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RECOGNIZING WEAKLY STABLE MATRICES*

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Abstract. A max-plus matrix A is called weakly stable if the sequence (orbit) $x, A\otimes x, A^2\otimes x, \dots$ does not reach an eigenvector of A for any x unless x is an eigenvector. This is in contrast to previously studied strongly stable (robust) matrices for which the orbit reaches an eigenvector with any nontrivial starting vector. Max-plus matrices are used to describe multiprocessor interactive systems for which reachability of a steady regime is equivalent to reachability of an eigenvector by a matrix orbit. We prove that an irreducible matrix is weakly stable if and only if its critical graph is a Hamiltonian cycle in the associated graph. We extend this condition to reducible matrices. These criteria can be checked in polynomial time.

Key words. matrix, steady regime, reachability, eigenspace, stability

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1. Introduction.

Consider the system in which processors $P_1, \dots, P_n$ work interactively and in stages [13] and [14]. In each stage all processors simultaneously produce components necessary for the work of some or all other processors in the next stage. Let $x_i(k)$ denote the starting time of the kth stage on P_i $(i = 1, \dots, n)$ and let a_{ij} denote the duration of the operation at which processor P_j prepares the component necessary for processor P_i in the $(k+1)^{st}$ stage $(i, j = 1, \dots, n)$. Then, avoiding any delay, we have

$$x_i(k+1) = \max(x_1(k) + a_{i1}, \dots, x_n(k) + a_{in}) \ (i = 1, \dots, n; k = 0, 1, \dots). \quad (1)$$

We refer to such a system as a *multiprocessor interactive system (MPIS)*. The matrix $A = (a_{ij})$ will be called the *production matrix*. In order to analyze such systems transparently and efficiently, we denote $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$. Then (1) gets the form

$$x_i(k+1) = a_{i1} \otimes x_1(k) \oplus \dots \oplus a_{in} \otimes x_n(k) \qquad (i = 1, \dots, n; k = 0, 1, \dots).$$

If, moreover, the pair of operations $(\oplus, \otimes)$ is extended to matrices and vectors in the same way as in linear algebra, then we obtain a compact expression

$$x(k+1) = A \otimes x(k) \ (k = 0, 1, \dots).$$

We say that an MPIS reaches a *steady regime* [14] if it eventually moves forward in regular steps, that is, for some λ and k_0 we have $x(k+1) = \lambda \otimes x(k)$ for all $k \geq k_0$.

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Equivalently, the time between the starts of consecutive stages will eventually be the same constant for every processor. If this happens, then we have

$$A \otimes x(k) = \lambda \otimes x(k) \text{ for all } k \geq k_0$$

and so $x(k)$ is an eigenvector of A with respect to the pair of operations $(\oplus, \otimes)$ ("max-eigenvector") with associated value λ ("max-eigenvalue"). Reaching stability is a desirable goal, and the task of achieving this is of fundamental importance for any MPIS. Since

$$x(k) = A \otimes x(k-1) = A^2 \otimes x(k-2) = \cdots = A^k \otimes x(0)$$

for every natural k, the questions that may be of operational interest are as follows:

Q1. Given $A \in \overline{\mathbb{R}}^{n\times n}$ and $x \in \overline{\mathbb{R}}^n$ is there a natural number k such that $A^k \otimes x$ is a max-eigenvector of A?

Q2. Characterize matrices A (strongly stable) for which $A^k \otimes x$ is an eigenvector for every nontrivial x and sufficiently large k.

Q3. Characterize matrices A (weakly stable) for which $A^k \otimes x$ is not an eigenvector for any x and any k unless x is an eigenvector itself.

Question Q1 for irreducible matrices can be answered using an $O(n^3 \log n)$ algorithm [32]; see also [6]. It is known that $A^k \otimes x$ always reaches a max-eigenvector of some power of A (see Corollary 3.1 below), and the algorithm in [32] in fact finds the smallest natural number s such that $A^k \otimes x$ is a max-eigenvector of A^s.

An efficient characterization of strongly stable matrices is known [8] (note that in that paper strongly stable matrices are called robust); see also [6]. The aim of the present paper is to provide a full and efficient characterization of weakly stable matrices, that is, to solve Q3. We start with an example of an MPIS to which we will return at the end of the paper in Examples 6.4 and 6.5.

Example 1.1. A vehicle manufacturer has three production units U1–U3. U1 produces cars, U2 trucks, and U3 motorcycles. These units also specialize in certain components, which they prepare for other units. U1 also specializes in windows, U2 in gearboxes, and U3 in wheels. U3 does not need windows and these are only prepared for U1 and U2. Production runs in stages and the delivery of windows, gearboxes, and wheels is done in batches, one for every new stage. In every stage each unit begins production as soon as the delivery of the batches of necessary components from all units is completed. The table below summarizes the delivery times of individual components between the units. The task is to find the vector of starting times of the work of individual units so that the production process is stable, that is, the duration of all stages at every unit is the same (except possibly for a few first stages):

	U1	U2	U3
U1	2	5	$-\infty$
U2	3	3	2
U3	3	5	1

The paper is organized as follows: We continue with the definitions of basic concepts of max-algebra and an overview of selected known results in this area (section 2). Section 3 presents elementary properties of attraction sets and weakly stable matrices. The main results are presented in sections 4 and 5, where we characterize irreducible and reducible weakly stable matrices, respectively. Finally, for completeness, a brief overview of the known results for strongly stable matrices is given in section 6.

2. Prerequisites.

2.1. Max-algebra: Basic definitions and properties. In this section we give the definitions and an overview of those results of max-algebra which will be instrumental for the formulations and proofs of the results of this paper. For the proofs and more information about max-algebra the reader is referred to [1], [3], [23], and [6].

We will use the following notation: As usual $\mathbb{R}$ is the set of real numbers and the symbol $\overline{\mathbb{R}}$ will stand for $\mathbb{R} \cup \{-\infty\}$. If $a, b \in \overline{\mathbb{R}}$, then we set

$$a \oplus b = \max(a, b)$$

and

$$a \otimes b = a + b.$$

Throughout the paper we denote $-\infty$ by ε (the neutral element with respect to $\oplus$), and for convenience we also denote by the same symbol any vector all of whose components are $-\infty$ or a matrix all of whose entries are $-\infty$. If $a \in \mathbb{R}$, then the symbol a^{-1} stands for $-a$. We assume everywhere that $n \geq 1$ is a natural number and denote $N = \{1, \dots, n\}$.

By max-algebra (recently also called "tropical linear algebra") we understand the analogue of linear algebra developed for the pair of operations $(\oplus, \otimes)$, extended to matrices and vectors as in conventional linear algebra. That is, if $A = (a_{ij}),\ B = (b_{ij})$, and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\overline{\mathbb{R}}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all $i, j \in N$ and $C = A \otimes B$ if

$$c_{ij} = \bigoplus_{k \in N} a_{ik} \otimes b_{kj} = \max_k (a_{ik} + b_{kj})$$

for all $i, j \in N$. If $\alpha \in \overline{\mathbb{R}}$, then $\alpha \otimes A = (\alpha \otimes a_{ij})$. Although the use of the symbols $\otimes$ and $\oplus$ is common in max-algebra we will use the usual convention of not writing the symbol $\otimes$. Thus in what follows the symbol $\otimes$ will not be used and unless explicitly stated otherwise, all multiplications indicated are in max-algebra.

A vector or matrix is called *finite* if all its entries are real numbers. A square matrix is called *diagonal* if all its diagonal entries are real numbers and off-diagonal entries are ε. A diagonal matrix with all diagonal entries equal to 0 is called the *unit matrix* and denoted I. Obviously, $AI = IA = A$ whenever A and I are of compatible sizes. A matrix obtained from a diagonal matrix (unit matrix) by permuting the rows or columns is called a *generalized permutation matrix* (*permutation matrix*). It is known that in max-algebra generalized permutation matrices is the only type of invertible matrices [14], [6].

If A is a square matrix, then the iterated product $AA \dots A$ in which the symbol A appears k-times will be denoted by A^k. By definition $A^0 = I$.

The following is easily proved: if $A, B \in \overline{\mathbb{R}}^{m \times n}$ and $x, y \in \overline{\mathbb{R}}^n$, then

$$\begin{aligned} A \geq B &\Longrightarrow Ax \geq Bx, \\ x \geq y &\Longrightarrow Ax \geq Ay. \end{aligned} \tag{2}$$

We will often benefit from the close link between matrices and digraphs. A *digraph* is an ordered pair $D = (V, E)$, where V is a nonempty finite set (of *nodes*) and $E \subseteq V \times V$ (the set of *arcs*). A *subdigraph* of D is any digraph $D' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. If $e = (u, v) \in E$ for some $u, v \in V$, then we say that e is *leaving* u and *entering* v. Any arc of the form (u, u) is called a *loop*.

Let $D = (V, E)$ be a digraph. A sequence $\pi = (v_1, \ldots, v_p)$ of nodes in D is called a *path* (in D) if $p = 1$ or $p > 1$ and $(v_i, v_{i+1}) \in E$ for all $i = 1, \ldots, p-1$. The node v_1 is called the *starting node* and v_p the *endnode* of π, respectively. The number $p - 1$ is called the *length* of π and will be denoted by $l(\pi)$. If u is the starting node and v is the endnode of π, then we say that π is a $u - v$ path. If there is a $u - v$ path in D, then v is said to be *reachable* from u (notation $u \longrightarrow v$). Thus $u \longrightarrow u$ for any $u \in V$.

A path $(v_1, \ldots, v_p)$ is called a *cycle* if $v_1 = v_p$ and $p > 1$, and it is called an *elementary cycle* if, moreover, $v_i \neq v_j$ for $i, j = 1, \ldots, p-1, i \neq j$. If there is no cycle in D, then D is called *acyclic*. An elementary cycle passing through all nodes of D is called *Hamiltonian*.

A digraph D is called *strongly connected* if $u \longrightarrow v$ for all nodes u, v in D. A subdigraph D' of D is called a *strongly connected component* of D if it is a maximal strongly connected subdigraph of D, that is, D' is a strongly connected subdigraph of D, and if D' is a subdigraph of a strongly connected subdigraph D'' of D, then $D' = D''$. All strongly connected components of a given digraph $D = (V, E)$ can be identified in $O(|V| + |E|)$ time [35]. Note that a digraph consisting of one node and no arc is strongly connected and acyclic; however, if a strongly connected digraph has at least two nodes, then it obviously cannot be acyclic.

If $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$, then the symbol F_A will denote the digraph with the node set N and arc set $E = \{(i,j); a_{ij} > \varepsilon\}$. If F_A is strongly connected, then A is called *irreducible*, and *reducible* otherwise. Note that every 1×1 matrix is irreducible, even if it is (ε). This is the only irreducible matrix with an ε column. We will assume in some statements that the irreducible matrix under consideration is different from ε; this is merely to exclude the 1×1 matrix (ε). The following three lemmas are easily seen.

LEMMA 2.1. *If $A \in \overline{\mathbb{R}}^{n\times n}$ is irreducible and $A \neq \varepsilon$, then A has no ε row and column.*

Note that a matrix may be reducible even if it has no ε row and column (e.g., I).

LEMMA 2.2. *If $A \in \overline{\mathbb{R}}^{n\times n}$ has no ε column and $x \neq \varepsilon$, then $A^k x \neq \varepsilon$ for every nonnegative integer k. Hence if $A \in \overline{\mathbb{R}}^{n\times n}$ has no ε column, then A^k also has no ε column for every k. This is true in particular when A is irreducible and $A \neq \varepsilon$.*

LEMMA 2.3. *If $A \in \overline{\mathbb{R}}^{n\times n}$ has no ε column or no ε row, then F_A contains a cycle.*

A *weighted digraph* is $D = (V, E, w)$, where (V, E) is a digraph and w is a real function on E. All definitions for digraphs are naturally extended to weighted digraphs. If $\pi = (v_1, \ldots, v_p)$ is a path in (V, E, w), then the *weight* of π is $w(\pi) = w(v_1, v_2) + w(v_2, v_3) + \cdots + w(v_{p-1}, v_p)$ if $p > 1$ and ε if $p = 1$. A path π is called *positive* if $w(\pi) > 0$. In contrast, a cycle $\sigma = (u_1, \ldots, u_p)$ is called a *zero cycle* if $w(u_k, u_{k+1}) = 0$ for all $k = 1, \ldots, p-1$. Since w stands for weight rather than length, we will use the term "heaviest path/cycle" instead of "longest path/cycle."

Given $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ the symbol D_A will denote the weighted digraph (N, E, w), where $F_A = (N, E)$ and $w(i,j) = a_{ij}$ for all $(i,j) \in E$. The digraph D_A is said to be *associated* with the matrix A. If $\pi = (i_1, \ldots, i_p)$ is a path in D_A, then we denote $w(\pi, A) = w(\pi)$ and it now follows from the definitions that $w(\pi, A) = a_{i_1 i_2} + a_{i_2 i_3} + \cdots + a_{i_{p-1} i_p}$ if $p > 1$ and ε if $p = 1$.

Let $S \subseteq \overline{\mathbb{R}}^n$. The set S is called a *max-algebraic subspace* if

$$\alpha u \oplus \beta v \in S$$

for every $u, v \in S$ and $\alpha, \beta \in \overline{\mathbb{R}}$. The adjective "max-algebraic" will usually be omitted.

A vector $v \in \overline{\mathbb{R}}^m$ is called a *max-combination* of a finite set $S \subseteq \overline{\mathbb{R}}^m$ if

$$v = \bigoplus_{x \in S} \alpha_x x, \alpha_x \in \overline{\mathbb{R}}. \tag{3}$$

The set of all max-combinations of S is denoted by $span(S)$. We set $span(\emptyset) = \{\varepsilon\}$. It is easily seen that $span(S)$ is a subspace. If $span(S) = T$, then S is called a *set of generators* for T. A subspace T is called *finitely generated* if there is a finite set of generators for T.

Let $v \in \overline{\mathbb{R}}^m$. The *max-norm* or just *norm* of v is the value of the greatest component of v (notation $\|v\|$); v is called *scaled* if $\|v\| = 0$. The set S is called scaled if all its elements are scaled.

The set $S = \{v_1, \dots, v_n\} \subseteq \overline{\mathbb{R}}^m$ is called *dependent* if v_k is a max-combination of $\{v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n\}$ for some $k \in N$. Otherwise S is *independent*.

Let $S, T \subseteq \overline{\mathbb{R}}^m$. The set S is called a *basis* of T if it is an independent set of generators for T. The following is of fundamental importance in max-algebra.

THEOREM 2.4 (see [37], [38], [6]). *If T is a finitely generated subspace, then there is a unique scaled basis for T.*

2.2. Max-algebra: The key tools. Given $A \in \overline{\mathbb{R}}^{n \times n}$, the symbol $\lambda(A)$ will stand for the *maximum cycle mean* of A, that is,

$$\lambda(A) = \max_{\sigma} \mu(\sigma, A), \tag{4}$$

where the maximization is taken over all elementary cycles in D_A and

$$\mu(\sigma, A) = \frac{w(\sigma, A)}{l(\sigma)} \tag{5}$$

denotes the *mean* of a cycle σ. Clearly, $\lambda(A)$ always exists since the number of elementary cycles is finite. However, it is easy to show [6] that $\lambda(A)$ remains the same if the word "elementary" is removed from the definition.

We use the convention $\max \emptyset = \varepsilon$. It follows that $\lambda(A) = \varepsilon$ if and only if D_A is acyclic.

Example 2.5. If

$$A = \begin{pmatrix} -2 & 1 & -3 \\ 3 & 0 & 3 \\ 5 & 2 & 1 \end{pmatrix},$$

then the means of elementary cycles of length 1 are $-2, 0, 1$, of length 2 are $2, 1, 5/2$, and of length 3 are 3 and $2/3$. Hence $\lambda(A) = 3$.

The maximum cycle mean of a matrix is of fundamental importance in max-algebra because for any square matrix A it is the greatest (max-algebraic) eigenvalue of A, and every eigenvalue of A is the maximum cycle mean of some principal submatrix of A. (See subsection 2.3 for details.)

Computation of the maximum cycle mean from the definition is difficult except for small matrices since the number of elementary cycles in a digraph may be prohibitively large in general. There are many algorithms of known or unknown computational complexity for finding the maximum cycle mean; for references see [6]. An algorithm with the lowest known computational complexity is probably *Karp's algorithm* [24], see also [6] for details, which finds the maximum cycle mean of an $n \times n$ matrix in $O(n|E|)$ time, where E is the set of arcs of D_A.

LEMMA 2.6 (see [6]). *If $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ has no ε row or column, then $\lambda(A) > \varepsilon$. This is true in particular when A is irreducible and $n > 1$.*

A matrix $A \in \overline{\mathbb{R}}^{n\times n}$ is called *definite* if $\lambda(A) = 0$ [9], [14]. Thus a matrix is definite if and only if all cycles in D_A are nonpositive and at least one has weight zero. It is easy to check that $\lambda(\alpha A) = \alpha\lambda(A)$ for any $\alpha \in \mathbb{R}$. Hence $(\lambda(A))^{-1}A$ is definite whenever $\lambda(A) > \varepsilon$. The matrix $(\lambda(A))^{-1}A$ will be denoted by A_λ.

Let $A \in \overline{\mathbb{R}}^{n\times n}$. A cycle σ in D_A is called *critical* if $\mu(\sigma, A) = \lambda(A)$. We denote by $N_c(A)$ the set of *critical nodes,* that is, nodes on critical cycles. These nodes play an essential role in solving the eigenproblem (subsection 2.3). If $i, j \in N_c(A)$ belong to the same critical cycle, then i and j are called *equivalent* and we write $i \sim j$. Clearly, $\sim$ constitutes a relation of equivalence on $N_c(A)$.

The *critical digraph* of A is the digraph C_A with the set of nodes N; the set of arcs (notation $E_c(A)$) is the set of arcs of all critical cycles. The equivalence classes with respect to $\sim$ coincide with the strongly connected components of C_A. A strongly connected component of C_A is called *trivial* if it consists of a single node without a loop, *nontrivial* otherwise. It is not difficult to prove that all cycles in a critical digraph are critical [3], [23], [6].

Given $A \in \overline{\mathbb{R}}^{n\times n}$ it is standard [14], [3], [23], [6] in max-algebra to define the infinite series

$$A^+ = A \oplus A^2 \oplus A^3 \oplus \cdots \tag{6}$$

and

$$A^* = I \oplus A^+ = I \oplus A \oplus A^2 \oplus A^3 \oplus \cdots \ . \tag{7}$$

The matrix A^+ is called the *weak transitive closure* of A, and A^* is the *strong transitive closure* of A. Notice that $AA^* = A^+ \leq A^*$ and $a^+_{ij} = a^*_{ij}$ if $i \neq j$.

The matrix A^+ is of fundamental importance in max-algebra as some of its columns form an essentially unique basis of the eigenspace if A is irreducible; A^* for finite subeigenvectors of A is similar (see subsection 2.3).

It is easily seen that the entries of A^2 are the weights of heaviest paths of length 2 for all pairs of nodes in D_A. Similarly the entries of A^k $(k = 1, 2, \dots)$ are the weights of heaviest paths of length k. Therefore diagonal entries of the matrix A^+ represent the weights of heaviest paths of any length, and finite off-diagonal entries represent the weights of heaviest paths of any positive length. Motivated by this fact A^+ is also called the *metric matrix* corresponding to A [14]. We deduce that A^+ has no ε entries if A is irreducible, $A \neq \varepsilon$. Note that A^* is often called the *Kleene star.*

It follows from the definitions that every entry of the matrix sequence

$$\left\{ \bigoplus_{j=0}^{k} A^j \right\}_{k=0}^{\infty}$$

is a nondecreasing sequence in $\overline{\mathbb{R}}$ and therefore either it is convergent (when bounded) or its limit is $+\infty$ (when unbounded). What matters is whether $+\infty$ is the limit of some entries.

LEMMA 2.7 (see [6]). *Let $A \in \overline{\mathbb{R}}^{n\times n}$. Then* (6) *converges to a matrix with no $+\infty$ entries if and only if $\lambda(A) \leq 0$. If $\lambda(A) \leq 0$, then*

$$A^+ = A \oplus A^2 \oplus \cdots \oplus A^k$$

and

$$A^* = I \oplus A \oplus A^2 \oplus \cdots \oplus A^{k-1}$$

for every $k \geq n$. If A is also irreducible and $A \neq \varepsilon$, then both A^+ and A^* are finite.

The following can also be immediately deduced.

LEMMA 2.8. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be a definite matrix and $A^+ = (\gamma_{ij})$. Then $\gamma_{ii} = 0$ if and only if $i \in N_c(A)$. Hence if $N_c(A) = N$, then $A^* = A^+$.*

A matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ is called *increasing* if $a_{ii} \geq 0$ for all $i \in N$. Obviously, $A = I \oplus A$ when A is increasing and so then there is no difference between A^+ and A^*. The following is readily seen.

LEMMA 2.9. *If $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ is increasing, then $x \leq Ax$ for every $x \in \overline{\mathbb{R}}^n$. Hence*

$$A \leq A^2 \leq A^3 \leq \cdots. \tag{8}$$

One of the most striking features of matrices in max-algebra is the periodicity of matrix powers, which is expressed using the concept of cyclicity. If D' is a strongly connected component of a digraph D, then the greatest common divisor of the lengths of all directed cycles in D' is called the *cyclicity* of D'. The cyclicity of D, notation $\sigma(D)$, is the least common multiple of the cyclicities of all strongly connected components of D. The cyclicity of a digraph consisting of a single node and no arc is 1 by definition. The cyclicity of a digraph can be found in linear time [16]. A digraph D is called *primitive* if $\sigma(D) = 1$ and *imprimitive* otherwise. The cyclicity of $A \in \overline{\mathbb{R}}^{n\times n}$, notation $\sigma(A)$, is the cyclicity of its critical digraph C_A. We will use the adjectives primitive and imprimitive for matrices in the same way as for their critical digraphs.

A matrix $A \in \overline{\mathbb{R}}^{n\times n}$ is called *ultimately periodic* if there exist natural numbers p and T such that

$$A^{k+p} = (\lambda(A))^p A^k \tag{9}$$

for every $k \geq T$. If A is ultimately periodic, then the smallest value of p for which a T satisfying (9) exists is called the *period of* A. If p is the period of A, then the smallest value of T for which (9) holds is called the *transient* of $\{A^k\}$ and will be denoted by $T(A)$.

THEOREM 2.10 (cyclicity theorem). *Every irreducible matrix $A \in \overline{\mathbb{R}}^{n\times n}$ is ultimately periodic, and its period is equal to the cyclicity of A.*

Theorem 2.10 has been proved for finite matrices in [14]. A proof for general matrices was presented in [10]; see also [11] for an overview without proofs. A proof in a different setting covering the case of finite matrices is given in [27]. The general irreducible case is also proved in [2], [3], [21], and [23].

2.3. Max-algebra: The eigenproblem. In this subsection we will give a brief overview of the methodology for solving the max-algebraic eigenproblem.

The max-algebraic eigenvalue-eigenvector problem (briefly, eigenproblem) is the following.

Given $A \in \overline{\mathbb{R}}^{n\times n}$, find all $\lambda \in \overline{\mathbb{R}}$ (eigenvalues) and $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$(eigenvectors) such that

$$Ax = \lambda x.$$

This problem has been studied since the work of Cuninghame-Green [13]. A full solution of the eigenproblem in the case of irreducible matrices has been presented by Cuninghame-Green [14], [15] and Gondran and Minoux [22]; see also Vorobyov [36]. The general (reducible) case was first presented by Gaubert [20] and Bapat, Stanford, and van den Driessche [4]. See also [8] and [6].

A key role in the solution of the eigenproblem is played by the maximum cycle mean. It is the biggest eigenvalue for any matrix (therefore called the *principal eigenvalue*), and although for an $n \times n$ matrix there may be up to n eigenvalues in total, each of them is the maximum cycle mean of some principal submatrix. The maximum cycle mean of a matrix is the only eigenvalue whose associated eigenvectors may be finite. Irreducible matrices have no eigenvalues other than the maximum cycle mean. If the maximum cycle mean is ε, then all eigenvectors can be described in a straightforward way; we will therefore usually assume that the maximum cycle mean is finite.

Let $A \in \overline{\mathbb{R}}^{n\times n}$. We denote by $V(A,\lambda)$ the set of all eigenvectors of A corresponding to $\lambda \in \overline{\mathbb{R}}$, by $V(A)$ the set of all eigenvectors of A, and by $\Lambda(A)$ the set of all eigenvalues of A. The set $V^+(A)$ consists of all finite eigenvectors of A. It is easily seen that $V(A,\lambda)\cup\{\varepsilon\}$ is a nontrivial subspace for each $\lambda \in \Lambda(A)$. We will therefore call $V(A,\lambda)\cup\{\varepsilon\}$ the *eigenspace of A associated with the eigenvalue λ.*

We first summarize how to find all eigenvectors associated with the principal eigenvalue (*principal eigenvectors*), and then we proceed with other eigenvalues. Note that A^+_λ stands for $(A_\lambda)^+$.

THEOREM 2.11 (see [14], [6]). *Let $A \in \overline{\mathbb{R}}^{n\times n}$. Then the following hold:*

- *$\lambda(A)$ is the greatest eigenvalue of A.*
- *If $\lambda(A) > \varepsilon$, then at least one column of A^+_λ has zero diagonal entry and every such column is an eigenvector of A with associated eigenvalue $\lambda(A)$.*
- *If A is irreducible, then there are no eigenvalues of A other than $\lambda(A)$, that is, $V(A) = V(A,\lambda(A))$.*

Any column of A^+_λ with zero diagonal entry will be called a *fundamental eigenvector* of A. Clearly, when considering all possible max-combinations of a set of fundamental eigenvectors (or, indeed, of any vectors), we may remove from this set vectors that are multiples of some other. To be more precise, we say that two fundamental eigenvectors g_i and g_j are *equivalent* if $g_i = \alpha g_j$ for some $\alpha \in \mathbb{R}$. We can easily characterize equivalent fundamental eigenvectors using the equivalence of critical nodes in the next statement. (Note that the relation $i \sim j$ has been defined in subsection 2.1.)

THEOREM 2.12 (see [14]). *Suppose that $A \in \overline{\mathbb{R}}^{n\times n}, \lambda(A) > \varepsilon$, and $g_1,\ldots,g_n$ are the columns of $A^+_\lambda = (\gamma_{ij})$. Then the following hold:*

- $i \in N_c(A)$ if and only if $\gamma_{ii} = 0$.
- If $i,j \in N_c(A)$, then $i \sim j$ if and only if $g_i = \alpha g_j$ for some $\alpha \in \mathbb{R}$.

The following theorem identifies an essentially unique basis of the eigenspace of A corresponding to the principal eigenvalue. Note that this statement for irreducible matrices was already proved in [14].

THEOREM 2.13 (see [2]). *Suppose that $A \in \overline{\mathbb{R}}^{n\times n}, \lambda(A) > \varepsilon$, and $g_1,\ldots,g_n$ are the columns of A^+_λ. Then*

$$V(A,\lambda(A)) = \left\{A^+_\lambda z; z \in \overline{\mathbb{R}}^n, z_j = \varepsilon \text{ for all } j \notin N_c(A)\right\}$$

and we obtain a basis of $V(A,\lambda(A))$ by taking exactly one g_i for each equivalence class in $(N_c(A),\sim)$.

If A is irreducible, $A \neq \varepsilon$, then A_λ^+ is finite and hence we also deduce the following.

COROLLARY 2.14 (see [14]). *If $A \in \overline{\mathbb{R}}^{n\times n}$ is irreducible, $A \neq \varepsilon$, and $g_1, \ldots, g_n$ are the columns of A_λ^+, then all eigenvectors of A are finite and*

$$V(A) = V^+(A) = \left\{ \bigoplus_{i\in N_c^*(A)} \alpha_i g_i; \alpha_i \in \mathbb{R} \right\},$$

where $N_c^*(A)$ is any maximal set of nonequivalent critical nodes of A.

COROLLARY 2.15. *If $A \in \overline{\mathbb{R}}^{n\times n}$ is irreducible, $A \neq \varepsilon$, and $N_c(A) = N$, then*

$$V(A) = V^+(A) = \left\{ A_\lambda^+ z; z \in \mathbb{R}^n \right\}.$$

We now discuss how to find all max-algebraic eigenvalues of a matrix. It is important that spectral properties of matrices are preserved by certain types of transformations. The first straightforwardly follows from the definitions.

LEMMA 2.16. *If $\alpha \in \mathbb{R}$ and A is a square matrix, then*

- $\Lambda(\alpha A) = \{\alpha\lambda; \lambda \in \Lambda(A)\}$ *and*
- $V(\alpha A) = V(A)$.

If $B = P^{-1}AP$ for some generalized permutation matrix P, then we say that A and B are *similar* (notation $A \sim B$). If, moreover, P is a permutation matrix, then A and B are said to be *directly similar* (notation $A \approx B$). If, on the other hand, P is a diagonal matrix, then we say that B has been obtained from A by *diagonal similarity scaling* (briefly *matrix scaling*) (notation $A \equiv B$). Clearly all these relations are relations of equivalence.

If $A \equiv B$, then $F_A = F_B$; if $A \approx B$, then F_A can be obtained from F_B by a renumbering of the nodes or, equivalently, A can be obtained from B by a simultaneous permutation of the rows and columns.

Let P be a generalized permutation matrix, $Ax = \lambda x, x \neq \varepsilon$. Then

$$\left(P^{-1}AP\right)\left(P^{-1}x\right) = \lambda\left(P^{-1}x\right)$$

and $P^{-1}x \neq \varepsilon$. Hence we deduce the following.

LEMMA 2.17. *If A and B are square matrices and $B = P^{-1}AP$, where P is a generalized permutation matrix, then we have*

- $\Lambda(A) = \Lambda(B)$ *and*
- $V(B) = \{P^{-1}x; x \in V(A)\}$.

The following standard notation will be useful: If

$$1 \leq i_1 < i_2 < \cdots < i_k \leq n, K = \{i_1, \ldots, i_k\} \subseteq N,$$

then $A[K]$ denotes the principal submatrix

$$\begin{pmatrix} a_{i_1 i_1} & \ldots & a_{i_1 i_k} \\ \ldots & \ldots & \ldots \\ a_{i_k i_1} & \ldots & a_{i_k i_k} \end{pmatrix}$$

of the matrix $A = (a_{ij})$ and $x[K]$ denotes the subvector $(x_{i_1}, \ldots, x_{i_k})^T$ of the vector $x = (x_1, \ldots, x_n)^T \in \overline{\mathbb{R}}^n$.

If $D = (N, E)$ is a digraph and $K \subseteq N$, then $D[K]$ denotes the *induced subgraph* of D, that is,

$$D[K] = (K, E \cap (K \times K)).$$

Every matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ can be transformed by simultaneous permutations of the rows and columns in linear time to a *Frobenius normal form* [28]

$$\begin{pmatrix} A_{11} & \varepsilon & \dots & \varepsilon \\ A_{21} & A_{22} & \dots & \varepsilon \\ \dots & \dots & \dots & \dots \\ A_{r1} & A_{r2} & \dots & A_{rr} \end{pmatrix}, \tag{10}$$

where $A_{11}, \dots, A_{rr}$ are irreducible square submatrices of A. If A is in a Frobenius normal form, then the corresponding partition subsets of the node set N of D_A will be denoted as $N_1, \dots, N_r$ and these sets will be called *classes (of* A). It follows that each of the induced subgraphs $D_A[N_i]$ $(i = 1, \dots, r)$ is strongly connected and an arc from N_i to N_j in D_A exists only if $i \geq j$. As a slight abuse of language we will also say for simplicity that $\lambda(A_{jj})$ is the eigenvalue of N_j.

If A is in the Frobenius normal form (10), then the *condensation digraph*, notation $C(A)$, is the digraph

$$(\{N_1, \dots, N_r\}, \{(N_i, N_j); (\exists k \in N_i)(\exists \ell \in N_j) a_{k\ell} > \varepsilon\}).$$

Observe that $C(A)$ is acyclic and represents a partially ordered set. Any class that has no incoming arcs in $C(A)$ is called *initial*, similarly for diagonal blocks.

Recall that the symbol $N_i \longrightarrow N_j$ means that there is a directed path from a node in N_i to a node in N_j in $C(A)$ (and therefore from each node in N_i to each node in N_j in D_A).

Due to Lemma 2.17 we may assume without loss of generality that A is in a Frobenius normal form, say (10). It is intuitively clear that all eigenvalues of A are among the unique eigenvalues of diagonal blocks. However, not all of these eigenvalues are also eigenvalues of A. The following key result appeared for the first time independently in the thesis [20] and [4]; see also [6] and [8].

THEOREM 2.18 (spectral theorem). *Let* (10) *be a Frobenius normal form of a matrix* $A \in \overline{\mathbb{R}}^{n\times n}$. *Then*

$$\Lambda(A) = \{\lambda; (\exists j)\, \lambda = \lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii})\}.$$

Note that if a diagonal block, say A_{jj}, has $\lambda(A_{jj}) = \lambda$, it still may not satisfy the condition $\lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii})$ and may therefore not provide any eigenvectors. It is therefore necessary to identify blocks that satisfy this condition: If

$$\lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii}),$$

then A_{jj} (and also N_j or just j) will be called *spectral*. Thus $\lambda(A_{jj}) \in \Lambda(A)$ if j is spectral but not necessarily the other way round. We can immediately deduce the following.

COROLLARY 2.19. *All initial blocks are spectral.*

Also, it follows that the number of eigenvalues does not exceed n and obviously, $\lambda(A) = \max_i \lambda(A_{ii})$ which confirms that $\lambda(A)$ is indeed always an eigenvalue.

LEMMA 2.20 (see [6]). *If x is an eigenvector with x_i finite for some $i \in N_s$ and some s, then x_j is finite for every $j, j \longrightarrow i$ and in particular for every $j \in N_s$.*

It remains to explain how to find a basis of the eigenspace associated with a general eigenvalue $\lambda \in \Lambda$. Let $A \in \overline{\mathbb{R}}^{n\times n}$ be in the Frobenius normal form (10) and

$N_1, \ldots, N_r$ be the classes of A and $R = \{1, \ldots, r\}$. Suppose $\lambda \in \Lambda(A), \lambda > \varepsilon$ and denote

$$I(\lambda) = \{i \in R; \lambda(N_i) = \lambda, N_i \text{ spectral}\}.$$

Note that $\lambda(\lambda^{-1}A) = \lambda^{-1}\lambda(A)$ may be positive since $\lambda \le \lambda(A)$, and thus $(\lambda^{-1}A)^+ = (\gamma_{ij})$ may now include entries equal to $+\infty$ (see Lemma 2.7). Let us denote

$$N_c(A, \lambda) = \bigcup_{i \in I(\lambda)} N_c(A_{ii}) = \left\{ j \in N; \gamma_{jj} = 0, j \in \bigcup_{i \in I(\lambda)} N_i \right\}.$$

Two nodes i and j in $N_c(A, \lambda)$ are called λ - *equivalent* (notation $i \sim_\lambda j$) if i and j belong to the same cycle of cycle mean λ.

THEOREM 2.21 (see [8], [6]). *Suppose that $A \in \overline{\mathbb{R}}^{n \times n}$ and $\lambda \in \Lambda(A), \lambda > \varepsilon$. Let $g_1, \ldots, g_n$ be the columns of $(\lambda^{-1}A)^+$. Then $g_j \in \overline{\mathbb{R}}^n$ (that is, g_j does not contain a $+\infty$ component) for all $j \in N_c(A, \lambda)$,*

$$V(A, \lambda) = \left\{ (\lambda^{-1}A)^+ z; z \in \overline{\mathbb{R}}^n, z \ne \varepsilon, z_j = \varepsilon \textit{ for all } j \notin N_c(A, \lambda) \right\},$$

and a basis of $V(A, \lambda)$ can be obtained by taking exactly one g_j for each $\sim_\lambda$ equivalence class.

COROLLARY 2.22. *The spectrum $\Lambda(A)$ and bases of $V(A, \lambda)$ for all $\lambda \in \Lambda(A)$ can be found using $O(n^3)$ operations.*

Note that if the set $I(\lambda)$ consists of only one index, then $V(A, \lambda)$ can alternatively be found as follows: If $I(\lambda) = \{j\}$, then define

$$M_2 = \bigcup_{N_i \to N_j} N_i, M_1 = N - M_2.$$

Hence $V^+(A[M_2]) \ne \emptyset$ and

$$V(A, \lambda) = \{x; x[M_1] = \varepsilon, x[M_2] \in V^+(A[M_2])\}.$$

2.4. Subeigenvectors and visualization scaling. If $A \in \overline{\mathbb{R}}^{n \times n}$ and $\lambda \in \overline{\mathbb{R}}$, then a vector $x \in \overline{\mathbb{R}}^n, x \ne \varepsilon$ satisfying

$$Ax \le \lambda x \tag{11}$$

is called a *subeigenvector of A with associated eigenvalue* λ and we denote

$$V_*(A, \lambda) = \left\{ x \in \overline{\mathbb{R}}^n; Ax \le \lambda x, x \ne \varepsilon \right\}.$$

Finite subeigenvectors are of particular importance and can easily be described.

THEOREM 2.23 (see [18], [6]). *Let $A \in \overline{\mathbb{R}}^{n \times n}, A \ne \varepsilon$. Then the following statements hold:*

(a) *$Ax \le \lambda x$ has a finite solution if and only if $\lambda \ge \lambda(A)$ and $\lambda > \varepsilon$.*

(b) *If $\lambda \ge \lambda(A)$ and $\lambda > \varepsilon$, then*

$$V_*(A, \lambda) \cap \mathbb{R}^n = \left\{ \left(\lambda^{-1}A\right)^* z; z \in \mathbb{R}^n \right\}.$$

LEMMA 2.24 (see [6]). *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be irreducible. If $Ax \leq \lambda x, x \neq \varepsilon$, then x is finite, that is,*

$$V_*(A,\lambda) = \{x \in \mathbb{R}^n; Ax \leq \lambda x\}.$$

COROLLARY 2.25. *If A is irreducible, and $A \neq \varepsilon$, then*

$$V_*(A,\lambda(A)) = \left\{A_\lambda^+ z; z \in \mathbb{R}^n\right\}.$$

Using matrix scaling [19], [29], [30], [31], [34] it is possible to simplify the structure of a matrix while preserving its spectral properties. In particular, it enables us to "visualize" some features, such as the entries corresponding to the arcs on critical cycles.

We say that $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ is *visualized* if

$$a_{ij} \leq \lambda(A) \quad \text{for all } i,j \in N. \tag{12}$$

Since for every $(i,j) \in E_c(A)$ the entry a_{ij} contributes to an average equal to $\lambda(A)$ it is easy to see that (12) actually implies

$$a_{ij} = \lambda(A) \quad \text{for all } (i,j) \in E_c(A). \tag{13}$$

If $x = (x_1,\ldots,x_n)^T \in \mathbb{R}^n$, then the symbol $\mathrm{diag}(x)$ stands for the diagonal matrix whose diagonal entries are $x_1,\ldots,x_n$.

THEOREM 2.26 (see [31], [34]). *If $A \in \overline{\mathbb{R}}^{n\times n}$, $\lambda(A) > \varepsilon, x \in V_*(A) \cap \mathbb{R}^n$, and $X = \mathrm{diag}(x)$, then $X^{-1}AX$ is visualized.*

As before, let $g_1,\ldots,g_n$ be the columns of A_λ^+. Note that due to Corollary 2.25 for the vector x in Theorem 2.26 we can take for instance $\bigoplus_{j\in N} g_j$.

A matrix A is called *strictly visualized* if it is visualized and

$$a_{ij} = \lambda(A) \quad \text{if and only if } (i,j) \in E_c(A).$$

THEOREM 2.27 (see [31], [34]). *If $A \in \overline{\mathbb{R}}^{n\times n}$ is irreducible, $A \neq \varepsilon, x = \sum_{j\in N} \alpha_j g_j$ (conventional linear combination), $\alpha_1,\ldots,\alpha_n > 0$, and $X = \mathrm{diag}(x)$, then $X^{-1}AX$ is strictly visualized.*

It follows from this theorem that the identification of the critical digraph can be done efficiently, in a polynomial number of steps.

Note that the strict visualization scaling for reducible matrices (not needed in this paper) is slightly more elaborate. The reader is referred to [34] and [6] for details.

Following Lemma 2.17 we will assume in some statements without loss of generality that the matrix is visualized.

2.5. Max-linear systems. One-sided max-linear systems $Ax = b$, where $A = (a_{ij}) \in \overline{\mathbb{R}}^{m\times n}$ and $b = (b_1,\ldots,b_m)^T \in \overline{\mathbb{R}}^m$, were historically probably the first problem studied in max-algebra [12]. We will only need the case when A has no ε columns and rows and $b \in \mathbb{R}^m$. The existence of a solution and unique solution in this case can easily be expressed in terms of set coverings and minimal set coverings. As before $N = \{1,\ldots,n\}$, and we now also denote $M = \{1,\ldots,m\}$,

$$S(A,b) = \left\{x \in \overline{\mathbb{R}}^n; Ax = b\right\},$$

and $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)^T$, where

$$\overline{x}_j = \left(\max_{i \in M} a_{ij} b_i^{-1} \right)^{-1}$$

for $j \in N$. Obviously, $\overline{x} \in \mathbb{R}^n$ and

$$\overline{x}_j = \min \left\{ b_i a_{ij}^{-1}; i \in M, a_{ij} \in \mathbb{R} \right\}$$

for $j \in N$. We will also denote

$$M_j = \left\{ i \in M; \overline{x}_j = b_i a_{ij}^{-1} \right\}$$

for $j \in N$.

THEOREM 2.28 (see [12], [6]). *If $A \in \overline{\mathbb{R}}^{m \times n}$ is a matrix with no ε columns and rows and $b \in \mathbb{R}^m$, then the following three statements are equivalent:*

1. $S(A, b) \neq \emptyset$,
2. $\overline{x} \in S(A, b)$,
3. $\bigcup_{j \in N} M_j = M$.

The set of all solutions can also be described in combinatorial terms.

THEOREM 2.29 (see [13], [39], [6]). *If $A \in \overline{\mathbb{R}}^{m \times n}$ is a matrix with no ε columns and rows and $b \in \mathbb{R}^m$, then $x \in S(A, b)$ if and only if $x \leq \overline{x}$ and*

$$\bigcup_{j : x_j = \overline{x}_j} M_j = M. \tag{14}$$

The combinatorial aspect of max-linear systems will become even more visible in the following criterion for unique solvability.

THEOREM 2.30 (see [6]). *If $A \in \overline{\mathbb{R}}^{m \times n}$ is a matrix with no ε columns and rows and $b \in \mathbb{R}^m$, then $S(A, b) = \{\overline{x}\}$ if and only if*

1. $\bigcup_{j \in N} M_j = M$ and
2. $\bigcup_{j \in N'} M_j \neq M$ for any $N' \subseteq N, N' \neq N$.

Note that finding minimal vectors in the solution set to $Ax = b$ reduces to the hypergraph transversal problem [26]. Using this observation efficient algorithms for finding all minimal solutions to $Ax = b$ have recently been published [17].

3. Attraction sets and weakly stable matrices.

Given $A \in \overline{\mathbb{R}}^{n \times n}$ and $x \in \overline{\mathbb{R}}^n$ the sequence of vectors $Ax, A^2x, A^3x, \ldots$ is called the *orbit* (*of A with starting vector x*).

For $A \in \overline{\mathbb{R}}^{n \times n}$ we denote by attr (A) the set of all starting vectors from which the orbit reaches an eigenvector, that is,

$$\operatorname{attr}(A) = \left\{ x \in \overline{\mathbb{R}}^n; (\exists k \geq 0)\, A^k x \in V(A) \right\}.$$

The set attr(A) is called the *attraction set of A*. We also denote for $\lambda \in \overline{\mathbb{R}}$

$$\operatorname{attr}(A, \lambda) = \left\{ x \in \overline{\mathbb{R}}^n; (\exists k \geq 0)\, A^k x \in V(A, \lambda) \right\}.$$

For stability it is important that once it is reached it can be guaranteed in subsequent stages, that is, $x \in V(A) \Longrightarrow Ax \in V(A)$ and hence also $x \in V(A) \Longrightarrow A^k x \in$

$V(A)$ for every $k \geq 0$. However, this is not true in general for eigenvectors associated with $\lambda = \varepsilon$: For instance, if

$$A = \begin{pmatrix} \varepsilon & \varepsilon \\ 0 & \varepsilon \end{pmatrix}$$

and $x = (0, \varepsilon)^T$, then $Ax = (\varepsilon, 0)^T \in V(A, \varepsilon)$ but $A^2 x = (\varepsilon, \varepsilon)^T \notin V(A, \varepsilon)$. Therefore we will assume everywhere that A has only finite eigenvalues. Equivalently, A has no ε columns.

On the other hand if $\lambda > \varepsilon$, then $x \in V(A, \lambda)$ implies $Ax \in V(A, \lambda)$ since $A(Ax) = A(\lambda x) = \lambda Ax$ and $Ax \neq \varepsilon$ since $Ax = \lambda x$ and $x \neq \varepsilon$. Hence if A has no ε columns we have

$$V(A) \subseteq \operatorname{attr}(A) \subseteq \overline{\mathbb{R}}^n - \{\varepsilon\}.$$

Following the definitions introduced in Q2–Q3 of section 1 we have that A is strongly stable (*robust* in [8]) if $\operatorname{attr}(A) = \overline{\mathbb{R}}^n - \{\varepsilon\}$ and it is weakly stable if $\operatorname{attr}(A) = V(A)$. Note that in the definition of attraction sets the exponent k starts from 0 and thus the 1×1 matrix $A = (\varepsilon)$ is both strongly and weakly stable since $V(A) = \overline{\mathbb{R}}^n - \{\varepsilon\} = \operatorname{attr}(A)$.

A remarkable feature of irreducible matrices is that although orbits with matrix A do not necessarily reach an eigenvector of A, they are guaranteed to reach an eigenvector of some power of A.

COROLLARY 3.1. *If $A \in \overline{\mathbb{R}}^{n\times n}$ is irreducible and $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$, then $A^k x \in V(A^{\sigma(A)})$ for some natural number k.*

Proof. By the cyclicity theorem we have $A^{k+\sigma(A)}x = (\lambda(A))^{\sigma(A)} A^k x$, which can also be written as

$$A^{\sigma(A)} \left(A^k x\right) = \left(\lambda\left(A\right)\right)^{\sigma(A)} \left(A^k x\right).$$

It remains to add that $A^k x \neq \varepsilon$ by Lemma 2.2 and $\lambda(A^r) = (\lambda(A))^r$ for any natural r (see [6], Theorem 4.5.10). ∎

The basic reachability question Q1 (see section 1) is solved by an $O(n^3 \log n)$ algorithm presented in [32]. In fact that algorithm finds the smallest value of s for which a given orbit reaches $V(A^s)$ by converting this question to finding the exact period of a periodic number sequence.

The aim of this paper is to characterize matrices whose attraction set coincides with the set of eigenvectors, that is, the weakly stable matrices. Note that a brief overview of the known characterization of strongly stable matrices [8], [6] is given in section 6.

Suppose that A is weakly stable. Let P be a generalized permutation matrix, B be the matrix $P^{-1}AP$, and suppose that $B^k z \in V(B)$. Then for $x = Pz$ we have

$$\begin{aligned} \left(P^{-1}AP\right)^k z &= \left(P^{-1}A^k P\right) P^{-1}x \\ &= P^{-1}A^k x \in V\left(B\right). \end{aligned}$$

Then by Lemma 2.17 $A^k x \in V(A)$. Hence $x \in V(A)$, and by the same lemma $P^{-1}x \in V(B)$. Thus $z \in V(B)$ and B is weakly stable. We have proved the following.

LEMMA 3.2. *If A and B are square matrices, $A \sim B$, then A is weakly stable if and only if B is weakly stable.*

At the same time $V(\alpha A) = V(A)$ for $\alpha \in \mathbb{R}$ by Lemma 2.17 and $(\alpha A)^k = \alpha^k A^k$ and so we deduce the following.

LEMMA 3.3. *If A and B are square matrices, $B = \alpha A$, then A is weakly stable if and only if B is weakly stable.*

When we investigate weak stability, due to the last two lemmas we may if necessary assume without loss of generality that the matrix under consideration is definite and visualized. Note that every 1×1 matrix A is weakly stable since then $V(A) = \mathbb{R}$.

LEMMA 3.4. *Let $A \in \overline{\mathbb{R}}^{n\times n}$. Then the following are equivalent:*

(a) *A is weakly stable.*

(b) *For any x, if $Ax \in V(A)$, then $x \in V(A)$.*

Proof. (a) $\Longrightarrow$ (b) Follows from the definition immediately.

(b) $\Longrightarrow$ (a) If $A^k x \in V(A)$, then $A^{k-1}x \in V(A), \ldots, x \in V(A)$, hence A is weakly stable. □

Recall that the set of subeigenvectors of A associated with $\lambda \in \overline{\mathbb{R}}$ is

$$V_*(A, \lambda) = \left\{ x \in \overline{\mathbb{R}}^n ; Ax \leq \lambda x, x \neq \varepsilon \right\}.$$

It will be helpful also to work with the set of *supereigenvectors* of A associated with $\lambda \in \overline{\mathbb{R}}$:

$$V^*(A, \lambda) = \left\{ x \in \overline{\mathbb{R}}^n ; Ax \geq \lambda x, x \neq \varepsilon \right\}.$$

LEMMA 3.5. $V(A, \lambda(A)) \subseteq V^*(A, \lambda(A)) \subseteq \operatorname{attr}(A, \lambda(A))$ *for every* $A \in \overline{\mathbb{R}}^{n\times n}$ *with* $\lambda(A) > \varepsilon$.

Proof. The first inclusion is trivial. For the second suppose that $Ax \geq \lambda(A)x, x \neq \varepsilon$. Then

$$x \leq A_\lambda x \leq A_\lambda^2 x \leq \cdots \leq A_\lambda^n x \leq A_\lambda^{n+1} x \leq \cdots .$$

Since A_λ is definite we have $A_\lambda^+ x = A_\lambda^n x = A_\lambda^{n+1} x \neq \varepsilon$, hence $A_\lambda^n x \in V(A_\lambda, 0) = V(A, \lambda(A))$ by Lemma 2.17. Therefore $A^n x = (\lambda(A))^n A_\lambda^n x \in V(A, \lambda(A))$ and so $x \in \operatorname{attr}(A, \lambda(A))$. □

LEMMA 3.6. *If $A \in \overline{\mathbb{R}}^{n\times n}$ is weakly stable and $\lambda(A) > \varepsilon$, then*

$$V(A, \lambda(A)) = V^*(A, \lambda(A)) = \operatorname{attr}(A, \lambda(A)).$$

Proof. Suppose $V(A) = \operatorname{attr}(A)$. Due to Lemma 3.5 we only need to prove $\operatorname{attr}(A, \lambda(A)) \subseteq V(A, \lambda(A))$. Let $x \in \operatorname{attr}(A, \lambda(A))$. Hence $A^k x \in V(A, \lambda(A))$ for some k, that is,

$$A^{k+1}x = \lambda(A) A^k x \text{ and } A^k x \neq \varepsilon. \tag{15}$$

Since $\operatorname{attr}(A, \lambda(A)) \subseteq \operatorname{attr}(A) = V(A)$ we also have that

$$Ax = \mu x$$

for some $\mu \in \mathbb{R}$. This implies that

$$A^{k+1}x = \mu A^k x$$

which together with (15) yields $\mu = \lambda(A)$ and thus $x \in V(A, \lambda(A))$. □

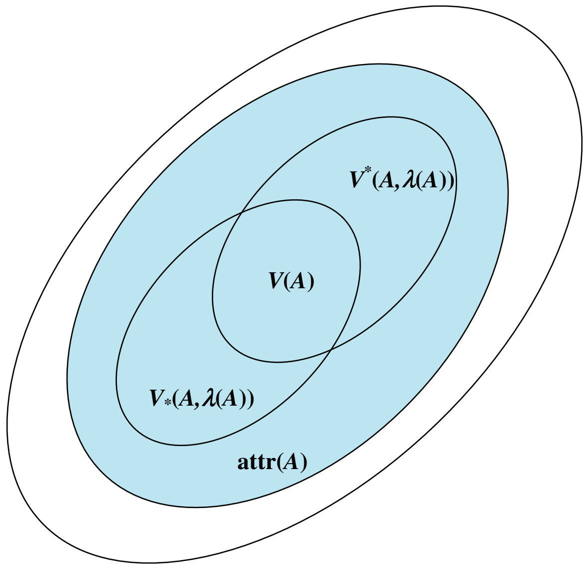

FIG. 1. *Sub- and supereigenvectors of irreducible matrices.*

4. Weakly stable matrices: The irreducible case. If A is irreducible, then $V(A) = V(A, \lambda(A))$ and so $\operatorname{attr}(A) = \operatorname{attr}(A, \lambda(A))$. Also, recall that A can only have finite subeigenvectors (Lemma 2.24), in particular $V_*(A, \lambda(A)) \subseteq \mathbb{R}^n$.

LEMMA 4.1. $V(A) \subseteq V_*(A, \lambda(A)) \subseteq \operatorname{attr}(A)$ *if* $A \in \overline{\mathbb{R}}^{n\times n}$ *is irreducible.*

Proof. The first inclusion is trivial. For the second suppose $Ax \leq \lambda(A)x, x \neq \varepsilon$, and then we have by the cyclicity theorem (for k big enough)

$$A^{p+k}x \leq \lambda(A)\, A^{p+k-1}x \leq \cdots \leq (\lambda(A))^p\, A^k x = A^{p+k}x.$$

Hence all inequalities are satisfied with equality and (using also Lemma 2.2) $A^{p+k-1}x \in V(A)$, thus $x \in \operatorname{attr}(A)$. ◻

The next statement follows from Lemmas 3.5 and 4.1.

COROLLARY 4.2. *If A is irreducible, then (see Figure 1)*

$$V(A) = V_*(A, \lambda(A)) \cap V^*(A, \lambda(A)) \subseteq V_*(A, \lambda(A)) \cup V^*(A, \lambda(A)) \subseteq \operatorname{attr}(A).$$

LEMMA 4.3. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be irreducible, $A \neq \varepsilon$. Then $V(A) = V_*(A, \lambda(A))$ if and only if every node in D_A is critical (that is, $N_c(A) = N$).*

Proof. Without loss of generality suppose that $A \neq \varepsilon$ and let A be definite.

Suppose that $V(A) = V_*(A, \lambda(A))$ and a node, say k, is not critical. Then $a^+_{kk} < 0 = a^*_{kk}$ and $AA^*_k \leq A^*_k \neq \varepsilon$ but

$$AA^*_k = A^+_k \neq A^*_k.$$

Hence $A^*_k \in V_*(A) - V(A)$.

If all nodes are critical, then $A^* = A^+$ and the statement follows from

$$V(A) = \left\{A^+z; z \in \mathbb{R}^n\right\},$$
$$V_*(A) = \{A^*z; z \in \mathbb{R}^n\}$$

due to Corollary 2.25 and Corollary 2.15, respectively. ◻

COROLLARY 4.4. *If $A \in \overline{\mathbb{R}}^{n\times n}$ is irreducible and weakly stable, then $N_c(A) = N$.*

LEMMA 4.5. *If $A \in \overline{\mathbb{R}}^{n\times n}$ is definite, $N_c(A) = N$, and $Ax \in V(A)$, then $Ax \geq x$.*

Proof. A^+ is increasing since $N_c(A) = N$. Also, $Ax \in V(A)$ implies $Ax = A^2x = A^3x = \cdots$ and hence by Lemma 2.9

$$x \leq A^+x = Ax \oplus A^2x \oplus A^3x \oplus \cdots = Ax. \qquad \square$$

LEMMA 4.6. *If $A \in \overline{\mathbb{R}}^{n\times n}$ is irreducible and weakly stable, then C_A is strongly connected.*

Proof. Suppose C_A is not strongly connected. Take any strongly connected component of C_A and renumber the nodes so that this component consists of the first, say k, nodes. By the corresponding simultaneous permutation of the rows and columns of A we obtain the matrix

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is $k \times k, k < n$, $\lambda(A_{11}) = \lambda(A)$ and $C_{A_{11}}$ is strongly connected. Let $z \in V(A_{11})$. Then

$$A \begin{pmatrix} z \\ \varepsilon \end{pmatrix} = \begin{pmatrix} A_{11}z \oplus A_{12}\varepsilon \\ A_{21}z \oplus A_{22}\varepsilon \end{pmatrix} = \begin{pmatrix} \lambda(A)\, z \\ A_{21}z \end{pmatrix} \geq \lambda(A) \begin{pmatrix} z \\ \varepsilon \end{pmatrix}.$$

Since $\binom{z}{\varepsilon} \notin V(A)$ (because $V(A)$ contains only finite vectors) and $\binom{z}{\varepsilon} \in V^*(A, \lambda(A))$, we have by Lemma 3.6 that A is not weakly stable. $\square$

A digraph with one node and no arc is called trivial, otherwise it is called nontrivial.

LEMMA 4.7. *If D is a nontrivial strongly connected digraph and D is not an elementary cycle, then there is a node in D with two or more incoming arcs. It is similar for outcoming arcs.*

Proof. If there is exactly one incoming arc at every node, then an elementary cycle can be constructed (following the arcs backwards) starting from any node. Since D is not an elementary cycle there is a node, say x, not on this cycle. But then the nodes on the cycle are not reachable from x as these nodes have no more than one incoming arc. This is a contradiction with the strong connectivity of D. $\square$

The next statement is a key result of this paper. In the proof $v(\alpha)$ denotes the vector whose every component is α.

THEOREM 4.8. *An irreducible matrix $A \in \overline{\mathbb{R}}^{n\times n}, A \neq \varepsilon$, is weakly stable if and only if its critical digraph is an elementary cycle containing all nodes of the associated digraph, that is, C_A is a Hamiltonian cycle in D_A.*

Proof. Due to Corollary 4.4 and Lemmas 3.4, 3.3, 3.2, and 4.6 it is sufficient to prove that if A is definite and visualized, $N_c(A) = N$, and C_A is strongly connected, then the following two statements are equivalent:

1. For any x, if $Ax \in V(A)$, then $x \in V(A)$.
2. C_A is an elementary cycle.

If C_A is strongly connected, $N_c(A) = N$, and A is visualized, $A \leq 0$, then all nodes in D_A are accessible from each other by a zero path. Hence the weights of heaviest paths between any pair of nodes is 0. Therefore $A^+ = 0$ and all eigenvectors are $v(\alpha), \alpha \in \mathbb{R}$.

Suppose that C_A is an elementary cycle, say σ. All arcs on σ have weight 0 since A is visualized. If any arc not on σ also had weight 0, then it would be on some critical cycle and consequently, C_A is not an elementary cycle. Hence A is actually strictly visualized, all column maxima are unique and of value 0, and column maxima in different columns are in different rows. Hence $Ax = v(\alpha)$ has a unique solution (Theorem 2.30), namely, $x = v(\alpha)$, yielding that $x \in V(A)$.

Suppose that C_A is not an elementary cycle. Since C_A is strongly connected by Lemma 4.7 there is a node with two or more incoming arcs, thus there is a column of A with two or more zeros, implying that the system of the sets of column maxima indices is a covering of the set of row indices but is not minimal. Hence there is a solution to $Ax = v(\alpha)$ whose components are $\leq \alpha$ with at least one equality and one strict inequality (Theorem 2.29). This solution is not in $V(A)$. $\square$

It follows from Theorem 2.27 that C_A can be found in a polynomial number of steps. It is then easy to check whether C_A is a Hamiltonian cycle in D_A. We can however formulate a problem in terms of weakly stable matrices equivalent to the Hamilton cycle problem, which thus embodies the hardness of the Hamilton cycle problem: Given a strongly connected digraph D, is it possible to assign the weights to the arcs of D so that the obtained weighted digraph is D_A for some weakly stable matrix A?

5. Weakly stable matrices: The reducible case. We will now suppose without loss of generality that $A \in \overline{\mathbb{R}}^{n\times n}$ is in the Frobenius normal form (10) and denote $R = \{1, \ldots, r\}$. Let $N_1, \ldots, N_r$ be the partition of N determined by the Frobenius normal form. Recall (see subsection 2.3) that $N_s, s \in R$ and in fact also the index s is called a class and that all initial classes are spectral (Corollary 2.19). A class $s \in R$ is called weakly stable if A_{ss} is weakly stable. As usual for $x \in \overline{\mathbb{R}}^n$ we denote $\mathrm{supp}(x) = \{j \in N; x_j \neq \varepsilon\}$.

If x is an eigenvector, then the set $T = \mathrm{supp}(x)$ is the union of certain classes, inaccessible from any $j \in N - T$ [6]. Clearly, $\lambda = \lambda(A[T])$. Any set $T = \mathrm{supp}(x)$ for some eigenvector x will be called spectral. (Note that every spectral set contains as a subset the union of a number of classes, including at least one spectral.) A set $T \subseteq N$ will be called weakly stable if $A[T]$ is weakly stable.

Lemma 5.1. *If $A \in \overline{\mathbb{R}}^{n\times n}$ is weakly stable, then every spectral set is weakly stable.*

Proof. Let $B = A[T]$, where $T = \mathrm{supp}(x)$ and x is an eigenvector. Suppose that $B^k u$ is an eigenvector of B. Since the indices in T are inaccessible from any $j \in N - T$, we can simultaneously permute the rows and columns of A so that we get

$$D = \begin{pmatrix} C & \varepsilon \\ \ldots & B \end{pmatrix}.$$

The corresponding permutation of x yields $y = \binom{\varepsilon}{u}$. We then have $D^k y = \binom{\varepsilon}{B^k u}$. The matrix D is weakly stable (Lemma 3.2). Also, $D^k y$ is an eigenvector of D since $B^k u$ is an eigenvector of B. Hence y is an eigenvector of D, and thus u is an eigenvector of B, and so B is weakly stable. $\square$

Lemma 5.2. *If $A \in \overline{\mathbb{R}}^{n\times n}$ is weakly stable, then every initial class is also weakly stable.*

Proof. Every initial class is spectral (Corollary 2.19) and so the statement immediately follows from Lemma 5.1. $\square$

Theorem 5.3. *Let $A \in \overline{\mathbb{R}}^{n\times n}$. Then A is weakly stable if and only if every spectral class is initial and weakly stable.*

Proof. In this proof we suppose without loss of generality (Lemma 3.2) that A is in the Frobenius normal form (10).

Suppose that A is weakly stable. Due to Lemma 5.2 for the first implication it is sufficient to prove that every spectral class is initial. To obtain a contradiction suppose that a class, say s, is spectral but not initial. Let λ be the associated eigenvalue and S be the set of all spectral classes from which s is accessible; thus $S - \{s\} \neq \emptyset$. Then there is an $x \in V(A,\lambda)$ such that $x[N_s] \neq \varepsilon$ and $x[N_i] = \varepsilon$ for $i \notin S$. By Lemma 2.20 every such eigenvector has $x[N_i]$ finite for all $i \in S$. Let us denote

$$T = \bigcup_{i \in S} N_i.$$

Clearly $\lambda = \lambda(A[T]) = \lambda(A[N_s])$,

$$A\,[T]\,x\,[T] = \lambda x\,[T]\,,$$

and T is weakly stable by Lemma 5.1. The matrix $A[N_s] = A_{ss}$ is irreducible. Let us take any vector $z \in \overline{\mathbb{R}}^n$ such that $z[N_s] \in V(A[N_s]) \subseteq \mathbb{R}^n$ and $z_i = \varepsilon$ for $i \notin N_s$. Then $A[N_s]z[N_s] = \lambda z[N_s]$, yielding $A[T]z(T) \geq \lambda z(T)$ and consequently, $z \in V^*(A[T], \lambda(A[T]))$. However, by Lemma 2.20 $z \notin V(A[T], \lambda(A[T]))$ since $z[N_s]$ is finite and $z_i = \varepsilon$ for at least one index $i \in T$ because $S - \{s\} \neq \emptyset$. This contradicts the weak stability of $A[T]$ since by Lemma 3.6 $V^*(A[T], \lambda(A[T])) = V(A[T], \lambda(A[T]))$ if $A[T]$ is weakly stable.

Suppose now that each spectral class of A is initial and weakly stable. If $Ax \in V(A,\lambda)$ for some λ, then $(Ax)[N_i]$ is finite for $i \in S \subseteq R$, where S is a set of spectral classes with eigenvalue λ. Also $(Ax)[N_i] = \varepsilon$ for $i \notin S$ since all spectral classes are initial. Then by Lemma 2.2 $x[N_i] = \varepsilon$ for $i \notin S$. Hence $(Ax)[N_i] = A_{ii}x[N_i] \in V(A_{ii}, \lambda)$ for all $i \in S$. Since A_{ii} is weakly stable $x[N_i] \in V(A_{ii}, \lambda)$ for every $i \in S$ and thus $x \in V(A,\lambda)$. Using Lemma 3.4 we conclude that A is weakly stable. □

COROLLARY 5.4. $A \in \overline{\mathbb{R}}^{n\times n}$ *is weakly stable if and only if every initial class is weakly stable and the (unique) eigenvalue of every noninitial class is strictly less than the eigenvalue of any initial class from which it is accessible. See Figure* 2.

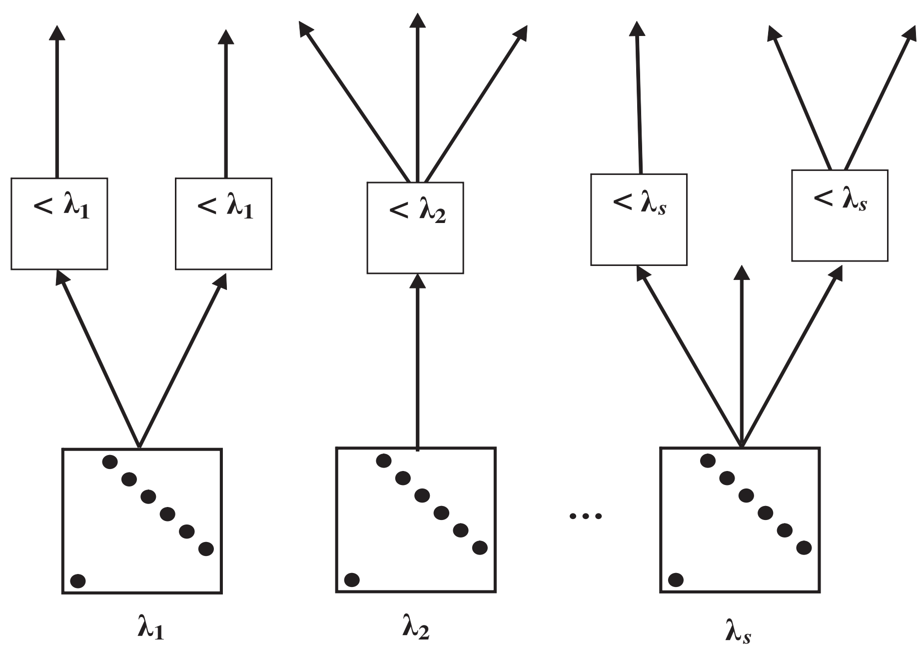

FIG. 2. *Condensation digraph of a weakly stable matrix.*

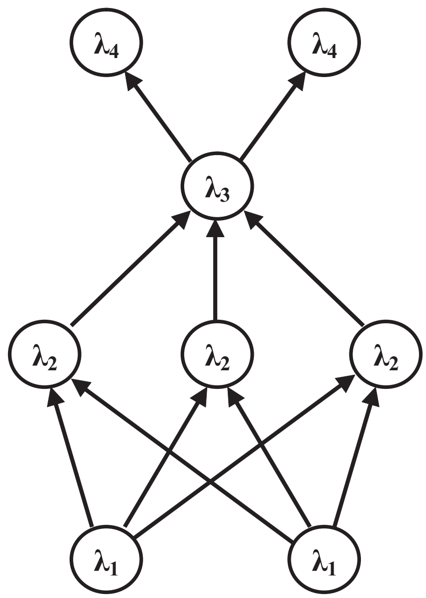

FIG. 3. *Condensation digraph of a strongly stable matrix.*

6. Strongly stable matrices. In order to provide a balanced picture on reachability in this section we present the main results on strongly stable matrices [8], [6], that is, the matrices for which $\mathrm{attr}(A) = \overline{\mathbb{R}}^n - \{\varepsilon\}$.

LEMMA 6.1. *If $A \sim B$, then A is strongly stable if and only if B is strongly stable.*

Due to this lemma we may without loss of generality investigate strong stability of the matrix arising from a given matrix by a simultaneous permutation of the rows and columns, that is, in a Frobenius normal form.

THEOREM 6.2 (see [7]). *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be irreducible. Then A is strongly stable if and only if A is primitive.*

A Frobenius normal form class is called trivial (primitive) if the corresponding diagonal block is the 1×1 matrix (ε) (primitive).

THEOREM 6.3. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be a matrix with no ε columns and in the Frobenius normal form* (10) *with classes $N_1, \ldots, N_r$ and $R = \{1, \ldots, r\}$. Then A is strongly stable if and only if the following hold:*

1. *All nontrivial classes $N_1, \ldots, N_r$ are spectral and primitive.*
2. *For any $i, j \in R$, if N_i, N_j are nontrivial mutually inaccessible classes, then $\lambda(N_i) = \lambda(N_j)$.*

A typical condensation digraph of a strongly stable matrix can be seen in Figure 3, where $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$. In general it consists of layers of classes. Classes in each layer have the same eigenvalues and these values are strictly increasing with layers. Also, any two consecutive layers form a complete bipartite digraph with all arcs directed from the layer with smaller eigenvalues to that with greater eigenvalues.

Example 6.4 (1.1 continued). Production matrix is

$$A = \begin{pmatrix} 2 & 5 & \varepsilon \\ 3 & 3 & 2 \\ 3 & 5 & 1 \end{pmatrix}.$$

It is easily seen that $\lambda(A) = 4$, that is, the duration of every stage at the periodic regime (if reached) is 4; there is only one critical cycle, namely, $(1,2,1)$ which is non-Hamiltonian, hence A is not weakly stable; $\sigma(A) = 2$ and so A is also not strongly stable; the dimension of the eigenspace is $d(A) = 1$, $A_\lambda^+ = \left(\begin{smallmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 0 & 1 & -1 \end{smallmatrix}\right)$, and all eigenvectors are multiples of $(1,0,1)^T$; orbits of A will reach the eigenspace with some but not all starting vectors outside $V(A)$ (for instance, $A \otimes x \in V(A)$ if $x = (1,0,0)^T$ but not with $x = (0,0,0)^T$).

Example 6.5. Let us examine some other matrices. If the production matrix is

$$B = \begin{pmatrix} 2 & 5 & \varepsilon \\ 2 & 3 & 4 \\ 3 & 2 & 1 \end{pmatrix},$$

then $\lambda(B) = 4$ and there is only one critical cycle, namely, $(1,2,3,1)$ which is Hamiltonian, and hence B is weakly stable; $\sigma(B) = 3$; $d(B) = 1$, $B_\lambda^+ = \left(\begin{smallmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{smallmatrix}\right)$, and all eigenvectors are multiples of $(1,0,0)^T$; orbits of B will not reach the eigenspace with any starting vector outside $V(B)$.

If the production matrix is

$$C = \begin{pmatrix} 2 & 5 & \varepsilon \\ 3 & 3 & 2 \\ 3 & 6 & 1 \end{pmatrix},$$

then $\lambda(C) = 4$, critical cycles are $(1,2,1)$ and $(2,3,2)$, $\sigma(C) = 2$, and hence C is neither weakly nor strongly stable; $d(C) = 1$, $C_\lambda^+ = \left(\begin{smallmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 1 & 2 & 0 \end{smallmatrix}\right)$, and all eigenvectors are multiples of $(1,0,2)^T$; orbits of C will reach the eigenspace with some but not all starting vectors outside $V(C)$ (for instance, $C \otimes x \in V(C)$ if $x = (1,0,0)^T$ but not with $x = (0,0,0)^T$).

If the production matrix is

$$D = \begin{pmatrix} 2 & 5 & \varepsilon \\ 2 & 3 & 4 \\ 3 & 2 & 4 \end{pmatrix},$$

then $\lambda(D) = 4$, critical cycles are $(1,2,3,1)$ and $(3,3)$, $\sigma(D) = 1$, and hence D is strongly stable; $d(D) = 1$, $D_\lambda^+ = \left(\begin{smallmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{smallmatrix}\right)$, and all eigenvectors are multiples of $(1,0,0)^T$; orbits of A will reach the eigenspace with every nontrivial starting vector.

If the production matrix is

$$E = \begin{pmatrix} 2 & 5 & \varepsilon \\ 3 & 3 & 2 \\ 3 & 5 & 4 \end{pmatrix},$$

then $\lambda(E) = 4$, critical cycles are $(1,2,1)$ and $(3,3)$, $\sigma(E) = 2$, and hence E is neither weakly nor strongly stable; $d(E) = 2$, $E_\lambda^+ = \left(\begin{smallmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 0 & 1 & 0 \end{smallmatrix}\right)$, and all eigenvectors are max-combinations of $(1,0,1)^T$ and $(1,0,2)^T$; orbits of E will reach the eigenspace with some but not all starting vectors outside $V(E)$ (for instance, $E \otimes x \in V(E)$ if $x = (1,0,0)^T$ but not with $x = (0,0,0)^T$).

7. Conclusions and further research. This paper fully characterizes weakly stable matrices, that is, matrices whose orbit never reaches an eigenvector unless it starts in one. The characterization enables us to check that a matrix is weakly stable in a polynomial number of steps. As a by-product the paper offers a new view of eigenspaces as intersections of super- and subeigenspaces, which proves to be a useful tool when considering periodic properties of matrices.

In general, the set of vectors x such that $A^k x$ is an eigenvector for a fixed k includes all eigenspaces, and the sequence of such sets grows with k, stabilizing at some stage if A is irreducible due to the cyclicity theorem. It may be interesting to ask for a combinatorial criterion characterizing sequences that would stabilize in no more than m steps (m-weakly stable matrices). The hypergraph transversal problem [17], [26] might be useful in answering this question.

Attraction sets are discussed in [6] and [33] in a more general setting where the orbit is required to have an ultimate period not exceeding a given number. These sets can be characterized as solution sets of max-algebraic two-sided systems. The concept of an attraction set was inspired by earlier works [5] and [25]. The concept in [25] is more general, asking for geometrical description and classification of all possible limit cycles and their attraction sets. These areas are still unexplored.

Supereigenspaces, in contrast to subeigenspaces, seem to be much harder to describe and to our knowledge no such description has been found yet.

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