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Characterizing isoclinic matrices and the Cayley factorization

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Abstract

Quaternions, particularly the double and dual forms, are important for the representation rotations and more general rigid-body motions. The Cayley factorization allows a real orthogonal 4×4 matrix to be expressed as the product of two isoclinic matrices and this is the key part of the underlying theory and a useful tool in applications. An isoclinic matrix is defined in terms of its representation of a rotation in four-dimensional space. This paper looks at characterizing such a matrix as the sum of a skew symmetric matrix and a scalar multiple of the identity whose product with its own transpose is diagonal. This removes the need to deal with its geometric properties and provides a means for showing the existence of the Cayley factorization.

Keywords

Quaternion, double quaternion, dual quaternion, Cayley factorization, rigid-body transform

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1 Introduction

Since their formulation by Hamilton in the 1800's, quaternions have proved to be an important means for representing rotations in three-dimensional space. However it is only comparatively recently that their use has been widespread, possibly prompted by Shoemake's work on rotations in animation [1].

In their basic form, quaternions represent rotations about an axis through the origin. Various attempts have been made to extend the ideas so that rotations about other axes and pure translations can be dealt with. The most successful of these have been double quaternions (where translations are approximated by rotations about distant axes) [2], and dual quaternions [3, 4]. These forms of quaternion are now used in a number of application areas including: representing the motion of mechanism systems [5, 6], and robotic systems [7, 8]; vision systems [9, 10]; creation of fair motions [11, 12]; manufacturing [13]; and "skinning" of computer animated characters [14].

Perhaps because researchers have approached quaternions from different points of view, the underlying ideas can be difficult to grasp. Certainly there is a relation between quaternions (in their various forms) and 4×4 matrices representing rigid-body motions [15]. For some applications, matrix exponentials need to be formed [16] which perhaps adds to the complication since this involves a move into Clifford (geometric) algebra [17].

The use of double quaternions for handling transforms and motions [15, 18] depends upon the idea of representing a transform as a pair of quaternions which are regarded as commuting. This corresponds to representing the transform by a 4×4 orthogonal matrix which is then factorized as a pair of commuting factors. This is the Cayley factorization. As noted in [19], the Cayley factorization is the key to linking homogeneous transformations and quaternions. It can be approached in a number of ways.

One approach uses the fact that the tensor product $\mathbb{H} \otimes \mathbb{H}$ of the ring of quaternions with itself is isomorphic to the ring of 4×4 real matrices [20]. For a pair of quaternions, (q_1, q_2) , in the tensor product, a map $F : \mathbb{H} \rightarrow \mathbb{H}$ is defined by $F(x) = q_1 x q_2^{-1}$ for $x \in \mathbb{H}$. If a quaternion is regarded as being a vector with four real components, then this map can be regarded as a linear transform of \mathbb{R}^4 to itself, and hence as a 4×4 matrix. When q_1 and q_2 are unit quaternions, they correspond to the factors in the Cayley factorization.

An alternative approach is introduced in [19], following [21]. This works directly with 4×4 matrices and so avoids the need to deal so explicitly with quaternions. The factors are "isoclinic" matrices.

An isoclinic matrix is one that represents a particular form of rotation in four-dimensional space. Its relationship with geometry is noted in [19, 21]. However it is not necessary to understand explicitly this geometric significance in order to use the Cayley factorization. It is this that is explored in this paper. In section 2, an extension of the idea of skew symmetric matrices is given. This is the pseudoskew form

(which is the replacement for the definition of isoclinic). It is shown that matrices M of this form for which $M^T M$ is diagonal fall into two sets. These correspond to the left and right isoclinic forms and hence provides a characterization of them. As noted in [19, 20], these two sets of matrices form division rings isomorphic to the ordinary quaternions.

Since the property of $M^T M$ being diagonal is preserved by orthogonal transformations, it is straightforward to derive the existence of the Cayley factorization. This is discussed in section 3, together with the uniqueness of the factorization. An example is given in section 4 and conclusions are drawn in section 5.

2 Pseudoskew matrices

Suppose that

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (1)$$

are two vectors and that c is a real number. This section considers matrices of the following form.

$$M = M(\mathbf{a}, \mathbf{b}, c) = \begin{bmatrix} c & -a_3 & a_2 & b_1 \\ a_3 & c & -a_1 & b_2 \\ -a_2 & a_1 & c & b_3 \\ -b_1 & -b_2 & -b_3 & c \end{bmatrix} \quad (2)$$

Such a matrix is called *pseudoskew*. If $c = 0$, then the matrix $S = M(\mathbf{a}, \mathbf{b}, 0)$ is *skew symmetric* meaning that $S^T = -S$. Furthermore, this is the most general 4×4 skew symmetric matrix. Hence the pseudoskew matrices comprise the sums of skew symmetric matrices and scalar multiples of the identity.

Lemma 2.1.

$$\det M(\mathbf{a}, \mathbf{b}, c) = \begin{vmatrix} c & -a_3 & a_2 & b_1 \\ a_3 & c & -a_1 & b_2 \\ -a_2 & a_1 & c & b_3 \\ -b_1 & -b_2 & -b_3 & c \end{vmatrix} = c^4 + (|\mathbf{a}|^2 + |\mathbf{b}|^2)c^2 + (\mathbf{a} \cdot \mathbf{b})^2$$

Proof. This can be checked by direct evaluation. □

Lemma 2.2. *The set of 4×4 pseudoskew matrices is a vector space over the real numbers with dimension 7.*

Proof. If α_1 and α_2 are real scalars, then

$$\alpha_1 M(\mathbf{a}_1, \mathbf{b}_1, c_1) + \alpha_2 M(\mathbf{a}_2, \mathbf{b}_2, c_2) = M(\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2, \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2, \alpha_1 c_1 + \alpha_2 c_2)$$

and hence the pseudoskew matrices form a vector space. Its dimension follows since there are three choices for each of the components of \mathbf{a} and \mathbf{b} , and one choice for c . \square

Clearly the identity is a pseudoskew matrix.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There are two triples of pseudoskew matrices (cf. [15, 19]) denoted by $\mathbf{i}_L, \mathbf{j}_L, \mathbf{k}_L$ and $\mathbf{i}_R, \mathbf{j}_R, \mathbf{k}_R$

$$\mathbf{i}_L = M(\mathbf{i}, -\mathbf{i}, 0) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{i}_R = M(\mathbf{i}, \mathbf{i}, 0) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad (3)$$

$$\mathbf{j}_L = M(\mathbf{j}, -\mathbf{j}, 0) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \mathbf{j}_R = M(\mathbf{j}, \mathbf{j}, 0) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad (4)$$

$$\mathbf{k}_L = M(\mathbf{k}, -\mathbf{k}, 0) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{k}_R = M(\mathbf{k}, \mathbf{k}, 0) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad (5)$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the standard unit vectors along the main axes.

The matrices within each triple have the following properties, where the subscripts have been omitted.

$$\begin{aligned} \mathbf{i}^2 &= -I & \mathbf{jk} &= \mathbf{i} = -\mathbf{kj} \\ \mathbf{j}^2 &= -I & \mathbf{ki} &= \mathbf{j} = -\mathbf{ik} \\ \mathbf{k}^2 &= -I & \mathbf{ij} &= \mathbf{k} = -\mathbf{ji} \\ \mathbf{ijk} &= -I \end{aligned}$$

So the matrices in each triple behave as the unit quaternions.

Further, the following result follows by inspection.

Lemma 2.3. *Each of $\mathbf{i}_L, \mathbf{j}_L, \mathbf{k}_L$ commutes with every one of $\mathbf{i}_R, \mathbf{j}_R, \mathbf{k}_R$.*

This second set of three skew symmetric matrices can be formed from the first by an orthogonal transformation. For example $Q^T \mathbf{i}_L Q = \mathbf{i}_R$ (and similarly for the others) where $Q = \text{diag}(1, 1, 1, -1)$. Equivalently, \mathbf{i}_R is obtained from \mathbf{i}_L by changing the signs in the last row and column.

Lemma 2.4. *The seven matrices $I, \mathbf{i}_R, \mathbf{j}_R, \mathbf{k}_R, \mathbf{i}_L, \mathbf{j}_L, \mathbf{k}_L$ form a basis for the vector space of 4×4 pseudoskew matrices. Hence, every 4×4 pseudoskew matrix can be uniquely expressed as a linear combination of these seven matrices, specifically*

$$M(\mathbf{a}, \mathbf{b}, c) = \frac{1}{2}(a_1 - b_1)\mathbf{i}_L + \frac{1}{2}(a_2 - b_2)\mathbf{j}_L + \frac{1}{2}(a_3 - b_3)\mathbf{k}_L \\ + \frac{1}{2}(a_1 + b_1)\mathbf{i}_R + \frac{1}{2}(a_2 + b_2)\mathbf{j}_R + \frac{1}{2}(a_3 + b_3)\mathbf{k}_R + cI$$

Proof. The seven matrices above are clearly linearly independent and hence form a basis for the vector space (cf. lemma 2.2). So any member of the space is a unique combination of them. The specific expression follows by inspection. \square

Consideration is now given to the case in which M is a pseudoskew matrix and $M^T M$ is diagonal. As the next result shows, this additionally means that $M^T M$ is a scalar multiple of the identity so that M^T is a scalar multiple of the inverse of M .

Lemma 2.5. *Suppose that $M = M(\mathbf{a}, \mathbf{b}, c)$ with $\mathbf{a} \cdot \mathbf{b} \neq 0$, then $M^T M$ is diagonal if and only if $\mathbf{b} = \alpha \mathbf{a}$ where $\alpha = \pm 1$.*

Further, if $M^T M$ is diagonal, then it is a multiple of the identity with $M^T M = (a_1^2 + a_2^2 + a_3^2 + c^2)I = (|\mathbf{a}|^2 + c^2)I$.

Proof. By direct multiplication

$$M^T M = \begin{bmatrix} c & a_3 & -a_2 & -b_1 \\ -a_3 & c & a_1 & -b_2 \\ a_2 & -a_1 & c & -b_3 \\ b_1 & b_2 & b_3 & c \end{bmatrix} \begin{bmatrix} c & -a_3 & a_2 & b_1 \\ a_3 & c & -a_1 & b_2 \\ -a_2 & a_1 & c & b_3 \\ -b_1 & -b_2 & -b_3 & c \end{bmatrix} \\ = \begin{bmatrix} a_2^2 + a_3^2 + b_1^2 + c^2 & -a_1 a_2 + b_1 b_2 & -a_1 a_3 + b_1 b_3 & -a_2 b_3 + a_3 b_2 \\ -a_1 a_2 + b_1 b_2 & a_1^2 + a_3^2 + b_2^2 + c^2 & -a_2 a_3 + b_2 b_3 & a_1 b_3 - a_3 b_1 \\ -a_1 a_3 + b_1 b_3 & -a_2 a_3 + b_2 b_3 & a_1^2 + a_2^2 + b_3^2 + c^2 & -a_1 b_2 + a_2 b_1 \\ -a_2 b_3 + a_3 b_2 & a_1 b_3 - a_3 b_1 & -a_1 b_2 + a_2 b_1 & b_1^2 + b_2^2 + b_3^2 + c^2 \end{bmatrix}$$

Suppose this is diagonal. The last row (or column) of $M^T M$ provides three relations which are equivalent to saying $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. Since their scalar product is non-zero, vectors \mathbf{a} and \mathbf{b} are both non-zero. Hence they are non-zero scalar multiples of each other, with say $\mathbf{b} = \alpha \mathbf{a}$.

The first row of $M^T M$ says that $a_1 a_2 = b_1 b_2$, so that $\alpha^2 = 1$, $\alpha = \pm 1$, and the diagonal entries are all equal to $a_1^2 + a_2^2 + a_3^2 + c^2 = (|\mathbf{a}|^2 + c^2)$.

This completes the proof starting with $M^T M$ being diagonal. The converse is straightforward. \square

Definition 2.6. A pseudoskew matrix $M = M(\mathbf{a}, \mathbf{b}, c)$ is said to be special if $\mathbf{a} \cdot \mathbf{b} \neq 0$ and $M^T M$ is diagonal.

A special pseudoskew matrix is said to be left special if it has the form $M(\mathbf{a}, -\mathbf{a}, c)$, and right special if its form is $M(\mathbf{a}, \mathbf{a}, c)$.

Lemma 2.7. Suppose that $M = M(\mathbf{a}, \mathbf{b}, c)$ is a pseudoskew matrix. Then

- (i) $\mathbf{a} \cdot \mathbf{b} \neq 0$ if and only if the corresponding skew symmetric matrix $M - cI$ is non-singular;
- (ii) if M is special then

$$\begin{array}{ll} \text{EITHER} & M = M(\mathbf{a}, -\mathbf{a}, c) \quad \text{that is} \quad M \text{ is left special} \\ \text{OR} & M = M(\mathbf{a}, \mathbf{a}, c) \quad \text{that is} \quad M \text{ is right special.} \end{array}$$

Proof. Part (i) follows from lemma 2.1. Part (ii) is a recasting of lemma 2.5. \square

Theorem 2.8. The properties of being a 4×4 pseudoskew matrix and being a special 4×4 pseudoskew matrix are preserved by orthogonal transformations in the sense that if P is a 4×4 orthogonal matrix then

- (i) M is a 4×4 pseudoskew matrix if and only if $P^T M P$ is;
- (ii) M is a special 4×4 pseudoskew matrix if and only if $P^T M P$ is.

Proof. For both parts, assume that $M = cI + S$ where S is skew symmetric. Then Hence $P^T M P = cI + (P^T S P)$.

Since $S^T = -S$, it is seen that $(P^T S P)^T = P^T S^T P = -(P^T S P)$, and so $P^T S P$ is also skew symmetric. Hence $P^T M P$ is pseudoskew. Conversely, if $P^T M P$ is pseudoskew, then $P(P^T M P)P^T = M$ is pseudoskew. This proves (i).

Now assume that M is also special. Then S is non-singular (lemma 2.7(i)), and so is $P^T S P$. Since $M^T M$ is diagonal and hence a multiple of the identity (lemma 2.5), it follows that $(P^T M P)^T (P^T M P) = P^T (M^T M) P$ is also diagonal. Conversely, if $P^T M P$ is special, then $P(P^T M P)P^T = M$ is special. This proves (ii). \square

Note that although this result says that if M is special then so is $P^T M P$, it does not say whether it is left or right special; M and $P^T M P$ may have the same or opposite ‘‘handedness’’.

Theorem 2.9. (i) Matrices $\mathbf{i}_L, \mathbf{j}_L, \mathbf{k}_L$ are left special pseudoskew matrices.

(ii) Matrices $\mathbf{i}_R, \mathbf{j}_R, \mathbf{k}_R$ are right special pseudoskew matrices.

(iii) Any left special 4×4 pseudoskew matrix can be uniquely expressed as a linear combination of $I, \mathbf{i}_L, \mathbf{j}_L, \mathbf{k}_L$; and any right special 4×4 pseudoskew matrix can be uniquely expressed as a linear combination of $I, \mathbf{i}_R, \mathbf{j}_R, \mathbf{k}_R$.

(iv) The span of $I, \mathbf{i}_L, \mathbf{j}_L, \mathbf{k}_L$ is a division ring isomorphic to the quaternions comprising all left special pseudoskew matrices and multiples of the identity. Similarly, the span of $I, \mathbf{i}_R, \mathbf{j}_R, \mathbf{k}_R$ is a (different) division ring also isomorphic to the quaternions comprising all right special pseudoskew matrices and multiples of the identity.

Proof. Parts (i) and (ii) follow from equations (3), (4), (5). Part (iii) follows from lemma 2.4.

The first set of matrices in (iv) are those of the form $M(\mathbf{a}, -\mathbf{a}, c)$ which are pseudoskew when \mathbf{a} is non-zero and multiples of the identity otherwise. That they form a division ring isomorphic to the quaternions follows from the multiplication rules for the basis matrices. The same argument applies for the second division ring. \square

Lemma 2.3 says that matrices from the different division rings commute. Consideration of commutivity more generally requires the following, the first part of which is a well known result for quaternions, and the second part confirms that the division rings identified in the last result are indeed closed under multiplication.

Lemma 2.10. (i) If $a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $b = b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ are two quaternions, then

$$ab = (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) + (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)\mathbf{i} \\ + (a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1)\mathbf{j} + (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\mathbf{k}$$

(ii) If M_1 and M_2 are two special pseudoskew matrices of the same kind, then they can be expressed as follows

$$M_1 = M(\mathbf{a}, \pm\mathbf{a}, c_1) = c_1I + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \\ M_2 = M(\mathbf{b}, \pm\mathbf{b}, c_2) = c_2I + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

where the same choice of signs is made and the subscripts L or R , which are the same throughout, have been omitted, and their product has the same form

$$M_1M_2 = M(c_1\mathbf{b} + c_2\mathbf{a} + \mathbf{a} \times \mathbf{b}, \pm(c_1\mathbf{b} + c_2\mathbf{a} + \mathbf{a} \times \mathbf{b}), (c_1c_2 - \mathbf{a} \cdot \mathbf{b}))$$

Proof. Part (i) follows from the multiplication rules for the unit quaternions. The expressions for M_1 and M_2 follow from the definitions, and then part (ii) (or direct multiplication) gives the expression for the product. \square

Theorem 2.11. (i) Any left special pseudoskew matrix commutes with every right special pseudoskew matrix.

(ii) If two special pseudoskew matrices commute then either they are not of the same kind, or a non-trivial linear combination of them is a scalar multiple of the identity.

Proof. Part (i) follows from lemma 2.3.

For part (ii), suppose that M_1 and M_2 are two special pseudoskew matrices of the same kind. Then, they can be expressed as in lemma 2.10(ii), which also shows that their products are

$$\begin{aligned} M_1 M_2 &= M(c_1 \mathbf{b} + c_2 \mathbf{a} + \mathbf{a} \times \mathbf{b}, \pm(c_1 \mathbf{b} + c_2 \mathbf{a} + \mathbf{a} \times \mathbf{b}), (c_1 c_2 - \mathbf{a} \cdot \mathbf{b})) \\ M_2 M_1 &= M(c_2 \mathbf{a} + c_1 \mathbf{b} + \mathbf{b} \times \mathbf{a}, \pm(c_2 \mathbf{a} + c_1 \mathbf{b} + \mathbf{b} \times \mathbf{a}), (c_1 c_2 - \mathbf{a} \cdot \mathbf{b})) \end{aligned}$$

So, M_1 and M_2 commute if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. Since the matrices are special, vectors \mathbf{a} and \mathbf{b} are both non-zero and their vector product is zero if and only if the vectors are non-zero scalar multiples of each other. If $\alpha \mathbf{a} = \beta \mathbf{b}$, where α and β are scalars which are not both zero, then $\alpha M_1 - \beta M_2 = (\alpha c_1 - \beta c_2)I$. \square

3 Cayley's factorization

Cayley's factorization allows any real 4×4 orthogonal matrix A with unit determinant to be written as the product of two matrices of a particular form which commute. The eigenvalues of A have unit modulus and appear in complex conjugate pairs. Suppose they are $\exp(\pm j\theta_1)$ and $\exp(\pm j\theta_2)$. Then it is well known that there exists a real orthogonal matrix Q which transforms A into the following canonical form.

$$Q^T A Q = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & & \\ \sin \theta_1 & \cos \theta_1 & & \\ & & \cos \theta_2 & -\sin \theta_2 \\ & & \sin \theta_2 & \cos \theta_2 \end{bmatrix} \quad (6)$$

As noted in [19], this means that the transform generated by A is a combination of two rotations in two mutually orthogonal planes through angles θ_1 and θ_2 . Such rotations are called *isoclinic* if $\theta_2 = \pm\theta_1$. The factors in Cayley's factorization are isoclinic. However it is not necessary to know this in order to be able to obtain and use the factorization.

Instead, the following definition is made.

Definition 3.1. *A left (right) special pseudoskew 4×4 matrix with unit determinant is said to be a left (right) isoclinic matrix. In addition, the matrices $+I$ and $-I$ are defined to be both left and right isoclinic.*

A *Cayley factorization* of a real orthogonal 4×4 matrix is its expression as the product of left and right isoclinic matrices.

Set

$$\begin{aligned}\phi_1 &= \frac{1}{2}(\theta_1 + \theta_2) \\ \phi_2 &= \frac{1}{2}(\theta_1 - \theta_2)\end{aligned}$$

then

$$\begin{aligned}\theta_1 &= \phi_1 + \phi_2 \\ \theta_2 &= \phi_1 - \phi_2\end{aligned}$$

and it is seen that

$$Q^T A Q = L R \tag{7}$$

where

$$\begin{aligned}L &= \begin{bmatrix} \cos \phi_1 & -\sin \phi_1 & & \\ \sin \phi_1 & \cos \phi_1 & & \\ & & \cos \phi_1 & -\sin \phi_1 \\ & & \sin \phi_1 & \cos \phi_1 \end{bmatrix} = (\cos \phi_1)I + (\sin \phi_1)\mathbf{k}_L \\ R &= \begin{bmatrix} \cos \phi_2 & -\sin \phi_2 & & \\ \sin \phi_2 & \cos \phi_2 & & \\ & & \cos \phi_2 & \sin \phi_2 \\ & & -\sin \phi_2 & \cos \phi_2 \end{bmatrix} = (\cos \phi_2)I + (\sin \phi_2)\mathbf{k}_R\end{aligned}$$

If neither of L and R is $\pm I$, then L and R are left and right special pseudoskew matrices and they have unit determinants; hence they are isoclinic. Further, by theorem 2.11, they commute. Hence equation (7) is Cayley's factorization of $Q^T A Q$, and rearrangement gives

$$A = (QLQ^T)(QRQ^T) \tag{8}$$

The factors QLQ^T and QRQ^T here have unit determinant and by theorem 2.8 they are special pseudoskew matrices: hence they are both isoclinic. It needs to be checked that they are of different kinds.

No non-trivial linear combination of L and R is a scalar multiple of the identity, and hence this is also true of QLQ^T and QRQ^T . Theorem 2.11(ii) shows that these new factors are isoclinic of different kinds. Hence equation (8) is Cayley's factorization of A .

Note that it is not necessarily the case that QLQ^T is left isoclinic and QRQ^T is right isoclinic. All that is known is that they are of different types. But this does not matter: since they commute the left isoclinic matrix can always be written first.

If one of L or R is $\pm I$, then the above is also trivially true and equation (8) is still the required factorization of A . The factors can be regarded as being of different kinds since QLQ^T or QRQ^T as appropriate is isoclinic of both kinds.

Now suppose that there are two factorizations.

$$A = L_1 R_1 = L_2 R_2$$

Since they have unit determinant, the matrices have inverses and so

$$L_2^{-1} L_1 = R_2 R_1^{-1}$$

This matrix lies in both division rings given by theorem 2.9. The only matrices common to both are multiples of the identity and since the determinant is unity, it is seen that $L_2^{-1} L_1 = R_2 R_1^{-1} = \pm I$, and so $L_2 = \alpha L_1$ and $R_2 = \alpha R_1$ where $\alpha = \pm 1$. Hence Cayley's factorization is unique except that the signs of both factors can be changed.

These observations have proved Cayley's result.

Theorem 3.2 (Cayley's factorization). *Any real orthogonal 4×4 matrix A with unit determinant can be factored as the product $A = LR$ of a left isoclinic matrix and a right isoclinic matrix which commute and which both have unit determinant. Further, this factorization is unique except that the signs of both L and R can be changed.*

4 Example

As an example consider the following orthogonal matrix.

$$A = \begin{bmatrix} 0.49639 & -0.25488 & 0.43233 & 0.70832 \\ 0.18943 & 0.48945 & 0.74516 & -0.41144 \\ -0.25022 & -0.80762 & 0.39674 & -0.35741 \\ -0.80938 & 0.20790 & 0.31689 & 0.44861 \end{bmatrix}$$

Its eigenvectors are

$$\begin{bmatrix} 0.18257 \mp 0.36515 \\ -0.27094 \mp 0.32900 \\ -0.56168 \pm 0.25761 \\ -0.27889 \mp 0.43825 \end{bmatrix} \quad \begin{bmatrix} 0.18257 \mp 0.54772 \\ -0.56123 \mp 0.05806 \\ 0.15203 \mp 0.30829 \\ 0.35857 \pm 0.31873 \end{bmatrix}$$

and the corresponding eigenvalues are $\exp(\pm j\theta)$ where $\theta = \theta_1 = -1.22173$ (-70°) and $\theta = \theta_2 = 0.95993$ (55°).

Taking the real and imaginary parts of the eigenvectors and making these of unit length creates the columns of the real orthogonal matrix Q

$$Q = \begin{bmatrix} 0.25820 & 0.51640 & 0.25820 & 0.77460 \\ -0.38317 & 0.46527 & -0.79370 & 0.08211 \\ -0.79433 & -0.36432 & 0.21501 & 0.43599 \\ -0.39441 & 0.61978 & 0.50709 & -0.45075 \end{bmatrix}$$

which brings A to the required form as in equation (6).

Taking $\phi_1 = -0.13090$ and $\phi_2 = -1.09083$ allows the factors in equation (8) to be created with

$$QRQ^T = \begin{bmatrix} 0.46175 & -0.28207 & 0.32828 & 0.77425 \\ 0.28207 & 0.46175 & 0.77425 & -0.32828 \\ -0.32828 & -0.77425 & 0.46175 & -0.28207 \\ -0.77425 & 0.32828 & 0.28207 & 0.46175 \end{bmatrix}$$

$$QLQ^T = \begin{bmatrix} 0.99144 & 0.12452 & 0.03422 & -0.01899 \\ -0.12452 & 0.99144 & 0.01899 & 0.03422 \\ -0.03422 & -0.01899 & 0.99144 & -0.12452 \\ 0.01899 & -0.03422 & 0.12452 & 0.99144 \end{bmatrix}$$

These are respectively the left and right isoclinic matrices providing the Cayley factorization of the example matrix A .

Note that this example is provided to illustrate the approach given in section 3 to establish the existence of the factorization. It requires knowledge of the eigenvectors of A or, at least, the ability to form matrix Q . A much better approach for finding the factors is that given in [19] which only requires the formation of sums of entries of A .

5 Conclusions

Quaternions are an important means for representing and manipulating rotations of three-dimensional space. Extensions, particularly double and dual quaternions allow also translations and hence rigid-body motions to be handled. They have found applications in a variety of areas including the design and analysis of mechanism and mechanical systems.

Fundamental to the underlying theory and important for some applications is the ability to move from a 4×4 orthogonal matrix representing a motion to a pair of “isoclinic” matrices corresponding to ordinary quaternions. This can be achieved using the Cayley factorization [19, 21] and this leads to a more direct presentation of the basic ideas based only on matrix methods.

A 4×4 isoclinic matrix can be regarded as the sum M of a skew symmetric matrix and a scalar multiple of the identity. This paper has shown that they can be characterized by the property that the product $M^T M$ is diagonal. This means they can be

defined without reference to their geometric properties in terms of rotations of four-dimensional space or their eigenvalues. The characterizing property is preserved by orthogonal transformations which means that the Cayley factorization can be shown to exist based on the factorization of a canonical form of the typical matrix M .

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