u}^{\mathrm{c}}(s,\tau))$$

$$\le \frac{p}{(1-p)k\tau}\|(\bar{u}^{\mathrm{c}}-\underline{u}^{\mathrm{c}})(\cdot,\tau)\|_B, \tag{4.24}$$

where T_k is defined in theorem 4.1 for $k\in(0,1)$, with k chosen so that

$$0<p<k<1. \tag{4.25}$$

On substituting (4.24) into (4.20), we have

$$(\bar{u}^{\mathrm{c}}-\underline{u}^{\mathrm{c}})(x,t)\le\frac{1}{\sqrt{\pi}}\int_0^t\int_{-\infty}^{\infty}\frac{p}{(1-p)k\tau}\|(\bar{u}^{\mathrm{c}}-\underline{u}^{\mathrm{c}})(\cdot,\tau)\|_B\,\mathrm{e}^{-\lambda^2}\,\mathrm{d}\lambda\,\mathrm{d}\tau\quad\forall(x,t)\in D_{T_k},$$

and so

$$\|(\bar{u}^{\mathrm{c}}-\underline{u}^{\mathrm{c}})(\cdot,t)\|_B\le\frac{p}{(1-p)k}\int_0^t\tau^{-1}\|(\bar{u}^{\mathrm{c}}-\underline{u}^{\mathrm{c}})(\cdot,\tau)\|_B\,\mathrm{d}\tau\quad\forall t\in[0,T_k], \tag{4.26}$$

on noting that the right-hand side of (4.26) is integrable via (4.22) and [21, corollary 5.16] with the limit of the right-hand side implied at $t=0$. Next, we define the function $w:[0,T_k]\to\mathbb{R}$ to be

$$w(t)=\begin{cases}\displaystyle\int_0^t\tau^{-1}\|(\bar{u}^{\mathrm{c}}-\underline{u}^{\mathrm{c}})(\cdot,\tau)\|_B\,\mathrm{d}\tau; & t\in(0,T_k]\\ 0; & t=0.\end{cases} \tag{4.27}$$

We note that w is non-negative, continuous and continuously differentiable (via [21, corollary 5.16]). The inequality (4.26) can be rewritten as

$$w'(s)-\frac{p}{k(1-p)s}w(s)\le 0\quad\forall s\in(0,T_k]. \tag{4.28}$$

This may be rewritten as

$$(w(s)s^{-p/k(1-p)})'\le 0\quad\forall s\in(0,T_k]. \tag{4.29}$$

We now integrate (4.29) from $s=\epsilon$ to $s=t$ (with $0<\epsilon<t\le T_k$) to obtain

$$w(t)\le w(\epsilon)\left(\frac{t}{\epsilon}\right)^{p/k(1-p)}\qquad\forall 0<\epsilon<t\le T_k. \tag{4.30}$$

Next, we substitute the bound in (4.22) into (4.27), which gives

$$w(\epsilon)=\int_0^\epsilon\tau^{-1}\|(\bar{u}^{\mathrm{c}}-\underline{u}^{\mathrm{c}})(\cdot,\tau)\|_B\,\mathrm{d}\tau$$

$$\le\int_0^\epsilon(1-p)^{1/(1-p)}\tau^{1/(1-p)-1}\,\mathrm{d}\tau$$

$$=(1-p)^{(2-p)/(1-p)}\epsilon^{1/(1-p)} \tag{4.31}$$

for $0<\epsilon<t\le T_k$. Finally, upon substituting (4.31) into (4.30), we obtain

$$w(t)\le(1-p)^{(2-p)/(1-p)}{T_k}^{p/k(1-p)}\epsilon^{1/(1-p)(1-p/k)}\quad\forall\,0<\epsilon<t\le T_k. \tag{4.32}$$

Now, via (4.25), upon letting $\epsilon\to 0$ in (4.32), we obtain

$$w(t)=0\quad\forall t\in[0,T_k]. \tag{4.33}$$

Therefore, via (4.33), (4.27) and (4.26), we have

$$\|(\bar{u}^{\mathrm{c}}-\underline{u}^{\mathrm{c}})(\cdot,t)\|_B=0\quad\forall t\in[0,T_k]$$

and, hence,

$$\bar{u}^{\mathrm{c}}(x,t)=\underline{u}^{\mathrm{c}}(x,t)\quad\forall(x,t)\in\bar{D}_{T_k}. \tag{4.34}$$

Now, let $T > T_k$. Consider the functions $\underline{u}^{\mathrm{c}}_{T_k}, \bar{u}^{\mathrm{c}}_{T_k} : \bar{D}_{T-T_k} \to \mathbb{R}$ defined as

$$\left.\begin{aligned} \underline{u}^{\mathrm{c}}_{T_k}(x,t) &= \underline{u}^{\mathrm{c}}(x, t+T_k) \\ \bar{u}^{\mathrm{c}}_{T_k}(x,t) &= \bar{u}^{\mathrm{c}}(x, t+T_k) \end{aligned}\right\} \quad \forall (x,t) \in \bar{D}_{T-T_k}. \tag{4.35}$$

Following from the definition of $\bar{u}^{\mathrm{c}}_{T_k}$ and $\underline{u}^{\mathrm{c}}_{T_k}$, theorem 4.1 and (4.34), we have, for $k \in (0,1)$ as in (4.25),

$$0 < ((1-p)kT_k)^{1/(1-p)} \le \underline{u}^{\mathrm{c}}_{T_k}(x,0) = \bar{u}^{\mathrm{c}}_{T_k}(x,0) \quad \forall x \in \mathbb{R}, \tag{4.36}$$

where $\underline{u}^{\mathrm{c}}_{T_k}(\cdot,0), \bar{u}^{\mathrm{c}}_{T_k}(\cdot,0) \in \mathcal{U}_{0+}$ via theorem 3.2 and [21, lemmas 5.12 and 5.15], because $f \in H_{\alpha'}$. Moreover, from theorem 3.2 and (4.35), it follows that

$$\underline{u}^{\mathrm{c}}_{T_k}(x,t) \le \bar{u}^{\mathrm{c}}_{T_k}(x,t) \quad \forall (x,t) \in \bar{D}_{T-T_k}. \tag{4.37}$$

Additionally, both $\underline{u}^{\mathrm{c}}_{T_k}$ and $\bar{u}^{\mathrm{c}}_{T_k}$ are bounded, twice continuously differentiable with respect to x and once with respect to t on $\bar{D}_{T-T_k}$. Now, because $\underline{u}^{\mathrm{c}}_{T_k}$ satisfies

$$\underline{u}^{\mathrm{c}}_{T_k t} - \underline{u}^{\mathrm{c}}_{T_k xx} = f(\underline{u}^{\mathrm{c}}_{T_k}) \ge 0 \quad \forall (x,t) \in D_{T-T_k}, \tag{4.38}$$

it follows from (4.36) and the extended maximum principle in [20, theorem 3.4] that

$$0 < ((1-p)kT_k)^{1/(1-p)} \le \underline{u}^{\mathrm{c}}_{T_k}(x,t) \le \bar{u}^{\mathrm{c}}_{T_k}(x,t) \quad \forall (x,t) \in \bar{D}_{T-T_k}. \tag{4.39}$$

Now, observe that because $\bar{u}^{\mathrm{c}}_{T_k}$ and $\underline{u}^{\mathrm{c}}_{T_k}$ solve (S) with initial data $u_0 = \underline{u}^{\mathrm{c}}(\cdot, T_k) \in \mathcal{U}_{0+}$, then, via (4.39),

$$\left.\begin{aligned} \underline{u}^{\mathrm{c}}_{T_k t} - \underline{u}^{\mathrm{c}}_{T_k xx} - f_\eta(\underline{u}^{\mathrm{c}}_{T_k}) &\ge 0 \\ \bar{u}^{\mathrm{c}}_{T_k t} - \bar{u}^{\mathrm{c}}_{T_k xx} - f_\eta(\bar{u}^{\mathrm{c}}_{T_k}) &\le 0 \end{aligned}\right\} \quad \forall (x,t) \in D_{T-T_k}, \tag{4.40}$$

where $f_\eta : \mathbb{R} \to \mathbb{R}$ is defined as in (3.1), with η chosen as

$$\eta = \min\{((1-p)kT_k)^{1/(1-p)}, \gamma\}.$$

Recall that $f_\eta \in L_u$, and also, via (4.40) and (4.36), $\underline{u}^{\mathrm{c}}_{T_k} : \bar{D}_{T-T_k} \to \mathbb{R}$ and $\bar{u}^{\mathrm{c}}_{T_k} : \bar{D}_{T-T_k} \to \mathbb{R}$ are a regular supersolution and a regular subsolution to ($\hat{\mathrm{S}}$) with $\hat{f} = f_\eta : \mathbb{R} \to \mathbb{R}$ and initial data $u_0 = \bar{u}^{\mathrm{c}}(\cdot, T_k) = \underline{u}^{\mathrm{c}}(\cdot, T_k) \in \mathcal{U}_{0+}$. It follows from a direct application of the comparison theorem in [20, theorem 4.4] that

$$\underline{u}^{\mathrm{c}}_{T_k}(x,t) \ge \bar{u}^{\mathrm{c}}_{T_k}(x,t) \quad \forall (x,t) \in \bar{D}_{T-T_k}. \tag{4.41}$$

It then follows from (4.37) and (4.41) that

$$\underline{u}^{\mathrm{c}}_{T_k}(x,t) = \bar{u}^{\mathrm{c}}_{T_k}(x,t) \quad \forall (x,t) \in \bar{D}_{T-T_k}. \tag{4.42}$$

Finally, equations (4.42), (4.35) and (4.34) give

$$\bar{u}^{\mathrm{c}}(x,t) = \underline{u}^{\mathrm{c}}(x,t) \quad \forall (x,t) \in \bar{D}_T.$$

This holds for any $T > 0$, and so $\underline{u}^{\mathrm{c}} = \bar{u}^{\mathrm{c}}$ on $\bar{D}_\infty$, as required. ■

It has now been established that problem (S), with $u_0 \in \mathcal{U}_{0+}$, has a unique global solution, and, therefore, that (P2) is satisfied. We next consider continuous dependence on initial data $u_0 \in \mathcal{U}_{0+}$.

5. Continuous dependence

Here, we obtain a continuous dependence result for (S) on the set of initial data $\mathcal{U}_{0+}$. Before we can proceed with an argument, we require a comparison theorem, which arises as a consequence of the uniqueness theorem established in §4.

Theorem 5.1. *Let $\bar{u}, \underline{u} : \bar{D}_T \to \mathbb{R}$ be a regular supersolution and a regular subsolution to* (S) *with $u_0 \in \mathcal{U}_{0+}$. Then, $\underline{u}(x,t) \le \bar{u}(x,t)$ for all $(x,t) \in \bar{D}_T$.*

Proof. Because $f \in H_{\alpha'}$, this follows directly from [21, proposition 8.26] together with theorem 4.2. ■

We can now consider continuous dependence of solutions to (S) with respect to the initial data $u_0 \in \mathcal{U}_{0+}$. We have the following.

Theorem 5.2 (Continuous dependence). *Given $\epsilon > 0$, $T \in (0, \infty)$ and $u_{10} \in \mathcal{U}_{0+}$, there exists $\delta > 0$, such that, for any $u_{20} \in \mathcal{U}_{0+}$ which satisfies $\|u_{20} - u_{10}\|_B < \delta$, the corresponding unique solutions $u_1, u_2 : \bar{D}_T \to \mathbb{R}$ to* (S) *are such that*

$$\|u_2 - u_1\|_A < \epsilon.$$

Proof. Consider $u_{30} \in \mathcal{U}_{0+}$, given by

$$u_{30}(x) = u_{10}(x) + \tfrac{1}{2}\delta \quad \forall x \in \mathbb{R}, \tag{5.1}$$

with $\delta > 0$. It follows from theorems 3.2 and 4.2 that there exists $u_3 : \bar{D}_T \to \mathbb{R}$ that uniquely solves (S) with initial data $u_{30} \in \mathcal{U}_{0+}$. Now, for any $u_{20} \in \mathcal{U}_{0+}$ such that $\|u_{20} - u_{10}\|_B < \frac{1}{2}\delta$, then

$$0 < u_{30}(x) - u_{i0}(x) < \delta \quad \forall x \in \mathbb{R}, \tag{5.2}$$

with $i = 1, 2$. It then follows from taking $u_3 : \bar{D}_T \to \mathbb{R}$ as a regular supersolution and $u_i : \bar{D}_T \to \mathbb{R}$ $(i = 1, 2)$ as a regular subsolution to (S) with initial data $u_{30} \in \mathcal{U}_{0+}$ in theorem 5.1 that

$$\max\{u_1(x,t), u_2(x,t)\} \le u_3(x,t) \quad \forall (x,t) \in \bar{D}_T. \tag{5.3}$$

Now, via the Hölder equivalence lemma in [21, lemma 5.10], (5.2), (5.3) and (2.4), for $i = 1, 2$, we have

$$\begin{aligned} 0 \le (u_3 - u_i)(x,t) &\le \delta + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} (f(u_3) - f(u_i))(x + 2\sqrt{t-\tau}\lambda, \tau)\, \mathrm{e}^{-\lambda^2}\, \mathrm{d}\lambda\, \mathrm{d}\tau \\ &\le \delta + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} (u_3 - u_i)^p (x + 2\sqrt{t-\tau}\lambda, \tau)\, \mathrm{e}^{-\lambda^2}\, \mathrm{d}\lambda\, \mathrm{d}\tau \\ &\le \delta + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} \|(u_3 - u_i)(\cdot, \tau)\|_B^p\, \mathrm{e}^{-\lambda^2}\, \mathrm{d}\lambda\, \mathrm{d}\tau \\ &\le \delta + \int_0^t \|(u_3 - u_i)(\cdot, \tau)\|_B^p\, \mathrm{d}\tau \end{aligned} \tag{5.4}$$

for all $(x,t) \in \bar{D}_T$. Therefore, because the right-hand side of (5.4) is independent of x, we have

$$\|(u_3 - u_i)(\cdot, t)\|_B \le \delta + \int_0^t \|(u_3 - u_i)(\cdot, \tau)\|_B^p\, \mathrm{d}\tau \quad \forall t \in [0, T], \tag{5.5}$$

from which we obtain (noting that $\|(u_3 - u_i)(\cdot, t)\|_B$ is continuous for $t \in [0, T]$ via [21, corollary 5.16])

$$\|(u_3 - u_i)(\cdot, t)\|_B \le (\delta^{(1-p)} + (1-p)t)^{1/(1-p)}, \quad (i = 1, 2)\ \ \forall t \in [0, T]. \tag{5.6}$$

Now, take δ sufficiently small so that $T_\delta = \delta^{(1-p)}/(1-p) < T$ and it follows from (5.6) that

$$\|(u_3 - u_i)(\cdot, t)\|_B \le (\delta^{(1-p)} + \delta^{(1-p)})^{1/(1-p)} \le 2^{1/(1-p)}\delta, \quad (i = 1, 2)\ \ \forall t \in [0, T_\delta]. \tag{5.7}$$

Next, fix $k \in (0, 1)$ such that $p < k < 1$, and it follows, via theorem 4.1, that there exists $T_k > 0$ which is *independent of* δ, such that

$$u_i(x,t) \ge ((1-p)kt)^{1/(1-p)}, \quad (i = 1, 2, 3)\ \ \forall (x,t) \in \bar{D}_{T_k}. \tag{5.8}$$

Now, take δ sufficiently small so that $T_\delta < T_k$, and set $T > T_k$. From (2.5), then (5.8) and (5.3) establish that, for $i = 1, 2$,

$$(f(u_3) - f(u_i))(s, \tau) \le p\theta_i^{p-1}(u_3 - u_i)(s, \tau) \tag{5.9}$$

for all $(s,\tau)\in D_{T_k}$, where $\theta_i(s,\tau)\in(u_i(s,\tau),u_3(s,\tau))$. Combining (5.9) with (5.8) we have, for $i=1,2$,

$$\begin{aligned}(f(u_3)-f(u_i))(s,\tau) &\le p((1-p)k\tau)^{(p-1)/(1-p)}(u_3-u_i)(s,\tau)\\ &= \frac{p}{k(1-p)\tau}(u_3-u_i)(s,\tau)\end{aligned}\tag{5.10}$$

for all $(s,\tau)\in D_{T_k}$. Now, the Hölder equivalence lemma in [21, lemma 5.10] gives (for $i=1,2$)

$$\begin{aligned}0\le(u_3-u_i)(x,t) &\le \delta+\frac{1}{\sqrt{\pi}}\int_0^{T_\delta}\int_{-\infty}^{\infty}(f(u_3)-f(u_i))(x+2\sqrt{t-\tau}\lambda,\tau)\,\mathrm{e}^{-\lambda^2}\,\mathrm{d}\lambda\,\mathrm{d}\tau\\ &\quad+\frac{1}{\sqrt{\pi}}\int_{T_\delta}^{t}\int_{-\infty}^{\infty}(f(u_3)-f(u_i))(x+2\sqrt{t-\tau}\lambda,\tau)\,\mathrm{e}^{-\lambda^2}\,\mathrm{d}\lambda\,\mathrm{d}\tau\\ &\le \delta+\frac{1}{\sqrt{\pi}}\int_0^{T_\delta}\int_{-\infty}^{\infty}(u_3-u_i)^p(x+2\sqrt{t-\tau}\lambda,\tau)\,\mathrm{e}^{-\lambda^2}\,\mathrm{d}\lambda\,\mathrm{d}\tau\\ &\quad+\frac{1}{\sqrt{\pi}}\int_{T_\delta}^{t}\int_{-\infty}^{\infty}\frac{p}{k(1-p)\tau}(u_3-u_i)(x+2\sqrt{t-\tau}\lambda,\tau)\,\mathrm{e}^{-\lambda^2}\,\mathrm{d}\lambda\,\mathrm{d}\tau\\ &\le \delta+\frac{1}{\sqrt{\pi}}\int_0^{T_\delta}\int_{-\infty}^{\infty}2^{p/(1-p)}\delta^p\,\mathrm{e}^{-\lambda^2}\,\mathrm{d}\lambda\,\mathrm{d}\tau\\ &\quad+\frac{1}{\sqrt{\pi}}\int_{T_\delta}^{t}\int_{-\infty}^{\infty}\frac{p}{k(1-p)\tau}\|(u_3-u_i)(\cdot,\tau)\|_B\,\mathrm{e}^{-\lambda^2}\,\mathrm{d}\lambda\,\mathrm{d}\tau\\ &\le \delta\left(1+\frac{2^{p/(1-p)}}{(1-p)}\right)+\int_{T_\delta}^{t}\frac{p}{k(1-p)\tau}\|(u_3-u_i)(\cdot,\tau)\|_B\,\mathrm{d}\tau\end{aligned}\tag{5.11}$$

for all $(x,t)\in\mathbb{R}\times[T_\delta,T_k]$, via (2.4), (5.10) and (5.7), respectively. It follows from (5.11) that (for $i=1,2$)

$$\|(u_3-u_i)(\cdot,t)\|_B\le\delta\left(1+\frac{2^{p/(1-p)}}{(1-p)}\right)+\int_{T_\delta}^{t}\frac{p}{k(1-p)\tau}\|(u_3-u_i)(\cdot,\tau)\|_B\,\mathrm{d}\tau\tag{5.12}$$

for all $t\in[T_\delta,T_k]$. Now, define $G:[T_\delta,T_k]\to\mathbb{R}^+$ to be

$$G(t)=\delta\left(1+\frac{2^{p/(1-p)}}{(1-p)}\right)+\int_{T_\delta}^{t}\frac{p}{k(1-p)\tau}\|(u_3-u_i)(\cdot,\tau)\|_B\,\mathrm{d}\tau\tag{5.13}$$

for all $t\in[T_\delta,T_k]$. It follows from (5.12), (5.13), [21, corollary 5.16] and the fundamental theorem of calculus that G is differentiable on $[T_\delta,T_k]$ and satisfies

$$\frac{1}{G(\tau)}\frac{\mathrm{d}G(\tau)}{\mathrm{d}\tau}\le\frac{p}{k(1-p)\tau}\quad\forall\tau\in[T_\delta,T_k].\tag{5.14}$$

Upon integrating both sides of (5.14) with respect to τ from T_δ to $t\in[T_\delta,T_k]$, we obtain

$$\ln\left(\frac{G(t)}{\delta(1+2^{p/(1-p)}/(1-p))}\right)\le\frac{p}{k(1-p)}\ln\left(\frac{t(1-p)}{\delta^{(1-p)}}\right)\le\ln\left(\frac{(T_k(1-p))^{p/k(1-p)}}{\delta^{p/k}}\right)\tag{5.15}$$

for all $t\in[T_\delta,T_k]$. Taking exponentials of both sides of (5.15) and rearranging gives

$$G(t)\le\delta^{(1-p/k)}\left(1+\frac{2^{p/(1-p)}}{(1-p)}\right)((1-p)T_k)^{p/k(1-p)}=l(p,k)\delta^{(1-p/k)}\tag{5.16}$$

for all $t\in[T_\delta,T_k]$, with

$$l(p,k)=\left(1+\frac{2^{p/(1-p)}}{(1-p)}\right)((1-p)T_k)^{p/k(1-p)},$$

which is *independent of* δ. It follows from (5.16), (5.13) and (5.12) that

$$\|(u_3-u_i)(\cdot,t)\|_B\le l(p,k)\delta^{(1-p/k)}\quad\forall t\in[T_\delta,T_k].\tag{5.17}$$

It remains to consider $t \in [T_k, T]$. Now, inequality (5.17) gives

$$\|(u_3 - u_i)(\cdot, T_k)\|_B \le l(p,k)\delta^{(1-p/k)} \quad (i = 1, 2). \tag{5.18}$$

In addition, via (5.8), we have

$$u_i(x, T_k) \ge ((1-p)kT_k)^{1/(1-p)} \quad \forall x \in \mathbb{R}, \quad (i = 1, 2, 3). \tag{5.19}$$

Now, consider $\tilde{u}_i : \bar{D}_{T-T_k} \to \mathbb{R}$ $(i = 1, 2, 3)$ given by

$$\tilde{u}_i(x, t) = u_i(x, t + T_k) \quad \forall (x, t) \in \bar{D}_{T-T_k}. \tag{5.20}$$

It follows that $\tilde{u}_i : \bar{D}_{T-T_k} \to \mathbb{R}$ are solutions to (S) with initial data $u_i(\cdot, T_k) \in \mathcal{U}_{0+}$, respectively (via [21, lemmas 5.12 and 5.15]), and hence, via (5.19) and the extended maximum principle in [20, theorem 3.4], we have

$$\tilde{u}_i(x, t) \ge ((1-p)kT_k)^{1/(1-p)} = l'(p,k) \quad \forall (x, t) \in \bar{D}_{T-T_k} \ (i = 1, 2, 3), \tag{5.21}$$

with $l'(p,k)$ being *independent of* δ. Additionally, via (5.18),

$$\|(\tilde{u}_3 - \tilde{u}_i)(\cdot, 0)\|_B \le l(p,k)\delta^{(1-p/k)} \quad (i = 1, 2). \tag{5.22}$$

It now follows from the Hölder equivalence lemma in [21, lemma 5.10], (5.22), (5.21), (5.3) and use of the mean value theorem (for f on $[0, 1]$ and $(1, \infty)$), with $\eta = \min\{l'(p,k), \frac{1}{2}\gamma\}$, which is *independent of* δ, that

$$\begin{aligned} 0 &\le (\tilde{u}_3 - \tilde{u}_i)(x, t) \\ &\le l(p,k)\delta^{(1-p/k)} + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} (f(\tilde{u}_3) - f(\tilde{u}_i))(x + 2\sqrt{t-\tau}\lambda, \tau)\, \mathrm{e}^{-\lambda^2}\, \mathrm{d}\lambda\, \mathrm{d}\tau \\ &\le l(p,k)\delta^{(1-p/k)} + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} f'(\eta)(\tilde{u}_3 - \tilde{u}_i)(x + 2\sqrt{t-\tau}\lambda, \tau)\, \mathrm{e}^{-\lambda^2}\, \mathrm{d}\lambda\, \mathrm{d}\tau \\ &\le l(p,k)\delta^{(1-p/k)} + \int_0^t f'(\eta)\|(\tilde{u}_3 - \tilde{u}_i)(\cdot, \tau)\|_B\, \mathrm{d}\tau \end{aligned} \tag{5.23}$$

for all $(x, t) \in \bar{D}_{T-T_k}$. Hence, via (5.23), [21, corollary 5.16] and the Gronwall inequality [21, proposition 5.6], we have $(i = 1, 2)$

$$\begin{aligned} \|(\tilde{u}_3 - \tilde{u}_i)(\cdot, t)\|_B &\le l(p,k)\delta^{(1-p/k)} + \int_0^t f'(\eta)\|(\tilde{u}_3 - \tilde{u}_i)(\cdot, \tau)\|_B\, \mathrm{d}\tau \\ &\le l(p,k)\delta^{(1-p/k)}\, \mathrm{e}^{f'(\eta)(T-T_k)} \end{aligned} \tag{5.24}$$

for all $t \in [0, T - T_k]$. Therefore, via (5.7), (5.17), (5.20) and (5.24), we have $(i = 1, 2)$

$$\|(u_3 - u_i)(\cdot, t)\|_B \le \begin{cases} 2^{1/(1-p)}\delta; & t \in [0, T_\delta] \\ l(p,k)\delta^{(1-p/k)}; & t \in [T_\delta, T_k] \\ l(p,k)\delta^{(1-p/k)}\, \mathrm{e}^{f'(\eta)(T-T_k)}; & t \in [T_k, T], \end{cases} \tag{5.25}$$

where $l(p,k) > 0$, $T_k > 0$ and $\eta > 0$ are all *independent of* δ. Now, given $\epsilon > 0$, we may choose δ sufficiently small in (5.25) to guarantee that $\|(u_3 - u_i)(\cdot, t)\|_B < \frac{1}{2}\epsilon$ for all $t \in [0, T]$, and hence that $\|u_3 - u_i\|_A < \frac{1}{2}\epsilon$ for $i = 1, 2$. Thus, $\|u_2 - u_1\|_A < \epsilon$, as required. ■

Here, we have established that the (unique) global solution to (S) when $u_0 \in \mathcal{U}_{0+}$ depends continuously on $u_0 \in \mathcal{U}_{0+}$. We are now in a position to establish that the problem (S) is globally well-posed on $\mathcal{U}_{0+}$.

6. Well-posedness and qualitative structure

We are now in a position to consider well-posedness of the problem (S) on $\mathcal{U}_{0+}$. First, we have the following.

Theorem 6.1. *The problem* (S) *is globally well-posed on* $\mathcal{U}_{0+}$.

Proof. It follows from theorem 3.2 that there exists a global solution to (S) for any initial data $u_0 \in \mathcal{U}_{0+}$, and, thus, (P1) is satisfied. Moreover, via theorem 4.2, this solution is unique, and, hence, (P2) is satisfied. Finally, theorem 4.2 exhibits that for any $\epsilon > 0$, $T > 0$ and $u_0 \in \mathcal{U}_{0+}$, there exists $\delta > 0$ (depending upon ϵ, u_0 and T) such that, for all $u_0' \in \mathcal{U}_{0+}$ that satisfy $\|u_0 - u_0'\|_B < \delta$, the corresponding solutions $u : \bar{D}_\infty \to \mathbb{R}$ and $u' : \bar{D}_\infty \to \mathbb{R}$ to (S) satisfy $\|u - u'\|_A < \epsilon$ on $\bar{D}_T$, and, therefore, (P3) is satisfied. We conclude that the problem (S) is globally well-posed on $\mathcal{U}_{0+}$. ■

To establish a uniform global well-posedness result for (S) on $\mathcal{U}_{0+}$, additional qualitative information is required. We have the following.

Proposition 6.2. *For any* $u_0 \in \mathcal{U}_{0+}$, *the corresponding unique solution* $u : \bar{D}_\infty \to \mathbb{R}$ *to* (S) *satisfies*

$$u(x,t) \geq 1 \quad \forall (x,t) \in \mathbb{R} \times [I_1, \infty),$$

where

$$I_1 = \int_0^1 \frac{1}{r^p(1-r)^q}\, \mathrm{d}r.$$

Proof. Consider the function $I : [0,1] \to \mathbb{R}$ given by

$$I(s) = \int_0^s \frac{1}{r^p(1-r)^q}\, \mathrm{d}r \quad \forall s \in [0,1], \tag{6.1}$$

where the improper integral is implied. It is readily established that I is continuous and bounded on $[0,1]$ and differentiable on $(0,1)$, with derivative given by

$$I'(s) = \frac{1}{s^p(1-s)^q} \quad \forall s \in (0,1). \tag{6.2}$$

It follows from (6.2) that I is strictly increasing for all $s \in [0,1]$, and, hence,

$$I : [0,1] \to [0, I_1] \text{ is a bijection.} \tag{6.3}$$

We conclude from (6.2), (6.3) and the inverse function theorem [29, pp. 221–222] that there exists a function $J : [0, I_1] \to [0,1]$ such that

$$J(I(s)) = s \;\; \forall s \in [0,1], \quad I(J(t)) = t \;\; \forall t \in [0, I_1], \quad J(0) = 0, \quad J(I_1) = 1. \tag{6.4}$$

Moreover, J is continuous and increasing on $[0, I_1]$ and differentiable on $[0, I_1]$ with derivative given by

$$J'(t) = (J(t))^p(1 - J(t))^q \quad \forall t \in [0, I_1]. \tag{6.5}$$

It follows from (6.5) and (6.4) that J' is continuous and therefore bounded on $[0, I_1]$ with

$$J'(0) = J'(I_1) = 0. \tag{6.6}$$

Now, consider $\underline{u} : \bar{D}_\infty \to \mathbb{R}$ given by

$$\underline{u}(x,t) = \begin{cases} J(t); & (x,t) \in \bar{D}_{I_1} \\ 1; & (x,t) \in \bar{D}_\infty \backslash \bar{D}_{I_1}. \end{cases} \tag{6.7}$$

It follows from (6.4)–(6.7) that $\underline{u}$ is continuous and bounded on $\bar{D}_\infty$, whereas $\underline{u}_t$, $\underline{u}_x$ and $\underline{u}_{xx}$ exist and are continuous on D_∞. Additionally, $\underline{u}$ satisfies

$$\underline{u}_t - \underline{u}_{xx} - f(\underline{u}) = 0 \leq 0 \quad \forall (x,t) \in D_\infty \tag{6.8}$$

and

$$\underline{u}(x,0) = 0 \quad \forall x \in \mathbb{R}, \tag{6.9}$$

via (6.5) and (6.4). It follows from (6.7)–(6.9) that $\underline{u}$ is a regular subsolution to (S), with any initial data $u_0 \in \mathcal{U}_{0+}$, on $\bar{D}_T$ for any $T > 0$. Also with $u : \bar{D}_\infty \to \mathbb{R}$ being the unique solution to (S) with

initial data $u_0 \in \mathcal{U}_{0+}$, we may take u as a regular supersolution to (S) with initial data $u_0 \in \mathcal{U}_{0+}$. An application of theorem 5.1 gives

$$\underline{u}(x,t) \le u(x,t) \quad \forall (x,t) \in \bar{D}_\infty. \tag{6.10}$$

The result follows from (6.7) and (6.10). ■

We can now establish a uniform global well-posedness result for (S) on $\mathcal{U}_{0+}$. Namely the following.

Theorem 6.3. *The problem* (S) *is uniformly globally well-posed on* $\mathcal{U}_{0+}$.

Proof. It follows from theorem 3.2 that there exists a global solution to (S) for any initial data $u_0 \in \mathcal{U}_{0+}$, and, thus, (P1) is satisfied. Moreover, via theorem 4.2, this solution is unique, and, hence, (P2) is satisfied. In addition, via theorem 4.2, for any $u_{10} \in \mathcal{U}_{0+}$ and any $\epsilon > 0$, there exists $\delta > 0$ such that, for all $u_{20} \in \mathcal{U}_{0+}$ that satisfy $\|(u_{10} - u_{20})\|_B < \delta$, the corresponding solutions $u_1, u_2 : \bar{D}_\infty \to \mathbb{R}$ to (S) satisfy

$$\|(u_1 - u_2)(\cdot,t)\|_B < \epsilon \quad \forall t \in [0, I_1], \tag{6.11}$$

with I_1 as in proposition 6.2. Now, consider the functions $\tilde{u}_1, \tilde{u}_2 : \bar{D}_\infty \to \mathbb{R}$ given by

$$\tilde{u}_i(x,t) = u_i(x, t + I_1) \quad (i = 1, 2) \ \forall (x,t) \in \bar{D}_\infty. \tag{6.12}$$

It follows from (6.11), (6.12), proposition 6.2 and (2.1) that

$$(\tilde{u}_1 - \tilde{u}_2)_t - (\tilde{u}_1 - \tilde{u}_2)_{xx} = 0 \quad \forall (x,t) \in D_\infty \tag{6.13}$$

and

$$\|(\tilde{u}_1 - \tilde{u}_2)(\cdot,0)\|_B < \epsilon. \tag{6.14}$$

A straightforward application of the extended maximum principle in [20, theorem 3.4] then gives

$$\|(\tilde{u}_1 - \tilde{u}_2)(\cdot,t)\|_B < \epsilon \quad \forall t \in [0, \infty). \tag{6.15}$$

It follows from (6.11), (6.12) and (6.15) that, for any $u_0 \in \mathcal{U}_{0+}$, there exists $\delta > 0$ (dependent on ϵ and u_0 only), such that, for all $u_0' \in \mathcal{U}_{0+}$ that satisfy $\|(u_0 - u_0')\|_B < \delta$, the corresponding solutions $u, u' : \bar{D}_\infty \to \mathbb{R}$ to (S) satisfy $\|(u - u')(\cdot,t)\|_B < \epsilon$ for all $t \in [0, \infty)$. Therefore, the problem (S) satisfies (P3) with a constant δ dependent on ϵ and u_0 only, and so problem (S) is uniformly globally well-posed on $\mathcal{U}_{0+}$. ■

We conclude this section by developing some qualitative properties of solutions to (S). First, we introduce the functions $w_+, w_- : [0, \infty) \to \mathbb{R}$ such that, with $M_0 \le 1$,

$$w_-(t) = \begin{cases} \phi_-(t); & 0 \le t \le t_- \\ 1; & t > t_- \end{cases}, \quad w_+(t) = \begin{cases} \phi_+(t); & 0 \le t \le t_+ \\ 1; & t > t_+, \end{cases}$$

where t_+ and t_- are given by

$$t_- = \int_{m_0}^{1} \frac{1}{s^p(1-s)^q}\,\mathrm{d}s, \quad t_+ = \int_{M_0}^{1} \frac{1}{s^p(1-s)^q}\,\mathrm{d}s, \tag{6.16}$$

and $\phi_+(t)$, $\phi_-(t)$ are defined implicitly by

$$\left.\begin{aligned} &\int_{m_0}^{\phi_-(t)} \frac{1}{s^p(1-s)^q}\,\mathrm{d}s = t \quad \forall t \in [0, t_-] \\ \text{and} \quad &\int_{M_0}^{\phi_+(t)} \frac{1}{s^p(1-s)^q}\,\mathrm{d}s = t \quad \forall t \in [0, t_+]. \end{aligned}\right\} \tag{6.17}$$

It follows from (6.16) and (6.17) that $w_+, w_- \in C^1([0, \infty))$, $w_+(t)$ and $w_-(t)$ are non-decreasing with respect to $t \in [0, \infty)$, $w_+(0) = M_0$ and $w_-(0) = m_0$ with $w_+(t) \ge w_-(t)$ for all $t \in [0, \infty)$. We now have the following (in what follows $\bar{D}_0 = \mathbb{R} \times 0$).

Theorem 6.4. *Let $u:\bar{D}_\infty \to \mathbb{R}$ be the unique solution to* (S) *with $u_0 \in \mathcal{U}_{0+}$ such that $M_0 \le 1$, then*

$$w_-(t) \le u(x,t) \le w_+(t) \quad \forall (x,t) \in \bar{D}_\infty.$$

Proof. This follows immediately from theorem 5.1, upon taking $\underline{u}^-, \bar{u}^- : \bar{D}_\infty \to \mathbb{R}$ and $\underline{u}^+, \bar{u}^+ : \bar{D}_\infty \to \mathbb{R}$ such that

$$\underline{u}^-(x,t) = w_-(t), \quad \bar{u}^-(x,t) = u(x,t),$$
$$\underline{u}^+(x,t) = u(x,t), \quad \bar{u}^+(x,t) = w_+(t)$$

for all $(x,t) \in \bar{D}_\infty$. ■

Corollary 6.5. *Let $u:\bar{D}_\infty \to \mathbb{R}$ be the unique solution to* (S) *with $u_0 \in \mathcal{U}_{0+}$ when $M_0 \le 1$, then $u(x,t) = 1$ for all $(x,t) \in \bar{D}_\infty \backslash \bar{D}_{t_-}$.*

Proof. Follows directly from theorem 6.4. ■

We next consider (S) when $u_0 \in \mathcal{U}_{0+}$ is such that $M_0 > 1$ and $m_0 \le 1$, with $\mathcal{S}_+ = \{x \in \mathbb{R} : u_0(x) > 1\}$ being bounded. We introduce $U^+ : \bar{D}_\infty \to \mathbb{R}$, such that

$$U^+(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u_0^+(x + 2\sqrt{t}\lambda)\, \mathrm{e}^{-\lambda^2}\, \mathrm{d}\lambda \quad \forall (x,t) \in \bar{D}_\infty, \tag{6.18}$$

with $u_0^+ : \mathbb{R} \to \mathbb{R}$ given by

$$u_0^+(x) = \begin{cases} u_0(x); & x \in \mathcal{S}_+ \\ 1; & x \in \mathbb{R} \backslash \mathcal{S}_+. \end{cases} \tag{6.19}$$

It follows from (6.18) and (6.19) that U^+ is continuous on $\bar{D}_\infty$, and U_t^+, U_x^+ and U_{xx}^+ exist and are continuous on D_∞, with

$$U_t^+ = U_{xx}^+ \quad \text{on } D_\infty,$$
$$U^+(x,t) \to 1 \quad \text{as } |x| \to \infty \text{ uniformly for } t \in [0,\infty),$$
$$1 < U^+(x,t) < 1 + \frac{L(M_0 - 1)}{\sqrt{\pi t}} \text{ for all } (x,t) \in D_\infty, \quad \text{where } L = \sup_{\lambda \in S_+} |\lambda|.$$

We now have the following.

Theorem 6.6. *Let $u:\bar{D}_\infty \to \mathbb{R}$ be the unique solution to* (S) *with $u_0 \in \mathcal{U}_{0+}$ when $M_0 > 1$, $m_0 \le 1$ and $\mathcal{S}_+$ is bounded. Then,*

$$w_-(t) \le u(x,t) \le U^+(x,t) \quad \forall (x,t) \in \bar{D}_\infty$$

and

$$1 \le u(x,t) < 1 + \frac{L(M_0 - 1)}{\sqrt{\pi t}} \quad \forall (x,t) \in \bar{D}_\infty \backslash \bar{D}_{t_-}.$$

Proof. Follows from theorem 5.1 with the properties of U^+ established above. ■

As a consequence of this we have the following.

Corollary 6.7. *Let $u:\bar{D}_\infty \to \mathbb{R}$ be the unique solution to* (S) *with $u_0 \in \mathcal{U}_{0+}$ and $\mathcal{S}_+$ is empty or bounded. Then,*

$$u(x,t) = 1 + O(t^{-1/2}) \quad \text{as } t \to \infty$$

uniformly for $x \in \mathbb{R}$.

Proof. Follows directly from theorems 6.4 and 6.6. ■

The above result establishes that there is a bifurcation in (S) across the boundary between $0 < p, q < 1$ and $p, q \ge 1$. In both cases, (S) is uniformly globally well-posed on $\mathcal{U}_{0+}$. However, for $p, q \ge 1$, $u(x,t) \to 1$ as $t \to \infty$ through the propagation of finite speed travelling wave structures [2–16], whereas, for $0 < p, q < 1$, $u(x,t) \to 1$ uniformly for $x \in \mathbb{R}$ (through uniform terms of $O(t^{-1/2})$ as $t \to \infty$), as demonstrated in this paper. In fact, we can now immediately infer stability

properties for the equilibrium solutions $u=0$ and $u=1$ to (S) with $u_0 \in \mathcal{U}_{0+}$ when $0<p,q<1$. In particular, $u=0$ is an unstable equilibrium solution to (S) with $u_0 \in \mathcal{U}_{0+}$, whereas $u=1$ is a Liapunov stable equilibrium solution to (S) with $u_0 \in \mathcal{U}_{0+}$, and an asymptotically stable equilibrium solution to (S) with $u_0 \in \mathcal{U}_{0+}$ when $\mathcal{S}_+$ is bounded.

7. Conclusion

In this paper, we have initially established, via results in [21], that for the semi-linear parabolic Cauchy problem (S) (detailed in §2) there exists a bounded global classical minimal solution for each initial data $u_0 \in \mathcal{U}_{0+}$ (see §3). Via the minimal property of these solutions to (S), and the approach in [28], we subsequently determined that, in fact, these solutions are the unique global bounded classical solutions to (S) for each initial data $u_0 \in \mathcal{U}_{0+}$ (see §4). Consequently, because the solution to (S) for each initial data $u_0 \in \mathcal{U}_{0+}$ is unique, then, via [21], there exists a comparison theorem for (S), which is then used, in conjunction with the approach in [28], to establish local continuous dependence of solutions to (S) on $u_0 \in \mathcal{U}_{0+}$ (see §5). After qualitative results for solutions to (S) for large t have been exhibited, we establish global continuous dependence of solutions to (S) on $u_0 \in \mathcal{U}_{0+}$. When combined, these results state that the problem (S) is uniformly globally well-posed on $\mathcal{U}_{0+}$. To conclude the paper, we have established additional qualitative results concerning the stability of the equilibrium solutions $u=0$ and $u=1$ to the problem (S) for certain classes of initial data in $\mathcal{U}_{0+}$ (see §6).

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