GROUPS OF EVEN TYPE WHICH ARE NOT OF EVEN CHARACTERISTIC, II

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1. Introduction

In this second part of the paper we continue the investigation started in the first part and finish the classification of the groups of even type, which are not of even characteristic. More precisely we prove:

Theorem 1.1. Let $G$ be a simple $K_2$-group of even type. Then either $G$ is of even characteristic or $G \cong J_1$, $Co_3$, $M(23)$, $A_{12}$, $\Omega_7(3)$ or $\Omega^-(8)$. 

Let us recall the notation used in the statement of the theorem.

Definition 1.2. A group $G$ is said to be of even type if the following hold:

(i) $\mathcal{L} \subseteq \mathcal{C}_2$, where $\mathcal{L}$ is the set of all components of $C_G(x)$ for all involutions $x \in G$.
(ii) $O(C_G(x)) = 1$ for every involution $x \in G$.
(iii) $G$ has 2-rank at least 3.

Here we denote by $\mathcal{C}_2$ the following set of components of $G$:

Definition 1.3. [GoLyS1, Definition (12.1)(1)] The set $\mathcal{C}_2$ consists of simple and quasisimple groups.

- The simple groups in $\mathcal{C}_2$ are $K \in \text{Chev}(2)$, $L_2(9)$, $L_2(p)$, $p$ a Fermat or Mersenne prime, $L_3(3)$, $L_4(3)$, $U_4(3)$, $G_2(3)$, $M_{11}$, $M_{12}$, $M_{22}$, $M_{23}$, $M_{24}$, $J_2$, $J_3$, $J_4$, $HiS$, $Suz$, $Ru$, $Co_1$, $Co_2$, $M(22)$, $M(23)$, $M(24)'$, $Th$, $F_2$, $F_1$.
- The groups $K \in \mathcal{C}_2$ with $K$ not simple are those for which $K/O_2(K)$ is a simple group in $\mathcal{C}_2$. But the following quasisimple groups are deleted, i.e. are not in $\mathcal{C}_2$: $SL_2(q)$, $q$ odd, $2A_8$, $SL_3(3)$, $SU_3(3)$, $Sp_4(3)$, and $[X]L_3(4)$, with $X \cong \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$.

Date: February 9, 2015.
Furthermore we call a group $G$ a $K_2$–group if any simple factor of any nontrivial 2–local subgroup of $G$ is either cyclic, a group of Lie type, an alternating group or one of the 26 sporadic groups.

We now define even characteristic.

**Definition 1.4.** A group $G$ is said to be of even characteristic, if for a Sylow 2-subgroup $S$ and all nontrivial 2-local subgroups $H$ of $G$ with $S \leq H$, we have that $C_G(O_2(H)) \subseteq O_2(H)$.

The main result of the first part of this paper was:

**Theorem 1.5.** Let $G$ be a simple $K_2$–group of even type. Then one of the following holds

- $G$ is of even characteristic; or
- $G \cong \Omega_7(3)$, $\Omega^-_7(3)$ or $A_{12}$; or
- There is a 2-central involution $z$ such that $C_G(z)$ possesses a standard subgroup $L$. Furthermore $C_G(L)$ is cyclic.

In this second part of the paper we start with the statement of Theorem 1.5. We assume that there is some 2-central involution $z \in G$ such that $C_G(z)$ possesses a standard subgroup $A_z$. Furthermore we assume that $G$ is not isomorphic to $J_1$, $Co_3$ or $M(23)$. We then first show that $Z(A_z) = 1$ and then that $A_z$ is a group of Lie type in characteristic two or is one out of a small list of sporadic groups (Proposition 5.1 and Proposition 5.2). For this we use some classifications of groups by standard subgroups. At this point our analysis moves away from $C_G(z)$ and we construct in Lemma 6.4 and Lemma 6.5 a subgroup $N$ of $G$ such that $N$ and $N_G(A_z)$ share a Sylow 2-subgroup $S$ of $G$, $C_N(O_2(N)) \leq O_2(N)$ and $N \not\leq N_G(A_z)$. By choosing $N$ minimal with these properties we achieve that $N$ is a minimal parabolic subgroup in the sense that we now describe.

We call a subgroup $P$ of a group $X$ a parabolic (subgroup) of $X$ if $1 \neq |X : P|$ is odd. A maximal parabolic is a parabolic which is maximal in the set of parabolics. In contrast a minimal parabolic $P$ is a parabolic which is not 2-closed such that there is exactly one class of maximal subgroups $M$ of $P$ such that $|P : M|$ is odd.

Now using the action of $O_2(C_{A_z}(x))$ on $\Omega_1(Z(O_2(N)))$ for some 2-central involution $x \neq z$ in $A_z$, we get results about the action of the group $N/C_N(\Omega_1(Z(O_2(N))))$ on $\Omega_1(Z(O_2(N)))$. Using this action we eventually are able to prove that there is some involution $t$ which is central in $N$ (Lemma 6.11, Lemma 6.12). This is the key for the final
contradiction. We are able to prove some similarity between $z$ and $t$. In particular in Lemma 6.16 we show that $C_G(t)$ also has a standard subgroup $A_t$ isomorphic to $A_z$, but $t \not\sim z$. Then we show that the group $N$ constructed above corresponds to a minimal parabolic in $A_z$ and $A_t$ as well. This at the end shows that $A_z = A_t$ is centralized by a unique involution, which would give $z = t$, the final contradiction. Hence for all 2-central involutions $z$ we have that $F^*(C_G(z)) = O_2(C_G(z))$. The theorem then follows from the following fact ([MaStr, Lemma 2.1]): Let $G$ be a group and $S$ be a Sylow 2-subgroup. Then $G$ is of even characteristic if and only if $C_G(O_2(C_G(x))) \leq O_2(C_G(x))$ for all involutions $x \in Z(S)$.

2. Preliminaries

In this chapter we collect some results in group theory of general nature and some properties of the groups involved in the proof of the main theorem. For convenience of the reader we will also state some of the preliminary lemmas from the first part, which are used quite frequently in this second part.

**Lemma 2.1.** [Glau] Let $G$ be a nonabelian simple group, $z$ an involution and $z \in S \in \text{Syl}_2(G)$. Then $z^G \cap S \neq \{z\}$.

**Lemma 2.2.** (Thompson transfer) [GoLyS2, Lemma 15.16]. Let $G$ be a group, $S \in \text{Syl}_2(G)$, $T \triangleleft S$ with $S = TA$, $A \cap T = 1$, $A$ cyclic. If $G$ has no subgroup of index two and $u$ is the involution in $A$, then there is some $g \in G$ with $u^g \in T$ and $C_S(u^g) \in \text{Syl}_2(C_G(u^g))$. In particular $|C_S(u)| \leq |C_S(u^g)|$.

**Lemma 2.3.** [GoLyS2, Lemma 24.1] Let $R$ be a $p$-group, $p$ odd, and $E$ be an elementary abelian 2-group, acting faithfully on $R$. Then there is a subgroup $U$ in $RE$, such that $U$ is a direct product of dihedral groups of order $2p$ and $E$ is a Sylow 2-subgroup of $U$.

**Lemma 2.4.** Let $Q$ be an extraspecial subgroup of a group $G$, which is normalized by some element $t \in G$. If $|Q : C_Q(t)| = 2$, then either $t \in QC_G(Q)$ or $[t, Q]$ is cyclic.

**Proof.** Assume $t \notin QC_G(Q)$. Let $[t, Q] = \langle s, Z(Q) \rangle$ be elementary abelian. In particular $Q \not\cong Q_s$ and so $Q$ is generated by involutions. Let $s_1$ be some involution in $Q \setminus C_Q(s)$. Assume $t^{s_1} = ts$. Then $t = t^{s_1} = tss^{s_1}$ and so $[s, s_1] = 1$, a contradiction. So $t$ centralizes modulo $Z(Q)$ any involution in $Q \setminus C_Q(s)$. As $Q$ is generated by such involutions, together with $s$, we get that $[Q, t] \leq Z(Q)$. Then $t$ induces an inner automorphism and so $t \in QC_G(Q)$, a contradiction. \hfill $\Box$
Lemma 2.5. Let \( G \cong L_2(p) \), \( p = 2^n \pm 1 > 5 \) a prime, \( A_6 \), \( L_3(3) \) or \( M_{11} \). Then a Sylow 2-subgroup of \( G \) is dihedral of order at least \( 8 \) or semidihedral of order \( 16 \).

Proof. This is [GoLyS3, Lemma 4.10.5] and [GoLyS3, Table 5.3a] for \( M_{11} \). \( \square \)

Lemma 2.6. [MaStr, Lemma 2.19] Let \( L = L_4(3) \), \( U_4(3) \) or \( 2U_4(3) \). Then the following holds:

(i) If \( z \in L \setminus Z(L) \) is a 2-central involution, then \( O_2(C_L(z)) \cong Q_8 \times Q_8 \) or \( O_2(C_L(z)) \cong \mathbb{Z}_2 \times Q_8 \times Q_8 \) in case of \( L \cong 2U_4(3) \). In all cases \( O_2(C_L(z)/O_2(C_L(z))) \) is elementary abelian of order 9 and \( C_L(z)/O_2(C_L(z)) \) acts faithfully on \( O_2(C_L(z)) \).

(ii) \( \text{Out}(U_4(3)) \cong D_8 \) and \( \text{Out}(L_4(3)) \) is elementary abelian of order 4.

(iii) If \( G \cong \text{Aut}(L) \), \( L \cong U_4(3) \) and \( x \) is an involution in \( G \) such that \( 2^6 \cdot 3^2 \) divides \( |C_L(x)| \) then one of the following holds:

\( (\alpha) \) \( x \) is contained in \( L \) and 2-central,

\( (\beta) \) \( C_L(x) \cong \text{PSp}_4(3) \), or

\( (\gamma) \) \( O_2(C_L(x)) \) is elementary abelian and \( |C_L(x)/O_2(C_L(x))| = 36 \) and \( C_L(x)/O_2(C_L(x)) \) acts faithfully on \( O_2(C_L(x)) \).

(iv) Let \( L \cong L_4(3) \) or \( U_4(3) \). Then \( |Z(T)| = 2 \) for \( T \) a Sylow 2-subgroup of \( L \). Let \( G \) be a subgroup of \( \text{Aut}(L) \) containing \( L \) and \( T_1 \) be a Sylow 2-subgroup of \( G \). If \( |\Omega_1(Z(T_1))| > 2 \), then \( L \cong L_4(3) \) and \( |G : L| = 2 \). Furthermore some element \( t \in \Omega_1(Z(T_1)) \) centralizes \( \text{PSp}_4(3) : 2 \) in \( L \).

Lemma 2.7. Let \( G \cong G_2(2)' \), \( G_2(3) \) or \( M_{22} \). Then \( G \) has exactly one conjugacy class or involutions with representative \( t \) and we have:

(i) \( O^2(C_G(t)) \cong SL_2(3) \) for \( G \cong G_2(2)' \);

(ii) \( O^2(C_G(t)) \cong SL_2(3) \times SL_2(3) \) for \( G \cong G_2(3) \) and

(iii) \( O^2(C_G(t)) \cong 2^{1+4}\mathbb{Z}_3 \) for \( G \cong M_{22} \).

(iv) If \( i \) is an outer automorphism of \( G \), then \( C_G(i) \cong SL_2(3) \) in case of \( G \cong G_2(2)' \) and \( L_2(8) : 3 \) in case of \( G_2(3) \).

(v) If \( i \) is an outer automorphism of \( G = M_{22} \), then \( C_G(i) \cong 2^3L_3(2) \) or \( 2^4F_{20} \).

Proof. As \( G_2(2)' \cong U_3(3) \), we get (i), (ii) and (iv) from [GoLyS3, Table 4.5.1]. The assertions (iii) and (v) follow from [GoLyS3, Table 5.3c]. \( \square \)

Lemma 2.8. Let \( G = M_{12} \). Then the following holds:

(i) \( G \) possesses two conjugacy classes of involutions with representatives \( t \) and \( u \).

(ii) \( O^2(C_G(t)) \cong 2^{1+4}\mathbb{Z}_3 \), \( C_G(u) \cong \mathbb{Z}_2 \times \Sigma_5 \).
(iii) $E(C_G(u))$ contains conjugates of $t$.
(iv) If $i$ is an outer automorphism of $G$, then $C_G(i) \cong \mathbb{Z}_2 \times A_5$.

Proof. (i), (ii), (iv) follow from [GoLyS3, Table 5.3b]. To prove (iii) let $T$ be a Sylow 2-subgroup of $C_G(u)$ and $T_1 \leq G$ with $|T_1 : T| = 2$. As $T' \leq E(C_G(u))$, we get that $Z(T_1) \cap E(C_G(u)) \neq 1$ and so $E(C_G(u))$ contains a 2-central involution.

Lemma 2.9. Let $G = 2F_4(2)$ and $i$ be an involution of $G$ which is not 2-central. Then $C_G(i)$ is of order $2^{10} \cdot 3$. If $T$ is a Sylow 2-subgroup of $C_G(i)$, then $|\Omega_1(Z(T))| = 4$.

Proof. By [Shi, Corollary 2] we just have two classes of involutions in $G$ and so $i$ is uniquely determined. By [Shi, Theorem 2.1] we see that $|C_{F_4(2)}(i)| = 2^{20} \cdot 3^2$ and so $|C_G(i)| = 2^{10} \cdot 3$. By the Borel-Tits-Theorem [MaStr, Lemma 2.13] we have that $C_G(i)$ is contained in the parabolic $P_1$ of $G$, with $P_1/O_2(P_1) \cong \Sigma_3$. Application of [MaStr, Lemma 2.31] shows that $i \in Z_3(S)$, where $S$ is a Sylow 2-subgroup of $P_1$ and so $|C_{O_2(P_1)}(i)| = 2^9$. Furthermore $Z_3(S) = Z(O_2(C_G(i)))$. As by [MaStr, Lemma 2.31] $C_G(i)$ induces $\Sigma_3$ on $Z_3(S)$, we see that $|\Omega_1(Z(T))| = 4$.

Lemma 2.10. Let $K \in C_2$ be a sporadic simple group and $N$ be a subgroup of $K$, $N \cong L_2(p)$, $p$ a Fermat or Mersenne prime, $p > 5$, $L_2(9)$, $L_3(3)$ or $L_4(3)$. Suppose that for a Sylow 2-subgroup $S$ of $K$ we have $S \leq M < K$ such that $F^*(M) = N$, then $N \cong L_2(9)$ and $K \cong M_{11}$.

Proof. If $N \cong L_2(p)$, then as $M$ is an automorphism group of $N$, we have that $S$ is dihedral. But there is no such sporadic group. Let $N \cong L_4(3)$. Then Lemma 2.6 implies $2^6 \leq |S| \leq 2^8$. Furthermore $3^6$ divides the order of $K$. Inspection of the list in [GoLyS3, Table 5.3] gives a contradiction. So we have $N \cong L_2(9)$ or $L_3(3)$ and then $|S| \leq 2^5$. As $K \in C_2$, we see $K \cong M_{11}$. As 13 does not divide $|M_{11}|$, we get $N \cong L_2(9)$.

Lemma 2.11. Let $F^*(G) \cong M(22)$ and $t \in F^*(G)$ be a 2-central involution. Set $Q_t = O_2(C_{F^*(G)}(t))$. Then $C_G(Q_t) = Z(Q_t)$. Furthermore $O_2(C_G(Z(Q_t))) = Q_t$.

Proof. This follows from [GoLyS3, Table 5.3t].

Lemma 2.12. Let $G \cong M_{11}$, $M_{23}$, $J_3$, $Th$, $Ru$, $M_{24}$, $J_4$, $Co_1$, $Co_2$, $F_2$ or $F_1$, then $G = \text{Aut}(G)$.

Proof. This can be found in [GoLyS3, Table 5.3].
Lemma 2.13. If $G \cong L_2(p)$, $p$ a Fermat or Mersenne prime, $p \neq 5$, $G \cong A_6$, $L_3(3)$ or $L_4(3)$ and $T$ is a Sylow 2-subgroup of $G$, then $|\Omega_1(Z(T))| = 2$.

Proof. For $G \cong L_4(3)$ this follows by Lemma 2.6. For the remaining groups it follows by Lemma 2.5

Lemma 2.14. Suppose that either $G \cong J_2$ or $G \cong M(24)'$. Let $S$ be a Sylow 2-subgroup of $G$. Then $N_G(Z_2(S))$ induces $\Sigma_3$ on $Z_2(S)$.

Proof. For $G \cong J_2$ the statement can be found in [GoLyS3, Table 5.3g]. So we assume $G \cong M(24)'$. Then by [Asch, chapter 19] there is a 2-local subgroup $P \cong 2^{11}M_{24}$ of $G$, where $O_2(P)$ is the irreducible part of the Todd-module. We may assume that $S \leq P$. Let $r$ be a 2-central involution in $S$, then by [GoLyS3, Table 5.3v] $C_G(r) \cong 2^{1+12}3U_4(3) : 2$. In particular by [Asch, (19.10)] we have that $C_P(r) \cong 2^{11}2^{6}3\Sigma_6$. According to [GoLyS3, Table 5.3e] there is some parabolic $P_1$ of $P$ containing $S$ with $P_1 \cong 2^{11}2^{1+6}L_4(2)$. Hence there is some minimal parabolic $P_2 \leq P_1$ such that $P_2/O_2(P_2) \cong \Sigma_3$ and $P_2 \not\cong C_G(r)$. Now $|\Omega_1(Z(O_2(P_2)))| = 4$, as $|\Omega_1(Z(S))| = 2$ by [MaStr, Lemma 2.33]. Hence $\Omega_1(Z(O_2(P_2))) = Z_2(S)$ by [MaStr, Lemma 2.35], the assertion follows.

Let us repeat the definition of a group of Lie type.

Definition 2.15. A genuine group of Lie type in characteristic $p$ is a group isomorphic to $O^p(\overline{C}_K(\sigma))$, where $\overline{K}$ is a semisimple $\overline{\text{GF}}(p)$-algebraic group, $\overline{\text{GF}}(p)$ is the algebraic closure of $\text{GF}(p)$, and $\sigma$ is the Steinberg endomorphism of $\overline{K}$, see [GoLyS3, Definition 2.2.2] for details. A simple group of Lie type in characteristic $p$ is a non-abelian composition factor of a genuine group of Lie type in characteristic $p$.

Hypothesis 2.16. [MaStr, Hypothesis 2.27] Let $G = G(q)$, $q = 2^n$, be a simple group of Lie type, $G \not\cong Sz(q)$, $L_2(q)$ or $2F_4(q)'$. Let $R$ be a long root subgroup of $G$ if $G \not\cong Sp_{2n}(q)$, and a short root subgroup if $G \cong Sp_{2n}(q)$. Set $X_R = C_G(R)$ and $Q_R = O_2(X_R)$.

Lemma 2.17. [MaStr, Lemma 2.28] Assume Hypothesis 2.16 with $G \not\cong L_3(q)$, $U_3(q)$, $Sp_{4}(2)'$ or $G_2(2)'$. Let $L$ be a Levi complement in $N_G(R)$. Then $Q_R/R$ has the following $L$-module structure:

(i) $G \cong L_n(q)$, $O^2(L) \cong SL_{n-2}(q)$, $Q_R/R = V_1 \oplus V_2$, $V_1$ is the natural $L$-module and $V_2$ its dual.

(ii) $G \cong \Omega_{2n}^+(q)$, $O^2(L) \cong \Omega_{2n-4}^\pm(q) \times L_2(q) = L_1 \times L_2$, $Q_R/R = V_1 \oplus V_2$, $V_i$, $i = 1, 2$, are natural $L_1$-modules and $[Q_R, L_2] = Q_R$.

(iii) $G \cong U_n(q)$, $O^2(L) \cong SU_{n-2}(q)$, $Q_R/R$ is the natural module.
(iv) $G \cong E_6(q)$, $O^2(L) \cong L_6(q)$, $Q_R/R$ is an irreducible module with $|Q_R/R| = q^{20}$.
(v) $G \cong 2E_6(q)$, $O^2(L) \cong U_6(q)$, $Q_R/R$ is an irreducible module with $|Q_R/R| = q^{20}$.
(vi) $G \cong E_7(q)$, $O^2(L) \cong \Omega^+_{12}(q)$, $Q_R/R$ is an irreducible module with $|Q_R/R| = q^{32}$.
(vii) $G \cong E_8(q)$, $O^2(L) \cong E_7(q)$, $Q_R/R$ is an irreducible module with $|Q_R/R| = q^{56}$.
(viii) $G \cong F_4(q)$, $O^2(L) \cong Sp_6(q)$, $Q_R/R$ is an extension of the natural module by a spin module, where the natural module is contained in $Z(Q_R)$, where the natural module is contained in $Z(Q_R)$. Finally $Z(Q_R)$ does not split over $R$.
(ix) $G \cong 3D_4(q)$, $O^2(L) \cong L_2(q^3)$, $Q_R/R$ is the 8-dimensional $GF(q)$-module for $L$.

Lemma 2.18. [MaStr, Lemma 2.29] Let $K \cong Sp_{2n}(q)$, $n \geq 3$, $q = 2^m$. We have two root groups $R_1$ and $R_2$, with

1. The Levi factor of $N_K(R_1)$ is $Sp_{2n-2}(q)$, $O_2(N_K(R_1))$ is elementary abelian and $O_2(N_K(R_1))/R_1$ is the natural module.
2. The Levi factor $L$ of $N_K(R_2)$ is $Sp_{2n-4}(q) \times L_2(q)$, furthermore $Z(O_2(N_K(R_2)))/R_2$ is the natural $L_2(q)$-module, and for $n > 2$, $O_2(N_K(R_2))/R_2$ and $O_2(N_K(R_2))/Z(O_2(N_K(R_2)))$ is a tensor product of the two natural modules for the two factors of $L$. If $q > 2$, then $Z(O_2(N_K(R_2)))$ does not split over $R_2$ as an $N_K(R_2)$-module.

Lemma 2.19. [DeSte, 10.10 and page 238] Assume Hypothesis 2.16 with $K \cong G(2^e)$, $e \neq 1$. Let $P$ be the normalizer of the root group $R$. Then $O'(P) \cong (2^e)^{1+4} : SL_2(2^e)$. If $e \neq 2$, then $O'(P)/Q_R$ acts irreducibly on $Q_R/R$. If $e = 2$, then $P$ acts irreducibly on $Q_R/R$ but $O'(P)/Q_R$ induces a direct sum of two permutation modules for $A_5$ on $Q_R/R$.

Let $S$ be a Sylow 2 subgroup of $P$, then $Z_2(S) \leq Q_R$ and $K$ induces the natural $L_2(q)$-module on $Z_2(S)$.

Lemma 2.20. [MaStr, Lemma 2.40] Let $G = L_3(q)$, $q = 2^n$, and $T$ be a Sylow 2-subgroup of $G$. Then $G$ possesses two parabolics $P_1$, $P_2$ which contain $T$, such that $U_i = O_2(P_i)$ is elementary abelian of order $q^2$ and $O^2(P_i/U_i) \cong L_2(q)$, for $i = 1, 2$. Furthermore $P_i$ induces the natural module on $U_i$, $i = 1, 2$, $T = U_1U_2$ and any involution of $T$ is contained in $U_1 \cup U_2$. Finally there is an automorphism $\alpha$ of $G$, which normalizes $T$ with $P_1^\alpha = P_2$. 
Lemma 2.21. [MaStr, Lemma 2.48] Let \( G = Sp_4(q) \), \( q = 2^n > 2 \), and \( T \) be a Sylow 2-subgroup of \( G \). Then \( G \) possesses two parabolics \( P_1, P_2 \) which contain \( T \), such that \( U_i = O_2(P_i) \) is elementary abelian of order \( q^3 \) and \( P_i/U_i \cong GL_2(q) \), for \( i = 1, 2 \). We have that \( U_i \) is an indecomposable module for \( P_i \), an \( Z(O^2(P_i)) = R_i \) is a root group. Furthermore \( Z(T) = R_1R_2 = T' \), \( T = U_1U_2 \) and any involution in \( T \) is contained in \( U_1 \cup U_2 \). There is an automorphism \( \alpha \) of \( G \) with \( R_1^{\alpha} = R_2 \) and \( P_1^{\alpha} = P_2 \).

Lemma 2.22. [GoLyS3, Theorem 2.5.1.] Let \( K \) be a group of Lie type over \( GF(p^e) \) and \( x \in \text{Out}(K) \). Then \( x = dfg \) with:

(a) \( d \) is a diagonal automorphism. In particular \( p \nmid o(d) \).

(b) \( f \) is a field automorphism. In particular if \( S \) is a Sylow \( p \)-subgroup of \( K \) normalized by \( f \), then \( X(t)^f = X(t^\sigma) \), where \( \sigma \) is a field automorphism of \( GF(p^e) \) and \( X(t) \) is a root group in \( S \). This implies that \( f \) also induces a field automorphism on any parabolic containing \( S \) and any Levi complement. Recall that twisted groups are not defined over \( GF(p^e) \) but over \( GF(p^{3e}) \) or \( GF(p^{3e}) \) and \( \sigma \) is an automorphism of this larger field, in particular \( f \) might be trivial on Levi factors, which are defined over \( GF(p^e) \).

(c) \( g \) is a graph automorphism, which comes from a symmetry of the corresponding Dynkin diagram. We have \( o(g) = 2 \) or \( 3 \). The case \( o(g) = 3 \) just occurs for \( K \cong \Omega_8^+(p^e) \). Further \( g = 1 \), if \( K \) is twisted.

Lemma 2.23. [MaStr, Lemma 2.25] Let \( G \) be a group and \( L = F^*(G) \) be a group of Lie type in characteristic two.

(1) If there is an outer automorphism of order 2 of \( L \), which centralizes a Sylow 2-subgroup of \( L \), then \( L \cong Sp_4(2)^t \).

(2) Assume that \( L \) is a central extension of \( Sp_{2n}(q) \), \( F_4(q) \), \( 2F_4(q)^t \) or \( Sz(q) \), \( q = 2^n \), and \( t \) is an involution in \( G \setminus Z(L) \).

(i) If \( C_L(t)/O(C_L(t)) \) has a component \( K \), then \( K \) is a central extension of \( Sp_{2n}(s) \), \( F_4(s) \), \( 2F_4(s)^t \), \( s = 2^h \), or in case of \( Sp_3(q) \) also \( Sz(q) \) is possible. Further \( F^*(L) \not\cong Sz(q) \) or \( 2F_4(2) \).

(ii) A Sylow 2-subgroup \( T \) of \( C_{F^*(G)}(t) \) is not abelian.

(3) Let \( L \cong PSL_3(4) \) and \( t \in G \) be an involution, which induces an outer automorphism on \( L \). Then \( C_L(t) \cong 3^2 : Q_8, PSL_2(7) \) or \( A_5 \).

Lemma 2.24. Let \( G \) be an automorphism group of a group \( H = G(q) \) of Lie type in characteristic two, \( G \not\cong L_2(q), Sp_{2n}(q), F_4(q), 2F_4(q)^t \) or
Let $S$ be a Sylow 2-subgroup of $G$. If $O_2(C_G(\Omega_1(Z(S)))) \not\leq H$, then $H \cong L_3(q)$ or $L_4(q)$.

Proof. We assume $H \not\cong L_3(q)$. Set $R = \Omega_1(Z(S \cap H))$. Then we have that $|R| = q$. By Lemma 2.23 we have $\Omega_1(Z(S)) \leq R$. Let now $t \in O_2(C_G(\Omega_1(Z(S))))$. Then we have that $[C_H(R),t] \leq O_2(C_H(R)) = Q_R$. If $H \cong U_3(q)$, then there is some element $\omega$ of order $q + 1$ in $H$, which centralizes $R$ and so also $\Omega_1(Z(S))$. As by Lemma 2.22 a Sylow 2-subgroup of the outer automorphism group of $H$ is cyclic and induces just field automorphism, we see that no such automorphism would centralize $\omega$ and so $S \cap H = O_2(C_G(\Omega_1(Z(S))))$. So we may assume that $H \not\cong U_3(q)$. Suppose that also $H \not\cong L_n(q)$. Then $N_H(R)$ is a maximal parabolic in $H$, whose structure is given by Lemma 2.17 or Lemma 2.19 in case of $H \cong G_2(q)$. Again by Lemma 2.22 we see that field automorphisms induce nontrivial automorphisms on the Levi factor of $N_H(R)$. As no graph automorphism can centralize the Levi factor, we have the assertion.

So we are left with $H \cong L_n(q)$. We now must have a graph automorphism, which centralizes the Levi factor, i.e. the Levi factor admits no nontrivial graph automorphism, which gives that it has to be $L_2(q)$ and so $H = L_4(q)$, the assertion. \hfill \square

Lemma 2.25. [MaStr, Lemma 2.45] Assume Hypothesis 2.16 with $G \not\cong G_2(2)'$. Let $t$ be a 2-element which induces an automorphism of $G$ such that $[t,Q_R] \leq Z(Q_R)$, then $t$ is induced by some element from $Q_R$, or $G \cong Sp_4(q)'$.

Lemma 2.26. Suppose Hypothesis 2.16 with $G \cong Sp_4(q)$ or $F_4(q)$, $q = 2^n$. Let $S$ be a Sylow 2-subgroup of $G$ with $R \leq Z(S)$. If $t$ is an automorphism of $G$ which normalizes $S$ with $R^t \neq R$ then $[Q_R,t]$ is not elementary abelian.

Proof. If $G \cong Sp_4(q)$, then by Lemma 2.21 $Q_R$ and $Q_{R^t}$ are the only maximal elementary abelian subgroups of $S$, so we are done.

Assume $G \cong F_4(q)$. Then $t$ normalizes $N_G(RR^t)$. We have that $Q_RQ_{R^t} = O_2(N_G(RR^t))$. Further $Q_R \cap Q_{R^t}$ is elementary abelian and $Q_RQ_{R^t}/Q_R \cap Q_{R^t}$ is a direct sum of two $Sp_4(q)$–modules which are both extensions of the trivial module by a natural module. Take the preimage $U$ of the two trivial modules. Then we have that $U = (Q_R \cap Q_{R^t})Z(Q_R)Z(Q_{R^t})$ and $Z(U) = Q_R \cap Q_{R^t}$. Further $Z(Q_{R^t})$ induces a group of $GF(q)$–transvections on $Z(U)Z(Q_R)$. This shows that $C_{Z(U)Z(Q_R)}(t) = Z(U)$ for all $t \in Z(Q_{R^t}) \setminus Z(U)$. In particular all involution are either in
induces a graph automorphism, \(L_G\). Let Lemma 2.28. Automorphisms act nontrivially on a graph/field automorphism, as this has to be nontrivial on type in characteristic 2. Then further that \(X\) is irreducible respectively the direct sum of two irreducible modules.

**Lemma 2.27.** Assume Hypothesis 2.16. Assume further that \(G \not\cong G_2(2)', L_3(2), L_3(4), L_3(16)\) or \(L_4(2)\). If \(t \in \text{Aut}(G)\) is an involution with \([t, X_R] \leq Q_R\), then \(t \in G\).

**Proof.** Suppose that \(t\) induces an outer automorphism on \(G\). Suppose further that \(X_R/Q_R\) has a normal subgroup \(L_R\), which is a group of Lie type in characteristic 2. Then \(t\) cannot induce a field automorphism or a graph/field automorphism, as this has to be nontrivial on \(L_R\). If \(t\) induces a graph automorphism, \(L_R\) must be of Lie rank at most 1. So we have that \(G \cong L_4(q)\), \(L_3(q)\) or \(U_3(q)\). In case of \(L_4(q)\) we have a cyclic group of order \(q - 1\), which is normal in \(X_R/Q_R\). As \(q > 2\) by assumption, we have that graph automorphisms act nontrivially on this group. So assume \(G \cong L_3(q)\). Now \(X_R/Q_R\) is cyclic of order \((q - 1)/d\), where \(d = \gcd(3, q - 1)\). Suppose \(d \neq q - 1\). Then both field- and graph automorphisms act nontrivially on \(X_R/Q_R\). By \([\text{AschSe}, (19.1)]\) graph automorphisms \(t\) invert \(X_R/Q_R\) and field automorphisms \(t\) centralize a subgroup of order \(r - 1\) for \(r^2 = q\). Hence we see that \(t\) must be a graph/field automorphism. Then \(t\) centralizes a group of order \(r + 1\) and inverts a group of order \(r - 1/d\). In particular we must have \(d = r - 1\), which is \(r = 4\), so \(q = 16\), a contradiction as \(G \not\cong L_3(4)\) or \(L_3(16)\).

Assume now \(G = U_3(q)\). Then \(X_R/Q_R\) is cyclic of order \(q + 1/d\), where \(d = \gcd(3, q + 1)\). As we now have \(q > 2\), we have that \(X_R/Q_R\) is nontrivial. Further by \([\text{AschSe}, (19.8)]\) we see that \([t, X_R] \not\leq Q_R\). □

**Lemma 2.28.** Let \(G = L_n(q)\) or \(U_n(q)\), \(S\) a Sylow 2-subgroup of \(G\), with center \(R\). Let \(V\) be normal in \(S\) with \(|V \cap Q_2(C_G(R))| = q^3\). If \(|S : C_S(V)| \leq q^2\) then \(V = Z(C_S(C_Q(Z_2(S))))\), where \(Q = O_2(C_G(R))\).

**Proof.** We start to prove that \(V = Z(C_S(C_Q(Z_2(S))))\). If \(G \cong U_n(q)\), then by Lemma 2.17 \(Q/Z(Q)\) is a module over \(GF(q^2)\). In particular \(|[V, Q]/Z(Q)| \geq q^2\). This shows \([V, Q] = V \cap Q\). If \(G \cong L_n(q)\), then again by Lemma 2.17 we have that \(Q/R\) is a direct sum of two irreducible modules over \(GF(q)\) and so again \([V, Q] = V \cap Q\). Hence in both cases we have that \(|Q : C_Q(V)| = q^2\). Furthermore as \(VQ/Q\) is normal in \(S/Q\), we see that \(S\) acts on \([V, Q]/R\) and so \([S, [V, Q]] \leq R\). This shows \(V \cap Q = Z_2(S)\). In particular \(V \leq C_S(Z_2(S))\). As \(C_Q(Z_2(S))/Z_2(S)\) is irreducible respectively the direct sum of two irreducible modules for \(N_{NC(Q)/Q}(VQ/Q)\), we see that \([V, C_Q(Z_2(S))] \leq R\). But as \(|Q : C_Q(V)| = q^2\), we get that \([C_Q(Z_2(S)), V] = 1\). So \(V \leq C_S(C_Q(Z_2(S)))\).

Let now \(t \in S\), with \([t, C_Q(Z_2(S))] = 1\). If \(t \not\in Q\), then \(t\) induces a
transvection to $Z_2(S)/R$. But this group of transvections in $U_{n-2}(q)$ and $L_{n-2}(q)$ is of order $q$ and so $t \in VQ$. Hence $V = Z(C_S(C_Q(Z_2(S))))$, the assertion.

A 2-local $N$ can fail to be of characteristic 2 in one of two ways. Either $E(N) \neq 1$ or $O(N) \neq 1$. The next lemma will become important when we show in Chapter 6 that $O(N) = 1$ for all 2-locals containing a Sylow 2-subgroup.

**Lemma 2.29.** Let $F^*(G)$ be a simple group, $F^*(G) \in \mathcal{C}_2$ and let $T$ be a Sylow 2-subgroup of $G$. Assume that $T$ normalizes a non-trivial subgroup $U$ of $G$ of odd order. Then $G \cong L_3(3)$ or $M_{11}$ and $|U| = 9$.

**Proof.** We always have that $T$ contains a fours group $V$. Then by co-prime action we have that

$$U = \langle C_{U}(v) \mid 1 \neq v \in V \rangle. \tag{1}$$

This we will use in what follows.

If $F^*(G)$ is a sporadic simple group we see that $F^*(G) \cong M_{11}$ by going over the groups in [GoLyS3, Table 5.3]. Suppose now that $F^*(G)$ is a group of Lie type in odd characteristic. If $F^*(G) \cong L_2(p), p$ a Fermat or Mersenne prime, or $L_2(9)$, we have that the centralizer of an involution is a 2-group, and so by (1) $T$ cannot act on $U$. If $F^*(G) \cong L_4(3), U_4(3)$ or $G_2(3)$, then by Lemma 2.6 and Lemma 2.7 centralizer of involutions are $\{2, 3\}$-groups. So by (1) $U$ is a 3-group. Then the Borel-Tits-Theorem [MaStr, Lemma 2.15] implies that $UT$ is contained in some parabolic subgroup. But obviously none of them contains a full Sylow 2-subgroup. Hence as $F^*(G) \in \mathcal{C}_2$ we are left with $F^*(H) \cong L_3(3)$.

So it remains to deal with groups of Lie type in characteristic 2. Then the assertion follows with [GoLyS3, Corollary 3.1.4].

**Lemma 2.30.** Let $G \cong L_4(3)$ or $U_4(3)$ and $t$ be a 2-central involution in $G$. Then $C_G(t)$ has no normal subgroup $Q \cong Q_8$.

**Proof.** For both groups the structure of $C_G(t)$ is described in Lemma 2.6. Hence we have that $O_2(C_G(t)) \cong Q_8 \ast Q_8$. Furthermore there is a subgroup $U \cong SL_2(3) \ast SL_3(3) = S_1 \ast S_2$ normal in $C_G(t)$, with $O_2(U) = O_2(C_G(r))$. Suppose $Q$ is normal in $C_G(r)$, then $Q \leq O_2(C_G(r))$ and is normal in $U$. So $Q$ is one of the two normal quaternion subgroups $O_2(S_i), i = 1, 2,$ of $O_2(U)$. But $C_G(t)$ contains some element $u$ with $S_1^u = S_2$, in particular $O_2(S_i), i = 1, 2,$ both are not normal in $C_G(t)$. \qed
Lemma 2.31. Let $G/Z(G) \in \mathcal{M}$ (see [MaStr, Definition 2.51]) with $Z(G) \neq 1$, and assume that $G$ has a 2–central involution $z$ such that $|C_G(z)| = 2^a \cdot 3^b$, with $b \leq 2$. Then $G \cong 2L_3(4), 2^2L_3(4), 2Sp_6(2), 2U_4(3), 2M_{12}, 2M_{22}, 4M_{22}, 2Sz(8)$ or $2^2Sz(8)$.

Proof. We have $z \not\in Z(G)$. Hence also $C_{G/Z(G)}(z)$ is a $\{2,3\}$–group. Now we just go over the groups in $\mathcal{M}$. Let us assume that $G$ is not one of the groups listed in the conclusion of the lemma. By inspection of [GoLyS3, Table 5.3] and Lemma 2.17 we see that $5||C_{G/Z(G)}(z)||$, or $G \cong \Omega_6^+(2)$ and $|C_{G/Z(G)}(z)| = 2^{12} \cdot 3^3$, which contradicts $b \leq 2$. Hence, $G/Z(G)$ is as claimed.

Lemma 2.32. Let $G \cong L_3(p), p$ an odd prime, $A_6, L_3(3), M_{11}, L_3(4)$ or $Sz(q), q = 2^m$. Then $G$ possesses exactly one conjugacy class of involutions.

Proof. If $G$ is isomorphic to $L_2(p)$, $A_6, L_3(3)$ or $M_{11}$, then by Lemma 2.5 a Sylow 2-subgroup of $G$ is dihedral or semidihedral. Now it is an easy application of Lemma 2.2 to see that these groups have precisely one class of involutions. For $G \cong L_3(4)$ the assertion follows from Lemma 2.20. For $G \cong Sz(q)$, we get the assertion with [GoLyS4, Lemma 4.3.4].

Lemma 2.33. Let $G \cong 2L_3(4), 2^2L_3(4), 2Sp_6(2), 2U_4(3), 2M_{12}, 2M_{22}, 4M_{22}, 2Sz(8)$ or $2^2Sz(8)$. If there is an element $x$ of order four in $G$ such that $x^2 \in Z(G)$, then $G \cong 2Sp_6(2), 2M_{12}$ or $4M_{22}$.

Proof. Let $S$ be a Sylow 2–subgroup of $G$. Suppose $G \not\cong 4M_{22}$. Then $Z(G)$ is elementary abelian. Further it is enough to deal with the case of $|Z(G)| = 2$. As $S$ is not a quaternion group, there are involutions in $S \setminus Z(G)$. Hence $G/Z(G)$ has more than one conjugacy class of involutions. But $U_4(3), L_3(4), M_{22}$ and $Sz(8)$ have just one class of involutions. So $G/Z(G) \cong Sp_6(2)$ or $M_{12}$.

Lemma 2.34. Let $G$ be a group, $L \leq G$, $L \cong L_4(3)$. Assume that $C_G(L)$ is a cyclic 2–group. Let $S$ be a Sylow 2–subgroup of $G$ and $|\Omega_1(Z(S))| = 8$. Then $C_G(L) \leq Z(S)$ and $S = C_S(L) \times ((S \cap L)\langle d \rangle)$ with $d \in Z(S)$.

Proof. By Lemma 2.6 we have that $Z(L \cap S) = \langle t \rangle$ and we may assume that there is $d \in S$, which centralizes in $L$ a group $PSp_4(3) : 2$. Furthermore $|G : LC_G(L)| = 2$ and $td \not\in d$, as $N_G(S)$ normalizes $S \cap L$ and so centralizes $Z(S)$. Then we have that $S = C_S(L) \times ((S \cap L)\langle d \rangle)$ and so $C_G(L) \leq Z(S)$.

Lemma 2.35. Let $G \cong M_{12}$ or $M_{22}$ and let $x$ be a 2–central involution in $G$. Then $|C_G(x)|$ is divisible by 3 but not by 9. Furthermore Out($G$) is of order 2.
Lemma 2.36. Let $G = Sp_6(2)$ and $x$ be a 2-central involution, which is centralized by an elementary abelian group $U$ of order 9. If there is an elementary abelian subgroup $E$ of order 32 in $C_G(x)$, which is normalized by $U$, then $x$ is a transvection on the natural module.

Proof. By the Borel-Tits-Theorem [MaStr, Lemma 2.15] $C_G(x)$ is contained in one of the parabolics $2^5Sp_4(2)$, $(2^2Q_8 * Q_8)(\Sigma_3 \times \Sigma_3)$, or $2^6L_3(2)$. As $|U| = 9$, we have that $C_G(x)$ is contained in one of the first two parabolics. By Lemma 2.18 we see that the centralizer of a group $U$ of order 9 in both cases is $U \langle x \rangle$. So we just have to eliminate the second case. Here $U$ normalizes $Q = Q_8 * Q_8$ and so it induces orbits of length 9 on the involutions in $Q \setminus Z(Q)$. In particular $U$ cannot normalize an elementary abelian group of order 32, as this group must contain all involutions of $O_2(C_G(x))$ and so equals to $O_2(C_G(x))$. □

Lemma 2.37. Let $G = L_2(p)$, $p$ an odd prime, $A_6$, $L_3(3)$, $M_{11}$, $Sz(q)$ or $L_3(4)$. Let furthermore $t$ be an involution in $G(t)$, which induces an outer automorphism on $G$ and $S$ be a Sylow 2-subgroup of $G(t)$. Then $t \sim tx$ for all $x \in \Omega_1(Z(C_S(t)))$, or $G \cong A_6$ and $t$ induces the $\Sigma_6$-automorphism.

Proof. If $G \cong M_{11}$, then by Lemma 2.12 there is no such automorphism $t$. The same is true for $G \cong Sz(q)$ by Lemma 2.23(2). If $G \cong L_2(p)$ then $Aut(G) \cong PGL_2(p)$ by [GoLyS3, Table 4.5.3]. Now $Aut(G)$ has a dihedral Sylow 2-subgroup and so all involutions in $Aut(G) \setminus G$ are conjugate anyway. If $G \cong L_3(3)$, then by [GoLyS3, Table 4.5.1], we see that $C_G(t) \cong \Sigma_4$ and so $C_S(t) = \langle t \rangle \times D$, where $D \cong D_8$. As $t$ obviously is not 2-central in $S$, we see that $t \sim tx$ with $\langle x \rangle = Z(D)$.

Let $G \cong L_3(4)$. Then by Lemma 2.23(3), $C_G(t) \cong L_2(4)$, $L_2(7)$ or $3^2Q_8$. In all cases $C_G(t)$ has just one class of involutions and as $t$ is not 2-central, the assertion follows.

So let finally $G \cong A_6$. As $t$ is an involution we get with [GoLyS4, Lemma 4.4.2] that $G(t)$ is isomorphic to $PGL_2(9)$ or $\Sigma_6$. In the former the assertion follows with [GoLyS4, Lemma 4.4.1]. □

Lemma 2.38. Let $G = A_8$. There is no subgroup $H$ of $G$, such that $H$ has abelian Sylow 2-subgroup and $|H|$ is divisible by $3 \cdot 5 \cdot 7$.

Proof. Assume false. Let $L$ be a Sylow 7-subgroup of $H$. The normalizer of $L$ in $G$ is of order 21. If $N_H(L) = L$, we have a normal 7–complement in $H$. But then $L$ centralizes a Sylow 5-subgroup, a contradiction. So we have $|N_H(L)| = 21$. We get with Sylow’s theorem that $|H| = 3^2 \cdot 5 \cdot 7$. 

\[ \blacksquare \]
or \(2^3 \cdot 3^2 \cdot 5 \cdot 7\). In the latter \(|G : H| = 8\) and so \(H \cong A_7\), which does not possess an abelian Sylow 2-subgroup. So we have \(|H| = 3^2 \cdot 5 \cdot 7\). Let \(T\) be a Sylow 5-subgroup of \(H\), then \(N_H(T) = C_H(T)\), as \(|H|\) is odd. Hence now \(H\) has a normal 5–complement and so again \(T\) centralizes a Sylow 7-subgroup, a contradiction. □

**Lemma 2.39.** [Asch1, Theorem A] Let \(G\) be a finite group with \(F^*(G) = L\) a simple group, \(T\) a Sylow 2-subgroup of \(G\) and \(z \in Z(T)\) be an involution. Assume that \(M = C_G(z)\) is the unique maximal subgroup of \(G\) which contains \(T\). Then one of the following holds:

1. \(L \cong L_2(q), q > 5\) odd.
2. \(q \equiv -1\ (\mod 4)\) and \(L \cong L_{2n+1}(q)\), or \(q \equiv 1\ (\mod 4)\) and \(L \cong U_{2n+1}(q)\), and \(M\) contains a normal subgroup \(SL_{2n}(q)\), \(SU_{2n}(q)\), respectively. In the first case \(S\) acts nontrivially on the Dynkin diagram.
3. \(L \cong \Omega_{2n+1}(q), q odd, n > 2, and M contains a normal subgroup SO_{2n}(q)\).
4. \(q \equiv -1\ (\mod 4)\) and \(L \cong \Omega_{2n+2}^+(q)\), or \(q \equiv 1\ (\mod 4)\) and \(L \cong \Omega_{2n+2}^-(q)\), and \(M\) contains a normal subgroup \(SO_{2n}^+(q)\). Further \(T\) is not contained in the group \(O_{2n+2}(q)\) extended by the group of field automorphisms.

### 3. Small Modules

In Chapter 6 we will construct a 2-local subgroup \(N\) of \(G\), which is not contained in \(C_G(z)\) (with \(z\) 2-central), such that \(N \cap C_G(z)\) contains a Sylow 2-subgroup \(S\) of \(C_G(z)\) and \(N \cap C_G(z)\) is the only maximal subgroup of \(N\) which contains \(S\). Finally we will have \(F^*(N) = O_2(N)\).

Then we will determine the action of \(N/O_2(N)\) on \(\Omega_1(Z(O_2(N)))\). This will be a so called small module for \(N/O_2(N)\). In this chapter we investigate small modules in generality. The results obtained will be applied to determine the action of \(N\) on \(\Omega_1(Z(O_2(N)))\).

**Definition 3.1.** Let \(X\) be a group, \(V\) be a faithful module over \(GF(p)\). We call \(V\) an

1. \(F\)-module if there is some nontrivial elementary abelian \(p\)-subgroup \(A\) of \(X\) such that \(|V : C_V(A)| \leq |A|\);
2. \(F + 1\)-module if there is some nontrivial elementary abelian \(p\)-subgroup \(A\) of \(X\) such that \(|V : C_V(A)| \leq 2|A|\);
3. \(2F\)-module if there is some nontrivial elementary abelian \(p\)-subgroup \(A\) of \(X\) such that \(|V : C_V(A)| \leq |A|^2\).
In all cases the group $A$ is called an offender. We call the module $V$ a sharp $F$-module, if for any offender $A$ we have that $|V : C_V(A)| = |A|$. We will call the modules defined in Definition 3.1 small modules. Here is a typical situation in which $F$-modules show up.

**Lemma 3.2.** Let $G$ be a group which acts on a $p$-group $X$ and $S$ be a Sylow $p$-subgroup of $XG$. Assume that $G$ acts faithfully on $W = \Omega_1(Z(X)) \neq \Omega_1(Z(S))$. Then either $J(S) \leq X$, and so $J(S)$ is normal in $XG$, or $W$ is an $F$-module for $G$.

**Proof.** We may assume that $J(S) \not\leq X$. Then there is a maximal elementary abelian subgroup $A$ of $S$, with $A \not\leq X$. Now $|A \cap X| |W|/|W \cap A| = |(A \cap X)W| \leq |A|$. This implies that $|W/W \cap A| \leq |A/A \cap X|$. As $W \cap A \leq C_W(A)$, we get that $W$ is an $F$-module with offender $A/A \cap X$. □

In the next two lemmas we give a classification of some of the small modules for simple groups using the classification of the finite simple groups. By a *full transvection group* we mean the unipotent radical of the stabilizer of a point or hyperplane of the natural module for $\text{SL}_n(q)$.

**Lemma 3.3.** Let $X$ be a group such that $F^*(X)$ is quasisimple and let $V$ be an irreducible $F^*(X)$-module over $GF(2)$ which is an $F$-module for $X$. Then $F^*(X)$ is classical, $G_2(q)$, $A_n$, or $3A_6$ and one of the following holds

1. $F^*(X)$ is classical and $V$ is the natural module, or $A_n$ and $V$ is the irreducible reduced permutation module.
2. $F^*(X) \cong SL_n(q)$ and $V$ is the exterior square of the natural module or its dual. Further this module is sharp.
3. $F^*(X) \cong Sp_6(q) \text{ or } \Omega_{10}^+(q)$ and $V$ is the spin module or half spin module, respectively. If $F^*(X) \cong \Omega_{10}^+(q)$, then this module is sharp.
4. $F^*(X) \cong G_2(q)$ and $V$ is the natural module or $F^*(X) \cong 3A_6$ and $V$ is the 6-dimensional module. In both cases this is sharp.
5. $X \cong A_7$ and $V$ is the 4-dimensional module over $GF(2)$.

**Proof.** [GM], [GM1], [GLM]. □

**Lemma 3.4.** Let $X$ be a group such that $F^*(X)$ is quasisimple and let $V$ be a faithful irreducible $X$-module over $GF(2)$. Suppose that $X$ is a minimal parabolic (i.e. a Sylow 2-subgroup of $X$ is not normal in
$X$ but contained in a unique maximal subgroup of $X$) and $V$ is a $2F$–module with offender $A$ such that $|V : C_V(A)| < |A|^2$. Then one of the following holds

(a) $V$ is an $F$–module, $F^*(X) \cong L_3(2^n)$ and $V$ is the natural module, or $F^*(X) \cong A_{2n+1}$ and $V$ is the irreducible section of the permutation module.

(b) $V$ is not an $F$–module and one of the following holds

1. $F^*(X) \cong SL_3(2^n)$ and $V$ is the direct sum of the natural module and its dual. Furthermore $X$ contains some element, which induces a graph or graph/field automorphism on $F^*(X)$.

2. $F^*(X) \cong L_2(2^{2n}) \cong \Omega^-_1(2^n)$ and $V$ is the orthogonal module.

3. $F^*(X) \cong Sp_4(2^n)$ and $V$ is a direct sum of the two 4–dimensional modules. Furthermore $X$ contains some element, which induces a graph automorphism on $F^*(X)$.

4. $F^*(X) \cong A_9$ and $|V| = 2^8$, $V$ is the spin module.

Proof. If $V$ is irreducible for $F^*(X)$ then we get (a), (b)(2) or (b)(4) by [GM], [GM1], [GLM]. If $V$ is not irreducible for $F^*(X)$, then there is a submodule $V_i$ such that $V = V_1 \oplus \cdots \oplus V_r$, $r > 1$ and $V_i$ are $X$–conjugate irreducible $F^*(X)$–modules.

We will show:

$V_i$ is an $F$–module for $F^*(X)\tilde{A}$, where

$\tilde{A}$ is an offender with $|V_i : C_{V_i}(\tilde{A})| < |\tilde{A}|$.

For this assume first that $A$ acts on each $V_i$. Then we see that it induces on at least one $V_i$ an $F$–module offender $A/C_A(V_i)$ such that $|V_i : C_{V_i}(A)| < |A/C_A(V_i)|$. We may assume $i = 1$. So we can set $\tilde{A} = A/C_A(V_1)$ to get (*). Now let $W = V_1^\tilde{A} = V_1 \oplus \cdots \oplus V_t$, $t > 1$. Then we have that $|A|^2 > |W : C_W(A)| = |V_1 : C_{V_1}(B)||V_1|^{t-1}$, where $B = N_A(V_1)$. Assume that $|V_1 : C_{V_1}(B)| \geq |B|$. Then $t^2|B| > |V_1|^t \geq (2|B|)^{t-1}$. This shows $t = 2$, $B \neq 1$ and $|V_1| = 2|B|$. In particular $B$ induces the full transvection group to a point on $V_1$. As $A \neq B$ and there is no outer automorphism of $L_n(2)$ centralizing a full transvection group this is not possible. Hence we have $|V_1 : C_{V_1}(B)| < |B|$. Now with $\tilde{A} = B$, we again have (*). This finally proves (*).

Using (*) an application of Lemma 3.3 shows that we have (b)(1) or (3) or $F^*(X) \cong A_n$. In case of $A_n$, as $X$ is a minimal parabolic, we have $n$ odd. Offenders are transvection groups and so they are sharp. Hence $F^*(X) \neq A_n$. 

$\square$
By Thompson replacement [GoLyS2, Theorem 25.2] $F$-modules are also quadratic modules. Hence we now turn to quadratic modules.

**Lemma 3.5.** [Cher, Theorem 3] Let $K$ be a component of a group $X$, $O_2(K) = 1$ and $V$ be a GF(2)-module for $X$ with $[V, K] \neq 1$. Suppose that $A \leq X$ and $[V, a, A] = 1$ for some $1 \neq a \in A$, then one of the following holds:

(i) $[K, A] \leq K$,
(ii) $K \cong SL_2(2^k)$, $|A/N_A(K)| = 2$ and $|A/C_A(K)| > 2$. Further $[V, (K^A)]$ is a direct sum of natural $\Omega^+_1(2^k)$-modules, or
(iii) $A \neq N_A(K)$, $|A/C_A(K)| = 2$.

If $[K, A] \not\leq K$, then $A$ does not act as a quadratic $F$-module offender on $[V, (K^A)]$.

**Lemma 3.6.** [Str2] Let $X \cong Sp_4(q)'$ or $2F_4(q)'$, $q = 2^n$, and $V$ be an irreducible GF(2)-module. Suppose there is a fours group $A$ in $X$ with $[V, A, A] = 1$. If $A$ intersects some root group $R$ nontrivially but $A \not\leq R$, then $X \cong Sp_4(q)'$ and $V$ is a natural module.

**Lemma 3.7.** Let $X$ be a group such that $F^*(X)$ is a perfect central extension of a finite simple group. Suppose there is some elementary abelian 2-subgroup $A$ of $X$, $|A| \geq 4$, such that for some irreducible nontrivial faithful module $V$ over GF(2) we have $[V, A, A] = 1$. Then:

(i) If $F^*(X)/Z(F^*(X))$ is sporadic, then $F^*(X)/Z(F^*(X)) \cong M_{12}$, $M_{22}$, $M_{24}$, $J_2$, $Co_1$, $Co_2$ or $Sz$. If $|A| \geq 8$, then $F^*(X) \cong 3 \cdot M_{22}$.
(ii) If $F^*(X)/Z(F^*(X))$ is a group of Lie type in odd characteristic which is not also a group of Lie type in even characteristic, then $F^*(X) \cong 3 \cdot U_4(3)$. Furthermore $V$ is the 12-dimensional module.
(iii) If $F^*(X)/Z(F^*(X))$ is alternating, then either $V$ is the reduced permutation module, a spin module or $F^*(X) \cong 3 \cdot A_6$ and $V$ is the 6-dimensional module or $F^*(X) \cong 3 \cdot A_7$ and $V$ is the 12-dimensional module. If $|A| > 8$, then $V$ is natural or $X \cong A_8$ and $|V| = 16$. If $V$ is the spinmodule and $|A| = 4$, then $A$ is conjugate to $\langle (12)(34), (13)(24) \rangle$ or $\langle (12)(34)(56)(78), (13)(24)(57)(68) \rangle$. If $|A| = 8$ then $A$ is conjugate to $\langle (12)(34)(56)(78), (13)(24)(57)(68), (14)(26)(37)(48) \rangle$ in $\Sigma_n$.

**Proof.** (i) This is [MeiStr2].
(ii) This is [MeiStr1].
(iii) The first assertion is [MeiStr2]. There the group $3 \cdot A_7$ was forgotten. But as J. Hall pointed out there is an embedding $3 \cdot A_7 \leq
$3 \cdot M_{22} \leq SU_6(2)$, which gives a 6-dimensional module over $GF(4)$ on which a fours group in $3 \cdot A_7$ acts quadratically.

For the proof of the second assertion suppose $|A| \geq 4$. Let $a \in A^2$ and $k$ be the number of fixed points of $a$. Then there is $K \leq C_X(a), K \cong \Sigma_k$.

Furthermore $C_{C_X(a)}(K')$ is an extension of a 2–group by $A_m$, $m = (n - k)/2$. Now choose $a \in A$ with $m$ as big as possible. Suppose first $m > 2$. By [MeiStr1, (4.3)] there is no $x \sim (12)(34)$ such that $[[V,a],x] = 1$. In particular $\langle A_{C_X(a)} \rangle$ does not contain such an element $x$.

Suppose first $[A,C_{C_X(a)}(K')] \neq 1$. If $m \geq 5$, then $A_m$ is nonsolvable and so $C_{C_X(a)}([V,a])$ contains an elementary abelian subgroup of $O_2(C_X(a))$ of order $2^{m-1}$. But then this group contains a conjugate $t$ of $(12)(34)$.

Now $\langle a,t \rangle$ acts quadratically, a contradiction.

Let $m = 4$. Then $a \sim (12)(34)(56)(78)$. Furthermore as we may assume that no $x \sim (12)(34)$ is contained in $\langle A_{C_X(a)} \rangle$ we see that $A$ is conjugate to a subgroup of $\langle (12)(34)(56)(78), (12)(24)(57)(68), (15)(26)(37)(48) \rangle$.

Let $m = 3$. Then $C_X(K') \leq \Sigma_6$ and $a \sim (12)(34)(56)$. We see that $\langle A_{C_X(a)} \rangle$ has to contain some $x \sim (12)(34)$, a contradiction.

So let $[A,O^2(C_{C_X(a)}(K'))] = 1$. If $[A,K'] \neq 1$, then $[K',[V,a]] = 1$. If $k \geq 4$, then $K'$ contains some $x \sim (12)(34)$, a contradiction. Let $k \leq 3$. As $[A,O^2(C_{C_X(a)}(K'))] = 1$ and $m > 2$, there is $x \sim (12)$ in $A$. But then $xa$ has fewer fixed points than $a$, a contradiction. So we are left with $[A,K'] = 1 = [A,C_{C_X(a)}(K')]$. But this is impossible with $m > 2$.

So we have $m \leq 2$ for all $a \in A^2$. As there is no fours group of transpositions we may assume $a = (12)(34) \in A$. Now $A \geq \langle a,b \rangle$, $b = (13)(24), (12)(56)$ or $(34)$. Let $b = (12)(56)$. If $[b,K'] \neq 1$ then $K'$ contains no involutions by [MeiStr1, (4.3)]. This shows $k \leq 3$ and so $A \leq \Sigma_7$. If $[b,K'] = 1$, then even $k \leq 2$ and so $A \leq \Sigma_6$. But for this group $A = \langle (12)(34), (12)(56) \rangle$ does not act quadratically on the four dimensional spin module. Recall that in case of $\Sigma_6$ the natural module is defined as the module on which $\langle (12)(34), (12)(56) \rangle$ acts quadratically.

Assume now $b = (34)$. Then $C_X(b) \cong \mathbb{Z}_2 \times \Sigma_{n-2}$. If $n - 2 > 3$, then $(12)(56) \in [a,C_X(b)]$. But then $\langle (34), (12)(56) \rangle$ acts quadratically, a
Lemma 3.8. Let $A \leq \Sigma_6$ be an elementary abelian subgroup of order $8$. Then $A$ does not act quadratically on both of the two 4-dimensional modules for $\Sigma_6$.

Proof. As the two 4-dimensional modules are interchanged by an outer automorphism of $\Sigma_6$, which also interchanges the two elementary abelian subgroups of order 8, it is enough to show that not both act quadratically on the irreducible part of the permutation module. But the fours subgroup $\langle (12)(34), (13)(24) \rangle$ does not act quadratically on the irreducible permutation module, as the commutator of $(12)(34)$ with the permutation module, which is $\langle v_1 + v_2, v_3 + v_4 \rangle$, is not centralized by $(13)(24)$.

For later applications we need some information about central extensions of some of the small modules.

Lemma 3.9. Let $X = A_n, n \geq 5, V$ be a $\text{GF}(2)X$-module with $[V, X]$ the natural irreducible permutation module. Assume $C_V(X) = 1$. Then $|V : [V, X]| \leq 2$, and $V = [V, X]$ if $n$ is odd. Furthermore $V$ is a factor of the reduced permutation module. In particular $V$ is of dimension $n - 1$ or $n - 2$.

Proof. This will be proved by induction on $n$. For $n = 5$ this is well known as the permutation module is injective. So let $n > 5, K \cong A_{n-1}, K \leq X$. If $n - 1$ is odd, then $[V, X] = [V, K]$ is the permutation module for $K$. By induction $V = [V, K] \oplus T$. Hence there is $v \in V \setminus [V, X], [v, K] = 1$, i.e. $\langle v^X \rangle = V_1$ is a factor of the permutation module. Let $K_1 \leq K$ such that $K_1 \cong A_{n-2}$. Then $|C_V(K_1) : T| = 2$. Now there is an involution $t \in X$ such that $t \not\in K$ but $t$ normalizes $K_1$. As $\langle K, t \rangle = X$, we get $C_T(t) = 1$ and so $T = \langle v \rangle$, i.e. $V_1 = V$.

Let $n - 1$ be even. Then we have a $K$-chain. $1 < T < T_1 < [V, X] < V$, with $|T| = 2, T_1/T$ the irreducible permutation module for $K$ and $|[V, X]/T_1| = 2$. Now by induction $C_{V/T}(K) \neq 1$. As $C_{V/T}(K) \not\leq [V, X]/T$, we again get some $v \in V \setminus [V, X], [v, K] = 1$, and so $V$ is a factor of the permutation module.

Lemma 3.10. Let $F^*(G) = L_2(2^n)$ and $V$ be a faithful $F$-module over $\text{GF}(2)$ for $G$ such that $C_V(G) = 1$. Then $V$ is irreducible.

Proof. If $n = 1$, then $V = [V, G'] \oplus C_V(G')$. As $C_V(G) = 1$ also $C_V(G') = 1$ and so $V = [G', V]$ is of order 4. So let $n > 1$. By Lemma 3.3 we have that there is an irreducible submodule $V_1$ in $V$ which is the natural $L_2(2^n)$-module or $n = 2$ and it is the permutation
module for $A_4$. In both cases we get $|V_1 : C_{V_1}(A)| = |A|$ for an offender $A$. Hence we see that $V = C_V(A)V_1$. In particular $V/V_1$ is a trivial $L_2(2^n)$–module. By Lemma 3.9 we may assume that $V_1$ is the natural $L_2(2^n)$–modules. Now $A$ is a Sylow $2$–subgroup of $L_2(2^n)$. Application of [Hu, (1.17.4)] gives $V = V_1$. □

Lemma 3.11. Let $X = \Omega_2^+(q)$, $q$ even, and $V$ be a module over $\text{GF}(2)$ with $[V, X]$ the natural module and $C_V(X) = 1$. Then $[V, X] = V$.

Proof. We have $X = X_1X_2$, $X_i \cong L_2(q)$, $i = 1, 2$. We may assume that $q > 2$, as the assertion is obvious for $q = 2$. There are $\omega_i \in X_i$ with $o(\omega_i) = q + 1$. If $C_{[V, X]}(\omega_1) \neq 1$, then as $X_2$ acts nontrivially on $C_{[V, X]}(\omega_1)$ we get $|C_{[V, X]}(\omega_1)| = q^2$ and so $|[V, X], \omega_1]| = q^2$. By Schur’s Lemma $[[V, X], \omega_1]$ is a $1$-dimensional module over $\text{GF}(q^2)$ for $X_2$ and so $X_2 \leq GL_1(q^2)$, a contradiction. Hence $\omega_i$ act fixed point freely on $[V, X]$ for both $i = 1, 2$. Now choose $v_1 \in V \setminus [V, X]$ with $[v_1, \omega_1] = 1$. Then $v_1$ is uniquely determined in the coset $[V, X]v_1$. Since $\omega_1$ and $\omega_2$ commute, we have $v_1$ is centralized by $\omega_2$. So $C_V(\omega_1) = C_V(\omega_2)$ is normalized by $\langle C_X(\omega_1), C_X(\omega_2) \rangle \geq \langle X_2, X_1 \rangle = X$, which is a contradiction. □

Lemma 3.12. Let $F^*(G) = A_{2^n+1}$ and $V$ be a module over $\text{GF}(2)$, which is an $2F$–module, with offender $A$ such that $|V : C_V(A)| < |A|^2$. Assume $C_V(G) = 1$ and $V$ involves just trivial and nontrivial irreducible parts of the permutation module. Then we have that $V$ is the irreducible part of the permutation module.

Proof. If we have just one irreducible part of the permutation module in $V$, the assertion follows by Lemma 3.9. So we may assume that we have at least two such modules involved. Let $W$ be the irreducible part of the permutation module. Then we have that $A$ is an $F$–module offender on $W$ with $|W : C_W(A)| < |A|$. Then by Thompson replacement [GoLyS1, Theorem 25.1] there is also a quadratic $F$–module offender with this property. Take an involution $x \in G$. On $W$ we have that $|[W, x]| = 2^u$, where $u$ is the number of transpositions in the cycle decomposition of $x$. We may assume that $\{1, 2, \ldots, m\}$ is the support of $A$. Then there is a subgroup $B$ of $A$ such that $|W : C_W(B)| = |W : C_W(R)|$, where $R = \langle (1, 2), (3, 4), \ldots, (m - 1, m) \rangle$. But then $|W : C_W(R)| = |R|$, a contradiction. □

Lemma 3.13. Let $G = A_{2^n+1}$ and $S$ be a Sylow $2$–subgroup of $G$. Let $V$ be the irreducible part of the permutation module over $\text{GF}(2)$ for $G$. Then $|C_V(S)| = 2$.

Proof. Let $W$ be the module with basis $v_i$, $i = 1, \ldots, 2^n+1$ with natural $G$-action on $W$. Then $W = V \oplus W_1$, $W_1$ the trivial module.
Choose $S \leq X \cong A_{2n}$, where $X$ is the stabilizer of $1$. Then we calculate immediately that $C_V(S) = \langle v_1, v_2 + \cdots + v_{2n+1} \rangle$. As $v_1 \not\in V$, we get the assertion.

**Lemma 3.14.** Let $G = L_2(2^n)$ or $A_{2n+1}$, $n \geq 2$. Let $H$ be a Borel subgroup in the first case and a subgroup isomorphic to $A_{2n}$ in the second case. Let $V$ be a $GF(2)$-module for $G$ such that $[V,G]$ is the natural module, or $G \cong A_9$ and $[V,G]$ is the 8-dimensional spin module. Then one of the following holds:

(i) $G = L_2(2^n)$ and $C_V(H) = C_V(G)$.
(ii) $G = A_{2n+1}$ and $C_V(H) = C_V(S)$, $S$ a Sylow 2-subgroup of $H$.
(iii) $G = A_9$, $[V,G]$ is the 8-dimensional spin module and $C_V(H) = C_V(G)$.

**Proof.** We may assume that in all cases $V = [V,G]C_V(H)$. As $H$ contains a Sylow 2-subgroup of $G$ we get that $V = [V,G]C_V(S)$. Now application of [Hu, (1.17.4)] shows that $V = [V,G] \oplus C_V(G)$. In case (i) and (iii) we have that $C_{[V,G]}(H) = 1$, so we have that $C_V(G) = C_V(H)$. In case (ii) by Lemma 3.13 we have that $C_{[V,G]}(H) = C_{[V,G]}(S)$, so we get $C_V(H) = C_V(S)$. □

**Lemma 3.15.** Let $G = E(G)T$, $T$ a Sylow 2-subgroup of $G$, $E(G) = G_1 \cdots G_r$, $G_1 \cong L_2(q)$, $q$ even, or $A_{2n+1}$. Assume that $T$ acts transitively on the $G_i$ and $C_G(E(G)) = 1$. Let $V$ be an irreducible faithful $F$-module over $GF(2)$ for $G$. Then $V = V_1 \oplus \cdots \oplus V_r$, $V_i$ the natural module for $G_i$, $i = 1, \ldots, r$, and $[V_j, G_i] = 1$ for $i \neq j$.

**Proof.** Let $A$ be an offender. We may assume $[V, A, A] = 1$ by Thompson replacement. Now choose $A$ with $|A|$ minimal. Set $A_1 = C_A(G_1)$. Then we may assume $A_1 = 1$ or $|V : C_V(A_1)| > |A_1|$. If $[G_1, A] \not\leq G_1$ we get with Lemma 3.5 that $G_1^A = G_1 G_1^a$ and $|A/C_A(G_1)| = 2$. In any case $\langle a \rangle$ has to be an $F$-module offender on $C_V(A_1)$. This shows $A_1 = 1$ and $\langle a \rangle = A$. But now $a$ inverts some element of prime order $p > 3$ in $E(G)$ and so cannot induce a transvection on $V$. So we have that $[G_1, A] \leq G_1$. Then $G_1$ induces an $F$-module in $C_V(A_1)$. By Lemma 3.3 we have that there is exactly one nontrivial module $W$ involved in $C_V(A_1)$, the natural one.

Assume that $A_1 \neq 1$. Let $B \leq A$ be a complement to $A_1$ and let $1 \neq a \in A_1$. As $A$ acts quadratically, we see that $[V, a, G_1] = 1$. This implies $[V, G_1] \leq C_V(A_1)$. If $A_1 = 1$, then also $[V, G_1] \leq C_V(A_1)$. Hence in any case $[V, G_1]$ involves just one nontrivial irreducible module. Now we have that $[V, G_1]$ is centralized by $G_2 \times \cdots \times G_r$. As $C_V(E(G)) = 1$, we get that $W = [V, G_1]$ is the natural module. But now $[V, G_i]$ is the
natural module for all \( i \), as \( T \) acts transitively. Hence \( V = V_1 \oplus \cdots \oplus V_r \) with \([V_i, G_j] = 1\) for \( i \neq j \) and \( V_i \) the natural \( G_i \)-module, the assertion. \( \Box \)

The next two lemmas deal with solvable groups having \( F \) or \( 2F \)-modules.

**Lemma 3.16.** Let \( G \) be a solvable group with Sylow 2-subgroup \( S \) and \( O_2(G) = 1 \). Assume that \( S \) is contained in a unique maximal subgroup of \( G \). Let \( V \) be a faithful \( GF(2) \)-module for \( G \). If \( V \) is an \( F \)-module, then \( G = O_3(G)S \).

**Proof.** If \( G \neq F(G)S \), then there are maximal subgroups containing \( F(G)S \) and \( N_G(S \cap O_{2^r}(G)) \), which are different. Hence \( G = F(G)S \). Further again by minimality \( F(G) = O_p(G) \) for some prime \( p \). By Lemma 2.3 we have a subgroup \( D = D_1 \times \cdots \times D_r \) of \( G \) such that the \( D_i \) are dihedral of order \( 2p \) and a Sylow 2-subgroup \( A \) of \( D \) is an \( F \)-module offender. Hence we have that \(|V/C_V(D)| \leq |A|^2\), as \( D \) is generated by two conjugates of \( A \). Now \( O_p(D) \) acts faithfully on \( V/C_V(D) \) and so \( p = 3 \). \( \Box \)

**Lemma 3.17.** Let \( G \) be a group and \( V \) be a faithful \( 2F \)-module over \( GF(2) \) with offender \( A \). Suppose \( G = O_p(G)A \) with \( O_p(G) = F(G) \) for some odd prime \( p \). Then \( p \leq 5 \) and in case of \( p = 5 \), we have that \(|V : C_V(A)| = |A|^2\). If \( A \) is an \( F \)-module offender, then \( p = 3 \) and \(|V : C_V(A)| = |A|\).

**Proof.** By the Dihedral Lemma 2.3, we may assume that 
\[ G = D_1 \times \cdots \times D_r, \]
\( D_i \) dihedral of order \( 2p \). Now as \(|V : C_V(A)| \leq |A|^2 \) or \(|A|\) we have that
\[ |V : C_V(G)| \leq |A|^4, |A|^2 \] respectively.
Hence \(|[V, O_p(G)]| \leq |A|^4 \leq 2^{4r} \), or \(|[V, O_p(G)]| \leq 2^{2r} \). In \( GL_{4r}(2) \) elementary abelian subgroups of order \( p^r \) just exist for \( p = 3 \) and \( p = 5 \), while in \( GL_{2r}(2) \) they just exist for \( p = 3 \). This shows that \( p \leq 5 \). If \( p = 5 \), then we must have that \(|V : C_V(G)| = 2^{4r} \) and so \(|V : C_V(A)| = |A|^2\). If \( p = 3 \) and \( A \) is an \( F \)-module offender then \(|V : C_V(G)| = 2^{2r} \) and so \(|V : C_V(A)| = |A|\). \( \Box \)

**Lemma 3.18.** Let \( X = Sz(q) \) or \( L_2(q) \), \( q > 2 \) even. Suppose that \( X \) acts on a 2-group \( U \). Let \( V \) be a normal subgroup of \( U \) of order 2 and \( U/V \) be the natural module for \( X \). In case of \( X \approx Sz(q) \) assume additionally that \( U \) contains an elementary abelian subgroup \( U_1 \) with \(|U_1|^2 = 2|U|\). Then \( U \) is abelian.
Proof. If $X \cong L_2(q)$, then $X$ acts transitively on $(U/V)^2$. As $q > 2$ we see that $U$ is not a quaternion group and so there are involutions in $U \setminus V$, so all elements in $U$ are involutions, the assertion.

So let $X \cong S_2(q)$. We may assume that $U$ is extraspecial. Now elements of order 5 act fixed point freely on $U/V$. The existence of $U_1$ guarantees that $U$ is extraspecial of $+$ type. As $q = 2^{2n+1}$, we get $|U/V| = 2^{8n+4}$ and so $U$ is a central product of $4n+2$ dihedral groups. But as an element of order 5 acts fixed point freely on $U/V$ the number of dihedral groups must be divisible by four by [MaStr, Lemma 2.9], a contradiction.

Lemma 3.19. Let $X = L_2(q)$ or $S_2(q)$, $q \geq 4$, $q$ even. Let $S \in Syl_2(X)$ and $A \leq \Omega_1(S), |A| \geq 4$. Then there is some $g \in X$ with $X = \langle A, A^g \rangle$.

Proof. We have that $X$ acts 2-transitively on a set $\Omega$ with $|\Omega| = q + 1$, $q^2 + 1$, respectively. For $1 \in \Omega$ we have that $X_1 = SK$, where $K$ is cyclic of order $q - 1$ and acts transitively on $\Omega_1(S)$. Further $K = X_{1,2}$, the stabilizer of two points. Finally the stabilizer of any three points is trivial.

This has the following consequences. Choose $1 \neq \rho \in K$. Then $\{1, 2\}$ are the two fixed points of $\rho$. Hence $N_X(\langle \rho \rangle)$ contains $K$ as a subgroup of index two. This shows that $K = C_X(\rho)$. Let $a \in S$ be an involution. Then $a$ has just one fixed point. This shows that $C_X(a) = S$, a 2-group.

Now choose $\langle t, a \rangle \leq A \leq \Omega_1(S), |A| \geq 4$. Choose $g \in X$ such that $N_X(K^g) = \langle a, b \rangle$ for some involution $b$. Then set $U = \langle a, b, t \rangle$. Let $T$ be a Sylow 2-subgroup of $U$ with $\langle a, t \rangle \leq T$. Then $T \leq C_X(a) = S$, so $T = U \cap S$. If $T = N_U(T)$, we get a normal 2-complement $W$ in $U$. But then one of $C_W(a), C_W(t), C_W(at)$ must be nontrivial, which contradicts the fact that centralizers of involutions are 2-groups. Hence we have that $K \cap U \neq 1$. Now choose $\rho \in K \cap U$ of prime order $p$. As $|K|$ is coprime to $|X : K|$ and $K^g \leq U$, there is some $x \in U$ with $\rho^x \in K^g$. Then $K^g = C_U(\rho^x)$. Now $K = C_X(\rho) = K^{q^{x-1}} \leq U$. This shows that $\langle \Omega_1(S), \Omega_1(S)^x \rangle \leq U$. Thus it is enough to show $\langle \Omega_1(S), \Omega_1(S)^x \rangle = X$.

We have that $Y = \langle \Omega_1(S), \Omega_1(S)^g \rangle$ contains at least $q + 1$ conjugates of $\Omega_1(S)$. Thus we are done if $X \cong L_2(q)$, as $\langle \Omega_1(S), \Omega_1(S)^h \rangle$ contains all conjugates.

So let $X \cong S_2(q)$. The number of conjugates of $\Omega_1(S)$ in $Y$ is $nq + 1$. But then $nq + 1 | q(q^2 + 1)$. Which gives $n = q$ and so $\langle \Omega_1(S)^x \rangle \leq Y$, hence $X = Y$.

The next two lemmas show how the $2F$-modules will appear later on.
Lemma 3.20. Let $G$ be a $K_2$-group with $F^*(G) = O_2(G) \neq 1$, $A \leq G$ be elementary abelian with $A \not\leq O_2(G)$ and $A \leq S$ for some Sylow $2$-subgroup $S$ of $G$. Then there is some $g \in G$ such that one of the following holds:

(i) $g^2 \in N_G(A)$, $A^g \leq S$, $1 \neq [A^g, A] \leq A \cap A^g$ and $|A : C_A(A^g)| = |A^g : C_{A^g}(A)|$.

(ii) With $X = \langle A, A^g \rangle$ the following hold:

1. $X/O_2(X) \cong L_2(q)$, $Sz(q)$ or $X/O_2(X)$ is a dihedral group of order $2u$, $u$ odd.
2. $S \cap X$ is a Sylow $2$-subgroup of $X$.
3. $Y = (A \cap O_2(X))(A^g \cap O_2(X)) \leq X$.
4. $Y \neq A \cap O_2(X)$.
5. $|A : C_A(Y)| \leq |Y : C_Y(A)| q \leq |Y : C_Y(A)|^2$, where $q = 2$ if $X/O_2(X)$ is dihedral. Further $[Y, a][A \cap A^g] = [Y, A](A \cap A^g)$ for all $a \in A \setminus O_2(X)$.
6. If $X/O_2(X)$ is not dihedral, then $Y/(A \cap A^g)$ is a direct sum of natural modules for $X/O_2(X)$.

Proof. We start the proof with some general remarks. Let $X$ be as in (ii) (1) and (2). Then obviously (3) follows. If (4) would be false, then as $[O_2(G), A] \leq O_2(G) \cap A \leq O_2(X) \cap A$, we get that $[O_2(G), X, X] = 1$ and so $[O^2(X), O_2(G)] = 1$, which contradicts $C_G(O_2(G)) \leq O_2(G)$.

Hence also (4) holds. Next we see that $C_Y(A) = A \cap Y$ and so we see that $C_{Y/(A \cap A^g)}(A) = (A \cap Y)/(A \cap A^g)$ and $Y/(A \cap A^g) = (Y \cap A)/(A \cap A^g) \oplus (Y \cap A^g)/(A \cap A^g)$. So the first assertion in (5) follows. Further we see that elements of odd order in $X$ act fixed point freely on $Y/(A \cap A^g)$.

Hence [Hi] and [Mar] yield (6) and the second assertion in (5). So to prove the lemma we may assume that (i) does not hold. Then to prove (ii) we just have to prove (1) and (2). In fact when constructing $X$ such that (1) holds, we immediately will see from this construction that also (2) holds.

Set $\bar{G} = G/O_2(G)$. We first prove

(*) Suppose there is a subgroup $L$ of $\bar{G}$ such that $|\bar{A} : C_{\bar{A}}(L)| = 2$ and $\bar{A} \not\leq O_2(\langle L, \bar{A} \rangle)$ then (ii) holds. In particular (ii) holds if $|\bar{A}| = 2$.

As $\bar{A} \not\leq O_2(\langle L, \bar{A} \rangle)$ there is some $\omega \in \langle L, \bar{A} \rangle$, $o(\omega)$ odd, which is inverted by some $\bar{a} \in \bar{A} \setminus C_{\bar{A}}(L)$. Then $\langle \bar{A}, \omega \rangle/O_2(\langle \bar{A}, \omega \rangle) \cong D_{2u}$, $u$ odd. Set $X = \langle A, \omega \rangle$. Then $X$ satisfies (ii)(1). Of course $S \cap X$ is a Sylow $2$-subgroup of $X$. So (ii)(2) is satisfied. Hence (*) is proved.

If $[F(\bar{G}), \bar{A}] \neq 1$ then $F(\bar{G}) = (C_{F(\bar{G})}(\bar{B}) | |\bar{A} : \bar{B}| = 2)$. Hence there is
some $B$ with $C_{F(G)}(B) \neq 1$ and $[C_{F(G)}(B), \bar{A}] \neq 1$. So by (*) (ii) holds.

For the remainder of this proof we will assume that $F^*(G) = E(G)$, $G = E(G)A$ and $|\bar{A} : C_A(L)| \geq 4$ for all components $L$. As $[S,A] \leq A$, we have that $A$ acts quadratically on $O_2(G)$. Hence by Lemma 3.5 we have $[L, \bar{A}] \leq L$ or $L \cong SL_2(q)$, $q$ even. In the latter there is some $a \in \bar{A}$ such that $C_{(LA)}(a) = L_1 \cong L_2(q)$ and as $\bar{A}$ is normal in a Sylow 2–subgroup of $\langle L, \bar{A} \rangle$, we have that $A_1 = L_1 \cap \bar{A}$ is a Sylow 2–subgroup of $L_1$. So $L_1 = \langle A_1, A_1^g \rangle$ for suitable $g \in L_1$. Hence $X = \langle A, A^g \rangle$ satisfies (ii)(1) and (2). So from now on we assume that $[L, \bar{A}] \leq L$. We collect this in

(\text{**}) $L = F^*(\bar{G})$ is a component, $|\bar{A}| \geq 4$ and if $\bar{A} \leq \bar{U} < \bar{G}$, with $S \cap \bar{U}$ a Sylow 2–subgroup of $\bar{U}$, then $\bar{A} \leq O_2(\bar{U})$.

Assume first that $L$ is of Lie type in odd characteristic, which is not also of Lie type in even characteristic. Then by Lemma 3.7 we have that $L/Z(L) \cong U_4(3)$. As $A \leq S$, there is some 2–central involution $s$ in $\bar{A}$. By (\text{**}) we have $\bar{A} \leq O_2(C_{\bar{A}}(s))$. As we may generate $C_{\bar{A}}(s)$ by elements $g$ with $g^2 \in O_2(C_{\bar{A}}(s))$, then if $\bar{A}$ is not normal in $C_{\bar{A}}(s)$ there is such a $g$ with $A^g \leq O_2(C_{\bar{A}}(s))$ and $1 \neq [\bar{A}, \bar{A}^g] \leq \bar{A} \cap \bar{A}^g$. Then also $1 \neq [A, A^g] \leq A \cap A^g$ and $A^{A^g} = A$. Obviously $|A : C_A(A^g)| = |A^g : C_{A^g}(A^{A^g})| = |A^g : C_{A^g}(A)|$. So we may assume $\bar{A} \leq C_{\bar{A}}(s)$. As $C_{\bar{A}}(s)$ contains a normal subgroup $U = SL_2(3) \ast SL_2(3)$ by Lemma 2.6(i) and $O_2(U) = O_2(C_{\bar{A}}(s))$, we see that $O_2(C_{\bar{A}}(s))$ cannot contain an elementary abelian subgroup of order at least four which is normal in $U$. So $\bar{A} \not\leq L$. In particular there is some $t \in \bar{A}$ such that $[U, t] \leq O_2(U)$. As $\langle s,t \rangle$ is normal in $C(\bar{A})$, we get that $|C_L(t)|$ is divisible by $2^6 \cdot 3$. Then by Lemma 2.6 we get $C_L(t) \cong PSp_4(3)$, contradicting (\text{**}).

Next let $L \cong G(r)$ be a group of Lie type in even characteristic. Suppose first that $\bar{A}$ acts nontrivially on the Dynkin diagram. If the rank is greater than two, then there is a parabolic $U$ of rank two of $L$ such that $\bar{A}$ acts nontrivially on $F^*(U/O_2(U))$. But this contradicts (\text{**}). So we may assume that $L/Z(L) \cong L_3(2)$ or $Sp_4(q)$. Let $B$ be a Borel subgroup of $L$, which is normalized by $\bar{A}$, then by (\text{**}) we have that $[B, \bar{A}] \leq O_2(B)$. This now gives $q = 2$. But then we easily see that $[S \cap L, \bar{A}]$ is not abelian, contradicting $\bar{A} \not\leq S$. So we have that $\bar{A}$ acts trivially on the Dynkin diagram.

Let $R$ be a root subgroup in $Z(\bar{S} \cap L)$. By (\text{**}) we have that $\bar{A} \leq O_2(N_G(R))$. If $C_L(R)$ is generated by elements $g$ with $g^2 \in O_2(N_L(R))$, then we either get (i), or $\langle \bar{A}^{N_L(R)} \rangle$ is abelian.
If even $\bar{A} \leq R$, then $\bar{A} \leq \bar{L} \leq L$, with $\bar{L} \cong L_2(r)$ or $Sz(r)$ and $S \cap \bar{L}$ is a Sylow 2–subgroup of $\bar{L}$.

Now we just have to handle rank 1 groups or groups $L$ in which $N_L(R)$ contains a normal elementary abelian subgroup different from $R$, in particular $N_L(R)$ does not act irreducibly on $O_2(N_L(R))/R$. Application of Lemma 2.17 shows $L/Z(L) \cong L_n(r)$, $Sp_{2n}(r)'$, $F_4(r)$ or $2F_4(r)$.

Suppose $L$ is a rank 1 group. We have that $\bar{A} \leq O_2(B\bar{A})$ for some Borel subgroup $B$ of $L$. Hence we have that $\bar{A} \leq L$. Then as $|A| \geq 4$, by Lemma 3.19 we get some $g \in L$ such that for $X = \langle A, A^g \rangle$. We have $X/O_2(X) \cong L_2(q)$ or $Sz(q)$ and a Sylow 2-subgroup of $X$ is contained in $\bar{S}$ and we are done. In particular from now on we may assume that $\bar{A} \not\leq R$.

Now assume that $L/Z(L) \cong L_n(r)$, $n \geq 3$. Let $P_1, P_{n-1}$ be the two parabolic subgroups of $L\bar{A}$ containing $\bar{S} \cap L$ which involve $L_{n-1}(r)$. We have that $\bar{A} \leq O_2(P_i)$ for both $i$. So we have $\bar{A} \leq O_2(P_1) \cap O_2(P_{n-1}) = R$, a contradiction.

Next let $L/Z(L) \cong Sp_{2n}(r)'$, $n \geq 2$. Now $C_L(R)$ is generated by elements $g$ with $g^2 \in O_2(C_L(R))$. By (**) we have $\bar{A} \leq Z(O_2(N_G(R)))$. We now may embed $\bar{A}$ into some $\bar{L} \cong Sp_4(r)'$ with $\bar{S} \cap \bar{L}$ a Sylow 2–subgroup of $\bar{L}$. Hence we may assume $L \cong Sp_4(r)'$. We apply Lemma 2.21. So we have two parabolics $P_1, P_2$ of $L\bar{A}$ containing $\bar{S} \cap L$. By (**) we have $\bar{A} \leq O_2(P_1) \cap O_2(P_2)$. As $\bar{A}$ is not contained in a root subgroup we see that $\langle \bar{A}^{P_i} \rangle = O_2(P_i)$ for $i = 1, 2$. Even in case of $r = 2$ this is true as $|\bar{A}| > 2$. Let $H_i$ be the preimage of $P_i$, i.e. $H_i/O_2(G) = P_i$. Now suppose that $\langle A^{O_2(H_i)} \rangle$ is not abelian. Then there is some conjugate $A^h$, $h \in O_2(H_1)$, with $1 \neq [A, A^h] \leq A \cap A^h$. As $O_2^2(H_1)$ is generated by elements $h$ with $h^2 \in N_{H_1}(A)$, we may even choose $h$ such that $A^{h^2} = A$, so (i) holds. Hence we may suppose that $\langle A^{O_2(H_i)} \rangle$ is abelian for both $i = 1, 2$. Then we see that $O_2(H_i) \leq C_S(A)O_2(G)$. As this is true for both $i$, we get $S \cap L = C_S(A)O_2(G)/O_2(G)$. As $A$ acts quadratically on $O_2(G)$ we may apply Lemma 3.6. Suppose there is a chief factor $V$ in $O_2(G)$ which is the natural module. We have $\langle [V, \bar{A}] \rangle = r^2$, while $|C_V(S \cap L)| = r$. As $[V, \bar{A}]$ is covered by $A$ this is a contradiction. So we have that $Z(L)$ is nontrivial and acts faithfully on $V$. This gives $q = 2$. By Lemma 3.7 we must have $L \cong 3 \cdot A_6$ and the 6–dimensional module is involved in $O_2(G)$. Then by quadratic action we get $A \leq L$. 


As \( \bar{A} \leq O_2(P_1) \cap O_2(P_2) \) and \( P_i \cap L \cong \Sigma_4 \), this implies \(|\bar{A}| = 2\), a contradiction.

Next let \( L \cong F_4(r) \). We have two root groups \( R_1 \) and \( R_2 \) in \( Z(\bar{S} \cap L) \) and by \((**)\) \( \bar{A} \leq Z(O_2(N_L(R_1))) \cap Z(O_2(N_L(R_2))) \). But this group is contained in some \( Sp_4(r) \) as can be seen in [Shi, (1.5), Proposition 2.2 and Theorem 2.1] and we get the assertion by induction.

Next let \( L \cong 2F_4(r)' \). As \( \bar{A} \) acts quadratically we get by Lemma 3.6 that \( \bar{A} \leq R \), a contradiction.

Now let \( L \cong A_n, n \geq 5 \). So we may assume \( n = 7 \) or \( n \geq 9 \). We have \( L\bar{A} \leq \Sigma_n \). If \( n \) is odd, then there is \( \bar{L} \leq L, \bar{L} \cong A_{n-1}, \) which is normalized by \( \bar{S} \). Hence we may assume \( n \) to be even right from the beginning. So \( n \geq 10 \). Let first \( n = 2^m \). Then there is a subgroup \( \bar{L} \leq L \) normalized by \( \bar{A} \) with \( S \cap L \leq \bar{L} \) and \( \bar{L} \) is a subgroup of index at most two in \( \Sigma_2 \wr \mathbb{Z}_2 \). As \( n \geq 16 \) we have \( O_2(\bar{L}) = 1 \) and so we get a contradiction with \((**))\). Let \( m_1, \ldots, m_r \) be the dyadic decomposition of \( n \). Let \( \bar{L} \) be the subgroup of \( L \) with \( S \cap L \leq \bar{L} = L \cap \Sigma_{m_1} \times \cdots \times \Sigma_{m_r} \). By \((**)\) \( \bar{A} \) centralizes any component \( X_1 \) of \( \bar{L} \). So as \(|\bar{A}| > 2\) by \((*)\) and \( \bar{A} \) acts nontrivially on \( \bar{L} \), we see that \( \bar{A} \leq \Sigma_4 \wr \mathbb{Z}_2 \). Now we can embed \( \bar{A} \) into some \( X_2 \cong \Sigma_6 \) or \( \Sigma_5 \), which contradicts \((**))\).

Finally let \( L \) be sporadic. By Lemma 3.7 we get that \( L/Z(L) \cong M_{12}, M_{22}, M_{24}, J_2, Co_1, Co_2, \) or \( Suz \), recall that by \((*)\) \(|\bar{A}| > 2\). Now we choose \( s \in Z(S \cap L \cap \bar{A}) \). By \((**)\) we have \( \bar{A} \leq O_2(C_G(s)) \). If there is some involution \( g \in C_L(s) \) with \( [\bar{A}, \bar{A}^g] \neq 1 \), we have \((i)\). So we may assume that \( \langle \bar{A}^{C_L(s)} \rangle \) is abelian. This gives \( L/Z(L) \cong M_{12}, i = 12, 22, 24 \).

If \( L \cong M_{24} \) there is a subgroup \( \bar{L} \leq L \) with \( S \cap L \leq \bar{L} \) and \( \bar{L} \cong 2^4A_4 \). Now by \((**)\) we have \( \bar{A} \leq O_2(\bar{L}) \). But there is no quadratic foursgroup in \( O_2(\bar{L}) \) according to [MeiStr2].

Next let \( L/Z(L) \cong M_{22} \). Then \( \bar{A} \) normalizes a subgroup \( P \) of \( G/Z(L) \) with \( 2^4A_6 \leq P \leq 2^4\Sigma_6 \). By \((**)\) we have that \( \bar{A} \leq O_2(P) \). Hence we may embed \( \bar{A} \) into a subgroup \( (S)L_3(4) \). But then \((**))\) gives a contradiction.

So we are left with \( L \cong M_{12} \). If \( \bar{A} \not\leq L \), then with [MeiStr2] we see that \( \bar{A} \) cannot be normalized by \( S \cap L \), so we have \( \bar{A} \leq L \). Now in \( L \) there are two parabolics \( P_1, P_2 \) such that \( P_i/O_2(P_i) \cong \Sigma_3 \). By \((**)\) we have that \( \bar{A} \leq O_2(P_1) \cap O_2(P_2) \) and so \( \langle \bar{A}^{C_L(s)} \rangle \) is elementary abelian.
of order 8. Then this group contains an involution $i$ which acts fixed point freely on the 12 points moved by $L$. So $C_L(i) \cong \mathbb{Z}_2 \times \Sigma_5$. Further $S$ contains a Sylow 2-subgroup of $C_L(i)$. As $A \leq C_L(i)$, we get a contradiction by (**).

**Lemma 3.21.** Suppose $M$ and $H$ are $K_2$-groups with $F^*(M) = O_2(M)$ and $F^*(H) = O_2(H)$, which are subgroups of some group $X$. Assume further that $M$ contains a Sylow 2-subgroup $S$ of $H$ and $O_2(M) \leq H$. Finally we assume that there is $Z \trianglelefteq M$, $Z \leq \Omega_1(Z(O_2(M)))$ and $Z \not\leq O_2(H)$. Then one of the following holds.

1. There is some $g \in H$, $g^2 \in N_H(Z)$ with $Z^g \leq S \leq M$, $Z \leq M^g$. Further $1 \neq |Z : C_Z(Z^g)| = |Z^g : C_{Z^g}(Z)|$. In particular $Z$ is an $F$-module.

2. There is some $g \in H$ such that for $L = (Z, Z^g)$ we have
   
   (i) $L/O_2(L) \cong L_2(q)$, $S_z(q)$, $q$ even, or $D_{2u}$, a dihedral group of order $2u$, $u$ odd. Set $q = 2$ in the latter.
   
   (ii) Set $B = Z^g \cap O_2(L) \leq S \leq M$. Then
       
       (a) For the action of $B$ on $Z$ we have $|Z : C_Z(B)| = |B/(B \cap Z)|^2$. If $x \in Z \setminus O_2(L)$, then $C_B(x) = B \cap Z$, $[x, B](Z \cap Z^g) = [Z, B](Z \cap Z^g)$ and $|Z : C_Z(B)| \leq q|B/(B \cap Z)|$.

   (b) In particular $Z$ is a 2$F$-module with offender $B/(B \cap Z)$ and an F+1-module in case of $q = 2$. In all cases we have $|Z : C_Z(B)| < |B/(B \cap Z)|^2$. Moreover if $B$ acts quadratically on $Z$, then $Z$ is an $F$-module.

**Proof.** Up to the last assertion that $|Z : C_Z(B)| < |B/(B \cap Z)|^2$, we find everything for (1) and (2) in Lemma 3.20 where $G = H$ and $A = Z$.

So assume $|Z : C_Z(B)| = |B/(B \cap Z)|^2$. Then $|(Z \cap O_2(L))(Z^g \cap O_2(L))/Z \cap Z^g| = q^2$. Hence we have that $L/O_2(L) \cong L_2(q)$ or $L$ induces $\Sigma_3$ on $(Z \cap O_2(L))(Z^g \cap O_2(L))/Z \cap Z^g$. In both cases $L$ acts transitively on $((Z \cap O_2(L))(Z^g \cap O_2(L))/Z \cap Z^g)^2$ and so $(Z \cap O_2(L))(Z^g \cap O_2(L))$ is abelian. But then $|Z : C_Z(B)| = |B/(B \cap Z)|$, a contradiction.

If $B$ acts quadratically we have that $|B, Z \cap O_2(L)| = 1$. If $L/O_2(L)$ is dihedral, we get that $B$ induces transvections. In the other case we see by Lemma 3.20(ii)(6) that $|B : B \cap O_2(H)| \geq q$. Then $Z$ is an $F$-module with offender $B$. \qed

The last lemma of this chapter is a generalization of Lemma 3.5 to 2$F$-modules.
Lemma 3.22. Let the notation be as in Lemma 3.21. Suppose we have the situation of Lemma 3.21(2). Set $B = B/C_B(Z)$ and suppose there is a component $K$ of $M/C_M(Z)$ with $[K, B] \not\subseteq K$. Then $|\bar{B}| > 4$ and $K \cong L_n(2)$ for some $n$. If $a \in \bar{B}$ with $K^a \neq K$, then $|[Z, a]| = 2^n$ and $\bar{B}$ induces the full transvection group on $[Z, a]$. In particular $|Z^g : B| = 2$. Further $KK^a = K^B$ and $\bar{B}$ acts faithfully on $KK^a$.

Proof. First we show
\[(*)\] $|\bar{B}| > q$.

For this assume $|\bar{B}| \leq q$. Then $|(O_2(L) \cap Z)(O_2(L) \cap Z^g)/Z \cap Z^g| \leq q^2$.

In particular $(Z, Z^g)$ induces $L_2(q)$ on this group, which gives that all elements in the factor group are conjugate. As $(O_2(L) \cap Z)(O_2(L) \cap Z^g)$ is generated by involutions, we get that this group is abelian. Furthermore $|\bar{B}| = q$ and so $\bar{B}$ is a quadratic $F$-module offender on $Z$. By Lemma 3.5 we get the contradiction that $\bar{B}$ has to normalize $K$. This proves $(*)$.

For $b \in \bar{B}$ set $K_b = K$ if $[K, b] = 1$. If $K^b \neq K$ set $K_b = C_{K \times K^b}(b)$.

Recall that there is always some $K_b$ as $K = K_b$ for $b = 1$. Hence this notation makes sense.

Suppose first $q > 2$. By Lemma 3.20 we know that $Y := (Z \cap O_2(L))(Z \cap O_2(L))/Z \cap Z^g$ is a direct sum of natural modules. So let $A_1 \leq Z^g$ such that $A_1 \geq Z \cap Z^g$, $|A_1 : Z \cap Z^g| = q$ and $A_1/Z \cap Z^g$ is contained in one of these modules $V_1$, say. We have $[Z, A_1, A_1] \leq Z \cap Z^g$.

Let $Z \cap Z^g \leq V_2 \leq O_2(L)$ with $V_2/Z \cap Z^g = V_1$. Let $R$ be any hyperplane in $Z \cap Z^g$. As $|(Z \cap V_2)/R|^2 = 2|V_2/R|$, we have the assumptions of Lemma 3.18, and so $V_2/R$ is abelian. Hence as $[A_1, Z] \leq V_2$, $[Z, A_1, A_1] \leq R$. As this is true for any hyperplane, we have that $A_1$ acts quadratically on $Z$. Note that $|A_1| = q > 2$, so by Lemma 3.5 we have three possibilities

1. $[K, A_1] \leq K$.
2. $|A_1 : C_{A_1}(K)| > 2$, $[K, A_1] \not\subseteq K$ and $K \cong L_2(2^n)$. Further $[Z, (K^{A_1})]$ is a direct sum of natural $\Omega^+_4(2^n)$-modules.
3. $|A_1 : C_{A_1}(K)| = 2$ and $[K, A_1] \not\subseteq K$.

We first show

4. $[K_b, A_1] \leq K_b$ for all $b \in \bar{B}$. In particular, taking $b = 1$, we have that $K$ is normalized by $A_1$. 


This will be done in several steps. We fix notation such that \([K_b, \bar{A}_1] \leq K_b\) for a certain \(b\). In particular \([K_b, \bar{B}] \leq K_b\).

(4.1) \([Z, K_b] \not\leq O_2(L)\).

By way of contradiction assume that \([Z, K_b] \leq O_2(L) \cap Z\). Then \(\bar{B}\) acts quadratically on \([Z, (K_b^B)]\). Hence we may apply Lemma 3.5 to \(K_b\) and \(\bar{B}\). Assume first \(|\bar{B} : C_{\bar{B}}(K_b)| = 2\). As \(q > 2\), we have \(C_{\bar{B}}(K_b) \neq 1\) and \(|Z : Z \cap O_2(L)| \geq 4\). Then by Lemma 3.21(2) we see that \(Z \cap O_2(L) = [Z, B](Z \cap Z^g) = [Z, C_{\bar{B}}(K_b)](Z \cap Z^g)\). We have that \(K_b\) acts on \([Z, C_{\bar{B}}(K_b)]\). By quadratic action we have that \([Z, K_b], C_{\bar{B}}(K_b), \bar{B}\) = 1. As \(K_b \leq \langle \bar{B}^{|K_b|} \rangle\), we get \([Z, K_b], C_{\bar{B}}(K_b), K_b\) = 1. Obviously we have \([C_{\bar{B}}(K_b), K_b, Z, K_b]\) = 1. So by the Three-Subgroups-Lemma we obtain \([K_b, Z, K_b], C_{\bar{B}}(K_b)\) = 1 and then also \([Z, K_b], C_{\bar{B}}(K_b)\) = 1, which again with the Three-Subgroups-Lemma implies \([Z, C_{\bar{B}}(K_b), K_b]\) = 1. As \([B, O_2(L) \cap Z] = [Z, C_{\bar{B}}(K_b)]\), we get \([Z \cap O_2(L), K_b]\) = 1. Now \([Z, K_b, K_b]\) = 1 and so \([Z, K_b]\) = 1, a contradiction.

So we have \(\langle K_b^B \rangle \cong \Omega_4^+(2^n)\). As \([Z, K_b] \leq O_2(L)\) by assumption, we get by Lemma 3.5 and Lemma 3.11 that \(Z = [Z, K_b]C(Z(K_b))\). Hence there is some \(y \in C_Z(K_b) \setminus O_2(L)\). For this \(y\) we see \([y, B](Z \cap B) = Z \cap O_2(L)\) Then \([Z, K_b, B] \leq [Z \cap O_2(L), B] = [y, B, B]\). But as \([y, K_b]\) = 1, also \([y, B, K_b]\) = 1 Hence \([Z, K_b, B, K_b]\) = 1, a contradiction as \([Z, K_b]\) contains natural \(\langle K_b^B \rangle\)-modules. So we have shown (4.1).

(4.2) \(C_{\bar{A}_1}(K_b) = 1\).

Assume there is \(1 \neq a \in C_{\bar{A}_1}(K_b)\). Then \(\langle K_b^{\bar{A}_1} \rangle\) acts on \([Z, a, K_b]\) = 1. By quadratic action of \(\bar{A}_1\) we have \([Z, a, K_b]\) = 1. By the Three-Subgroups-Lemma we get that \([Z, K_b]\) is centralized by \(a\) and so by Lemma 3.21 as \(C_{Z}(a) \leq O_2(L), [Z, K_b] \leq O_2(L)\), a contradiction to (4.1). This proves (4.2).

Now as \(|\bar{A}_1| > 2\) we get with (4.2) that we have (2), so \(\langle K_b^{\bar{A}_1} \rangle \cong \Omega_4^+(2^n)\).

In particular by Lemma 3.5 \([Z, \langle K_b^{\bar{A}_1} \rangle]\) is a direct sum of natural modules for \(\Omega_4^+(2^n)\). Let \(W\) be the sum of all such modules which are in \(O_2(L)\). Then \(W\) is a \(\langle K_b, \bar{A}_1 \rangle\)-module. As \([Z, K_b] \not\leq O_2(L)\) by (4.1) there is some module \(V\) for \(\langle K_b, \bar{A}_1 \rangle\) in \(Z\) such that \(V \not\leq O_2(L)\) and \(V/W\) is the natural \(\Omega_4^+(2^n)\)-module. Choose \(y \in V \setminus O_2(L)\). We have \([y, A_1](Z \cap Z^g) = [Z, A_1](Z \cap Z^g)\). As \(|[V/W, \bar{A}_1]| > |\bar{A}_1|\) by Lemma 3.5, we see that \(V \cap B \not\leq W\).

(4.3) \(b = 1\) and \(C_{\bar{B}}(K_b) = 1\). In particular \(K_b = K\).

Suppose there is some \(1 \neq a \in C_{\bar{B}}(K_b)\). Then \([a, V \cap B] = 1\). Hence \(a\) centralizes some element in \(V \setminus W\). As \(a\) normalizes \(O_2(L)\) and \(\langle K_b, \bar{A}_1 \rangle\),
we see that $a$ normalizes also $\langle K_b, A_1 \rangle$-submodules of $V$, which are in $O_2(L)$. Hence $a$ normalizes $W$. So we get $[V, a] \leq V$. But now $[V/W, a] < V/W$, and so $[V, a] \leq W$. As $[W, a] \leq Z \cap Z^a$, $[W, a]$ is a sum of natural modules for $\langle K_b^{A_1} \rangle$ and $[Z \cap Z^a, A_1] = 1$, we see that $[W, a] = 1$. As $V \not\leq O_2(L)$ and so $V/C_V(a) \cong [V, a]$, we have that $[V, a]$ is a natural module for $\langle K_b^{A_1} \rangle$. This gives that $|[V, a] : [[V, a], A_1]| = 2^{2n}$. As $[[V, a], A_1] = [V, a] \cap Z^g$, we see that $|V : V \cap O_2(L)| = 2^{2n}$. In particular $q \geq 2^{2n}$.

But as $C_{\tilde{A}_1}(K_b) = 1$, we have that $q = |\tilde{A}_1| \leq 2^{n+1}$. As $n \geq 2$, this is a contradiction. This proves (4.3).

(4.4) $\langle K^B \rangle = \langle K^{A_1} \rangle$.

Let $B_1 \leq Z$, $Z \cap Z^g \leq B_1$ such that $B_1$ covers another natural module in $Y/Z \cap Z^g$. Let $b \in \tilde{B}_1$. If $K^b \neq K$, then $\tilde{A}_1$ normalizes $K_b$ by (4.3) and so $K_b \leq \langle K^{A_1} \rangle$. Hence $\tilde{B}_1$ normalizes $\langle K^{A_1} \rangle$. As $\tilde{B}$ is generated by such groups, we get (4.4).

Define $W$ and $V$ as above. If $W \neq 1$, then $|W/C_W(A_1)| \geq 2^{2n}$. So $|Y : C_Y(Z^g \cap Y)| \geq 2^{2n}$ and then also $|Y : C_Y(Z \cap Y)| \geq 2^{2n}$, hence $|\tilde{B}| \geq 2^{2n}$. As $C_{\tilde{B}}(K) = 1$ by (4.3) we have $|\tilde{B}| \leq 2^{n+1}$ and $n > 1$, which is not possible. So we have that $W = 1$. Hence $V$ is the natural $\Omega^+_1(2^n)$-module. Let $a \in \tilde{A}_1$ such that $K^a \neq K$. Then $C_{K \times K_a}(a) = K_a \cong K$. Further a Sylow 2-subgroup of $K_a$ together with $a$ acts quadratically on $V$. As $\tilde{A}_1$ acts quadratically we have that $\tilde{A}_1$ projects onto $K_a \times \langle a \rangle$. So we have that $\tilde{B}$ centralizes $a$ and then acts on $K_a$. As $C_{\tilde{B}}(K) = 1$ by (4.3), we see that $\tilde{B}$ contains a subgroup $\tilde{B}$ of index two containing $\tilde{A}_1$, which acts quadratically on $V$. As $V \not\leq O_2(L)$, we have that $|Y : \tilde{B}| \leq 4$. As $[[\tilde{B}, V] \tilde{B}, \tilde{B}] = 1$, we see that $\tilde{B}$ centralizes a subgroup of index at most $2q$ in $V$. Now as $V$ is not an $F$-module for $\tilde{B}$ by Lemma 3.5, we get $|\tilde{B}| \leq q$, which gives $|\tilde{B}| \leq 2q$. But as $q > 2$, and $|\tilde{B}|$ is a power of $q$, we would get $\tilde{A}_1 = \tilde{B}$ and then $\tilde{B}$ acts quadratically on $Z$, which by Lemma 3.21(2) gives that $\tilde{B}$ is an $F$-module offender on $V$, contradicting Lemma 3.5. So we have proved (4).

From (4) we now get that $\langle K^{\tilde{A}_1} \rangle = K$. As $\tilde{B}$ is generated by such subgroups $\tilde{A}_1$, we have the contradiction $[K, \tilde{B}] \leq K$. This shows

(5) $q = 2$.

(5.1) There is some $K_b$ such that $[K_b, \tilde{B}] \nsq K_b$.

Otherwise, if there is no such $K_b$ then for $b = 1$ we have $K_b = K$ and so $[K, \tilde{B}] \leq K$, a contradiction.
For the remainder of the proof we fix $K_b$ such that it satisfies (5.1).

(5.2) $[Z, K_b] \not\leq O_2(L)$. 

If $[Z, K_b] \leq O_2(L)$, then again $B$ acts quadratically on $[Z, K_b]$. Hence by Lemma 3.5 we have one of the cases (2) or (3) above with $A_1$ replaced by $B$. Assume $|B : C_B(K_b)| = 2$. As $|\bar{B}| > q = 2$ by (*) we can choose $1 \neq a \in C_B(K_b)$. Then $\langle K_b^B \rangle$ acts on $[Z, a]$. As $|Z : Z \cap O_2(L)| = 2$, we have that $|[Z, a] : [Z, a] \cap Z^g| = 2$. Therefore $|[Z, a] : C_{[Z, a]}(B)| \leq 2$. If $\bar{B}$ does not centralize $[Z, a]$, then $\bar{B}$ induces transvections on $[Z, a]$.

As $\bar{B}$ does not normalize $K_b$ this is impossible by Lemma 3.5. Hence $\bar{B}$ centralizes $[Z, a]$ and so $|[Z, a], K_b| = 1$ for all $a \in C_B(K_b)$. We have that $[Z, C_B(K_b)](Z \cap Z^g) \cap [Z, K_b]$ is a subgroup of index at most four in $[Z, K_b]$. So $\bar{B}$ centralizes a subgroup of index two in $[Z, K_b]/[Z, K_b] \cap [Z, C_B(K_b)]$, which gives $[K_b, Z] \leq [Z, C_B(K_b)]$ and then $[Z, K_b] = 1$, a contradiction.

Hence we are in case (2), i.e. $|\bar{B} : C_B(K_b)| > 2$. As before by Lemma 3.5 and Lemma 3.11 there is some $y \in C(Z(K_b)) \setminus O_2(L)$. This shows $[y, B](Z \cap B) = Z \cap O_2(L)$. Now $[Z, K_b, B] \leq [Z \cap O_2(L), B] = [y, B, B]$. But as $[y, K_b] = 1$, also $[y, B, K_b] = 1$. In particular $[Z, K_b, B, K_b] = 1$, a contradiction. So we have (5.2).

Fix $a \in \bar{B}$ with $K_b^a \neq K_b$.

(5.3) $[Z, a, \bar{B}] \neq 1$.

Assume $[Z, a, \bar{B}] = 1$. Then by Lemma 3.5 either $|\bar{B} : C_B(K_b)| = 2$, or $K_b \cong L_2(2^r)$ and $[Z, K_b]$ is a direct sum of orthogonal $\Omega^+_4(r)$–modules, $r = 2^n$.

Suppose the latter. As before let $W$ be the sum of all natural modules in $[Z, K_b]$, which are contained in $O_2(L)$ and $V/W$ be a natural $\Omega^+_4(r)$–module. Then there is $y \in V \setminus O_2(L)$ and $[B, y]|(B \cap Z) = [Z, B]/(B \cap Z)$. In particular as $|V : V \cap O_2(L)| = 2$, we see that $V \cap B \not\leq W$. This shows that $B$ normalizes $V$. Now let $c \in C_B(K_b)$. Then we have that $[V, c] \leq W$. As $[V, c] \leq Z \cap Z^g$ and $[B, Z \cap Z^g] = 1$, we get that $[W, c] = 1$ or $[W, c]$ is the natural module. But $[\bar{B}, [W, c]] = 1$ and so $[K_b, [W, c]] = 1$, hence $[W, c] = 1$. If $c \neq 1$, then $[V, c]$ is the natural module. But we have that $|[V, c] : [V, c] \cap Z^g| = 2$ and so $B$ induces transvections on $[V, c]$, a contradiction. So we have that $C_B(K_b) = 1$, i.e. $b = 1$ and $K_b = K$. Assume $W \neq 1$. In the natural module the centralizer of a quadratic fours group is just the commutator of this fours group. Hence we have that $C_B(K) = W \cap Z^g$. So $|W : W \cap Z^g| = |W \cap Z^g| = |W \cap Z|$ and then $|\bar{B}| \geq |W/W \cap Z| \geq r^2$. As the largest quadratic group
in $O^+_2(r)$ is of order $2r$ we have $|B| \leq 2r$, a contradiction. This implies $W = 1$. So we have $V$ is the natural module and then $B$ acts quadratically on $V$. But $[y, B](Z \cap Z^g) = Z \cap O_2(L)$. As $[y, B] \leq [V, B]$, $[B, Z \cap O_2(L)] = 1$, and so $B$ induces transvections on $Z$, a contradiction as $B$ does not normalizes $K$.

So we have $|\tilde{B} : C_{\tilde{B}}(K_b)| = 2$. As $[Z, a, B] = 1 = [Z, B, a]$ by the Three-Subgroups-Lemma, we see that $[Z, C_{\tilde{B}}(K_b), K_b] = 1$. By the Three-Subgroups-Lemma again we get $[K_b, Z, C_{\tilde{B}}(K_b)] = 1$. But as $[Z, K_b] \not\leq O_2(L)$ by (5.2) this shows $C_{\tilde{B}}(K_b) = 1$ and then $|B : B \cap Z| = 2$. Now $B$ induces transvections on $Z$ and so by Lemma 3.5 $B$ has to normalize $K_b$, a contradiction. This proves (5.3).

(5.4) We have that $C_{\tilde{B}}(K_b) = 1$ and then $b = 1$ and $K_b = K$.

By (5.3) $|\tilde{B}| \geq 4$. As $|[Z, a] : [Z, a] \cap Z^g| = 2$, $\tilde{B}$ induces transvections on $[Z, a]$ to a hyperplane. Choose $1 \neq c \in C_{\tilde{B}}(K_b)$ and assume that $[Z, c, K_b] = 1$. Then also $[Z, K_b, c] = 1$ and so $[Z, K_b] \leq O_2(L)$, a contradiction. Hence $K_b$ acts nontrivially on $[Z, c]$. But $a$ induces a transvection on $[Z, c]$, a contradiction as $K_b^a \neq K_b$. This proves (5.4).

In particular we get

(5.5) If $b \neq 1$, then $[K_b, \tilde{B}] \leq K_b$.

Let $b \in \tilde{B}$ with $(KK^a)^b \neq KK^a$. Then $a$ does not normalize $K_b$, a contradiction to (5.5). So we have that $KK^a = \langle K\tilde{B} \rangle$. As $[Z, a, \tilde{B}] \neq 1$ by (5.3), we see that $K_a = C_{K \times K^a}(a) \cong K$ acts faithfully on $[Z, a]$, and so, as $\tilde{B}$ induces transvections to a hyperplane, we get by Lemma 3.3 that $K \cong L_n(2), Sp_{2n}(2), \Omega^{±}_{2n}(2)$ or $A_n$. We further have that $C_{\tilde{B}}(K_a) = \langle a \rangle$ as $C_{\tilde{B}}(K) = 1$ by (5.4).

(5.6) $|\tilde{B}| > 4$.

Assume $|\tilde{B}| \leq 4$. Then $|[Z, a]| \leq 4$, but $K_a$ has to act nontrivially on $[Z, a]$, a contradiction.

By (5.6) $|\tilde{B}| > 4$ and $\tilde{B}$ induces at least a fours group of transvections on $[Z, a]$. This gives

(5.7) $K \cong L_n(2)$.

It remains to prove that $[Z, a]$ is the natural module. In fact we know that $[Z, a]/C_{[Z, a]}(K_a)$ is the natural module. We have that $|\tilde{B}| \leq 2^n$. Then as $[Z, a, a] = 1$ and $|Z : Z \cap O_2(L)| = 2$ we see that $|[Z, a]| \leq |\tilde{B}|/|a| = |\tilde{B}| \leq 2^n$. This shows that $|[Z, a]| = 2^n$ and $\tilde{B}$ induces the full transvection group on $[Z, a]$. \qed
4. Examples

In this chapter we show under which circumstances the examples $M(23)$, $Co_3$, $\Omega_7(3)$, $\Omega^-_7(3)$ and $J_1$ in the main theorem appear. The group $A_{12}$ already appeared in [MaStr, Theorem 1.4].

**Lemma 4.1.** [MaStr, Lemma 4.15] Let $G$ be a group of even type, which is not of even characteristic. If $G$ has standard subgroup with $L \cong 2M(22)$, then $G \cong M(23)$.

**Lemma 4.2.** [Se] Let $G$ be a group of even type, which is not of even characteristic. Let furthermore $L \in \mathcal{L}$ be a standard subgroup with $L \cong 2Sp_6(2)$. If $C_G(L)$ has cyclic Sylow 2–subgroups, then $G \cong Co_3$.

**Lemma 4.3.** Let $G$ be a group of even type, which is not of even characteristic. Let furthermore $L$ be a standard subgroup of $G$. Assume that the following hold:

1. $L \cong L_4(3)$, $U_4(3)$ or $2U_4(3)$ and $C_G(L)$ is a cyclic $2$–group.

Then $G \cong \Omega_7(3)$ or $\Omega^-_7(3)$.

**Proof.** Suppose false. We have that $C_G(L)$ is normal in $S$, $S$ as in (2), and so contains a 2-central involution $z$. By Lemma 2.6 we have that for an involution $t$ in $L \setminus Z(L)$ we get $O_2(C_{L/(z)}(t)) \cong \mathbb{Z}_2 \times Q_8 \ast Q_8$. Now we choose $t$ such that $t \in O_2(C_L(t))^\prime$. Again by Lemma 2.6 we see that $O_3(N_L(O_2(C_L(t)))/O_2(C_L(t)))$ is elementary abelian of order 9. Let $U$ be the full preimage. Then $[U, O_2(C_L(t))] = Q \cong Q_8 \ast Q_8$. In particular $Q \leq C_{C_G(z)}(t)$ and so we may assume that $[S, t] = 1$, i.e. $t \in Z(S)$.

We have that $S$ centralizes $(z, t)$ and so normalizes $U$. Now the Frattini argument provides us with a Sylow 3–subgroup $U_1$ of $U$ such that

(*) \[ S = Q N_S(U_1). \]

Next we try to determine $O_2(C_G(t))$. For this we assume that $C_G(t) \not\leq N_G(L)$. Furthermore we first assume that $C_{C_G(t)}(O_2(C_G(t))) \leq O_2(C_G(t))$. Suppose additionally that there is some $1 \neq u \in U_1$, with $[u, Q \cap O_2(C_G(t))] = 1$. We have that $O_2(C_G(t)) \leq S$, so $[U_1Q, O_2(C_G(t))] \leq O_2(C_G(t)) \cap U_1Q \leq Q$. Hence $[u, O_2(C_G(t))] \leq Q$ and we get

\[ [O_2(C_G(t)), u] = [O_2(C_G(t)), u, u] \leq [Q \cap O_2(C_G(t)), u] = 1, \]

contradicting $C_{C_G(t)}(O_2(C_G(t))) \leq O_2(C_G(t))$.

This shows that $U_1$ acts faithfully on $Q \cap O_2(C_G(t))$. Then $Q \leq O_2(C_G(t))$. By Lemma 2.6 we have $|C_{L/(z)}(Q)| = 4$. Furthermore we
have that \(\text{Out}(L)\) does not contain an elementary abelian group of order 8 by Lemma 2.6. Hence we see that

\[
|\Omega_1(Z(O_2(C_G(t))))| \leq 16.
\]

We set \(Z = \Omega_1(Z(O_2(C_G(t))))\). If \((z, t) = Z\), we get \(N_G((z, t)) = C_G((z, t))\) and then the contradiction \(C_G(t) \leq C_G(z) \leq N_G(L)\). So we conclude that \(|Z| \geq 8\). Then \([Z, U_1] \leq Z\). As \(Z \leq S\) we see \([Z, U_1] \leq Q\) and so \([Z, U_1] \leq Q \cap Z\). As \([Q \cap Z, U_1] = 1\), we have \([Z, U_1] = 1\). Furthermore \([O_2(C_G(t)), U_1] \leq Q\). As \(C_G(Z) \leq N_G(L)\), we now see that \(U_1O_2(C_G(t))/O_2(C_G(t)) \leq C_G(t)/O_2(C_G(t))\). Hence

\[
U_1O_2(C_G(t)) \leq C_G(t).
\]

Let \(U_2 \leq U_1, |U_2| = 3\), with \([U_2, Q] \cong Q_8\). Then \(([Q, U_2], U_2) = X \cong SL_2(3)\). Further \(t \in Z(X)\). As \([O_2(C_G(t)), U_2] = [U_2, Q]\), we see \(X \leq U_1O_2(C_G(t)) \leq C_G(t)\) and so \(X \leq C_G(t)\) and for \(g \in C_G(t)\) we have either \(X^g \cap X = \langle t \rangle\) or \(X = X^g\). The assertion now follows with [MaStr, Lemma 3.2].

So we may assume that \(C_G(O_2(C_G(t))) \not\leq O_2(C_G(t))\). Then

\[
F = E(C_G(t)) \neq 1
\]

as \(O(C_G(t)) = 1\) by the general assumption. Set \(T = S \cap F\). Assume there is \(1 \neq u \in U_1\) with \([F \cap Q, u] = 1\). Then \(Q \neq F \cap Q\) and \(F \cap Q\) is normal in \(C_L(t)\). By Lemma 2.30 \(T \cap Q \leq \langle t \rangle\). We also have \([T, Q] \leq T \cap Q \leq \langle t \rangle\). So \([U_1, S] \leq U_1Q\) and then \([T, U_1] \leq U_1(Q \cap T) \leq U_1\langle t \rangle\). This shows \([T, U_1] = [T, U_1, U_1] \leq [U_1, U_1\langle t \rangle] = 1\). Now \(T \leq C_G([U_1]) \leq C_G(Q)\) and then again \(T/T \cap \langle t \rangle\) has a cyclic normal subgroup \(C_L(t) \cap T/\langle t \rangle\) of index at most 4. This shows that \(F\) is quasissimple.

Assume first \([C_{C_G(t)}(F), U_1] = 1\). As \(C_{C_G(t)}(F)\) is normal in \(C_G(t)\), we get that \([Q, C_{C_G(t)}(F)] = 1\). Hence \(QU_1\) induces an outer automorphism group on \(F\), which centralizes a Sylow 2-subgroup, contradicting [GoLyS4, Lemma 4.1.1]. So we have that \([C_{C_G(t)}(F), U_1] \neq 1\). If \([Q, F] \neq 1\) then we get by Lemma 2.30 that \(C_Q(F) \leq \langle t \rangle\) and then \(Q \cap C_{C_G(t)}(F)F = \langle t \rangle\). But then we have the same contradiction as before. So we have that \([Q, F] = 1\). The Frattini argument now implies that \(C_G(t) = FN_{C_G(t)}(T)\). Further we have \((C_S(Q)/\langle t \rangle)/\langle t \rangle \leq C_S(L)/\langle t \rangle\) as \(N_S(L)/S \cap L\) is abelian. So if \(F(t)/\langle t \rangle\) has nonabelian Sylow 2–subgroups, we get that \(\langle z, t \rangle \leq F(t)\) is centralized by \(N_{C_G(t)}(T)\) and so

\[
C_G(t) = C_{N_G(L)}(t)F.
\]
We are going to prove the same result if $F(t)/\langle t \rangle$ has abelian Sylow 2–subgroups. As $F \in C_2$ we have, that $F$ itself has abelian Sylow 2–subgroups. In particular $t \notin F$. If $|\Omega_1(Z(S))| = 4$, then again $z \in F \langle t \rangle$ and so $N_{C_G(t)}(T) \leq C_G(z)$, as all involutions in $F$ are conjugate. If $|\Omega_1(Z(S))| > 4$, then by application of Lemma 2.6 we see $L \cong L_4(3)$ and $|\Omega_1(Z(S))| = 8$. By Lemma 2.34 we have that $S \cap C_G(L) \leq Z(S)$ and $S = C_S(L) \times ((S \cap L)\langle u \rangle)$. Hence $C_S(Q) = (S \cap C_G(L))\langle t, u \rangle$. If $C_{G(t)}(z) \neq \langle z \rangle$, then $z \in Z(N_G(Z(S)))$ and so $z^G \cap Z(S) = \{ z \}$, contradicting Lemma 2.1 and Lemma 2.2. So $Z(S) = \langle z, t, u \rangle = C_S(Q)$. Hence a Sylow 2–subgroup of $F$ is contained in $Z(S)$. Now we have that $N_{C_G(t)}(T) = N_F(T)C_{C_G(t)}(T)$, which gives again
\[ C_G(t) = C_{N_G(L)}(t)F. \]
As $Q \nsubseteq F$ and $[U_1, S] \leq QU_1$ we have $U_1 \cap F = 1$ and then $C_F(z) = S \cap F$. Hence $U_1$ cannot induce nontrivial inner automorphisms on $F$, so $[F, U_1] = 1$. This now gives $[F, U] = 1$. As $C_G(t) = C_{N_G(L)}(t)F$ by (**) we see that $Q \leq O_2(C_{C_G(t)}(F))$ and then $U$ is normal in $C_{C_G(t)}(F)$. Hence as above we construct a subgroup $X \cong SL_2(3)$ in $U$, with $X \leq C_G(t)$. Again the assertion follows with [MaStr, Lemma 3.2].

So we may assume
\[ Q \leq F. \]

Let first $N$ be a component with $N \cap Q = 1$, then $[S \cap N, Q] \leq S \cap N \cap Q = 1$. As $[S \cap F, Q] \neq 1$, there is at least one component $N$ with $Q \cap N \neq 1$. We now fix such a component $N$ and set $F_1 = \langle N^{U_1} \rangle$. As $Q$ normalizes $N$ we have $F_1 = N_1 * N_2 * \cdots * N_x$, where $x$ divides $|U_1| = 9$. If $x = 9$, then, as any $N_i$ has an elementary abelian section of order 4, we have an elementary abelian section of order $2^{18}$ in $F_1$, which contradicts the structure of $S$. Let $x = 3$. As $U_1$ acts on $S \cap F_1$ and $[S \cap F_1, U_1] \leq [Q, U_1] = Q$, we see by Lemma 2.30 that $Q \leq F_1$. As $x = 3$ there is $1 \neq u \in U_1$, with $[N_i, u] \leq N_i$ for all $i$. Furthermore we have some element $u_1 \in U_1$, which acts transitively on the $N_i$ and normalizes $S \cap F_1$. As $Q \leq F_1$ we see $[u, N_i \cap S] \neq 1$. As $S \cap N_i \langle u_1 \rangle$ is a subgroup, we get that $1 \neq [S \cap N_i, u_1] \leq Q$. But then $|\langle(Q \cap N_i)\langle u_1 \rangle\rangle| \geq 2^6$, a contradiction. So we have $x = 1$ and then $U_1$ normalizes $N_1 = N$. Then $(S \cap N)U_1$ is a subgroup of $G$. As $U_1$ cannot centralizes all components $N$ with $N \cap Q \neq 1$, we get that there is a component $N$ with $N \cap Q > \langle t \rangle$.

The action of $U_1$ on $Q$ shows that either $Q \cap N = Q$ or $Q \cap N$ is a quaternion group. Suppose first that $Q \cap N$ is a quaternion group.
Then by Lemma 2.30 there is some \( s \in S \cap L \) such that \( N^s \neq N \) and \( N^s \cap Q \) also is a quaternion group. As \( Z(N) \geq \langle t \rangle \) [MaStr, Lemma 2.53] implies that \( N \in \mathcal{M} \). In particular the same lemma implies that \( |N/Z(N)| \geq 2^6 \), and hence \( |[s,T]| \geq 2^6 \). As \( |S \cap L : O_2(C_L(t))| \leq 4 \), we now see that \( |[T,s] \cap O_2(C_L(t))| \geq 2^8 \) and then \( |[T,s] \cap Q| \geq 2^4 \). Now \( Q \cap N \) is a quaternion group, which implies that \( [T,s] \cap N \neq \langle t \rangle \), a contradiction. So we have \( Q \leq N \) for some component \( N \). Further \( C_{C_G(z)}(t) \) normalizes \( N \) as it normalizes \( Q \). Now \( z \) induces some automorphism on \( N \), which centralizes a Sylow 2–subgroup and has a solvable centralizer in \( N \) of order \( 2^a \cdot 3^b \), \( b \leq 2 \). As \( N \in \mathcal{M} \) Lemma 2.31 implies that \( N \cong 2L_3(4), 2^2L_3(4), 2SP_6(2), 2U_4(3), 2M_{12}, 2M_{22}, 4M_{22}, 25z(8) \) or \( 2^2Sz(8) \). As \( Q \leq N \), there are involutions in \( N/Z(N) \) which become elements of order 4 in \( N \). So by Lemma 2.33 we are left with \( 2SP_6(2), 2M_{12}, 4M_{22} \). If \( N \cong 2M_{12} \) or \( 4M_{22} \), then by Lemma 2.35 \( |U_1| \) induces an inner automorphism of order at most three. On the other hand by the same lemma \( N \) has no outer automorphism of order three, so \( C_{U_1}(N) \neq 1 \), which contradicts \( C_{U_1}(Q) = 1 \). So we have \( N \cong 2SP_6(2) \).

Now \( U_1 \) has to induce a group of inner automorphisms of order 9. We have that \( Q/U_1 \) is elementary abelian of order 16. Hence let \( \tilde{z} \) be the inner automorphism induced by \( z \), then we see that \( \langle Q/U_1, \tilde{z} \rangle/t \rangle \) is elementary abelian of order 32. By Lemma 2.36 we have that \( \tilde{z} \) corresponds to a transvection in \( Sp_6(2) \). But then a group isomorphic to \( \Sigma_6 \) would be in \( C_G(\langle z, t \rangle) \), a contradiction to \( C_G(z) \leq NG(L) \).

So we have shown that \( C_G(t) \leq NG(L) \). But then \( C_G(t) \) has a subnormal subgroup \( SL_2(3) \). [MaStr, Lemma 3.2] now yields the assertion. \( \square \)

**Lemma 4.4.** Let \( G \) be of even type but not of even characteristic. Let \( L \cong L_2(q) \), \( q \) even, be a standard subgroup with \( C_G(L) \) cyclic. Assume that \( C_G(L) \) contains a 2–central involution \( z \). Then \( q = 4 \) and \( G \cong J_1 \).

**Proof.** Let \( S \) be a Sylow 2–subgroup of \( NG(L) \) containing \( z \). In \( \Omega_1(Z(S)) \) there are three \( NG(L) \)–classes of involutions, \( \{z\}, \Omega_1(Z(S)) \cap L \), and \( z\Omega_1(Z(S)) \cap L \). Hence either \( z^G \cap \Omega_1(Z(S)) = \Omega_1(Z(S)) \) or \( z^G \cap \Omega_1(Z(S)) = \{z\} \). Set \( U = L \cap S \). Then there are at most two abelian subgroups of \( S \) which have the same order as \( E = C_{C_G(L) \cap S}(z) \times U \). In particular conjugacy takes place in \( NG(E) \).

Assume first that \( z^G \cap \Omega_1(Z(S)) \neq \{z\} \). Then in particular \( C_{C_G(L) \cap S}(z) = \langle z \rangle \). As \( Out(L) \) is cyclic, we have that \( z \notin S' \). So we conclude \( \Omega_1(Z(S)) \cap S' = 1 \) and then \( C_G(z) \cong \langle z \rangle \times L_2(q) \). By O’Nan’s lemma [MaStr, Lemma 2.6] we obtain \( q = 4 \) and so \( G \cong J_1 \) by [Ja].
So we may assume that \( z^G \cap \Omega_1(Z(S)) = \{ z \} \). As \( L \) has just one class of involutions, we have that \( z^G \cap (L \times C_{CG(L)\cap S}(z)) = \{ z \} \). By Lemma 2.1 \( L \) must possess some outer automorphism \( u \) with \( u \sim z \) in \( G \). Obviously \( u \not\in C_S(u)' \). In particular also \( z \not\in C_S(u)' \). Hence \( C_{CG(L)\cap S}(u) \leq Z(C_S(u)) \). As \( u \) is not a square in \( Z(C_S(u)) \), we get that the same holds for \( z \). In particular \( C_{CG(L)\cap S}(z)(u) = \langle z \rangle \). If \( z \not\in S' \), then in particular \( S \cap C_G(L) = \langle z \rangle \) and we get a contradiction by Lemma 2.2. So we may assume that \( z \in S' \). Then \( u \sim zu \) by some element in \( C_S(L) \). As \( C_{S\cap CG(L)}(u) \leq Z(C_S(u)) \), we see that \( C_S(u) = \langle u, z, C_U(u) \rangle \). Further \( u\langle z, C_U(u) \rangle \subseteq u^G \). Now we may assume that \( u \sim z \) in \( N_G(C_S(u)) \). In particular \( C_S(u) \) contains a hyperplane \( H \) with \( z \not\in H \) but \( zH \leq z^G \). Choose \( u_1 \in C_U(u)^2 \). Then neither \( u_1 \) nor \( zu_1 \) are in \( zH \), so both are in \( H \) and so \( z \in H \), a contradiction. \( \Box \)

5. The central case

In this chapter we fix a Sylow 2–subgroup \( S \) of \( G \) and assume that \( G \) is of even type but not of even characteristic. Furthermore we assume that \( G \) is not one of the exceptional groups in the main theorem, i.e. \( G \not\in \Omega_7(3), \Omega_8(3), A_{12}, Co_3, M(23) \) or \( J_1 \).

This means by [MaStr, Theorem 1.4] that there is some \( 1 \neq z \in \Omega_1(Z(S))^r \), which possesses a standard component \( A_z \). Furthermore \( C_G(A_z) \) has cyclic Sylow 2–subgroups.

We will prove:

**Proposition 5.1.** \( z \not\in A_z \).

and

**Proposition 5.2.** \( A_z \) is a simple group of Lie type in characteristic two or isomorphic to \( J_2 \) or \( M(24)' \). Further \( A_z \) is not isomorphic to \( L_2(q), S_2(z), 2F_4(q)', q \) even, \( L_3(4), Sp_{2n}(2), G_2(2)', L_4(2), U_4(2), A_6 \) or \( L_3(2) \).

We first are going to prove Proposition 5.1. For this until further notice we assume \( z \in A_z \) and aim for a contradiction. By [MaStr, Lemma 2.53] we have that \( A_z/Z(A_z) \in \mathcal{M} \). For the proof we consider the various groups in \( \mathcal{M} \).

**Lemma 5.3.** \( A_z/Z(A_z) \not\cong S_z(8) \).

**Proof.** Assume \( A_z/Z(A_z) \cong S_z(8) \). Let \( 1 \neq x \in S, x^2 = 1 \). Then, as \( C_S(A_z) \cap A_z = \langle z \rangle \), we see that \( x = ab, a \in C_S(A_z) \) and \( b \in A_z \), where
\(a^2, b^2 \in \langle z \rangle\). By Lemma 2.33 \(z\) is not a square in \(S \cap A_z\). In particular \(b^2 = 1\). But then also \(a^2 = 1\), which shows that \(\Omega_1(S) = \Omega_1(S \cap A_z)\). Hence \(\Omega_1(S)\) is elementary abelian of order 16. Furthermore \(\Omega_1(S) = J(S)\). So \(N_G(J(S))\) controls \(G\)-fusion of involutions in \(S\).

If \(z^G \cap S \neq \{z\}\), then all involutions in \(S\) are conjugate. But then
\[|N_G(J(S)) : N_G(J(S))| = 15,\]
and \(N_G(J(S))/C_G(J(S))\) is a subgroup of \(GL_4(2) \cong A_8\) of order divisible by \(3 \cdot 5 \cdot 7\). As \(S/C_S(J(S))\) is abelian, we get a contradiction by Lemma 2.38. So \(z^G \cap S = \{z\}\) which contradicts Lemma 2.1.

Lemma 5.4. [Se] \(A_z/Z(A_z) \not\cong F_4(2)\) or \(G_2(4)\).

Lemma 5.5. [EgaYo] \(A_z/Z(A_z) \not\cong \Omega_8^+(2)\).

Lemma 5.6. \(A_z/Z(A_z) \not\cong U_6(2)\).

Proof. [DaSo, Theorem 3.1]. In fact there is shown that \(G \cong M(22)\). But then \(z \notin Z(S)\).

Lemma 5.7. \(A_z/Z(A_z) \not\cong 2E_6(2)\).

Proof. [Str1]. In fact in [Str1, (2.2)] it is shown that \(z \notin Z(S)\).

Lemma 5.8. \(A_z/Z(A_z) \not\cong HiS, M_{12}, M_{22}, J_2, Suz, Co_1\) or \(Ru\).

Proof. Suppose false. Application of [So] shows \(A_z/Z(A_z) \not\cong HiS\). In the cases of \(A_z/Z(A_z) \cong M_{12}\) or \(M_{22}\) we get a contradiction with [HaSo]. The remaining cases are treated in [Fin1] and [Fin2].

Lemma 5.9. \(A_z/Z(A_z) \not\cong F_2\).

Proof. If \(A_z \cong 2F_2\) then by [DaSo, (5.5)] we get \(z \notin Z(S)\).

Lemma 5.10. \(A_z/Z(A_z) \not\cong L_3(4)\).

Proof. Suppose \(A_z/Z(A_z) \cong L_3(4)\). As \(A_z \in C_2\) we have by [MaStr, Definition 1.1] that \(Z(A_z) = \langle z \rangle\). According to Lemma 2.20 there are exactly two elementary abelian groups of order 16 in \((S \cap A_z)/\langle z \rangle\). Let \(E\) be the preimage of such a group. Again by Lemma 2.20 \(A_5\) acts transitively on \((E/\langle z \rangle)^3\). So we see that \(E\) is elementary abelian of order 32. Let \(C_S(A_z)\) be cyclic of order \(2^n\), then by Lemma 2.20 there are exactly two abelian subgroups of type \((2,2,2,2,2^n)\) in \(S \cap A_z C(A_z)\). Let \(F\) be an elementary abelian group of order 32 in \(S\). Assume there is some \(t \in F \setminus A_z C(A_z)\). As \(m_2(C_{A_z/\langle z \rangle}(t)) \leq 2\) by Lemma 2.23(3), we get that \(|F \cap A_z C(A_z)| \leq 8\). But then \(F\) has to induce a fours group of outer automorphisms on \(A_z/\langle z \rangle\). Choose \(f_1 \in F\) such that \(f_1\) centralizes \(A_5\) in \(A_z/\langle z \rangle\). Then \(F\) induces an outer automorphism on \(A_5\), which gives the contradiction \(m_2(C_{A_z/\langle z \rangle}(F)) \leq 1\). Hence any elementary abelian
subgroup of order 32 in $S$ is contained in $A_zC_G(A_z)$. By Lemma 2.20 there are exactly two abelian groups $E_1$, $E_2$ of type $(2,2,2,2^n)$ contained in $S$. Set $E_3 = \Omega_1(E_1 \cap E_2)$. As again by Lemma 2.20 $E_1E_2$ is a Sylow 2–subgroup of $A_zC_G(A_z)$, we have $E_3 = Z(S \cap A_z)$ and $|E_3| = 8$.

Suppose $z^G \cap A_zC(A_z) \neq \{z\}$. Let $t \in A_zC(A_z)$, $t \neq 1$, $t \sim z$ in $G$. By Lemma 2.20 any involution in $A_zC_G(A_z)$ is conjugate in $A_z$ to some involution in $E_i$, $i = 1, 2$. On $\Omega_1(E_i)$ we have that $N_{C_G(z)}(E_i)$ induces orbits of length 1, 15 and 15. Hence we see that $z^{N_G(E_i)} \neq \{z\}$. This implies that $C_S(A_z)$ is of order two and so both $E_i$ are elementary abelian. We have that $N_G(E_1) \not\leq C_G(z)$. As $|z^{N_G(E_1)}| = 31$, this shows that $|N_G(E_1) : N_{C_G(z)}(E_1)| = 31$. Now all involutions in $A_z$ are conjugate in $G$. As $N_{C_G(z)}(E_1)/E_1 \cong A_5$, $A_5 \times \mathbb{Z}_3$, $\Sigma_5$ or $(A_5 \times \mathbb{Z}_3) : 2$, we get that $N_G(E_1)/E_1$ has the order $2^{2} \cdot 3 \cdot 5 \cdot 31$, $2^{3} \cdot 3 \cdot 5 \cdot 31$, $2^{2} \cdot 3^{2} \cdot 5 \cdot 31$ or $2^{3} \cdot 3^{2} \cdot 5 \cdot 31$, respectively. As the normalizer of a Sylow 31–subgroup in $GL_5(2)$ has order $31 \cdot 5$ we get a contradiction with Sylow’s theorem. So we have shown

$$z^G \cap A_zC(A_z) = \{z\}.$$

Again let $t \in z^G \cap S$, $z \neq t$ and $E_1$, $E_2$ as above. By Lemma 2.20 we have that $N_{A_z}(E_1E_2) = E_1E_2(\rho)$, where $o(\rho) = 3$ and $\rho$ acts fixed point freely on $E_i/C_S(A_z)$ for $i = 1, 2$. By Lemma 2.20 we have that $t$ normalizes $E_1E_2$. By (1) and the Frattini argument we may assume that $t$ normalizes $\langle \rho \rangle$.

Suppose first $\rho^t = \rho^{-1}$. Assume further that $[E_i, t] \leq E_i$ for both $i = 1, 2$. As $\rho$ acts fixed point freely on $E_i/E_i \cap E_2$ for both $i$, there is $e_i \in E_i$ with $t^e_i = f_i t$, $f_i \in \Omega_1(E_i) \setminus E_3$, $i = 1, 2$. So we have that $[f_1 f_2, t] = 1$. Now $t^{e_1e_2} = f_1^{e_2} f_2 t$. Further $f_1^{e_2} = f_1^{e_2} r$, with $1 \neq r \in E_3 \setminus \langle z \rangle$. By Lemma 2.20 $f_1 f_2$ is of order four. So $1 \neq u = (f_1 f_2)^2 = (f_1^{e_2} f_2)^2$. Hence $t \sim u t$. This shows that

$$\langle z, t, u \rangle \cap \Phi(C_S(t)) = \langle u \rangle.$$

By Lemma 2.22 we see that $\Phi(S) \leq S \cap A_zC_G(A_z)$. So we have that $t, zt \not\in \Phi(C_S(t))$. As $z^G \cap \Phi(S) \subseteq \{z\}$ by (1), we see that $z \not\in \Phi(C_S(t))$, which shows that

$$\langle z, t, u \rangle \cap \Phi(C_S(t)) = \langle u \rangle.$$

Assume now that $E_1 = E_2$. We will show that also in this case (2) and (3) hold. Choose $e_1 \in E_1 \setminus E_3$. Then $e_2 = e_1^r \in E_2$. Now $t \sim (e_1 e_2)^2 t$. This shows that $E_3 = \langle z, (e_1 e_2)^2, r \rangle$, with $x = [t, r] \neq 1$, as $[\rho, t] \neq 1$.
and \(C_E(\rho) = \langle z \rangle\). In particular \(x \neq z\).

Suppose that \(\langle x, (e_1e_2)^2 \rangle = \langle z, (e_1e_2)^2 \rangle\). Then \(t \sim zt\). Again we have that \(\Omega_1(\langle Z(C_S(t)) \rangle) = \langle z, t, (e_1e_2)^2 \rangle\). In \(G\) we have that \(t \sim tz \sim t(e_1e_2)^2 \sim tz(e_1e_2)^2 \sim z\). Further neither \((e_1e_2)^2\) nor \(z(e_1e_2)\) are conjugate to \(z\) in \(G\). This shows that \(N_G(C_S(t))\) normalizes \(\langle (e_1e_2)^2, z(e_1e_2)^2 \rangle\). But as \(z^G \cap \langle (e_1e_2)^2, z \rangle = \{z\}\), we have that \(N_G(C_S(t)) \leq C_G(z)\), and so \(C_S(t)\) is a Sylow 2-subgroup of \(C_S(t)\). As \(C_S(t) \neq S\), \(t\) cannot be conjugate to \(z\) in \(G\), a contradiction.

So we have that \((e_1e_2)^2 = x\). Hence \(e_1e_2r \in C_S(t)\). As \((e_1e_2)^2 = (e_1e_2)^2\), we again get (2) and (3) with \(u = (e_1e_2r)^2\).

Now we show that \([\rho, t] = 1\). Otherwise (2) and (3) hold. We have that \(\langle u \rangle\) is normalized by \(N_G(C_S(t))\). Let \(T \leq C_G(t)\) with \(|T : C_S(t)| = 2\). Then we obtain for \(g \in T \setminus C_S(t)\) that \([g, \langle u, t \rangle] = 1\) and so \(z^g = zt\) or \(ztu\). But in \(G\) we have \(zt \sim ztu\). Now \(z^G \cap \langle z, u, t \rangle = \{z, t, tu, zt, ztu\}\) and so \(\langle u, zu \rangle\) is normal in \(T\), which shows \(z \in Z(T)\), a contradiction.

So we have shown that
\[
[\rho, t] = 1.
\]

Set \(E_3 = \langle z, r, s \rangle\), where we choose notation such that \([E_3, \rho] = \langle r, s \rangle\). As \(t\) and \(\rho\) normalize \(E_3\) and \([t, \rho] = 1\) we force \([E_3, t] = 1\). Set \(F = \langle E_3, t \rangle\). Then \(F\) is elementary abelian of order 16. Further we have that \(N_{A_4}(F)\) is the preimage of \(C_{(S \cap A_4)/E_3}(t)\). Hence \(N_{A_4}(F)\) induces \(A_4\) on \(F\). We first show
\[
(4) \quad z^{N_G(F)} = \{z\}.
\]

Suppose false. We have that \(N_{A_4}(F)\) induces orbits of length 1 \((z)\), 3 \((r\) and \(zr)\) and length 4 \((t, zt)\). As \(z\) is not conjugate to \(r\) or \(zr\) by (1), we see that \(z\) has 5 or 9 conjugates under \(N_G(F)\). If \(z\) has 9 conjugates, then all the other elements generate \(\langle z, s, r \rangle\), a contradiction. So we see that \(z\) has 5 conjugates. In particular all \(N_G(F)\)-orbits have a length divisible by 5, so we must have an orbit of length 10. This shows that \(r \sim zr\) in \(G\). As \(z, r \in \Omega_1(\langle Z(S) \rangle)\), we have that \(zr \sim r\) in \(N_G(S)\). But \(\Omega_1(\langle Z(S) \rangle) \leq A_2\) and so \(z^G \cap \Omega_1(\langle Z(S) \rangle) = \{z\}\) by (1). Hence \(N_G(S) \leq C_G(z)\), contradicting \(zr \not\sim r\) in \(C_G(z)\). So we proved (4).

We have that \(F \cap A_2 = C_{S \cap A_2}(t)\) and so \(F \cap A_2 = \Omega_1(C_{S \cap A_4}C_G(A_2))(t)\). As \(N_G(C_S(t)) \not\leq C_G(z)\), we conclude from (4) that \(N_G(C_S(t)) \not\leq N_G(F)\). Hence we get that \(|C_S(t) : C_S(F)| = 2\) and \(C_S(t) = C_S(F)F^g\), for
some \( g \in N_G(C_S(t)) \). So we have that \( \Omega_1(Z(C_S(t))) = \langle t, z, u \rangle \), where \( \langle u \rangle = \Omega_1(Z(C_S(t))) \cap \Phi(C_S(t)) \). Further it shows that there are exactly two conjugates of \( F \) in \( C_S(t) \). In particular \( O^2(N_G(C_S(t))) \) normalizes \( F \) and so is contained in \( C_G(z) \). Hence \( |z^{N_G(C_S(t))}| \) is a power of two. Now we may assume that \( z \sim t \) in \( N_G(C_S(t)) \). As \( z \not\sim zu \) in \( G \) by (1), we have that also \( t \not\sim tu \) in \( N_G(C_S(t)) \) \( \leq C_G(u) \). As \( N_{C_G(z)}(C_S(t)) \not\leq C_G(t) \), we obtain that \( t \sim tz \) or \( tzu \) in \( N_{C_G(z)}(C_S(t)) \). So as \( |z^{N_G(C_S(t))}| \) is even and \( z \not\sim u \), we get that both \( zt \) and \( ztu \) have to be conjugate to \( z \) in \( N_G(C_S(t)) \) \( \leq C_G(u) \), but this again would imply \( z \sim zu \), a contradiction to (1). This final contradiction proves the lemma.

We are going to prove Proposition 5.1. By [MaStr, Lemma 2.53] we have that \( A_z/Z(A_z) \in \mathcal{M} \). The groups in \( \mathcal{M} \) are given in [MaStr, Definition 2.51(a)]. According to Lemma 5.3 through Lemma 5.10 we are left with \( A_z/Z(A_z) \cong Sp_6(2), M(22) \) or \( U_4(3) \). By Lemma 4.2 \( A_z/Z(A_z) \not\cong Sp_6(2) \), by Lemma 4.1 \( A_z/Z(A_z) \not\cong M(22) \) and finally by Lemma 4.3 \( A_z/Z(A_z) \not\cong U_4(3) \). This proves Proposition 5.1.

Next we will prove Proposition 5.2. For this we first go over all components \( A_z \), which are not of Lie type in characteristic two or \( J_2 \) or \( M(24) \). We furthermore show that the groups of Lie type in characteristic two, which were excluded in Proposition 5.2 also do not appear. The main ingredient of the proof is the interplay between Glauberman’s \( Z^* \)-theorem and Thompson’s transfer lemma.

We begin by eliminating the sporadic groups and some groups in characteristic three.

**Lemma 5.11.** \( A_z \not\cong M_{23}, J_3, Th, Ru, M_{24}, J_4, Co_1, Co_2, F_2 \) or \( F_1 \).

**Proof.** By Lemma 2.12 in all cases we have \( \text{Out}(A_z) = A_z \). So \( C_G(z) = C_{C_G(A_z)}(z) \times A_z \). Further by [MaStr, Lemma 2.34] \( |Z(S \cap A_z)| = 2 \). Hence either \( z^G \cap Z(S) = \{z\} \) or all involutions in \( Z(S) \) are conjugate in \( G \). But \( z \not\in S' \) and as \( S \) is not abelian, we have \( Z(S) \cap S' \neq \emptyset \). So we get \( z^G \cap Z(S) = \{z\} \). As \( S = (S \cap A_z)C_S(A_z) \) this by Lemma 2.2 contradicts the simplicity of \( G \). \( \square \)

**Lemma 5.12.** \( A_z \not\cong HiS, Suz \) or \( M(22) \).

**Proof.** Suppose false. Let \( S \) be a Sylow 2–subgroup of \( N_G(A_z) \). Then we have by [MaStr, Lemma 2.34] \( \Omega_1(Z(S)) = \langle z, t \rangle \), with \( t \in A_z \). Further by [GoLyS3, Table 5.3m, 5.3o, 5.3t] we see that

\[
C_G(\Omega_1(Z(S)))/O_2(C_G(\Omega_1(Z(S)))) \cong \Sigma_5, U_4(2) \text{ or } U_4(2) : 2.
\]
So $\langle t \rangle = C_G(\Omega_1(Z(S)))^{(\infty)} \cap \Omega_1(Z(S))$. In particular

\begin{align*}
(1) \quad z^G \cap \Omega_1(Z(S)) &= \{z\}.
\end{align*}

Next we show that

\begin{align*}
(1) \quad z^G \cap A_zG(A_z) &= \{z\}.
\end{align*}

Let first $A_z \cong HiS$ or $Suz$. Choose $u \in A_z$, $u \not\approx t$. Then by [GoLyS3, Table 5.3m], [GoLyS3, Table 5.3o] we have that $C_{A_z}(u) = \langle u \rangle \times PGL_2(9)$ or $(V_4 \times L_3(4)) : 2$, respectively. As again by [GoLyS3, Table 5.3m] or [GoLyS3, Table 5.3o] no outer automorphism of $HiS$ centralizes $PGL_2(9)$ and no outer automorphism of $Suz$ centralizes $L_3(4)$ we see that $\Omega_1(Z(C_S(u))) = \langle z, t, u \rangle$. Assume that $z$ is conjugate to $u$ or $zu$ in $G$. We will denote this element by $v$. So let $g \in G$ with $z^g = v$. Then obviously $z$ centralizes in $A_z$, a subgroup $PGL_2(9)$, $L_3(4)$, respectively. So $z \in A_zC(A_z)$.

Hence $E(C_G(z) \cap C_G(v)) = F$ is normalized by $g$. We now show that we may assume $t \in F$. For this we choose a Sylow $2$-subgroup $T$ of $C_{A_z}(v)$ and $T_1 \leq A_z$ with $|T_1 : T| = 2$. In the first case, $A_z \cong HiS$, we have that $T' \leq F$, and so we have a $2$-central involution in $F$, in particular we can assume that $t \in F$. In the second case, $A_z \cong Suz$, we have by Lemma 2.20 exactly two elementary abelian subgroups $F_1$, $F_2$ of order $64$ in $T$ and $|F_1, F_2| \leq F$. Hence again $F$ contains a $2$-central involution and we may assume $t \in F$. As all involutions in $A_6$ and $L_3(4)$ are conjugate, we may assume that $t^g = t$. But in $C_G(z)$ we have that $u \sim ut$ and so $v \sim vt$, while $z \not\sim zt$ by $(*)$, a contradiction. This proves (1) in these cases.

Assume finally $A_z \cong M(22)$. By [GoLyS3, Table 5.3t] we have a subgroup $H \cong 2^{10}M_{22}$ in $A_z$. By Lemma 3.3 the group $M_{22}$ does not possess an $F$-module. Hence $O_2(H) = J(S \cap A_z)$ and so $J(S \cap A_z)$ is the only elementary abelian subgroup of order $2^{10}$ in $S \cap A_z$. In particular in $S$ there is exactly one abelian subgroup $E$, which is a direct product of an elementary abelian group of order $2^{10}$ and a cyclic group of order $2^m$, where $|C_S(A_z)| = 2^m$. We see from [GoLyS3, Table 5.3t] that involutions of type $2A$ of $A_z$ are centralized by $L_3(4)$ in the group $M_{22}$ in $H$ above. Hence $H$ induces one orbits of length $22$. The product of two involutions in this orbit gives an orbit of length $231$. As $A_z$ has exactly three classes of involutions and $H$ controls fusion in $J(A_z \cap S)$, we have a third orbit of length $770$. Further any involution of $A_zC_G(A_z)$ is conjugate to one inside of $E$. So we get that $N_G(E)$ controls fusion in $A_zC_G(A_z)$. In particular if $\langle z \rangle \neq C_S(A_z)$ then $\Phi(E) = \langle z \rangle$ and we have (1). So we may assume that $\langle z \rangle = C_S(A_z)$. By Lemma 2.2 and $(*)$ we have that $C_G(z)/\langle z \rangle \cong \text{Aut}(M(22))$. We now obtain that $z \not\in C_G(z)'$.
and so \( z \not\in u \in A_z \) with \( C_{A_z}(u) \cong 2U_6(2) \). In particular there is at least one orbit of length 22, which cannot be fused with \( z \). As \( |z^{N_G(E)}| \) is odd, we get, just by checking all possibilities, that \( |z^{N_G(E)}| = 23, 771, 793, 1541 \) or 1563. As \( N_G(E)/E \) is a subgroup of \( GL_{11}(2) \) and 771, 7793, 1541 and 1563 do not divide the order of \( GL_{11}(2) \), we conclude \( |z^{N_G(E)}| = 23 \). As \( N_{N_G(A_z)}(E)/E \cong Aut(M_{22}) \) and \( |z^{N_G(E)}| = 23 \), we obtain \( |N_G(E)/E| = 2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \). As \( 2^{11} - 1 = 23 \cdot 89 \), we see that a Sylow 23 subgroup of \( N_G(E)/E \) is just centralized by itself. Now with Sylow’s theorem we receive that a Sylow 23 subgroup is normalized by a cyclic group of order 22. Hence this group acts on \( z^{N_G(E)} \) by fixing a point. In particular \( N_{N_G(A_z)}(E)/E \) contains a cyclic group of order 22. Then \( Aut(M_{22}) \) contains a cyclic group of order 22, which contradicts \([GoLyS3, Table 5.3c]\). Hence (1) holds.

By Lemma 2.1 there is some involution \( u \sim z \), which induces an outer automorphism on \( A_z \). If \( C_S(A_z) = \langle z \rangle \) we get a contradiction with Lemma 2.2. Hence

\[
(2) \quad C_S(A_z) > \langle z \rangle.
\]

As \( |Aut(A_z)| = 2 \), we have that \( \Phi(C_S(u)) \leq A_z C_S(A_z) \) and \( u \not\in \Phi(C_S(u)) \). As \( N_G(C_S(u)) \not\leq C_G(z) \), we see by (1) that \( z \not\in \Phi(C_S(u)) \). This implies \( C_{C_S(A_z)}(u) = \langle z \rangle \). In particular \( u \sim uz \) by (2). Now there is a fours group \( V = \langle z, s \rangle \leq C_S(u) \), \( s \in A_z \), not containing \( u \) such that \( uV \subseteq u^G \). Hence there must be another fours group \( W \) such that \( z \not\in W \) and all involutions in \( zW \) are conjugate. We see that \( W \cap C_G(A_z)A_z \not= 1 \), which contradicts \( z^G \cap A_z C_G(A_z) = \{z\} \). This contradiction combined with Lemma 2.1 proves the lemma.

\[ \square \]

**Lemma 5.13.** \( A_z \not= G_2(3), G_2(2)' \), \( M_{12} \) or \( M_{22} \).

**Proof.** By [MaStr, Lemma 2.35] we have \( \Omega_1(Z(S)) = \langle z, t \rangle \), with \( t \in A_z \). We first show that

\[
(1) \quad z^G \cap \Omega_1(Z(S)) = \{z\}.
\]

Otherwise under \( N_G(C_G(\Omega_1(Z(S)))) \) all elements in \( \Omega_1(Z(S)) \) are conjugate. Let \( P \) be a Sylow 3-subgroup of \( C_G(\Omega_1(Z(S))) \). By Lemma 2.7 and Lemma 2.8 we have that \( t \in W = [O_2(C_G(\Omega_1(Z(S)))), P] \) while \( z \not\in W \) as \( W \leq A_z \), a contradiction. This proves (1).

Next we show

\[
(2) \quad z^G \cap (A_z C_G(A_z)(z)) = \{z\}.
\]
Assume false. If $A_z \not\cong M_{12}$ then by Lemma 2.7 all involutions in $C_G(A_z) \times A_z$ are conjugate to $z, t \in A_z$ or $zt$ and so are conjugate into $\Omega_1(Z(S))$. Hence (2) follows from (1). So we may assume $A_z \cong M_{12}$. Let $i \in C_S(A_z)A_z$, $i \neq z$ and $i \sim z$ in $G$. We have $\Omega_1(Z(C_S(i))) = \langle z, i, t \rangle$. By (1) and Lemma 2.8(ii) we see $C_{A_z}(i) \cong \mathbb{Z}_2 \times \Sigma_5$. Let $z^g = i$. Then $z$ is some involution in $C_G(i)$ which centralizes $\Sigma_5$ there. By Lemma 2.8(iv) this shows that $z \in \langle i, A_i \rangle$. And so $i \sim z$ in $N_G(E(C_E((i, z))))$, i.e. $g$ normalizes $E(C_E((i, z)))$. By Lemma 2.8(iii) we may assume $t^g = t$. Further $i \sim it$ under the action of $S$, while $z \not\sim zt$ by (1), a contradiction.

Suppose now that there is some outer automorphism $i$ of $A_z$ with $i \sim z$ in $G$. As $i \not\in \Phi(C_S(i))$, we get by (2) that also $z \not\in \Phi(C_S(i))$, which implies that $\langle z \rangle = C_{C_S(A_z)}(i)$. Further by Lemma 2.2 $C_{C_S(A_z)}(z) > \langle z \rangle$. Hence $i \sim iz$.

Let now first $A_z \cong G_2(2)'$ or $M_{22}$. Then application of Lemma 2.7 shows $C_{A_z}(i) \cong SL_2(3)$, $E_6L_3(2)$, or $2^4F_{20}$. So in all cases we see $\Omega_1(Z(C_S(i))) = \langle i, z, t \rangle$. Furthermore we notice that $i \sim iz \sim it \sim itz$.

Now $\langle t, zt \rangle$ is generated by the involutions in $\Omega_1(Z(C_S(i)))$ which are not conjugate to $z$ in $G$. Then $\langle z, t \rangle \leq N_G(C_S(i))$. By application of (1) we get $N_G(C_S(i)) \leq C_G(z)$ but $C_S(i) \neq S$, contradicting $i \sim z$.

So we have $A_z \cong G_2(3)$ or $M_{12}$. By Lemma 2.7 and Lemma 2.8 we get $C_{A_z}(i) \cong L_2(8) : 3$ or $\mathbb{Z}_2 \times A_5$, respectively. In both cases $C_S(i)$ is elementary abelian and all involutions in $i(C_S(i) \cap (C_G(A_z)A_z))$ are conjugate to $i$ in $C_G(z)$. As $z \sim i$ in $N_G(C_S(i))$ there is some elementary abelian group $E \leq C_S(i)$ of order 8 with $z^G \cap zE = zE$, $z \not\in E$. Hence we have that $|E \cap A_z| \geq 2$. But this contradicts $z^G \cap zA_z = \{z\}$ by (2). This final contradiction by Lemma 2.1 proves the lemma.

We now start to exclude the exceptional cases in Proposition 5.2.

**Lemma 5.14.** $A_z \not\cong 2F_4(q)'$.

*Proof.* Suppose false and assume first $O^2(C_G(z)) = A_z \times C_G(A_z)$. By [MaStr, Lemma 2.31] we see that $Z(S) \cap A_z \leq S'$. In particular $z^G \cap \Omega_1(Z(S)) = \{z\}$ as $z \not\in S'$. But then by Lemma 2.2 we get $z \not\in G'$, a contradiction.

So we have $O^2(C_G(z)) \neq A_z \times C_G(A_z)$. If $q \neq 2$ then by [MaStr, Lemma 2.24] $A_z$ has just outer automorphisms of odd order. Hence we have $q = 2$. Further we have that $N_G(A_z)/C_G(A_z) \cong 2F_4(2)$. By [MaStr,
Lemma 2.24] we know that there are no involutions in $^2F_4(2) \setminus ^2F_4(2)'$. In particular $Z(S) \cap A_z \leq \Omega_1(S)'$, while $z \not\in \Omega_1(S)'$. Hence again

\[ z^G \cap \Omega_1(Z/S) = \{z\}. \]

As $|\Omega_1(Z(S))| = 4$ and fusion in this group is controlled by $N_G(\Omega_1(Z(S)))$, we get with (*)&

(1) No two involutions in $\Omega_1(Z(S))$ are conjugate in $G$.

Let $i \in C_G(z) \setminus \langle z \rangle$, $i \sim z$ in $G$. Then $i \in C_G(A_z)A_z$. Furthermore by Lemma 2.9 $\Omega_1(Z(C_S(i))) = \langle z, i, r \rangle$, where $r$ is 2-central in $A_z$. In the notation of [MaStr, Lemma 2.31] we have that $C_{A_z}(i) \leq P_1$. This shows

\[ \Omega_1(Z(O_2(C_G(i)))) = \langle z, i, r, r_1 \rangle, \] where $\langle r, r_1 \rangle = Z_2(S \cap A_z)$.

Thus

\[ r \sim r_1 \sim rr_1 \text{ in } A_z. \]

Additionally

(3) \[ i \sim ir \sim ir_1 \sim irr_1 \text{ and } zi \sim zir \sim zir_1 \sim zirr_1. \]

Let now $g \in G$ with $z^g = i$. We have that $z$ is an involution in $C_G(A_i)A_i$, which is centralized by $C_{C_G(z)}(i)$. Furthermore $i^g$ also is contained in $C_G(A_z)A_z$ and centralized by $C_{C_G(z)}(i)^g$. As all involutions in $A_i$ centralizing a subgroup isomorphic to $C_{A_i}(z)$ are conjugate, we may choose $g$ such that

\[ C_G(\langle i, z \rangle)^g = C_G(\langle i, z \rangle). \]

Hence we have that $i \sim z$ in $H = N_G(\langle i, z, r, r_1 \rangle)$. Application of (1), (2) and (3) show that $|z^H| = 5$ or 9. In the latter case $\langle zr, r, r_1 \rangle$ is the subgroup generated by all those involutions, which are not conjugate to $z$. But then (*)& implies $H \leq C_G(z)$, a contradiction.

Thus $|z^H| = 5$. Let $\omega$ be an element of order 5 in $H$. Then $\omega$ acts fixed point freely on $\langle z, r, r_1, i \rangle$. Hence all orbits have a length divisible by 5. Now by (2) and (3) there are $H \cap C_G(z)$-orbits of length 3,3,4 left. This shows that we must have an orbit of length 10. But then $r \sim rz$ in $G$, which contradicts (1).

So we have shown that $z^G \cap C_G(z) = \{z\}$, which contradicts Lemma 2.2. This proves the lemma.

\[ \square \]

**Lemma 5.15.** $A_z \not\leq Sp_{2n}(2)$, $n \geq 3$. 
Lemma 5.16. $A_z \not\cong A_8$ or $U_4(2)$.

Proof. Suppose false. As a Sylow 2-subgroup of Aut($U_4(2)$) is isomorphic to one of $\Sigma_8$, treat $A_8$ and $U_4(2)$ using similar argument. Set $\langle t \rangle = Z(S \cap A_z)$, then $\Omega_1(Z(S)) = \langle z, t \rangle$. We have that $C_G(\Omega_1(Z(S))) \cong (S \cap C_G(A_z)) \times ((Q_8 \ast Q_8)\Sigma_3)$ or $(S \cap C_G(A_z)) \times ((Q_8 \ast Q_8)\Sigma_3) \cdot 2$ depending on whether $C_G(z)/C_G(A_z) \cong A_8$ or $\Sigma_8$ and $C_G(\Omega_1(Z(S))) \cong (S \cap C_G(A_z)) \times ((Q_8 \ast Q_8)\Sigma_3 \times \mathbb{Z}_3)$ or $(S \cap C_G(A_z)) \times ((Q_8 \ast Q_8)\Sigma_3 \times \mathbb{Z}_3) \cdot 2$ depending on whether $C_G(z)/C_G(A_z) \cong U_4(2)$ or $U_4(2) : 2$. Now $z \not\in [C_G(\Omega_1(Z(S))), O_2(C_G(\Omega_1(Z(S))))]^t$ while $t$ is. This shows

\[ z^G \cap \Omega_1(Z(S)) = \{ z \}. \]

If $C_G(z) \cong C_G(A_z) \times A_z$ or $C_G(A_z) \times A_z : 2$, we get a contradiction by application of Lemma 2.2. So we have

\[ C_G(z)/C_G(A_z) \cong \Sigma_8 \text{ or } U_4(2) : 2. \]

Furthermore there is no involution in $C_G(z) \setminus A_z C_G(A_z)$, which centralizes $C_G(A_z)$.

Let $F$ be the elementary abelian subgroup of $S \cap A_z$ corresponding to $\langle (12)(34), (13)(24), (56)(78), (57)(68) \rangle$. Then this is the only elementary abelian subgroup of order 16 in $S \cap A_z$. Set $E = (S \cap C_G(A_z)) \times F$, then $E$ is an abelian subgroup of $S$ of type $(2^n, 2, 2, 2, 2)$, where $2^n = |C_S(A_z)|$. As $\Sigma_8$ and $U_4(2) : 2$ possess no elementary abelian subgroups of order 32, and no involution in $C_G(z) \setminus A_z C_G(A_z)$ centralizes $C_S(A_z)$, we see that $E$ is the only abelian subgroup of this type in $S$. Hence $N_G(E)$ controls fusion in $E$. As all involution in $A_z C_G(A_z)$ are conjugate into $E$ in $C_G(z)$, we see that $N_G(E)$ controls fusion of involutions in $A_z C_G(A_z)$. We are going to show

\[ z^G \cap C_S(A_z)A_z = \{ z \}. \]

If $n > 1$, then we have that $\langle z \rangle = \Phi(E)$ and so $N_G(E) \leq C_G(z)$, which implies (2). So we may assume that $C_S(A_z) = \langle z \rangle$ and so $E$ is elementary abelian. We have that $N_{A_8}(F)$ induces two orbits on $F^2$ of length 6 and 9 in case of $A_8$ and of length 5 and 10 in case of $U_4(2)$. Hence $N_{A_8}(E)$ induces orbits of length $1, 6, 6, 9, 9$ or $1, 5, 5, 10, 10$ on $E^2$. As $N_G(E)/E$ is a subgroup of $GL_5(2)$ and neither 11 nor 13 divides the order of $GL_5(2)$, we see from (*) that $z$ has one or seven conjugates under $N_G(E)$ in the case of $A_8$ and one or 21 conjugates in the case of $U_4(2)$. So assume first that $z$ has 7 conjugates. Then $|N_G(E)/E| = 2^3 \cdot 3^2 \cdot 7$. 

Proof. Suppose $A_z \cong Sp_{2n}(2)$. Then by [MaStr, Lemma 2.21] and Lemma 2.22 we see that $C_G(z) = C_S(A_z) \times A_z$. By Lemma 2.18 we see that $\Omega_1(Z(S)) \cap A_z \leq S'$. As $z \not\in S'$, we have that $z^G \cap \Omega_1(Z(S)) \cap A_z = \emptyset$. Application of Thompson transfer Lemma 2.2, now yields the contradiction $z \not\in G'$. 

\[ \square \]
As this is a subgroup of $GL_5(2)$, and as the normalizer of a Sylow 7–
subgroup in $GL_5(2)$ is isomorphic to $\Sigma_3 \times F_{21}$, we see that a Sylow
7–subgroup of $N_G(E)$ is centralized by some element of order three in
$N_G(E)$. As $|z^{N_G(E)}| = 7$, we see that this 3–element has to centralize
$z^{N_G(E)}$. But this orbit generates $E$, a contradiction. So we have again
$N_G(E) \leq C_G(z)$, which proves (2). If $z$ has 21 conjugates, then by (*)
we have two orbits of length 5 under $N_G(E)$. But one of theses orbits
generates $F$ and so $F$ is normal in $N_G(E)$. This contradicts the fact
that $z$ is conjugate to elements in the orbit of length 10 in $F$. Hence
also in this case we have (2).

Suppose now that $z^G \cap S \neq \{z\}$. Then there is some $i$, $i \sim z$
which induces an outer automorphism on $A_z$. From (1) we get that $C_S(A_z) >\langle z\rangle$ and so $i \sim iz$. Now conjugacy happens in $N_G(E_1)$, where $E_1 =\langle z\rangle \times \langle (1, 2), (3, 4), (5, 6), (7, 8)\rangle$. In both case $A_z$ induces a group of order $2^b \cdot 3$ on $E_1$. We have that $N_{C_G(z)}(E_1)$ induces two orbits of length 8 on $E_1 \setminus A_zC_G(A_z)$. Hence by (2) we get that $|z^{N_G(E_1)}| = 9$. Then $|N_G(E_1)/C_G(E_1)| = 2^a \cdot 3^3$, but $3^3$ does not divide the order of $GL_5(2)$.
This shows $z^G \cap S = \{z\}$, contradicting Lemma 2.1, which proves the
lemma. □

**Lemma 5.17.** We have $A_z \not\cong L_2(p)$, $p$ a prime, $p > 5$, $A_6$, $Sz(q)$, $q$
even, $L_3(4)$, $L_3(3)$ or $M_{11}$.

**Proof.** Suppose false. If $\Omega_1(Z(S)) \leq C_S(A_z)A_z$, then $\langle z, \Omega_1(Z(S)) \cap
A_z \rangle = \Omega_1(Z(S))$. If $\Omega_1(Z(S)) \not\leq C_S(A_z)A_z$, then $A_z$ possesses an
involuntary outer automorphism, which centralizes a Sylow 2-subgroup of
$A_z$. Application of [MaStr, Lemma 2.26] shows $C_G(z) \cong C_G(A_z) \times \Sigma_6$.
In this case we have $\Omega_1(Z(S)) = \langle z, x, t \rangle$, with $x \in A_z$, where $t$
induces the $\Sigma_6$–automorphism.

First we show

(1) \[ z^G \cap (\Omega_1(Z(S)) \cap A_zC_G(A_z)) = \{z\}. \]

If $\Omega_1(Z(S)) = \langle z, x, t \rangle$, then $C_G(z) \cong C_{C_G(z)}(A_z) \times \Sigma_6$ and so $\langle x \rangle = S' \cap \Omega_1(Z(S))$. In particular $z \not\sim x$ in $N_G(S)$, as $z \not\in S'$. This shows
that $|z^{\Omega_1(Z(S))}| = 1$ or $3$, as $|z^{N_G(\Omega_1(Z(S)))}|$ has to be odd. Suppose
that we have three conjugates. Let $\rho$ be some element in $N_G(\langle z, t, x \rangle)$
which induces an element of order three. Then $\langle x \rangle$ is fixed by $\rho$ and so
$\rho$ acts fixed point freely on $\langle z, x, t \rangle/\langle x \rangle$. This implies that $z \not\sim x$. In
particular $z^G \cap \langle z, x \rangle = \{z\}$, which is (1). Of course (1) also holds if
$z^G \cap \Omega_1(Z(S)) = \{z\}$. By Lemma 2.32

So we may assume that $\Omega_1(Z(S)) = \langle z, \Omega_1(Z(S)) \cap A_z \rangle$. By Lemma 2.32
all involutions in $\Omega_1(Z(S)) \cap A_z$ are conjugate in $A_z$. Hence we may assume that $z^G \cap \Omega_1(Z(S)) = \Omega_1(Z(S))^z$.

First let $A_z \cong L_2(p)$, $A_6$, $L_3(3)$, or $M_{11}$. Lemma 2.5 implies that $\Omega_1(Z(S)) = \langle x, z \rangle$. Suppose that there is some automorphism $g$ of $S$ of order three, with $z^g \in A_z$. We have $S \cap A_z \leq S$. Hence $(S \cap A_z)^g \cap (S \cap A_z) \leq S \cap A_z$. Assume that $(S \cap A_z)^g \cap (S \cap A_z) \neq 1$. Then $\Omega_1(Z(S)) \cap A_z = \langle x \rangle \leq (S \cap A_z)^g$ and the same applies to $x^g$. In particular $\Omega_1(Z(S)) \leq S \cap A_z$, a contradiction. So we have that $(S \cap A_z)^g \cap (S \cap A_z) = 1$. Now we get $(S \cap A_z)^g \leq C_G(S \cap A_z)$. As there is no subgroup isomorphic to $(S \cap A_z) \times (S \cap A_z)^g$ in $A_zC_G(A_z)$, recall that $C_S(A_z)$ is cyclic, we have that $(S \cap A_z)^g$ contains some outer automorphism of $A_z$ which centralizes $S \cap A_z$. By [MaStr, Lemma 2.26] we get $A_z \cong A_6$ and this automorphism is a $\Sigma_6$–automorphism. As $S \cap A_z \cong D_8$, we now get that $C_S(A_z) \cong Z_4$ and then $S \cong D_8 \times D_8$, but this group has no automorphism of order three and the order of $g$ was three.

Let now $A_z \cong Sz(q)$, $q = 2^{2n+1}$. Then by Lemma 2.22 we get $S = (S \cap A_z) \times C_S(A_z)$. But $\Omega_1(Z(S)) \cap A_z \leq S'$, as $S$ is not abelian. As $z \not\in S'$ we get (1).

Let now finally $A_z \cong L_3(4)$. By Lemma 2.23(3) any elementary abelian subgroup of order 16 in $\text{Aut}(L_3(4))$ is contained in $L_3(4)$. According to Lemma 2.20 there are exactly two elementary abelian subgroups $U_1$, $U_2$ of order 16 in $S \cap A_z$. Hence in $S$ there are exactly two abelian subgroups of type $(2^n, 2, 2, 2, 2)$, where $|C_S(A_z)| = 2^n$. Then the conjugacy in $\Omega_1(Z(S))$ takes place in the normalizer of $C_S(A_z) \times U_1$. As $A_z$ induces an orbit of length 15 on the involutions of $U_1$ and $|z^{N_G(C_S(A_z) \times U_1)}|$ is odd, we may assume that $z$ possesses 31 conjugates. This then would imply that $|N_G(\langle z, U_1 \rangle)/C_G(\langle z, U_1 \rangle)| = 2^n \cdot 3 \cdot 5 \cdot 31$, where $a = 2$ or 3. So by Sylow’s theorem $N_G(\langle z, U_1 \rangle)/C_G(\langle z, U_1 \rangle)$ must have a normal subgroup of order 31, a contradiction to the structure of $GL_3(2)$. So we have proved we have that $z^G \cap C_S(A_z) \times U_1 = \{z\}$, which is (1). In particular (1) holds in all cases.

As $A_z$ has just one class of involutions, we have that

$$z^G \cap (A_z \times C_{C_G(A_z)}(z)) = \{z\}.$$

By Lemma 2.1 we get some $t \in S$, $t \neq z$ with $t \sim z$ in $G$. So by (2) $t$ has to induce an outer automorphism on $A_z$. By Lemma 2.22 and Lemma 2.12 we see that $A_z \not\cong M_{11}$ or $Sz(q)$. 

Let first $t$ induces the $\Sigma_6$-automorphism on $A_2$. Then we have that
\[ C_S(t) = C_{C_S(A_2)}(t) \times (S \cap A_2) \times \langle t \rangle. \]
As $z \sim t$ and $t \not\in \Phi(C_S(t))$, we see that $z \not\in \Phi(C_S(t))$, so
\[ C_{C_S(A_2)}(t) = \langle z \rangle. \tag{*} \]
This now shows that $C_S(t) = E_1E_2$, where $E_i$ are elementary abelian of order 16, $i = 1, 2$, and $C_{A_2}(t) \cong \Sigma_4$. We choose notation such that
\[ E_1 = \langle z, t, r, s \rangle, \langle r, s \rangle = E_1 \cap A_2, E_1 \not\subseteq C_{C_G(z)}(t). \]
Then $N_{A_2}(E_1)$ induces in $E_1^2$ orbits of length 1, 3, 3, 3, 3, 3, 1, 1 with representatives $z, t, zt, r, zr, tr, ztr$, respectively.

Suppose first that $z^{N_G(E_1)} \neq \{z\}$. Assume further that $t \sim zt$ in $C_G(z) \cap N_G(E_1)$. As $N_{Aut(A_2)}(E_1) \leq \Sigma_6$, there is some $u \in C_S(A_2)$ with $t^u = tz$. This shows that $N_{C_G(z)}(E_1)$ induces on $E_1$ orbits of length $1, 3, 3, 3, 3, 3, 1, 1$ with representatives $z, t, r, zr, tr$, respectively. By (2) we have that $z \not\sim r$ and $z \not\sim zr$. Hence $\langle z, r, s \rangle$ is generated by involutions which are not conjugate to $z$ in $G$. As $z^G \cap \langle z, r, s \rangle = \{z\}$, we see that $\langle z, r, s \rangle$ must not be $N_G(E_1)$-invariant. In particular $tr \not\sim z$, too. Now $z$ has seven conjugates under $N_G(E_1)$. As this number is odd, we have that $N_S(E_1)$ is a Sylow 2-subgroup of $N_G(E_1)$. In particular as $r \in Z(N_S(E_1))$, we have that both $|r^{N_G(E_1)}|$ and $|(zr)^{N_G(E_1)}|$ are odd. As $z \in \langle zr, zs, zrs \rangle$, we see that $|r^{N_G(E_1)}| = 3$ and so $\langle r, s \rangle$ is normal in $N_G(E_1)$. Let $\nu$ be an element of order 7 in $N_G(E_1)$. Then $[\nu, \langle r, s \rangle] = 1$. Furthermore also $[E_1/\langle r, s \rangle, \nu] = 1$, which gives the contradiction $[E_1, \nu] = 1$, but $z^{\nu} \neq z$. But as $z^G \cap z\langle r, s \rangle = \{z\}$ and $t^G \cap t\langle r, s \rangle$ contains $t$ and $ts$, we get a contradiction.

So we have that $t \not\sim tz$ in $N_{C_G(z)}(E_1)$. In particular $[t, C_S(A_2)] = 1$ and so $C_S(A_2) = \langle z \rangle$ by (\*). This again shows that $C_G(z)$ contains a Sylow 2-subgroup of $N_G(E_1)$, i.e. $|z^{N_G(E_1)}|, |r^{N_G(E_1)}|$ and $|(zr)^{N_G(E_1)}|$ are all odd. Further $z \not\sim r \not\sim zr \not\sim z$ by (2). In particular $t$ is not conjugate to $r$ or $zr$. Counting orbits we see again that either $|r^{N_G(E_1)}| = 3$ or $|(zr)^{N_G(E_1)}| = 3$. As above we see the later is not possible, so $zr$ has 5 or 7 conjugates and $\langle r, s \rangle \not\subseteq N_G(E_1)$. But then $p$-elements, $p = 5$ or 7, have to centralize $E_1$, a contradiction.

So we have shown that
\[ z^{N_G(E_1)} = \{z\}. \]
Assume now that \( N_G(C_S(t)) \not\subseteq C_G(z) \). Then we have that \( N_G(C_S(t)) \not\subseteq N_G(E_1) \). This means that there is some \( g \in N_G(C_S(t)) \) with \( E_1^g = E_2 \). In particular \( (z^g)^{N_G(E_2)} = \{ z^g \} \). We have that \( E_2 \) is normal in \( C_G(z)(t) \). So \( N_{A_z}(E_2) \) induces orbits of length three and exactly three orbits of length 1 with representatives \( z, t \) and \( zt \). If \( t \sim zt \) in \( N_{C_G(z)}(t) \), then there is exactly one \( N_G(E_2) \)-orbit of length 1, which is \( \{ z \} \). But then \( z^g = z \), a contradiction. This again shows that \([t, C_S(A_z)] = 1 \) and then \( C_S(A_z) = \langle z \rangle \). As there are exactly two elementary abelian subgroups of order 16 in \( S \), we see that \( o(g) \) cannot be odd. This implies \( t \not\in Z(S) \). Hence we get that there is some \( u \in C_G(z) \), which induces an additional outer automorphism on \( A_6 \), in particular we may assume that \( E_6^g = E_1 \). Now \( gu \in N_G(E_1) \leq C_G(z) \). But then \( g \in C_G(z) \), a contradiction. So we have shown

(3) If \( A_z \cong A_6 \), then \( t \) does not induce a \( \Sigma_6 \)-automorphism.

Now (3) together with (1) imply that

(4) \( z^G \cap \Omega_1(Z(S)) = \{ z \} \).

Again by Lemma 2.1 we get

(5) \( z \in S' \).

By Lemma 2.22 or Lemma 2.12 we have that \( t \) is not a square in \( C_S(t) \). Hence we also have that \( z \) is not a square in \( C_S(t) \). This gives that

(6) \( C_{C_S(A_z)}(t) = \langle z \rangle \).

We next show

(7) \( t \sim tz \) in \( C_G(z) \).

This is true if \( C_S(A_z) > \langle z \rangle = C_{C_S(A_z)}(t) \) by (6). So we may assume that \( C_S(A_z) = \langle z \rangle \). By (5) we have \( z \in S' \). In particular there is some \( s \in S \) with \( t^s = tzj \), where \( j \in A_z \cap S \). As by (3) all involutions in \( A_z t \) are conjugate to \( t \), there is some \( g \in A_z \) with \( (tj)^g = t \). Hence (7) holds.

We now come to the final contradiction. We have that \( \Omega_1(Z(C_S(t))) = \langle z, t, X \rangle = F \), where \( X \leq A_z \). Assume first that \( z^{N_G(F)} = \{ z \} \). Then \( C_S(t) \) is a Sylow 2–subgroup of \( C_G(t) \) and so \( t \in Z(S) \), as \( t \sim z \) in \( G \). But then \( A_z \cong A_6 \) and \( t \) induces a \( \Sigma_6 \)-automorphism, contradicting (3).

So we have \( z^{N_G(F)} \neq \{ z \} \). By (2) we have that \( \langle z, X \rangle \) is generated by involutions which are not conjugate to \( z \) in \( G \). Hence \( t\langle z, X \rangle \) must also contain such involutions, as \( z^G \cap \langle z, X \rangle = \{ z \} \). But by (3) and
Lemma 2.37 all involutions in $tX$ are conjugate and by (7) $t \sim tz$ in $C_G(z)$, so all involutions in $t\langle z, X \rangle$ are conjugate to $t$ and thus to $z$, a contradiction. This final contradiction proves the lemma. □

Now we are going to prove Proposition 5.2. Besides the groups we have excluded in Lemma 5.14 through Lemma 5.17 we just have to exclude the groups $A_z \cong L_2(3), U_4(3), L_2(q), q$ even and $M(23)$. The first three cases have been handled in Lemma 4.3 and Lemma 4.4 where groups show up which are in the statement of our theorem, so $G$ is not a counterexample. The last has been handled in [MaStr, Lemma 4.14].

6. SOME 2-LOCAL SUBGROUPS

We continue with the assumption that $G$ is a counterexample to the main theorem. Hence there is some $z \in \Omega_1(Z(S))$ such that $A_z$ is standard. By Proposition 5.1 we have that $A_z$ is simple. Furthermore by Proposition 5.2 we have that $A_z$ is a group of Lie type of characteristic two or $J_2$ or $M(24)'$. Remember that among the groups of Lie type in characteristic two the group $A_z$ is not isomorphic to one of $L_2(q), Sz(q), 2F_4(q)', q$ even, $L_3(4), G_2(2)', L_4(2), A_6$ or $L_3(2)$. The aim of this chapter is to derive a contradiction, which then proves the main theorem.

For this chapter we fix the following notation. We denote by $S$ a Sylow 2–subgroup of $G$ with $z \in Z(S)$. By $R$ we denote a fixed root group in $\Omega_1(Z(S \cap A_z))$ if $A_z$ is of Lie type and just $\Omega_1(Z(S \cap A_z))$ if $A_z$ is sporadic. By $Q_R$ we denote $O_2(C_{A_z}(R))$. As $A_z$ is normal in $C_G(z)$ we see that

**Lemma 6.1.** $|\Omega_1(Z(S))| \geq 4$.

The first step towards deriving a contradiction it to show the existence of a group $N$ such that $S \leq N$, $N \not\leq C_G(z)$ and $F^*(N) = O_2(N)$ (Lemma 6.5 and Lemma 6.6). Among these groups we choose $N$ minimal with this property. In Lemma 6.11 and Lemma 6.12 we determine the structure of $N$. Here Lemma 3.20 and Lemma 3.21 come into the game. The key fact for us will be to show that there is some $t \in Z(N)$, $t \neq z$ and $t \not\in A_z$. Furthermore we will see that $A_z$ is one of the two sporadic groups or defined over GF(2). In particular we get that $Q_R$ is extraspecial.

At this point we turn our attention to $C_G(t)$. We show that also $C_G(t)$ has a standard component $A_t$. Then we can show that $Q_R \leq A_z \cap A_t$. 
This is sufficient to show that eventually \( A_t \) will be isomorphic to \( A_z \). With this information we get that \( N \) is isomorphic to a minimal parabolic in \( A_z \) and \( A_t \) as well. Now both of these groups induce some action on \( \Omega_1(Z(O_2(N))) \). This together with the fact that \( t \not \sim z \) in \( G \) eventually yields the desired contradiction.

Now we are going to show the existence of a suitable \( N \). But first a technical lemma.

**Lemma 6.2.** Let \( x \in \Omega_1(Z(S)) \setminus C_S(A_z) \) and \( K \leq C_{C_G(z)}(x) \) such that \( K = D_1 \times D_2 \times \cdots \times D_m, m \geq 1, D_1 \) dihedral of order \( 2^n \), quaternion of order 8 or isomorphic to \( SL_2(3) \ast SL_2(3) \) and there are \( s_2, \ldots, s_m \in S \) such that \( D_i = D_i^q, i = 2, \ldots, m \). Then \( K \) is not normal in \( C_{C_G(z)}(x) \).

**Proof.** Suppose false. We first will treat the case of \( A_z \cong Sp_4(q), q > 2 \), as in this group there is some \( x \) such that \( C_{A_z}(x) \) is a 2-group. We fix the following notation. According to Lemma 2.21 there are two elementary abelian subgroups \( E_1, E_2 \) in \( A_z \cap S \) of order \( q^3 \) such that \( E_1 E_2 = S \cap A_z \) and \( E_1 \cup E_2 = \Omega_1(S \cap A_z) \). Furthermore \( E_1 \cap E_2 = R_1 R_2 \), where \( R_1, R_2 \) are the two root subgroups such that \( R_1 R_2 = \Omega_1(Z(S \cap A_z)) \). We now set

\[ F_i = \langle z, E_i \rangle, i = 1, 2, \text{ and } S_1 = S \cap A_z C_S(A_z). \]

We first show

\[ D_1 \text{ is dihedral.} \]

Obviously \( D_1 \not \cong SL_2(3) \ast SL_2(3) \). So let \( D_1 \) be quaternion. Then \( \Omega_1(K) = Z(K) \). Assume \( D_1 \leq S_1 \). Then \( [E_1, D_1] \leq E_1 \) and so \( [D_1, E_1] \leq Z(K) \). As \( |D_1 : D_1 \cap E_1| \geq 4 \), we see with [MaStr, Lemma 2.67] that \( Z(S \cap A_z) \leq [D_1, E_1] \). As \( |K| = |Z(K)|^3 \), we now get \( |K| \geq q^6 \). But \( |\Omega_2(S_1)| \leq 4q^3 \), a contradiction. So we have that \( D_1 \not \leq S_1 \). Choose \( u \in D_1 \setminus S_1 \). If \( [u, E_1] \leq E_1 \), then by Lemma 2.21 and Lemma 2.22 we see that \( u \) induces a field automorphism on \( A_z \) and so \( [[E_1, u]] = r^3 \), where \( q = r^2 \). Again \( [E_1, u] \leq Z(K) \) and so \( K \cap A_z \leq C_{A_z}([E_1, u]) \). As \( [E_1, u] \not \leq Z(A_z \cap S) \), we see that \( C_{S \cap A_z}([E_1, u]) = E_1 \). Hence we have that \( [E_1, u] = K \cap A_z \). But the same applies to \( E_2 \). So \( [E_1, u] = [E_2, u] \), which is impossible as \( E_1 \cap E_2 = Z(S \cap A_z) \). This shows that \( E_1^u = E_2 \). Then \( [[E_1, u] : [E_1, u] \cap Z(S \cap A_z)] = q \). Again by [MaStr, Lemma 2.67] we get that \( Z(S \cap A_z) \leq [[E_1, u], S \cap A_z] \) and so \( Z(S \cap A_z) \leq Z(K) \). But then \( [R_1, u] = 1 \), while we have \( R_1^u = R_2 \), a contradiction.

So we have shown that \( D_1 \) is dihedral. We fix the following notation:

\[ D_1 = \langle x_1, x_2 \rangle, \text{ where } x_1^2 = x_2^3 = 1. \]
Let first $m = 1$. We may assume that $[E_1, x_1] \neq 1$. In particular $x_1 \not\in E_1$. If $[E_1, x_1] \leq E_1$ then $|[\langle E_1, x_1 \rangle, x_1]| \geq 2q \geq 8$. But there are no elementary abelian subgroups of order 8 in $D_1$. So we have that $R^2_1 = R_2$ and again $|[\langle R_1 R_2, x_1 \rangle, x_1]| \geq 2q$ and this group is elementary abelian.

So we have proved that $m > 1$. Now we set $$D_2 = \langle y_1, y_2 \rangle,$$ where we choose notation such that $x_1^{y_2} = y_i, i = 1, 2$.

Suppose first that $D_1 \leq S_1$. Then as $S_1$ is normal in $S$, we have $K \leq S_1$. As $\Omega_1(S_1) = F_1 \cup F_2$, we may assume that $x_1y_1 \in F_1$. As $[x_1, x_2] \neq 1$, we get $x_2y_2 \in F_2$. Now we consider the involution $x_2y_2 \in K$. We have $[x_1y_1, x_2y_2] \neq 1 \neq [x_2y_2, x_2y_1]$, so $x_2y_1 \not\in F_1 \cup F_2$, a contradiction.

So we may assume that $x_1 \not\in S_1$. By Lemma 2.22 we have that $S/S_1$ is abelian. Hence $x_1x_1^{y_2} = x_1y_1 \in S_1$. Furthermore also $x_2y_2 \in S_1$. So we may assume that $x_1y_1 \in F_1$ and $x_2y_2 \in F_2$. As $[x_1y_1, x_2y_2] \neq 1$, we see that $x_1y_1, x_2y_2$ both are not in $Z(S_1)$. As $[x_1, x_1y_1] = 1$, we see that $x_1$ normalizes $E_1$ and induces a field automorphism on $A_2$. In particular it also normalizes $E_2$ and so we get that $K$ normalizes $E_i, i = 1, 2$. As the group of field automorphisms is cyclic, we get $|K : K \cap S_1| = 2$. We consider the involution $x_1y_2$. As above we get that $x_1y_2 \not\in S_1$. But then $y_2 = x_1x_1y_2 \in S_1$ and so also $x_2 \in S_1$. In particular $\langle x_2, Z(D_2) \rangle$ is normal in $D_2$, which shows that $D_1$ is dihedral of order 8. As $[x_2, x_2y_2] = 1$, we have $x_2, y_2 \in E_2$. As $[x_1y_1, x_2] \neq [x_1y_1, y_2]$, we see that $|\langle x_2, y_2, Z(S_1) \rangle / Z(S_1)| = 4$. Now application of [MaStr, Lemma 2.67] shows that $[\langle x_2, y_2 \rangle, E_1] = R_1R_2$ and so $R_1R_2 \leq K$. As $x_1$ induces a field automorphism on $A_2$, we have that $|R_1R_2 : C_{R_1R_2}(x_1)| = q > 2$.

On the other hand $|K : C_K(x_1)| = 2$, a contradiction. So we have shown

$$(1) \quad A_2 \not\cong Sp_4(q).$$

By Lemma 2.23 we have that $x \in C_S(A_2) \times A_2$. Hence $x = x^i r$ where $1 \neq r \in Z(S \cap A_2)$ and $i = 0, 1$.

We assume first that $r \in R$ and show

$$(*) \quad O_2(K) \leq A_2 C_S(A_2).$$

Suppose false. As $[O_2(K), C_{A_2}(r)] \leq Q_R$, we see from [GoLyS3, Table 5.3] for the two sporadic groups and by application of Lemma 2.27 in the case $A_2$ a group of Lie type that $A_2 \cong L_3(16)$ and some element $k \in K$ induces a graph/field automorphism on $A_2$. In particular $K = O_2(K)$. So $C_{A_2}(k) \cong U_3(4)$. As $Z(K) \leq K'$, we have
\[ Z(K) \leq C_5(A_2)A_2 \text{ and } Z(K) \leq C(k). \] Hence \(|Z(K)| \leq 8. \text{ In } C_{A_2}(k)\]

we have some element \( \omega \) of order 5, which centralizes \( Z(K) \) and \( x \).

Hence this element has to normalize any quaternion group or dihedral group in \( K \) modulo \( Z(K) \) and so it has to centralize \( K \). As \( \omega \) acts fixed

point freely on \( Q_2/R \), we see that \( K \cap A_2C_5(A_2) \leq Z(K)C_5(A_2) \). But then \( K \) cannot be normal in \( S \). So we have (\( * \)).

As \( Z(K) \leq K' \) and \( C_5(A_2) \) is cyclic, we get by (\( * \)) that \( K \cap C_5(A_2) = 1. \)

As \( C_5(A_2)A_2/C_5(A_2) \cong A_2 \), we may assume \( x = r \) and \( O_2(K) \) is a subgroup of \( A_2 \). Now \( O_2(K) \leq O_2(C_{A_2}(r)) \), as \( K \) is normal in \( C_{A_2}(R) \),

which gives that \( O_2(K) \) is of class two and \( Z(K) \leq O_2(C_{A_2}(r))' = R. \)

But then any \( O_2(D_4) \) has to be normal modulo \( R \), which gives that \( C_{A_2}(r) \) has a normal dihedral group, quaternion group or \( Q_8 \times Q_8 \). For \( A_2 \not\cong L_3(q) \) we receive from Lemma 2.17, Lemma 2.18, Lemma 2.19 or

\([MaStr, \text{Lemma 2.10}]\) that \( C_{A_2}(r) \) induces at most two nontrivial

modules on \( O_2(C_{A_2}(r))/Z(O_2(C_{A_2}(r))) \). We conclude that we must have

exactly two such modules and \( C_{A_2}(r) \) induces \( \Omega_3 \) or \( \Omega_3 \), or \( m = 1 \) and

\( O_2(C_{A_2}(r)) \) is dihedral of order eight or isomorphic to \( Q_8 \times Q_8 \). This

then implies that we are over \( GF(2) \). Hence we just have the groups

excluded by Proposition 5.2. In case of \( A_2 \cong L_3(q) \), \( q > 2 \), by \([MaStr, \text{Lemma 2.39}]\), we have that \( R = Z(K) \) and so as \( |O_2(K)| \geq |Z(K)|^2 \),

we see \( K = Q_2 \). As \( q > 2 \), we have \( m > 1. \) But \( Q_2 \) is not a direct

product of \( m \) dihedral groups.

So we may assume that \( r \) is not a root element. In particular \( A_2 \cong \)

\( F_4(q) \) or \( Sp_{2n}(q) \). In the latter by (1) we have \( n > 2. \) We first show

that (\( * \)) holds again. Set \( X_2 = C_{A_2}(Z(S \cap A_2)) \). Then we have that

\([O_2(K), X_2] \leq O_2(X_2) \). Assume that there is some \( t \in O_2(K) \) such

that \( t \) induces an outer automorphism on \( A_2 \). As \( A_2 \not\cong Sp_4(q) \), we

have that \( E(X_2/O_2(X_2)) \) is a nonsolvable group and by Lemma 2.22

any outer automorphism of \( A_2 \) induces a nontrivial automorphism

on this group. Hence (\( * \)) holds. So as above we may assume that

\( \langle x, K \rangle \leq A_2. \) As \( O_2(X_2)/O_2(C_{A_2}(R)) \) is elementary abelian, we see

that \( Z(K) \leq O_2(C_{A_2}(R)) \).

Let first \( A_2 \cong Sp_{2n}(q) \). Assume furthermore \( Z(K) \not\leq Z(O_2(C_{A_2}(R))) \). Then

\( Z(K)/Z(K) \cap Z(O_2(C_{A_2}(R))) \) is a natural \( Sp_{2n-4}(q) \)-module. We have that

\( C_{O_2(C_{A_2}(R))}(Z(K)) = Z(K)Z(O_2(C_{A_2}(R))) \geq K \cap O_2(C_{A_2}(R)). \)

Hence

\[
\frac{|O_2(K)Z(O_2(C_{A_2}(R))) : Z(K)Z(O_2(C_{A_2}(R)))|}{|O_2(X_2) : O_2(C_{A_2}(R))|} = q = 2^t.
\]
This shows that \(m \leq t\). As \(|Z(K)/Z(O_2(C_{A_z}(R)))| = 2^{t(2n-4)} \geq 2^t\), as \(n > 2\), and \(2^t > 2^m\), we get a contradiction to \(|Z(K)| = 2^m\). Hence \(Z(K) \leq Z(O_2(C_{A_z}(R)))\). But now \(O_2^2(X_z)\) centralizes \(Z(K)\), i.e. \(O_2(K) \leq C_{O_2^2(X_z)}(O_2^2(X_z))\), or \(O_2^2(X_z)\) induces a 3-group on \(O_2(X_z)\).

Assume first that \([O_2^2(X_z), O_2(K)] = 1\). Then \(K \nleq O_2(C_{A_z}(R))\), as \(C_{O_2^2(C_{A_z}(R))}(O_2^2(X_z)) = Z(O_2^2(C_{A_z}(R)))\). Take \(u \in K \setminus O_2(C_{A_z}(R))\). Then \([u, O_2^2(C_{A_z}(R))] \leq O_2^2(C_{A_z}(R)) \cap K \leq Z(O_2(C_{A_z}(R)))\). But this contradicts Lemma 2.18.

So assume that \(O_2^2(X_z)\) induces a 3-group on \(O_2(X_z)\). Then application of Lemma 2.18 yields \(A_z \cong Sp_6(2)\). But this contradicts Proposition 5.2.

So we are left with \(A_z \cong F_4(q)\). As there is no 3-group, which centralizes \(C_{A_z}(Z(S \cap A_z))/O_2^2(C_{A_z}(Z(S \cap A_z)))\), we see that \(K = O_2^2(K)\). By Lemma 2.17 \(C_{A_z}(O_2(C_{A_z}(R))/Z(O_2(C_{A_z}(R))) \leq O_2^2(C_{A_z}(R))\). Now assume that \(O_2(C_{A_z}(R)) \cap K \leq Z(O_2(C_{A_z}(R)))\). As \([K, O_2^2(C_{A_z}(R))] \leq K \cap O_2^2(C_{A_z}(R))\), we get \(K \leq O_2^2(C_{A_z}(R))\). But then \(K\) would be elementary abelian, a contradiction. Hence

\[
(**) \quad O_2(C_{A_z}(R)) \cap K \nleq Z(O_2^2(C_{A_z}(R)))
\]

and so as \(O_2^2(C_{A_z}(R)) \cap K\) is normal in \(O_2^2(C_{A_z}(R))\) we get \(R \leq K\). But in case of \(F_4(q)\) we have two roots with isomorphic centralizers. Then a similar argument shows that \(Z(S \cap A_z) \leq K\). As \(O_2(X_z)/O_2^2(C_{A_z}(R))\) is elementary abelian and \(Z(K) \leq K'\), we have \(Z(K) \leq O_2^2(C_{A_z}(R))\).

Assume first \(Z(K) = Z(S \cap A_z)\). Then \(O_2^2(X_z)\) centralizes \(K\). But by Lemma 2.17 we have that \(O_2(X_z)/Z(S \cap A_z)\) has a normal subgroup which is a direct sum of two natural \(Sp_4(q)\)-modules whose factor group is a direct sum of two natural \(\Omega_5(q)\)-modules. This implies that \(C_{O_2^2(C_{A_z}(R))/Z(C_{A_z}(R))})(O_2^2(X_z)) = 1\), which contradicts (**) So we have that \(Z(K) > Z(A_z \cap S)\). Hence by Lemma 2.17 we have that either \(|Z(K)/Z(K) \cap Z(O_2^2(C_{A_z}(R)))| = q^4\) or \(|Z(K) \cap Z(O_2^2(C_{A_z}(R)))| \geq q^6\). In both cases we have that \(|Z(K)| \geq q^6\) and so as \(|K| \geq |Z(K)|^3\), we get \(|K| \geq q^{18}\). In particular there is some proper normal subgroup of order at least \(q^{18}\). But now the structure of \(X_z\) as described before shows that \(K = O_2^2(X_z)\). Then \(Z(K) = Z(O_2^2(X_z)) = Z(A_z \cap S)\), a contradiction.

Next we set

\[
\mathcal{N} = \{ N \mid N \leq G, \Omega_1(Z(S)) \leq N \nleq C_G(z), 1 \neq O_2(N) \leq S \}.
\]
The group $N$ we are looking for will be in this set $\mathcal{N}$. So we first show that $\mathcal{N}$ is not empty.

**Lemma 6.3.** There exists $1 \neq S_1 \subseteq S$ such that $N_G(S_1) \not\subseteq C_G(z)$. Among those choose $S_1$ such that $|N_G(S_1) \cap C_G(z)|_2$ is maximal. Then

(i) $N_G(S_1) \in \mathcal{N}$, in particular $\mathcal{N} \neq \emptyset$.

(ii) $N_G(S_1) \cap S \in \text{Syl}_2(C_{N_G}(S_1)(z)) \subseteq \text{Syl}_2(N_G(S_1))$.

(iii) If $N_G(S_1) \cap S$ is not a Sylow 2-subgroup of $G$, then $N_G(S \cap N_G(S_1)) \leq C_G(z)$.

**Proof.** As $C_G(z)$ cannot control fusion in $C_G(z)$ by Lemma 2.1 we have that $C_G(z)$ is not strongly 2-embedded in $G$. Hence there is some $1 \neq S_1 \subseteq S$ with $N_G(S_1) \not\subseteq C_G(z)$. Now we choose $S_1$ such that $|N_G(S_1) \cap C_G(z)|_2$ is maximal. Obviously $\Omega_1(Z(S)) \leq N_G(S_1)$. Set $T = N_S(S_1)$. Then $S_1 \leq T$. Let $T_1$ be a Sylow 2-subgroup of $C_{N_G(S_1)}(z)$, which contains $T$. Then there is some $g \in C_G(z)$ with $T_1^g \leq S$. We have $|S \cap N_G(S_1)^g| \geq |S \cap N_G(S_1)|$. As $S_1^g \leq S$ and $N_G(S_1)^g \not\subseteq C_G(z)$, we have by the choice of $N_G(S_1)$ that $T = T_1$ is a Sylow 2-subgroup of $C_{N_G(S_1)}(z)$. If $T = S$, we have the assertion (ii). So assume $T \neq S$. In particular $N_S(T) > T$. Hence by the choice of $S_1$ we have that $N_G(T) \leq C_G(z)$, which is (iii). As $T$ is a Sylow 2-subgroup of $C_{N_G(S_1)}(z)$, this shows that $T$ is a Sylow 2-subgroup of $N_G(S_1)$, which finishes the proof of (ii). In particular $O_2(N_G(S_1)) \leq T \leq S$, which shows that $\mathcal{N} \neq \emptyset$, which proves (i). \qed

**Lemma 6.4.** Set $\mathcal{N}_1 = \{U \mid U \in \mathcal{N} \text{ with } |U \cap S| \text{ maximal}\}$. Choose $N \in \mathcal{N}_1$ minimal by inclusion. Then $N$ is a minimal parabolic where $C_N(z)$ is the unique maximal subgroup of $N$ containing $N \cap S$. Furthermore we have:

(i) If $E$ is normal in $N$ and $E \leq C_G(z)$, then $S \cap E$ is also normal in $N$.

(ii) $E(N) = 1$.

(iii) $O(N) \leq C_G(z)$ and $O_{2r,2}(N) = O_2(N)O(N)$.

**Proof.** Recall that by Lemma 6.3 there is such an $N \in \mathcal{N}$. Further we have that $T = S \cap N$ is a Sylow 2-subgroup of $N$.

The minimality of $N$ then shows, that for $M < N$ and $T \leq M$ we have $M \leq C_G(z)$. Therefore $N \cap C_G(z)$ is the only maximal subgroup of $N$ containing $T$, which means that

$N$ is a minimal parabolic with respect to $T$.

Let now $E$ be normal in $N$. Then we have that $N = N_{N}(E \cap T)E$. If $E \leq C_G(z)$, then $N_{N}(E \cap T) \not\subseteq C_G(z)$ and so by minimality we have
that \( N = N_N(E \cap T) \), which is (i).

Assume there is some involution \( x \in Z(N) \). Then \( C_G(x) \nleq C_G(z) \). Let \( T_1 \leq C_G(x) \) with \(|T_1 : T| = 2\). Then as \( N_G(T) \leq C_G(z) \) by Lemma 6.3, we see \( T_1 \leq C_G(z) \). This implies that there is some \( g \in C_G(z) \) with \( T_1^g \leq S \). In particular \(|S \cap C_G(x^g)| > |S \cap N|\). So we may apply Lemma 6.3 with \( \langle x^g \rangle \). This implies the existence of some \( S_1 \leq S \) such that \( N_G(S_1) \in \mathcal{N} \) and \(|N_G(S_1) \cap C_G(z)|_2 \geq |C_S(x^g)| > |N \cap S|\), which contradicts the choice of \( N \). So we have that \( T \) is a Sylow 2–subgroup of \( C_G(x) \), in particular \( O_2(C_G(x)) \leq O_2(N) \). We collect:

(1) If \( 1 \neq x \in Z(N) \) is an involution then 
\( T \) is a Sylow 2-subgroup of \( C_G(x) \).

Assume now \( E(N) \neq 1 \). Then by (i) we have \( E(N) \nleq C_G(z) \) and so \( N = E(N)T \). Let \( E(N) = N_1 \cdots N_r \). As \( E(N)T \) is a minimal parabolic we have that \( N_1N_T(N_1) \) is a minimal parabolic with respect to \( N_T(N_1) \). As \([O_2(N), E(N)] = 1\), we have \( z \nleq O_2(N) \). So the maximal subgroup containing the Sylow 2-subgroup is \( C_{N_1}(z)N_T(N_1) \), the centralizer of an involution. Hence by Lemma 2.39 we get that

(2) \( N_1 \) is a group of Lie type in odd characteristic.

Choose \( x \in Z(N) \) an involution, which exists as \( O_2(N) \neq 1 \). By (1) we have that \( T \) is a Sylow 2–subgroup of \( C_G(x) \), in particular \( O_2(C_G(x)) \leq O_2(N) \) and then \([E(N), O_2(C_G(x))] = 1\). This shows that \( E(N) \leq E(C_G(x)) \) (recall that \( O(C_G(x)) = 1 \) by the general assumption).

Assume first that \( N_1 \) is not conjugate to \( L_2(p) \), \( L_3(9) \), \( L_3(3) \), \( L_4(3) \), \( U_4(3) \) or \( PSp_4(3) \). It follows that \( N_1 \) is not a component of \( C_G(x) \), as now by (2) \( N_1 \nleq C_2 \). Furthermore from Lemma 2.39 we get that \( C_{N_1}(z) \) has a component \( K_1 \), which is a group of Lie type in odd characteristic. Let \( K \) be some component of \( C_G(x) \) with \( N_1 \leq K \). As by (*) we see that \( N \) contains a Sylow 2-subgroup of \( C_G(x) \), we have that \([K_1, O_2(C_{C_G(x)}(x))] = 1\). This shows that also \( C_K(z) \) has a component. As \( K \in C_2 \) and \( z \) centralizes a Sylow 2-subgroup of \( K \), we get with [MaStr, Lemma 2.26] that either \( z \) induces an inner automorphism on \( K \) or \( K \cong L_4(3) \) and then \( z \) has to induce an outer automorphism, which then is a graph automorphism, which centralizes \( L \cong PSp_4(3) \) in \( K \). Now \( K_1 \leq L \). But as \( PSp_4(3) \cong \Omega^-(6) \) we get with the Borel-Tits-Theorem, that all subgroups of \( L \) containing a Sylow 2-subgroup of \( L \) are constrained, in particular do not have components, a contradiction. So we may assume that \( z \) induces an inner automorphism on
K. With [MaStr, Lemma 2.22] we see that \(K\) cannot be a group of Lie type in characteristic two. Further as centralizers of involutions in \(L_3(3), G_2(3), U_4(3)\) are solvable by [MaStr, Lemma 2.20], Lemma 2.7, Lemma 2.6 respectively, these groups are also not possible. The centralizers of 2-central involutions in the sporadic groups are given by [GoLyS3, Table 5.3]. From there we see that only \(M(23)\) possesses a 2-central involution, whose centralizer has a component. In \(M(23)\) this component would be \(2M(22)\), which is not a group of Lie type in odd characteristic. Hence \(K_1 \neq 2M(22)\). But then \(M(22)\) must contain a subgroup \(L\), which contains a Sylow 2-subgroup and a normal subgroup which is a product of groups of Lie type in odd characteristic, contradicting [GoLyS3, Table 5.3].

Hence we have that

\[(3) \quad N_1/Z(N_1) \cong L_2(p), p > 5, L_2(9), L_3(3), L_4(3), U_4(3) \text{ or } PSp_4(3).\]

In particular \(N_1 \in \mathcal{C}_2\). By Lemma 2.39 we get that \(N_1/Z(N_1) \not\cong U_4(3) \text{ or } PSp_4(3)\). If \(N_1/Z(N_1)\) is isomorphic to \(L_2(p)\) or \(L_3(3)\), then \(\Omega_1(Z(N_T(N_1))/C_T(N_1))) \leq N_1\). If \(N_1 \cong A_6\), then as \(N_T(N_1)N_1\) is a minimal parabolic there is some element in \(N_T(N_1)\), which interchanges the two subgroups isomorphic of \(\Sigma_4\). So also \(\Omega_1(Z(N_T(N_1))/C_T(N_1))) \leq N_1\).

In case of \(N_1/Z(N_1) \cong L_4(3) \cong \Omega^+_6(3)\), we see from Lemma 2.39 that also a graph automorphism is induced by \(T\). This then again implies \(\Omega_1(Z(N_T(N_1))/C_T(N_1))) \leq N_1\). As \(N_1 \not\cong L_2(5)\), we have by Lemma 2.13 that \(|\Omega_1(Z(N_T(N_1))/C_T(N_1))| = 2|\).

We have \(\Omega_1(Z(S)) \leq N\) by the definition of \(\mathcal{N}\). Further we have \(|\Omega_1(Z(S))| \geq 4\) by Lemma 6.1. As \(Z(S)\) centralizes \(T \cap N_1\) it normalizes \(N_1\), we get that \(\Omega_1(Z(S)) \cap C(N_1) \neq 1\). As \(T\) acts transitively on the components of \(N\), we get that \(\Omega_1(Z(S)) \cap C(E(N)) \neq 1\).

So we may assume

\[(4) \quad x \in Z(S) \text{ and then } T = S. \text{ Further } N_1 \leq K \text{ for } K \text{ some component of } C_G(x).\]

Now we show that

\[(5) \quad K = N_1 \text{ or } K \cong M_{11}.\]

There is \(T_1 \leq T\) such that \(M_1 = \langle N_1^{T_1} \rangle T_1 \leq K\) and \(M_1\) contains a Sylow 2-subgroup of \(K\). If \(K\) is a group of Lie type in characteristic 2, then by [MaStr, Lemma 2.15] we have that \(K = N_1\), as \(M_1\) would be a parabolic subgroup. This proves (5). If \(K\) is a group of Lie type in odd
characteristic, the list $C_2$ shows that $K/Z(K) \cong L_2(p)$, $L_2(9)$, $L_3(3)$, $L_4(3)$, $U_4(3)$ or $G_2(3)$. As the centralizer of a 2-central involution in $K$ is solvable ([MaStr, Lemma 2.20], Lemma 2.7, Lemma 2.6), we see that $O_2(M_1) = 1$. Now the order of $L_4(3)$ is divisible by $3^6 \cdot 5 \cdot 13$, so we get $N_1 = K$ in case of $N_1 \cong L_4(3)$ or $G_2(3)$. Also the order of $L_3(3)$ is divisible by 13, which shows $N_1 = K$ or $M_1 \leq L_4(3)$ in case of $N_1 \cong L_3(3)$. Suppose the latter. By [MaStr, Lemma 2.21] and Lemma 2.22 we have that $|\Aut(L_3(3))| = 2^5 \cdot 3^3 \cdot 13$, which contradicts $|K|_2 = 2^6$ and $O_2(M_1) = 1$. So it remains $N_1 \cong L_2(q)$. If $N_1 \neq K_1$, we see that $K \cong L_3(3)$, $L_4(3)$, $U_4(3)$ or $G_2(3)$. As $p$ is a Fermat or Mersenne prime and $p > 5$, we see that $N_1 \cong L_2(7)$ or $L_2(9)$. As neither 5 nor 7 divides the order of $L_3(3)$, we get $K \ncong L_3(3)$. As $2^6$ does not divide $|\Aut(N_1)|$, we get that $N_1 \times N_1 \leq K$. But then $5^2$ or $7^2$ has to divide $|K|$, which is not the case. This proves (5) in case of $K$ a group of Lie type in odd characteristic.

So we are left with $K$ a sporadic simple group. Suppose first that $C_{M_1}(N_1) = 1$. Then by Lemma 2.10 we get $M_1 \cong M_{10}$ and $K \cong M_{11}$, which is (5). So assume $K \ncong M_{11}$. Then $C_{M_1}(N_1) \neq 1$. If $M_1 = \langle N_1^T \rangle T_1$ has $2^n$, $n \geq 1$, many components isomorphic to $N_1$, there is some involution $y \in M_1$, which centralizes in $T_1$ a subgroup of index two and $2^{n-1}$ of these components. In particular in both cases $C_K(y)$ possesses a component $\tilde{K}$. By [GoLyS3, Table 5.3] we see that $K \cong M_{22}$ or $M_{23}$. The same is true if $O_2(M_1) \neq 1$, where $y \in Z(M_1)$. Now the situation in $\tilde{K}$ is the same as in $K$ and so $\tilde{K}/Z(\tilde{K})$ cannot be a group of Lie type in characteristic 2. As $\tilde{K}/Z(\tilde{K}) \cong U_6(2)$ for $K \cong M_{22}$, we get $K \cong M_{23}$ and $\tilde{K}/Z(\tilde{K}) \cong M_{22}$. The odd part of the order of $M_{23}$ implies that there are at most two components $N_1$, $N_1^T$ in $M_1$. If there are two of them, we have with [GoLyS3, Table 5.3] and (1) that $N_1 \cong L_3(9)$. Now in any case we see that $|M_1/O_2(M_1)| \geq 2^{11}$. Hence $|O_2(M_1)| \geq 4$. So we may choose $y \in Z(T_1)$ and then we have the same situation in $C_K(y)/\langle y \rangle \cong M_{22}$. Now we get a contradiction with the same arguments as for $K \cong M_{22}$.

So we have shown that either $K = N_1$ or $K \cong M_{11}$. In any case we have that $C_K(z)$ is dihedral or isomorphic to $GL_2(3)$ or in case of $L_4(3)$ contains a normal subgroup $SL_2(3) \times SL_2(3)$. This shows that $C_G(z) \cap C_G(x)$ has a normal subgroup which is a direct product of dihedral, quaternion groups or groups isomorphic to $SL_2(3) \times SL_2(3)$, which are permuted by $S$. This now contradicts Lemma 6.2 and so we have (ii).
Assume now \( O(N) \neq 1 \). We first show \( O(N) \leq C_G(z) \). So assume false. Then by the minimality of \( N \) we have that \( N = O(N)T \). Again there is some involution \( x \in Z(N) \). By \((*)\) \( T \) is a Sylow 2-subgroup of \( C_G(x) \). As \([O_2(C_G(x)),O(N)] = 1\) and \( O(C_G(x)) = 1\), we must have that \( E(C_G(x)) \neq 1 \). As \([T,O(N)] \leq O(N)\), we see that \( O(N) \) normalizes any component \( K \) of \( C_G(x) \). Further a Sylow 2-subgroup of \( K \) has to normalize some nontrivial group of odd order of its automorphism group. As \( K \in C_2 \) we get by Lemma 2.29 that \( K \cong L_3(3) \) or \( M_{11} \).

Set \( K_1 = \langle K^T \rangle \) and \( K_2 = K_1T \). Then by Lemma 2.29 we have that \( \Omega_1(Z(T)) = \Omega_1(Z(T)) \cap C_T(K_1) \times \Omega_1(Z(T)) \cap K_1 \). Further \( |\Omega_1(Z(T)) \cap K_1| = 2 \). As \( |\Omega_1(Z(S))| \geq 4 \), we see that \( \Omega_1(Z(S)) \cap C_T(K_1) \neq 1 \).

Hence we have that \( \Omega_1(Z(S)) \cap O_2(N) \neq 1 \), and so we may choose \( x \in \Omega_1(Z(S)) \), which gives \( S = T \). As \( C_K(z) \) is a direct product of groups isomorphic to \( GL_2(3) \), we see that \( C_{C_G(z)}(x) \) contains a normal subgroup, which is a direct product of quaternion groups, contradicting Lemma 6.2.

So we have that \( O(N) \leq C_G(z) \). Further by (i) we get that \( T \cap O_{2',2}(N) \) must be normal in \( N \), so we have (iii).

**Lemma 6.5.** There is some subgroup \( N \in \mathcal{N} \) with \( S \leq N \).

**Proof.** Assume false. Then in particular by Lemma 6.3 we see \( C_G(x) \leq C_G(z) \) for all \( x \in Z(S)^\# \). By Lemma 6.3 we can pick some \( N \in \mathcal{N} \) with \( |N \cap S| \) maximal. Among all such \( N \) choose \( N \) minimal. Set \( T = S \cap N \).

By Lemma 6.3 \( T \) is a Sylow 2-subgroup of \( N \). As \( S \neq T \) and by the maximal choice of \( N \cap S \) we see \( N_G(T) \leq C_G(z) \) and then that no nontrivial characteristic subgroup of \( T \) is normal in \( N \). Set \( W = \Omega_1(Z(O_2(N))) \).

By Lemma 6.4 we have \( C_N(O_2(N)) \leq O_2(N)O(N) \), and so \( \Omega_1(Z(T)) \leq W \). Hence \( W \neq \Omega_1(Z(T)) \) and then as \( J(T) \leq O_2(N) \), we have by Lemma 3.2 that \( W \) is an \( F \)-module for \( N \). As by Lemma 6.4 \( N \) is a minimal parabolic with respect to \( T \), this now gives with Lemma 3.4 that any component of \( N/C_N(W) \) is isomorphic to \( L_2(2^n) \) or \( A_{2n+1} \), for suitable \( n \), or \( N/C_N(W) \) is solvable.

First assume that \( N/C_N(W) \) is not solvable, i.e. \( 1 \neq E(N/C_N(W)) = N_1 \times \cdots \times N_r \). Then by Lemma 3.15 \( W/C_W(E(N/C_N(W))) = V_1 \oplus \cdots \oplus V_r \), where each \( V_i \) is a natural \( N_i \)-module. By the choice of \( N \in \mathcal{N} \), we have that the maximal subgroup \( M \) in \( E(N/C_N(W)) \) containing \( TC_N(W)/C_N(W) \) centralizes \( z \). If \( N_i \cong L_2(2^n) \), then \( M \) is the normalizer of a Sylow 2-subgroup in \( E(N/C_N(W)) \) and so has no fixed point in \( V_1 \oplus \cdots \oplus V_r \). This shows \( [z,E(N/C_N(W))] = 1 \), which contradicts
Lemma 6.4(i).

So let $N_i \cong A_{2^{n+1}}$. We have that in each module $V_i$, which is the irreducible part of the permutation module, a Sylow 2-subgroup of $N_i$ just centralizes a 1-space. As $T$ acts transitively on the components $N_i$, since $N$ is a minimal parabolic, we get that $|C_{W/CW(N)}(T)| = 2$. As $\left|\Omega_1(Z(S))\right| \geq 4$, we must have some $1 \neq t \in \Omega_1(Z(S))$ with $[E(N/C_N(W)), t] = 1$. But for any such $t$ we know that $C_G(t) \leq C_G(z)$ and so again $[E(N/C_N(W)), z] = 1$, a contradiction to Lemma 6.4(i).

So we have that $N/C_N(W)$ is solvable. As $C_N(W) \leq C_G(z)$, we get by application of Lemma 6.4(i) that $T \cap C_N(W)$ is normal in $N$, so $N$ is solvable. Set $\tilde{N} = N/O(N)$. As $W = \Omega_1(Z(O_2(N)))$ is an $F$–module, we have by Lemma 3.16 that $\tilde{N}/C_{\tilde{N}}(W) = O_{3,2}(\tilde{N})$. As $C_{\tilde{N}}(W)$ is 2-closed and $\tilde{N}$ is a minimal parabolic we get that $\tilde{N} = O_{2,3,2}(\tilde{N})$. Let $P$ be a Sylow 3–subgroup of $\tilde{N}$. Then obviously $P$ is not contained in $C_N(z)/O(N)$. If $C$ is a proper characteristic subgroup of $P$, then $CT < \tilde{N}$ and so by minimality of $N$, we have that $[C, z] = 1$. In particular $[\Phi(P), z] = 1$. Set $W_1 = \langle z^N \rangle$. Then $[W_1, \Phi(P)] = 1$. As $T$ acts irreducibly on $P/\Phi(P)$ we see that $P/\Phi(P)$ acts faithfully on $W_1$ and so also on $W_2 = [C_W(\Phi(P)), P]$. Let $A$ be an $F$–module offender on $W$. As $A$ acts faithfully on $P/\Phi(P)$, we see that $A$ also acts faithfully on $W_2$ and induces an $F$–module offender there. Then by the Dihedral Lemma 2.3 we get some $\rho \in P \setminus \Phi(P)$ with $|[W_2, \rho]| = 4$. Further $|W_2 : C_{W_2}(A)| = |A|$. So let $|P/\Phi(P)| = 3^n$, then $|W_2| = 4^n$. We also have that $W_2 = U_1 \oplus \cdots \oplus U_n$, where $|U_i| = 4$ and $T$ acts transitively on the $U_i$. As we may choose $U_1 = [W_2, \rho]$, where $\rho$ is inverted by some element in $A$, we see that $|C_{W_2}(T)| = 2$. We have $\left|\Omega_1(Z(S))\right| \leq W$ and $|\Omega_1(Z(S))| \geq 4$ by Lemma 6.1. Further as $C_G(t) \leq C_G(z)$ for all involutions $t \in \Omega_1(Z(S))$ and $P \leq C_G(z)$, we have $\left|\Omega_1(Z(S))\right| \cap C_N(P) = 1$. Assume $[W, \Phi(P)] = 1$. Then $W = W_2 \oplus C_W(P)$. As $|\Omega_1(Z(S)) \cap W_2| \leq 2$ and $|\Omega_1(Z(S))| \geq 4$, we get $C_{\Omega_1(Z(S))}(P) \neq 1$. But as $C_G(x) \leq C_G(z)$ for all $1 \neq x \in \Omega_1(Z(S))$, we now get $N \leq C_G(z)$, a contradiction. Therefore $W = [W, \Phi(P)] \oplus C_W(\Phi(P))$, with $W_3 = [W, \Phi(P)] \neq 1$. As $A$ is an $F$–module offender and $|W_2 : C_{W_2}(A)| = |A|$, we must have that $[A, W_3] = 1$. Now $[A, P] \leq C_N(W_3)$. But as $A \not\leq O_2(N)$, we have that $[A, P] \not\leq O_2(N)\Phi(P)$. Hence by the irreducible action of $T$ on $P/\Phi(P)$ we get $P = C_P(W_3)\Phi(P) = C_P(W_3)$, which contradicts $[W_3, \Phi(P)] \neq 1$. $\square$
We now set
\[ \mathcal{N}_S = \{ N \in \mathcal{N} \mid N \text{ is minimal with respect to } S \leq N \not\leq C_G(z) \} \]

By Lemma 6.5 \( \mathcal{N}_S \) is not empty.

**Lemma 6.6.** For \( N \in \mathcal{N}_S \) we have \( C_N(O_2(N)) \leq O_2(N) \).

**Proof.** By Lemma 6.4(iii) \( O(N) \leq C_G(z) \) and is normalized by \( S \). As \( O(C_G(z)) = 1 \), we get the assertion from Lemma 2.29 as \( A_z \not\leq L_3(3) \) or \( M_{11} \) by Proposition 5.2. \( \square \)

We recall some notation which will be maintained until the end of this chapter.

**Notation 6.7.** If \( A_z = G(q) \), \( q = 2^j \), is a group of Lie type not isomorphic to \( Sp_{2n}(q) \), we denote by \( R \) a long root group in \( Z(S \cap A_z) \). In the case of \( A_z \cong Sp_{2n}(q) \) we take a short root group. If \( A_z \) is a sporadic simple group we choose \( R = Z(S \cap A_z) \). Further we denote the group \( O_2(C_{A_z}(R)) \) by \( Q_R \). The structure of \( Q_R \) is given in Lemma 2.17 and Lemma 2.19. In all cases but \( A_z \cong Sp_{2n}(q) \) or \( F_4(q) \) we have that \( R = \Omega_1(Z(Q_R)) \). If \( A_z \) is a sporadic simple group we have by [MaStr, Lemma 2.10] that \( Q_R \) is extraspecial. If \( A_z \not\cong Sp_4(q) \) then \( R = Q'_R \).

Finally we always have that \( C_{A_z}(Q_R) = Z(Q_R) \) by Lemma 2.11.

**Lemma 6.8.** Let \( A_z \cong Sp_4(q) \) or \( F_4(q) \) and assume that \( S \) induces a graph automorphism on \( A_z \). Set \( X = \langle z, R_1, R_2 \rangle \), where \( R_1R_2 = Z(S \cap A_z) \). Then \( N_G(X) \leq C_G(z) \).

**Proof.** We have \( O_2(C_G(X)) = C_S(A_z)Q_{R_1}Q_{R_2} \), so \( Z(O_2(C_G(X))) = R_1R_2C_S(A_z) \). In particular \( \Phi(C_S(A_z)) \) is invariant under \( N_G(X) \). So if \( C_S(A_z) \geq \langle z \rangle \), we get that \( N_G(X) \leq C_G(z) \), the assertion.

Assume now \( C_S(A_z) = \langle z \rangle \). As \( Z(Q_{R_1}Q_{R_2}) = X \cap ((z)Q_{R_1}Q_{R_2})' \), we have that \( N_G(X) \) acts on \( Z(Q_{R_1}Q_{R_2}) \). We have that \( N_{C_S(z)}(X) \) induces two orbits of length \( 2(q - 1) \) and \( (q - 1)^2 \) in \( (R_1R_2)^2 \) (recall that there is a graph automorphism in \( S \), so \( R_1 \) is conjugate to \( R_2 \) in \( S \)). Further \( \emptyset = z^{N_G(X)} \cap Z(Q_{R_1}Q_{R_2}) \). As the \( |z^{N_G(X)}| \) is odd, we get that either \( N_G(X) \leq C_G(z) \) or \( z \) has precisely \( 2q - 1 \) conjugates under \( N_G(X) \), which are \( zR_1 \cup zR_2 \).

By way of contradiction we assume that \( z \) has precisely \( 2q - 1 \) conjugates. Then \( N_G(X) \) acts 2-transitively on \( z^{N_G(X)} \). In particular all \( z^g, z^h, g, h \in N_G(X), z^g \neq z^h \), are conjugate. Choose \( r_1, \tilde{r}_1 \in R_1, r_2, \tilde{r}_2 \in R_2 \) with \( r_1r_2 \neq 1 \neq \tilde{r}_1\tilde{r}_2 \). Then \( (zr_1)(z\tilde{r}_2) \) is conjugate to \( (z\tilde{r}_1)(z\tilde{r}_2) \). This shows that all elements in \( Z(Q_{R_1}Q_{R_2})^2 \) are conjugate in \( N_G(X) \). As
\[ Z(Q_{R_1}Q_{R_2}) \] contains involutions \( x \) which are centralized by \( S \), we see that \( |x^{N_G(x)}| \) is odd. Hence \( x \) has exactly \( q^2 - 1 \) conjugates. This gives that \( q^2 - 1 \) divides \( |N_G(X)/C_G(X)| \).

Assume there is a Zsigmondy prime \( p \) dividing \( q^2 - 1 \) and let \( \omega \) be some element in \( N_G(X) \) with \( \omega \not\in C_G(X) \) but \( \omega^p \in C_G(X) \). Suppose first that \( p \) does not also divide \( 2q - 1 \), then we may assume that \( [\omega, z] = 1 \). But \( |N_G(x)/C_G(x)| \) divides \( (q - 1)^2u \), where \( q = 2^u \). As \( p \) is a Zsigmondy prime, it does not divide \( (q - 1) \). Hence \( p \) divides \( u \). By the little Fermat Theorem we have that \( p \) divides \( 2^{p-1} - 1 \) which is smaller than \( q - 1 = 2^u - 1 \), but this contradicts \( p \) being a Zsigmondy prime. Hence we may assume that \( p \) divides \( 2q - 1 \) which gives \( q = 2 \) and \( p = 3 \). By Proposition 5.2 we have \( A_2 \cong F_4(2) \). As \( Q_{R_1} \cap Q_{R_2} = (Q_{R_1}Q_{R_2})' \), we have that \( \omega \) normalizes \( Q_{R_1} \cap Q_{R_2} \). Further it acts on \( C_{Q_{R_1}Q_{R_2}}(Q_{R_1} \cap Q_{R_2}) = (Q_{R_1} \cap Q_{R_2})Z(Q_{R_1})Z(Q_{R_2}) = Y \). As \( q = 2 \) and \( Z(Q_{R_1}) \) induces a transvection on \( Z(Q_{R_2}) \), we see \( |Y'| = 2 \), and so \( C_{R_1R_2}(\omega) \neq 1 \). As \( |R_1R_2| = 4 \), we get \( [\omega, R_1R_2] = 1 \), which then gives the contradiction \( [X, \omega] = 1 \).

So we have that there is no Zsigmondy prime which divides \( q^2 - 1 \). Hence \( q = 8 \). By Lemma 2.22 we see \( |\text{Out}(F_4(8))| = |\text{Out}(Sp_4(8))| = 2 \cdot 3 \cdot 5 \cdot 7^2 \). This implies \( |S : C_S(X)| = 2 \). In particular \( N_G(X)/C_G(X) \) has a normal 2–complement \( K \). As \( |z^{N_G(X)}| = 15 \), we get that \( |K| = 3 \cdot 5 \cdot 7^2 \) or \( 3^2 \cdot 5 \cdot 7^2 \), as \( 7^2 = |N_{A_2}(X)/C_{A_2}(X)| \). In both cases with the Burnside lemma we get a normal 5–complement in \( K \). Hence a Sylow 5–subgroup centralizes a Sylow 7–subgroup and then we have a normal Sylow 7–subgroup \( P \) in \( K \). As \( 7^2 \) divides \( |N_{A_2}(X)| \), we have \( P \leq C_G(z) \) and \( P \) acts as the Borel subgroup on \( X \). This gives \( C_X(P) = \langle z \rangle \). But then \( \langle z \rangle \leq N_G(X) \), a contradiction. \( \square \)

**Lemma 6.9.** \( N_G(S) \leq C_G(z) \). In particular \( z^G \cap \Omega_1(Z(S)) = \{ z \} \).

**Proof.** Set \( N = N_G(S) \) and assume that \( N \not\leq C_G(z) \). We first show that

\[ Z(Q_R) \neq R. \]

(1)

Suppose false. Assume first that \( O_2(C_G(\Omega_1(Z(S)))) = Q_R \times C_S(A_2) \). Set \( M = N_G(Q_R \times C_S(A_2)) \). Then \( N \leq M \). If \( Z(Q_R) = R \), then \( M \) acts on \( \langle z, R \rangle \). As all elements in \( R^2 \) are conjugate in \( M \), and \( |z^M| \) is odd, we would get that \( z^M = \langle z, R \rangle^2 \). But \( R \leq (C_S(A_2) \times Q_R)' \), while \( z \) is not, a contradiction. So we have that \( O_2(C_G(\Omega_1(Z(S)))) / Q_R \times C_S(A_2) \).

By Lemma 2.24 we get that \( A_2 \cong L_3(q) \) or \( L_4(q) \). By Proposition 5.2 we have \( q > 2 \). If \( A_2 \cong L_3(q) \), then by Lemma 2.20 \( S \) contains exactly
two abelian groups isomorphic to \( \mathbb{Z}_{2^n} \times E_{q^2} \), where \(|C_S(A_z)| = 2^n\). If \( A_z \cong L_4(q) \), then \( S \) contains exactly one abelian group isomorphic to \( \mathbb{Z}_{2^n} \times E_{q^4} \). This shows that elements of odd order in \( N \) normalize these groups. As \( N \not\subseteq C_G(z) \), we see that \( n = 1 \). If \( A_z \cong L_3(q) \), then the product of these two elementary groups is just \( \langle z \rangle Q_R \), which now is normal in \( N \). But \( z \not\in \langle \langle z \rangle Q_R \rangle \), a contradiction as before. So assume \( A_z \cong L_4(q) \). Then some graph automorphism is contained in \( O_2(C_G(\Omega_1(Z(S)))) \). In particular this group contains \( \langle z \rangle \times Q_R \) of index two. Then again \( z \not\in O_2(C_G(\Omega_1(Z(S)))) \) but \( R \) is, a contradiction as before. This proves (1)

With [MaStr, Definition 2.32] and (1) we now have that \( A_z \cong Sp_{2n}(q) \) or \( F_4(q) \). We next show

\[
R \cap \Omega_1(Z(S)) = 1.
\]

Suppose false and let first \( A_z \not\cong Sp_4(q) \), i.e. \( Q_R \) is not abelian. Set \( X_z = O_2(C_G(\Omega_1(Z(S)))) \). Then \( X_z = C_S(A_z) \times (X_z \cap A_z) \). Further \( Z(X_z \cap A_z) \) is elementary abelian. As \( N \) normalizes \( X_z \), we get \( C_S(A_z) = \langle z \rangle \) again. We see that \( |X_z \cap A_z : Q_R| = q \) in case of \( Sp_{2n}(q) \) and \( X_z \cap A_z = Q_{R_1}Q_{R_2} \) in case of \( F_4(q) \), where \( R_1, R_2 \) are the two root groups in \( Z(S \cap A_z) \). Now in both cases \( Z(X_z \cap A_z) = X_z', \) while \( z \not\in X_z' \). Let \( K \) be a 2–complement of \( S \) in \( N_G(S) \). Then \( K \) acts on \( \Omega_1(Z(S)) \cap A_z \) and \( \Omega_1(Z(S))/\Omega_1(Z(S)) \cap A_z \). If \( |\Omega_1(Z(S)) \cap A_z| > 4 \), then \( q > 2 \), and so \( \Omega_1(Z(S)) \cap A_z = [\Omega_1(Z(S)), N_{NG(A_z)}(z)] \) again. Hence \( \Omega_1(Z(S)) \cap A_z = [\Omega_1(Z(S)), K] \) and we see that \( \Omega_1(Z(S)) = (\Omega_1(Z(S)) \cap A_z) \times C_{\Omega_1(Z(S))}(K) \). As \( C_{\Omega_1(Z(S))}(N_{NG(A_z)}(z)) = \langle z \rangle \), we get \( [z, K] = 1 \), a contradiction.

So we have \( |\Omega_1(Z(S)) \cap A_z| = 4 \). If \( A_z \cong Sp_{2n}(q) \), then \( q > 2 \) by Proposition 5.2. So we receive \( O_2(C_G(\Omega_1(Z(S))))^{(\infty)} \leq Q_R \) is nonabelian. Hence \( R = O_2(C_G(\Omega_1(Z(S))))^{(\infty)} \). But then \( |K, R \cap \Omega_1(Z(S))| = 1 \). So we are left with \( A_z \cong F_4(q) \). Now with [MaStr, Definition 2.32] and Lemma 2.17 we receive \( O_2(C_G(\Omega_1(Z(S)))) = Q_{R_1}Q_{R_2} \). We further have that \( Q_{R_1} \cap Q_{R_2}/R_1R_2 \) just involves two natural \( Sp_4(q) \)–modules and \( Q_{R_1}Q_{R_2}/Q_{R_1} \cap Q_{R_2} \) is a direct sum of two modules which are non split extensions of the trivial module by the natural module. As \( Q_{R_1} \cap Q_{R_2} = (Q_{R_1}Q_{R_2})' \), we have that \( K \) normalizes \( Q_{R_1} \cap Q_{R_2} \) and then \( Y_z = (Q_{R_1} \cap Q_{R_2}) \) is the sum of the trivial modules in \( Q_{R_1}Q_{R_2}/(Q_{R_1} \cap Q_{R_2}) \), i.e. \( Y_z = (Q_{R_1} \cap Q_{R_2})Z(Q_{R_1})Z(Q_{R_2}) \). Hence \( Y_z' = [Z(Q_{R_1}), Z(Q_{R_2})] \). We have \( |K, \Omega_1(Z(S))| \leq A_z \). As \( |\Omega_1(Z(S)) \cap A_z| = 4 \), there is a field automorphism \( \nu \) of \( A_z \) possibly trivial, such that \( Z_z = C_{Y_z/Q_{R_1} \cap Q_{R_2}}(\nu) \) is
of order 4. As then $Z_z = C_{Y_z/Q_{R_1} \cap Q_{R_2}}(S)$, we have that $K$ normalizes $Z_z$. For the preimage $Z_z$ we have $|Z_z'| = 2$. Hence $[K, Z_z'] = 1$. As $|\Omega_1(Z(S)) \cap A_z| = 4$, this yields $[K, \Omega_1(Z(S)) \cap A_z] = 1$ and so as $|\Omega_1(Z(S)) : \Omega_1(Z(S)) \cap A_z| = 2$, the contradiction $[K, \Omega_1(Z(S))] = 1$.

To complete the proof of (2) we have to treat $A_z \cong Sp_4(q)$. Then by Proposition 5.2 $q > 2$. We have two root groups $R_1, R_2$ in $Z(S \cap A_z)$. Let $|C_S(A_z)| = 2^n$. By Lemma 2.21 we have exactly two abelian subgroups $C_S(A_z) \times Q_{R_1}$ and $C_S(A_z) \times Q_{R_2}$ of type $Z_{2^n} \times E_{q^3}$ in $S$. Hence $N$ normalizes both and so $C_S(A_z) = \langle z \rangle$. Now $N$ normalizes a Sylow 2–subgroup of $A_z \times \langle z \rangle$, which is $\langle z \rangle \times Q_{R_1}Q_{R_2}$.

We have $(Q_{R_1}Q_{R_2})' = R_1R_2$. Let $K$ be as before a 2–complement in $N$. Then $K$ acts on $R_1R_2$. If $|Z(S) \cap R_1R_2| > 4$, we may argue as before. So we may assume that $|Z(S) \cap R_1| = |Z(S) \cap R_2| = 2$. Then again we must have some element $\nu \in S$, which induces a field automorphism on $A_z$ such that $|C_S(\nu)| = 2$. By Lemma 2.22 $\nu$ acts in the same way on $Q_{R_1}/R_1R_2$, $i = 1, 2$. Hence $\overline{Z}_z = C_{Q_{R_1}Q_{R_2}/R_1R_2}(\nu)$ is of order 4. This shows $|\overline{Z}_z| = 2$, and so $[N, Z_z] = 1$. But then also $[N, Z(S) \cap R_1R_2] = 1$ and so $[\Omega_1(Z(S)), K] = 1$, a contradiction. This proves (2).

By (2) we have that $R$ does not contain 2–central elements of $C_G(z)$. Then $A_z$ admits a graph automorphism in $C_G(z)$. So $A_z \cong Sp_4(q)$ or $F_4(q)$. Set $X = \langle z \rangle Z(Q_{R_1}Q_{R_2})$, $R_i$ as above. Now $\Omega_1(Z(S)) \leq X$. As before we see that $\langle z \rangle = C_S(A_z)$. Now by Lemma 2.21 $\langle z \rangle \times Q_{R_1}Q_{R_2}$ is the group generated by the elementary abelian subgroups of $O_2(N)$ of order $2q^2$ for $A_z \cong Sp_4(q)$ and $\langle z \rangle \times Q_{R_1}Q_{R_2} = O_2(C_G(\Omega_1(Z(S))))$ if $A_z \cong F_4(q)$. Hence $N$ normalizes $\langle z \rangle Q_{R_1}Q_{R_2}$ in both cases. So $N \leq N_G(X)$. By Lemma 6.8 we have $N_G(X) \leq C_G(z)$ and so also $N_G(S) \leq C_G(z)$, the assertion. □

**Lemma 6.10.** If $N \in N_S$, then $Q_R \not\leq O_2(N)$.

**Proof.** Suppose $Q_R \leq O_2(N)$. Assume first that we have $\Omega_1(Z(Q_R)) = R$. Then we have that $\Omega_1(Z(O_2(N))) = \langle z, R_1 \rangle$ with $R_1 \leq R$. But then all elements in $\Omega_1(Z(O_2(N)))$ are 2–central in $G$. By Lemma 6.9 we then have $z^N \cap Z(O_2(N)) = \{z\}$ and so the contraction $N \leq C_G(z)$.

By [MaStr, Definition 2.32] we are left with $A_z \cong Sp_{2n}(q)$ or $F_4(q)$. Then all involutions in $Z(Q_R)$ are 2–central in $A_z$. If this is also true in $C_G(z)$, then again all involutions in $\Omega_1(Z(O_2(N)))$ are 2–central and so again $N \leq C_G(z)$. 

So $S$ must contain some element which induces a graph automorphism on $A_z$. This implies $A_z \cong Sp_4(q)$ or $F_4(q)$. In both cases we have that $Q_{R_1}$ and $Q_{R_2}$ both are contained in $O_2(N)$, where $R_1$, $R_2$ are the two root subgroups with $R_1R_2 = Z(A_z \cap S)$. Set $X = \langle z \rangle Z(Q_{R_1}Q_{R_2})$, $\Omega_1(Z(O_2(N))) \leq X$. If $A_z \cong F_4(q)$, we see that $C_S(A_z) \times Q_{R_1}Q_{R_2} = O_2(C_G(X))$. If $A_z \cong Sp_4(q)$, we see by Lemma 2.22 and Lemma 2.21 that $\langle C_S(A_z), Q_{R_1} \rangle, \langle C_S(A_z), Q_{R_2} \rangle$ are the only two abelian subgroups of order $2^aq^3$, $|C_S(A_z)| = 2^a$, in $S$. Hence in any case we see that $C_S(A_z) \times Q_{R_1}Q_{R_2}$ is normal in $N$. As $N \not\leq C_G(z)$, we get $C_S(A_z) = \langle z \rangle$. Now $N$ normalizes $Z(\langle z, Q_{R_1}, Q_{R_2} \rangle) = \langle z, R_1, R_2 \rangle$. Application of Lemma 6.8 gives the final contradiction.

The next two lemmas are of central importance for the proof of the main theorem. These describe the structure of $N \in \mathcal{N}_S$. Moreover we show that $q = 2$, if $A_z$ is a group of Lie type over $GF(q)$, and finally that there is some involution $t \in Z(N)$. In what follows we then determine the centralizer of this involution $t$, which eventually will yield the final contradiction.

**Lemma 6.11.** Let $N \in \mathcal{N}_S$ with $U = \Omega_1(Z(O_2(N))) \leq C_G(A_z) \times Q_R$, then $|\Omega_1(Z(S)) \cap (A_z \times C_G(A_z))| = 4$, $|R| = 2$ and there is some $t \in \Omega_1(Z(S)) \setminus \langle z \rangle$ such that $t \not\in A_z$ and $t \in Z(N)$. Further one of the following holds:

- (i) $N/C_N(U) \cong \Sigma_3$ and $Q_R \leq S$; or
- (ii) $N/C_N(U) \cong \Sigma_3 \rtimes \mathbb{Z}_2$, $A_z \cong F_4(2)$ and $Q_R \not\leq S$

**Proof.** By Lemma 6.6 $z \in U$, so $C_G(U) \leq C_G(z)$. We have

$$1 \leq Z(Q_R) \times C_G(A_z).$$

Otherwise $\langle Q_N^N \rangle \leq C_N(U)$, so $\langle Q_N^N \rangle \leq C_G(z).$ By Lemma 6.4(i) we have $N = N_N(S \cap \langle Q_N^N \rangle)$. Hence $\langle Q_N^N \rangle \leq O_2(N)$, contradicting Lemma 6.10.

In particular by (1) $Q_R$ is not abelian, hence $A_z \not\cong Sp_4(q)$. As by (1) $[U, Q_R] = R \leq U$, we have that $Q_R$ induces an elementary abelian group $Q_R/Q_R \cap O_2(N)$ on $U$.

Let $H$ be a hyperplane in $Z(Q_R)$ not containing $Q_R$. Then by [MaStr, Lemma 2.36] $Q_R/H$ is extraspecial. Hence we receive that $|Q_R/H : C_{Q_R/H}(UH/(\langle z \rangle H))| \geq |UH/(\langle z \rangle H)\cap Z(Q_R)/H|$. So we have that $|Q_R : C_{Q_R}(U)| \geq |U : C_U(Q_R)|$. In particular $U$ is an $F$-module with quadratic offender $A = Q_R/C_{Q_R}(U)$.

Suppose that $N/C_N(U)$ is nonsolvable. We have that $O^2(N/C_N(U)) = N_1 \cdots N_r$, $S$ acts transitively on the $N_i$ and by Lemma 6.4 induces
on each a minimal parabolic. By Lemma 3.5 A normalizes each \( N_i \) and so induces with some \( N_i \) an \( F \)-module on \( U \). Hence by Lemma 3.3 and [Asch1, Theorem A] we have that \( N_i \cong SL_2(2^n) \) or \( A_{2n+1} \). Let \( V_i = [U, N_i] \), then also \([V_i, Q_R] \leq V_i \) and then we may assume that \( R \leq V_i \). But as \( S \) acts transitively on the \( N_i \), we get \( r = 1 \) if \( R \) is normalized by \( S \). If there is \( t \in S \) with \( R' = R_tR \), then \( R \leq V_2 \) and we have \( r \leq 2 \). If \( r = 2 \) then \([Q_R, V_2] = 1 = [Q_R, V_i] \). In any case we have that \( V_i/C_{V_i}(N_1) \) is the natural module by Lemma 3.3, Lemma 3.9 and Lemma 3.10. As \([N_1, V_2] = 1 \), we get in any case that \( U/C_U(N_1) \) is the natural module.

Let \( N_1 \cong SL_2(2^n) \). Then first of all, as \(|Q_R : C_{Q_R}(U)| \geq q \) and \(|[U, Q_R]| = q \), we have \( 2^n = q \). Then as \( N \) is a minimal parabolic such that the unique maximal subgroup containing \( S \) is \( C_N(z) \), we have that a Borel subgroup \( B \) of \( N_1 \) centralizes \( z \). But we have that \( U/C_U(N_1) \) is the natural module. Now \( C_U(B) = C_U(N_1) \) by Lemma 3.14, and so \( z \in Z(N) \), a contradiction.

Let \( N_1 \cong A_{2n+1} \), \( n > 1 \). Then \( U/C_U(N_1) \) is the permutation module. We have again that \( z \) is centralized by some subgroup \( L \cong A_{2n} \) in \( N_1 \). By Lemma 3.13 we see that \( \Omega_1(Z(S)) \leq C_U(C_{N_1}(z)) \). Hence \([C_N(z), \Omega_1(Z(S))] = 1 \). So \( C_N(z) \leq C_N(\Omega_1(Z(S))) \) and then \( O_2(C_N(z)) \leq O_2(C_N(\Omega_1(Z(S)))) \). In particular we have that \( Q_R \leq O_2(C_N(z)) \). As \( O_2(C_N(z)) \), we have that \( 2^n = 4 \). So we have that \( N_1/C_N(U) \cong A_5 \) and \( Q_R \) projects onto a subgroup of a Sylow 2-subgroup of \( N_1 \). As \([N_1, U] \) is the permutation module now \( Q_R \) cannot be an offender.

Assume next that \( N/C_N(U) \) is solvable. Set \( U_1 = \langle z^N \rangle \). By Lemma 3.16 we have that \( C_N(U_1) \) is 2-closed and \( N/C_N(U_1) = O_{3,2}(N/C_N(U_1)) \).

Hence as \( N \) is a minimal parabolic, we receive \( N = SP \), where \( P \) is a Sylow 3-subgroup of \( N \). Further by the minimal choice of \( N \) we have that \( \Phi(P) \) centralizes \( z \), so \([\Phi(P), U_1] = 1 \). Let \( A \) be an \( F \)-module offender, which normalizes \( P \). As \( O_2(N) \) is a Sylow 2-subgroup of \( C_N(U_1) \), we have that \( A \) exists and \([a, P] \not\leq \Phi(P) \) for \( a \in A^2 \).

Hence \( A \) induces an \( F \)-module offender on \( U_1 \) too. By Lemma 3.17 we get that \(|U_1 : C_{U_1}(A)| = |A| \). As \(|U : C_U(A)| \leq |A| \), we see that \(|U, A| = U_1 \). As \( S \) acts irreducibly on \( P/\Phi(P) \), we see that \(|U, P| \leq U_1 \) and so \(|U, \Phi(P)| \leq C_U(\Phi(P)) \). Hence \([U, \Phi(P)] = 1 \). This shows that \( C_N(U_1) = C_N(U) \) and so \( P \) induces an elementary abelian group on \( U \).
By Lemma 2.3 we get a direct product \( M = M_1 \times \cdots \times M_r \) of dihedral groups \( M_i \) of order 6 contained in \( N/C_N(U) \) with \( Q_R/C_{Q_R}(U) \) as a Sylow 2-subgroup. As \( U \leq Q_R \times C_S(A_z) \) we see \( [U, Q_R] \leq R \) and so \( Q_R \) acts quadratically on \( U \). We get that \( [U, O_3(M)] = V_i \oplus \cdots \oplus V_r \), with \([O_3(M_i), V_i] = V_i \) and \([O_3(M_i), V_j] = 1, i, j = 1, \ldots, r, i \neq j \). As \([Q_R, V_i] \leq R \) and \([V_1, Q_R] = R \), we get \( r = 1 \) and \( |Q_R/C_{Q_R}(U)| = 2 \). Then also \([|U, Q_R]| = 2 \). If \( R \) is normalized by \( S \), \( Q_R \) must invert \( P/C_P(U) \), so \([|P/\Phi(P), Q_R]| = 3 \) and \( N/C_N(U) \cong \Sigma_3 \), which is (i). In the other case \([P/C_P(U)] = [P/C_P(U), Q_RQ_s] \) for some \( t \) in \( S \setminus N_S(Q_R) \). Hence \([P/C_P(U)] = 9 \). We have a fours group acting on \( P \) and \( U/C_U(P) \) is the natural \( O_4(2) \)-module. In both cases \([Q_R : C_{Q_R}(U)] = 2 \). This shows that \( Q_R \) is extraspecial or isomorphic to \( E \times Q \), with \( Q \) extraspecial and \( E \leq Z(Q_R) \). In particular if \( Q_R \not\cong S \), we get that \( A_z \cong F_4(2) \), which is (ii).

Suppose that \( |Z(S) \cap E \times Q| > 2 \). Then by Proposition 5.2 we have that \( A_z \cong F_4(2) \). Further \( Q_R \) is normal in \( S \) and so \( C_G(z) = A_z \times C_G(A_z) \). As \( z \notin Z(N) \), we have \( C_S(A_z) = \langle z \rangle \). Then Lemma 2.2 and Lemma 6.9 give a contradiction. So we have

\[
|\Omega_1(Z(S)) \cap A_z \times C_S(A_z)| = 4.
\]

To prove the lemma, we just have to show the existence of the involution \( t \).

In any case we have that \( U = [O_{2,3}(N), U] \times C_{U}(O_{2,3}(N)) \). Furthermore \( |\Omega_1(Z(S))| = 4 \) and \( |C_{[O_{2,3}(N), U]}(S)| = 2 \). This implies \( C_U(O_{2,3}(N)) \neq 1 \). Hence there is some \( t \in \Omega_1(Z(S)) \setminus \langle z \rangle \) with \([t, N] = 1 \). We have that \([O_{2,3}(N), U], Q_R] \neq 1 \). In particular we have that \( R \leq [O_{2,3}(N), U] \). Hence \( \Omega_1(Z(S)) \cap A_z \leq [O_{2,3}(N), U] \). As \( t \not\in [O_{2,3}(N), U] \) we get \( t \notin A_z \).

**Lemma 6.12.** Let \( N \in \mathcal{N}_S \) with \( U = \Omega_1(Z(O_2(N))) \not\leq C(A_z) \times Q_R \). Then \( |\Omega_1(Z(S)) \cap (A_z \times C_G(A_z))| = 4, \ |R| = 2 \) and \( Q_R \leq S \). Further \( E(N/C_N(U)) \cong A_5 \) and induces just one nontrivial irreducible module in \( U \), the permutation module, or \( N/C_N(U) \cong O_4^+(2) \) and just the natural module is induced in \( U \). Further there is some \( t \in \Omega_1(Z(S)) \setminus \langle z \rangle \) such that \( t \notin A_z \) and \( t \in Z(N) \).

**Proof.** We first show

\[(1) \quad U \text{ normalizes } Q_R.\]

If \( U \) does not normalize \( Q_R \) we get \( A_z \cong Sp_4(q) \) or \( F_4(q) \). We have \([U, Q_R] \leq U \). In particular \([U, Q_R] \) is abelian. From Lemma 2.26 we get
$A_z \not
sim Sp_4(q)$ or $F_4(q)$. This proves (1).

Next we show

\[ O_2(N) \leq N_{N_G(A_z)}(R). \]  

Otherwise $R \cap U = 1$. Hence by (1) $[[U, Q_R], Q_R] = 1$, which shows that $[U, Q_R] \leq Z(Q_R)$. As $U \not
leq Q_R \times C_G(A_z)$, we get from Lemma 2.25 that $A_z \cong Sp_4(q)$ and $Q_R$ is elementary abelian. Let $R_1, R_2$ be the two root groups in $Z(S \cap A_z)$. As $U$ is elementary abelian, we have $R_1 \cap R_2 = Q_R$. Then we also see that $U$ cannot contain elements which induce field automorphisms on $A_z$, otherwise for such $u \in U$, we have that $1 \neq [u, R] \leq R$, contradicting $R \cap U = 1$. Hence $U \leq R_1 R_2 \times C_G(A_z)$, contradicting $U \not
leq Q_R \times C_G(A_z)$. So we have (2).

Now we apply Lemma 3.21 with $N$ in the role of $M$ and $N_{N_G(A_z)}(R)$ in the role of $H$. Suppose that $X = O_2(N_{N_G(A_z)}(R)) \not
leq C_G(A_z) \times Q_R$. Then there is some $x \in X$ inducing an outer automorphism on $A_z$; in particular $A_z$ is of Lie type in characteristic two. If $x$ is a field automorphism it acts on a group of order $q - 1$ which acts nontrivially on $R$, so $x$ cannot be contained in $X$. Hence $x$ acts nontrivially on the Dynkin diagram and so has to centralize the Levi factor. This shows $A_z \cong L_4(q)$. By Proposition 5.2 we have $q > 2$. But then $x$ acts nontrivially on a group $Z_{q-1} \times Z_{q-1}$ in $N_{A_z}(R)$, a contradiction. So we have that $X \leq C_G(A_z) \times Q_R$ and then $U \not
leq X$. Hence from Lemma 3.21 we get

\[ U \text{ is a } 2F \text{-- module.} \]  

We show

(3) If $\tilde{U}$ is a $Q_R$--invariant submodule of $U$ with $[Q_R, \tilde{U}] \neq 1$.

(U) Then either $Q_R$ is abelian or $R \leq \tilde{U}$.

Further $O_2(N) \leq N_N(Q_R)$.

Suppose $Q_R$ to be nonabelian. By (1) $[\tilde{U}, Q_R] \leq Q_R$. Then $[\tilde{U} \cap Q_R, Q_R] = 1$ or $[\tilde{U} \cap Q_R, Q_R] = R$. In the latter $R \leq \tilde{U}$. In the former we have $[Q_R, \tilde{U}] \leq Z(Q_R)$. Hence by Lemma 2.25 $\tilde{U} \leq Q_R \times C_S(A_z)$ and so $[U, Q_R] = R$ and we are done. The second statement in (U) follows by (2).
We now first work under the assumption:

(A) Assume $N$ is nonsolvable.

Let $E(N/C_N(U)) = N_1 \ast \cdots \ast N_r$. Assume first that for an offender $A$ as a $2F$-module which is given by Lemma 3.21 we have that $[A, N_1] \nsubseteq N_1$. If $A$ acts quadratically then by Lemma 3.21 we have that $A$ induces an $F$-module offender. But this contradicts Lemma 3.5. So $A$ cannot be quadratic on $U$. By Lemma 3.22 we get that $N_1 \cong L_n(2)$, and for some $a \in A$ with $N_1^a \neq N_1$ we have that $A$ induces the full transvection group on $[U, a]$. Hence $C_{N_1N_1}(a)$ induces the natural module on $[U, a]$. As $N_1N_1(N_1)$ is a minimal parabolic by Lemma 6.4, we have $n = 3$ and with the natural module also the dual module is involved. Hence for $C_{N_1N_1}(a)$ we have a natural and a dual module involved in $U$, which contradicts that $C_{N_1N_1}(a)$ induces just the natural module.

So we have that

(A.1) The offender $A$ from Lemma 3.21 normalizes all components.

We have $|U : C_U(A)| < |A : C_A(U)|^2$ by Lemma 3.21. Now we choose $A$ minimal such that $|U : C_U(A)| < |A : C_A(U)|^2$. Set $A_1 = C_A(N_1)$. Then we have that $|U : C_U(A_1)| \geq |A_1 : C_{A_1}(U)|^2$. In particular for a complement $B$ of $A_1$ in $A$ we have that $|C_U(A_1) : C_{C_U(A_1)}(B)| < |B : C_B(C_U(A_1))|^2$, which yields:

(A.2) Let $T = C_S(N_1)$, then we have that $V = C_U(T)$ is a $2F$-module for $N_1$ with offender $B$ such that $|V : C_V(B)| < |B|^2$.

Application of Lemma 6.10 yields $Q_R \nsubseteq O_2(N)$. By Lemma 6.4(i) we have that $O_2(N)$ is normal in $C_N(U)$. This shows $[Q_R, U] \neq 1$. Hence $[N_1 \ast \cdots \ast N_r, Q_R] \neq 1$. So we may assume that $[N_1, Q_R] \neq 1$. Set $U_1 = [N_1, V]$.

Now by (A.2) $N_1$ and $U_1$ are given in Lemma 3.4. As $[z, N_1] \neq 1$ we get $C_V(C_{N_1}(z)) \neq C_V(N_1)$. If the irreducible $N_1$-modules in $V$ are $F$-modules, we have that $N_1 \cong L_2(2^n)$ and $C_{N_1}(z)$ is a Borel subgroup, a contradiction to Lemma 3.14, or $N_1 \cong A_{2n+1}$ and $C_{N_1}(z) \cong A_{2n}$. By Lemma 3.12 we see that $V/C_V(N_1)$ is the permutation module. Then Lemma 3.14 shows that $C_{N_1}(z)$ centralizes $\Omega_1(Z(S))$. So we see that $Q_R \leq O_2(C_N(z))$. We get $N_1 = A_5$ and then $U_1$ is the permutation module.
So assume now that we have Lemma 3.4(b). In the first three cases always $C_{N_1}(z)$ is a Borel subgroup, which has a fixed point on the corresponding modules exactly when $r = 2$, so $N_1 \cong L_3(2)$ or $Sp_4(2)'$. In both cases we have that $U_1/C_{U_1}(N_1)$ is a direct sum of a natural module and its dual. Assume now $N_1 \cong A_9$ and $|U_1/C_{U_1}(N_1)| = 2^8$. Then $z \in C_U(N_1)$ by Lemma 3.14, a contradiction.

So we collect:

$$N_1 \cong L_2(4), L_3(2), \text{ or } A_6. \text{ In the last two cases we have}$$

$$U_1/C_{U_1}(N_1) = U_{11} \oplus U_{12}, \text{ where } U_{11} \text{ and } U_{12} \text{ are dual modules for } N_1. \text{ In the first case we have that } U_1$$

$$\text{is the permutation module.}$$

Next we show

$$A_z \not\cong F_4(q). \text{ Further if } A_z \cong Sp_4(q), \text{ then } Q_R \text{ is elementary}$$

$$\text{abelian and } Q_R \text{ acts quadratically on } U.$$ 

The second statement follows from (1). So assume $A_z \cong F_4(q)$. Suppose first $\langle U^S \rangle \leq Q_R \times C_S(Q_R)$. Then $[(U^S), Z(Q_R)] = 1$. Hence $[(N_1^S), Z(Q_R)] \leq O_2(N)$. This shows that $Z(Q_R) \leq O_2(N)$ and so $[U, Z(Q_R)] = 1$, which by Lemma 2.17 shows $U \leq Q_R \times C_S(Q_R)$, a contradiction. Thus we may assume that $U_1 \not\leq Q_R \times C_S(Q_R)$. Then for $u \in U_1 \setminus Q_R \times C_G(A_z)$, we obtain that $|[Q_R, u]| \geq q^4$. As $R \leq U$ by (U) we receive that $Q_RO_2(N)/O_2(N)$ is elementary abelian. Hence we get from (A.3) that $|Q_R : C_{Q_R}(U_1)| \leq 8$ if $Q_R$ normalizes $N_1$, a contradiction to $|[Q_R, u]| \geq q^4 \geq 16$. So we have that $Q_R$ does not normalize $N_1$. Then at least a subgroup $T$ of index two in $S \cap N_1$ normalizes $Q_R$, as this is true in $\text{Aut}(F_4(q))$, and then $[Q_R, T]$ is abelian and centralized by $Q_R$. Hence we get that $|N_1^{Q_R}| = 2$ and then $|Q_R : C_{Q_R}(U_1)| \leq 16$. In particular $q = 2$. Further $U_1$ does not induce transvections on $Z(Q_R)$, as for any transvection $u \in U_1$ we have $|[Q_R/Z(Q_R), u]| = 16$ by Lemma 2.17. This implies $N_1 \cong A_6$ and further $Sp_4(2)$ is induced. Now $Z(Q_R)$ acts quadratically on $U$ and so we have by Lemma 3.5 that $Z(Q_R)$ normalizes $N_1$. Then it acts quadratically on $U_1$. As $U_1$ involves the natural module and the dual as well, we see that $Z(Q_R)$ induces a group of order at most four which is in the center of a Sylow 2–subgroup of $Sp_4(2)$. But then $U_1$ contains some $u$ which induces a transvection on $Z(Q_R)$, a contradiction. This proves (A.4).

$$[Q_R, N_1] \leq N_1.$$ 

Suppose false. If $A_z \cong Sp_4(q)$, then by Proposition 5.2 $q > 2$. Hence $|Q_R : C_{Q_R}(U_1)| \geq 4$. By (A.4) $Q_R$ acts quadratically on $U$. We get
by Lemma 3.5 that \( N_1 \cong L_2(4) \). Further \( \langle N_1^R \rangle \) induces just natural \( \Omega_4^+(4) \)-modules in \( U \), contradicting the fact that by (A.3) \( N_1 \) induces an \( \Omega_4^- \)-module. So we have that \( A_z \not\equiv Sp_4(q) \) and by (A.4) \( Q_R \not\leq S \). We further have that \( R \leq U \) by (U) and so \( Q_R O_2(N) / O_2(N) \) is elementary abelian. Hence \( N_1 \) has elementary abelian Sylow 2-subgroups, as for \( t \in Q_R \), with \( N_1^t \neq N_1 \), we have that \([N_1,t]\) has a Sylow 2-subgroup contained in \( Q_R O_2(N) / O_2(N) \) and so is abelian. Then \( N_1 \cong L_2(4) \) again. We further have \( |Q_R : N_{Q_R}(N_1)| = 2 \). Set \( W_1 = [U_1, N_{Q_R}(N_1)] \).

As \( N_{Q_R}(N_1) \) projects onto a Sylow 2-subgroup of \( N_1 \) and \( N_1 \) induces an \( \Omega_4^+(2) \)-submodule, we have \( [W_1, N_{Q_R}(N_1)] \neq 1 \). As \( U_1 \) normalizes \( Q_R \), we have that \( W_1 \leq Q_R \) and so \( |R \cap U_1| = 2 \). Set \( W_1 = \langle U_1 \cap R, x_1, y_1 \rangle \) and choose \( x \in Q_R \) with \( N_1^x = N_2 \). We have \( |Q_R : C_{Q_R}(x_1)| \leq 4 \). From Lemma 2.17 we see that \( |Q_R : C_{Q_R}(x_1)| \neq q \). Hence \( |R| = q \leq 4 \).

As \( [W_1, x] \leq R \) and \( |R : R \cap U_1| \leq 2 \), we may assume that \( [x_1, x] \in U_1 \cap R \). Hence \( |Q_R : C_{Q_R}(x_1)| = 2 \), which gives \( |R| = 2 = q \) and \( R \leq U_1 \). But then \( [Q_R : W_1] \leq R \leq W_1 \). This shows \( W_1^x = W_1 \).

Set \( M = N_{N_1}(W_1) N_{N_1}(W_1)^x \), which is isomorphic to \( A_4 \times A_4 \). Then \( M \) acts on \( W_1 \). Hence there is some element of order three in \( M \) which centralizes \( W_1 \). But then \( O_2(M) \) centralizes \( W_1 \) too, which contradicts the action of \( N_{Q_R}(N_1) \) on \( W_1 \). This proves (A.5)

Next we show

(A.6) \[ N_1 \cong A_5 \text{ and } U_1 \text{ is the irreducible part of the permutation module.} \]

According to (A.3) we may assume \( N_1 \cong L_2(2) \) or \( A_6 \). Assume further \( A_z \not\equiv Sp_4(q) \). If \( Q_R \) normalizes both modules \( U_{11} \) and \( U_{12} \) given in (A.3) then by (U) \( R \leq U_{11} \cap U_{12} \), a contradiction. Hence there is \( x \in Q_R \) with \( U_{11}^x = U_{12} \). But then \( x \) induces an outer automorphism of \( L_2(2) \) or \( \Sigma_6 \) and then \( [x, S/O_2(N)] \) is not abelian. By (U) we have \( R \leq O_2(N) \) and so \( Q_R / Q_R \cap O_2(N) \) is elementary abelian. This contradicts \( Q_R \not\leq S \) and \( [x, S/O_2(N)] \) being not abelian.

So we have that \( A_z \cong Sp_4(q) \), \( q \geq 4 \). Then by (A.4) \( Q_R \) is elementary abelian and acts quadratically on \( U \). As \( |Q_R / O_2(N) \cap Q_R| \geq 4 \), we see that \( Q_R \cap N(U_{11}) \not\leq C(U_{11}) \). By quadratic action we get that \( Q_R \) normalizes \( U_{11} \) and \( U_{12} \). This even shows \( Q_R / Q_R \cap O_2(N) \cap N_1 \neq 1 \).

In particular \( |U_1 : C_{U_1}(Q_R)| \geq 16 \). As by Lemma 3.8 in \( \Sigma_6 \) no subgroup of order 8 acts quadratically on both modules, we get that \( Q_R \) induces a foursgroup on \( N_1 \) and so \( q = 4 \). But then \( N_{C_{G(z)}}(Q_R) / C_{C_{S_2}(z)}(Q_R) \) is isomorphic to a subgroup of \( (A_5 \times \mathbb{Z}_3) : \mathbb{Z}_2 \) and so contains no elementary abelian subgroup of order 16, but \( U_1 / C_{U_1}(Q_R) \) contains such an
elementary abelian subgroup. This proves (A.6).

Next we are going to describe the structure of $N$. We have $N_1 \cong L_2(4)$. Further we have that $U_1$ is the permutation module. As before we see by (U) that $R \leq U_1$ for $Q_R$ not abelian. This shows that $Q_R$ acts quadratically on $U/U_1$ in any case. As by Lemma 3.14 $Q_R \leq O_2(C_N(z))$, we see that $Q_R$ projects into a subgroup of $C_N(z) \cong A_4$. If this projection is of order two, we get that $U_1$ induces transvections on $Q_R$. In particular $A_z \not\cong Sp_4(q)$. This now shows that $U_1 \leq Q_RC_G(Q_R)$. We have that $|U_1 : C_{U_1}(Q_R)| = 4$ and so $|Q_R : C_{Q_R}(U_1)| \geq 4$. Hence $Q_R/Q_R \cap O_2(N)$ acts as a Sylow 2-subgroup of $A_5$, which is not quadratic on the permutation module. In particular $Q_R \leq S$ and $A_z \not\cong Sp_4(q)$ by (A.4). Hence $U_1$ is the only permutation module for $N_1$ involved in $U$. This shows that $[U_1, N_i] = 1$ for $i = 2, \cdots, r$. Choose $s \in S$ with $N_i^s = N_2$. Then by (U) we have that $R \leq U_1 \cap U_2 = 1$, a contradiction. This shows $r = 1$. Now we have that $U = U_1 \oplus U_2$, with some $N$-module $U_2$. As $R \leq U_1$, we get from (U) that $U_2$ is a trivial $E(N/O_2(N))$-module. Hence

$$U = U_1 \oplus C_U(N_1).$$

So we have shown

If $N/C_N(U)$ is nonsolvable, then $E(N/C_N(U)) \cong A_5$ and

(A.7) $U = U_1 \oplus C_U(E(N/C_N(U)))$, where $U_1$ is the permutation module. Further $|R| = 2, R \leq U_1$ and $Q_R \leq S$.

Now we assume:

(B) Assume $N$ is solvable.

By Lemma 6.4 $N = O_{2,2',2}(N)$. As by Lemma 3.21(2) offenders are not exact provided $U$ is not an $F$-module, we get with Lemma 3.17 that $N/C_N(U)$ is a $\{2, 3\}$-group. As $N$ is a minimal parabolic we have $N = O_{2,3,2}(N)$. By minimality we have that $\Phi(O_{2,3}(N)/O_2(N)) \leq C_N(z)/O_2(N)$. So $\Phi(O_{2,3}(N)/O_2(N))$ centralizes

$$\langle z^N \rangle = U_1,$$

which gives that $S$ acts irreducibly on $O_3(N/C_N(U_1))$.

We show

(*) $C_{O_{2,3}(N)}(U) = C_{O_{2,3}(N)}(U_1)$ and so $[O_{2,3}(N)', U] = 1$.

For this let $P$ be a Sylow 3-subgroup of $N$ such that $O_2(N) N_N(P) = N$. In particular $P/C_P(U_1)$ is elementary abelian. Hence we may assume that a $2F$-module offender $A$ with $|U : C_U(A)| < |A|^2$ acts on $P$. We have that $|U_1 : C_{U_1}(A)| \geq |A|$ by Lemma 3.17. Hence we conclude
\( |U/U_1 : C_{U/U_1}(A)| < |A| \). So by Lemma 3.17, we get some \( 1 \neq a \in A \), which acts trivially on \( U/U_1 \). This gives that \( C_P(U/U_1) \not\cong \Phi(P) \). As \( N_N(P) \) acts irreducibly on \( P/\Phi(P) \), we get that \( P = \Phi(P)C_P(U/U_1) \) and then \( [P, U] \leq U_1 \). In particular \( [C_P(U_1), U] = 1 \). This is (*).

Application of (U) shows that for \( A_z \not\cong Sp_4(q) \) we have \( Q_R = R \leq U_1 \).

So we have
\[
(B.1) \quad Q_RC_N(U)/C_N(U) \text{ is abelian.}
\]

Let \( |Q_R : C_{Q_R}(U)| = 2 \). Then \( U \) induces a transvection on \( Q_R \) with elementary abelian commutator, so \( U \leq Q_RC_S(Q_R) \), a contradiction.

We receive
\[
(B.2) \quad |Q_R : C_{Q_R}(U)| \geq 4.
\]

By the Dihedral Lemma 2.3 we have a subgroup \( D_1 \times \cdots \times D_s, s \geq 2 \) in \( N/C_N(U) \), \( D_i = \langle x_i, \rho_i \rangle, x_i \in Q_R, o(\rho_i) = 3, D_i \cong \Sigma_3, i = 1, \ldots, s \).

Set \( W = [[\rho_1, U], x_1] \). We have \( W \leq Q_R \). If \( [Q_R : W] = 1 \), then \( W \leq Z(Q_R) \). As \( (W^{\rho_1}) = [\rho_1, U] \), we get \( [x_i, [\rho_1, U]] = 1, i = 2, \ldots, s, \) and so \( [[[\rho_1, U], Q_R] \leq Z(Q_R) \). Now the elements in \( [\rho_1, U] \) induce transvections on \( Q_R \), which gives that \( A_z \not\cong Sp_4(q) \), \( q > 2 \). Application of Lemma 2.25 shows \( [\rho_1, U] \leq Q_R \) and so \( [[[\rho_1, U], Q_R] \leq R \). As \( [\langle x_2, \ldots, x_s \rangle, [\rho_1, U]] = 1, \) we see that \( |Q_R : C_{Q_R}([\rho_1, U])| = 2 \) and so we have \( q = 2 \), and \( W = R \) is of order 2, further \( |U, [\rho_1]| = 4 \). Set \( T = N_S(Q_R) \) and let \( t \in T \). Then \( R \leq [U, \rho_1] \cap [U, \rho_1] \). But as \( [\rho_1, \rho_1^s] \in C_N(U) \) by (*), we have \( [U, \rho_1, \rho_1^s] \leq [U, \rho_1] \). This yields \( \langle \rho_1 \rangle C_N(U) = \langle \rho_1^s \rangle C_N(U) \). Now also \( \langle \rho_1^s \rangle C_N(U)/C_N(U) = \langle \rho_1 \rangle C_N(U)/C_N(U) \). By (B.2) we have that \( O_{2,3}(N)/C_N(U) \) contains an elementary abelian group of order 9. So we get that \( |S : T| = 2 \) and \( O_3(N/C_N(U)) = \langle \rho_1, \rho_1^s \rangle C_N(U)/C_N(U) \), for some \( s \in S \setminus T \). This shows \( |[U, O_{2,3}(N)]| = 16 \) and so \( N/C_N(U) \) is a subgroup of \( GL_4(2) \), which gives that \( S/C_S(U) \) is contained in a dihedral group. But as \( |Q_R/C_{Q_R}(U)| = 4 \), this shows that \( Q_R \) is normal in \( S \), a contradiction.

So we have
\[
(B.3) \quad [Q_R, [[U, \rho_1], x_i]] \neq 1 \text{ for all } i = 1, \ldots, s.
\]

As by (B.3) \( Q_R \) does not act quadratically, we have that \( Q_R \) is not abelian and so
\[
(B.4) \quad A_z \not\cong Sp_4(q).
\]
By (B.3) and (U) we have $R \leq [U, \rho_1]$. Hence $R \cap C_U(\rho_1) = 1$. So by (U) we get that $[Q_R, C_U(\rho_1)] = 1$. In particular $[C_U(\rho_1), \rho_2] = 1$. By choosing $\rho_1$ with $C_U(\rho_1)$ maximal we obtain

$C_U(\rho_1) = C_U(\rho_i)$ and $[U, \rho_1] = [U, \rho_i]$ for $i = 1, \ldots, s$.

Now we consider $(\rho_1, \rho_2)$. We have $(\rho_1\rho_2)^x = \rho_1\rho_2^{-1}$. Then $[U, \rho_1] = C_{[U, \rho_1]}(\rho_1\rho_2) \times C_{[U, \rho_1]}(\rho_1\rho_2^{-1})$. Set $V_1 = C_{[U, \rho_1]}(\rho_1\rho_2)$. We have that $x_1x_2$ normalizes $V_1$ and $[V_1, x_1x_2] \leq Q_R$. Let $V_2 = (V_1)^x$, then we obtain $1 \neq [V_1, x_1x_2, x_2] \leq R$. Further $|[V_1, x_1x_2]| = |V_1, x_1x_2|$. As $x_1x_2$ inverts $\rho_1\rho_2^{-1}$ and $\rho_1\rho_2^{-1}$ acts fixed point freely on $V_1$, we get that $|V_1| = |[V_1, x_1x_2]|^2 \leq |R|^2 = q^2$. This gives

(B.6) \[ |[U, \rho_1]| \leq q^4. \]

Suppose $s \geq 3$. Now $x_3$ centralizes $\rho_1\rho_2$ and so normalizes $V_1$ and $[V_1, x_1x_2]$. This gives $|[V_1, x_1x_2], x_3| \leq Q \cap V_1$. As $Q \cap V_1 = (Q \cap V_2)^x = Q \cap V_2$ and $V_1 \cap V_2 = 1$, we get $|[V_1, x_1x_2], x_3| = 1$. But as $[x_3, \rho_1\rho_2^{-1}] = 1$ and $V_1 = (V_1, x_1x_2)\rho_1\rho_2^{-1}$ we then have $[x_3, V_1] = 1$ and also $[x_3, V_1^x] = 1$. This gives $|[U, \rho_1], x_3| = 1$. But then $|[U, \rho_1], \rho_3| = 1$, a contradiction to (B.5). So we have

(B.7) \[ s = 2. \]

Suppose that $[V_1, x_1x_2] \leq C_G(Q_R)$. Then

$|Q_RC_S(Q_R) / C_S(Q_R) : C_{Q_RC_S(Q_R) / C_S(Q_R)}(V_1)| \leq 2.$

By (B.4) and Lemma 2.4 we see that $V_1 \leq Q_RC_S(Q_R)$. Now also $V_2 = V_1^x \leq Q_RC_S(Q_R)$, which gives $[U, \rho_1] \leq Q_RC_S(Q_R)$. This shows $|[U, \rho_1], x_1, Q_R| = 1$, which contradicts (B.3). Hence we have that $[V_1, x_1x_2]$ centralizes a subgroup of index two in $Q_R$, which implies

(B.8) \[ q = 2. \]

Assume now $|S : T| = 2$, $T = N_S(Q_R)$. Then by (B.4) and (B.8) $A_S \cong F_2(2)$. As $[U, \rho_1] \notin Q_R$, we have for $1 \neq u \in [U, \rho_1]$ that $|Z(Q_R) : C_{Z(Q_R)}(u)| \geq 2$ and $|Q_R/Z(Q_R) : C_{Q_R/Z(Q_R)}(u)| \geq 4$. In particular $|Q_R : C_{Q_R}(u)| \geq 8$, which contradicts $|Q_R : C_{Q_R}(u)| = 4$ by (B.7).

So we have that $Q_R \leq S$. Further $\langle \rho_1, \rho_2 \rangle, U$ is of order 16 by (B.6) and (B.8). As above we see that $\langle \rho_1, \rho_2 \rangle, U = [(\rho_1, \rho_1), U]$ for all $s \in S$. In particular $O_{2,3}(N)/C_N(U)$ is of order 9.
So we have shown
\begin{equation}
Q_R \leq S, |R| = 2, N/C_N(U) \cong O_4^+(2) \text{ and } \left[ U, O_3(N/C_N(U)) \right] \text{ is the natural module.}
\end{equation}

As $R \leq [U, O_3(N/C_N(U))]$ and $[Q_R, O_3(N/C_N(U))] = O_3(N/C_N(U))$, we get
\begin{equation}
U = [U, O_3(N/C_N(U))] \times C_U(O_3(N/C_N(U))).
\end{equation}
Hence in both cases, $N$ solvable and nonsolvable, by (A.7) and (B.9) we just need to prove the existence of $t$ and determine the size of $|\Omega_1(Z(S))|$. For the remainder $N$ might be solvable or not. Assume $|\Omega_1(Z(S)) \cap A_z| > 2$. By (B.8) and (A.7) $q = 2$. So we have that $A_z \cong S_{p_{2n}(2)}$ or $F_4(2)$. By Proposition 5.2 we have $A_z \not\cong S_{p_{2n}(2)}$. Now in $[U, O^2(N)]$, we have some $x$ such that $x \notin Q_R$ but $|[Q_R/R, x]| = 4$. As $A_z \cong F_4(2)$, then by Lemma 2.17 $Q_R/R$ involves two non isomorphic modules for $N_{A_z}(R)$ on one there are transvections on the other not, a contradiction. So we have $|\Omega_1(Z(S)) \cap A_z| = 2$. As $|\Omega_1(Z(S))| \geq 4$, we see $|\Omega_1(Z(S))| = 4$ and from (A.7) and (B.10) we get that $C_U(N) \neq 1$ and so there is some $1 \neq t \in \Omega_1(Z(S))$, which is central in $N$. By (U) $R \leq [U, F^*(N/C_N(U))]$. As $|\Omega_1(Z(S)) \cap A_z| = 2$ we have that $\Omega_1(Z(S)) \cap A_z \leq \langle R^S \rangle$ and so $\Omega_1(Z(S)) \cap A_z \leq [U, F^*(N/C_N(U))].$ Hence $t \notin A_z$.

We now can get further restrictions on the structure of $A_z$.

**Lemma 6.13.** $A_z \not\cong F_4(2)$. Further $Q_R$ is extraspecial with center $R$, normal in $S$ and $N_{G(A_z)}(Q_R)$ acts irreducibly on $Q_R/R$.

**Proof.** Suppose $A_z \cong F_4(2)$. By Lemma 6.11 and Lemma 6.12 we have that $|\Omega_1(Z(S)) \cap A_z| = 2$. Hence there is some $u \in C_G(z)$, which induces a graph automorphism on $A_z$. In particular $Q_R \not\leq S$. This shows by Lemma 6.12 that Lemma 6.11(ii) holds. In particular $U = \Omega_1(Z(O_2(N)) \leq Q_RC_S(A_z).$ As $z^G \cap U \neq \{z\}$ also $z^G \cap \langle z \rangle \times Q_R \neq \{z\}$. Let $r_1, r_2$ be the two root elements such that $\langle r_1, r_2 \rangle = Z(S \cap A_z).$ Then $C_S(\langle z, r_1, r_2 \rangle) = C_S(A_z) \times (S \cap A_z).$ As $Q_R \leq O_2(C_G(\langle z, r_1, r_2 \rangle), we get from Lemma 6.10 that $N_G(\langle z, r_1, r_2 \rangle)$ does not contain some element in $N_S$. Hence $N_G(\langle z, r_1, r_2 \rangle) \leq N_G(A_z).$ Let $v \in \langle z, r_1, r_2 \rangle$ such that $|S : C_S(v)| = 2$. So $\Omega_1(Z(C_S(v))) = \langle z, r_1, r_2 \rangle.$ As $N_G(\Omega_1(Z(C_S(v)))) \leq A_z,$ we see that $C_S(v)$ is a Sylow 2-subgroup of $C_G(v)$ and so $v \not\sim z$ in $G$. As $z^G \cap \Omega_1(Z(S)) = \{z\}$ by Lemma 6.9 we get that $z^G \cap \langle z, r_1, r_2 \rangle = \{z\}$.

In particular $z^G \cap Z(\langle z, Q_R \rangle) = \{z\}$. On the other hand we have some
v \in U, z \neq v \sim z$. This implies $v \notin Z(Q_R)\langle z \rangle$. By Lemma 6.11 we see $|S : C_S(v)| \leq 8$. As $F_4(2)$ has four classes of involutions by [Shi, Theorem 2.1] three of them are 2-central and the forth has centralizer of order $2^{20} \cdot 3^2$, we see that $v$ must be conjugate to some element in $\langle z, r_1, r_2 \rangle$ in $N_G(A_z)$. As $z^G \cap \langle z, r_1, r_2 \rangle = \{z\}$, this is impossible. So $A_z \not\cong F_4(2)$.

As by Proposition 5.2 $A_z \not\cong G_2(2)'$ and $A_z \not\cong Sp_{2n}(2)$, we have that $Q_R$ is extraspecial. Further by Proposition 5.2 $A_z \not\cong L_3(2)$ or $L_4(2)$. If $N_{NG(A_z)}(Q_R)$ is not irreducible on $Q_R/R$, then by [MaStr, Lemma 2.33] $A_z \cong L_n(2)$ and no graph automorphism is involved. Now $N_G(A_z) = C_G(A_z) \times A_z$. From Lemma 2.2 and Lemma 6.9 we get a contradiction.

For the remainder of this chapter we fix $t$ as in Lemma 6.11 or Lemma 6.12. We will prove that $C_G(t)$ has a standard subgroup $A_t$, which is isomorphic to $A_z$.

**Lemma 6.14.** $A_t = E(C_G(t))$ is simple, $Q_R \leq A_t$ and $C_S(A_t)$ is cyclic. In particular $A_t$ is a standard subgroup.

**Proof.** By Lemma 6.13 we have that $Q_R$ is extraspecial, $R = \langle r \rangle$ and $C_G(\langle z, t \rangle) = C_G(\langle z, r \rangle)$ acts irreducibly on $Q_R/R$.

We first prove:

(A) Let $H \leq C_G(t)$ with $N_{C_G(z)}(Q_R) \leq N_G(H)$ and let $T = S \cap H$ be a Sylow 2-subgroup of $H$, then $Q_R \leq H$, or $T \leq C_S(Q_R)$.

For this suppose $Q_R \not\leq H$. As by Lemma 6.13 $N_{C_G(z)}(Q_R)$ acts irreducibly on $Q_R/R$, we see that $H \cap Q_R \leq R$. Hence $[T, Q_R] \leq H \cap Q_R \leq R$. Then we have by Lemma 2.25 that $T \leq C_S(Q_R)^r Q_R$. As $N_{C_G(z)}(Q_R)$ normalizes $H$ and $C_S(Q_R) Q_R$, it also normalizes $T = H \cap C_S(Q_R) Q_R$. As $N_{C_G(z)}(Q_R)$ has no fixed point in $Q_R/R$ we see that $T \leq C_S(Q_R)$, the assertion (A).

Suppose first $C_{C_G(t)}(O_2(C_G(t))) \leq O_2(C_G(t))$. Then set $H = O_2(C_G(t))$. As $N_{C_G(z)}(Q_R) \leq C_G(\langle z, t \rangle)$ we see that $N_{C_G(z)}(Q_R)$ normalizes $H$. As $t \in Z(S)$, we also have $H \leq S$. Now (A) implies that either $Q_R \leq O_2(C_G(t))$ or $O_2(C_G(t)) \leq C_S(Q_R) \leq C_S(A_z) \times \langle r \rangle$. But the latter contradicts $C_{C_G(t)}(O_2(C_G(t))) \leq O_2(C_G(t))$. So we have $Q_R \leq O_2(C_G(t))$.

By Lemma 6.10 we see that $C_G(t)$ contains no $M \in N_S$. This implies $C_G(t) \leq C_G(z)$, contradicting the choice of $N$. 
So we have that $E(C_G(t)) \neq 1$ (recall that $O(C_G(i)) = 1$ for all involutions $i \in G$). Now set $H = E(C_G(t))$ in (A). If $Q_R \not\leq E(C_G(t))$ then as $C_S(Q_R)/t$ is cyclic, we get a cyclic Sylow 2–subgroup of $E(C_G(t))$, a contradiction. Hence $Q_R \leq E(C_G(t))$.

Now let $N_1$ be some component of $C_G(t)$ and set $T = S \cap N_1$. If $[T, Q_R] = 1$, then $T/t/\langle t \rangle$ is cyclic, which cannot be a Sylow 2-subgroup of $N_1$. So $1 \neq [T, Q_R]$. In particular $R \leq N_1$. If $[R, N_1] = 1$ we get as $\langle t, R \rangle = \langle z, R \rangle$ that $N_1 \leq C_G(z)$. Now $N_1$ normalizes $O_2(C_{A_t}(\langle z, R \rangle)) = Q_R$. But $[Q_R, N_1] \leq Q_R \cap N_1 \leq R \leq Z(N_1)$, a contradiction. So we have that $R \not\leq Z(N_1)$. In particular $N_{C_G(z)}(Q_R) \leq N_G(N_1)$. Application of (A) now shows that $Q_R \leq N_1$. As this is true for any component $N_i$, we get that $E(C_G(t))$ is quasisimple.

Next we show

$$E(C_G(t)) \text{ is simple.}$$

Otherwise some $1 \neq u \in Z(S)$ is contained in $Z(E(C_G(t)))$. Suppose $u \neq t$. We then have that $\Omega_1(Z(S)) = \langle r, z \rangle = \langle u, t \rangle$. Hence $E(C_G(t)) \leq C_G(z)$, a contradiction. So we must have $t \in Z(E(C_G(t)))$.

Now $C_G(E(C_G(t))) \leq C_{C_G(t)}(Q_R) = C_{C_G(z)}(Q_R)$. As $r \in E(C_G(t)) \setminus Z(E(C_G(t)))$, we see that $C_G(E(C_G(t)))$ has a cyclic Sylow 2–subgroup and so in particular $E(C_G(t))$ is standard. But this contradicts Proposition 5.1. Hence $E(C_G(t))$ is simple.

As $Q_R \leq E(C_G(t))$ we see that $C_S(E(C_G(t))) \leq C_S(Q_R)$ is cyclic. In particular $E(C_G(t))$ is standard. $$

We have $\langle z, t \rangle = \Omega_1(Z(S))$ and $r = zt \in A_z \cap A_t$. Now everything we proved for $A_z$ applies for $A_t$ too. This shows that both groups are isomorphic to one of the following groups: $J_2$, $M(24)'$, $L_n(2)$, $U_n(2)$, $n \geq 5$, $\Omega_{2n}^+(2)$, $E_6(2)$, $E_7(2)$, $E_8(2)$, $^2E_6(2)$, $^3D_4(2)$.

**Lemma 6.15.** We have that $O_2(C_{A_t}(R)) = O_2(C_{A_z}(R))$. Further let $H_t$ be the preimage of $E(N_{A_t}(O_2(C_{A_t}(R)))/O_2(C_{A_t}(R)))$ and $H_z$ the preimage of $E(N_{A_z}(Q_R)/Q_R)$. Then $H_t = H_z$.

**Proof.** By Lemma 6.14 we have $Q_R \leq A_t$. Further we have that $H_z \leq C_G(t)$. As $H_t^z = H_t^z$ by Lemma 6.13 and $C_G(t)/A_t$ is solvable, we get that $H_z \leq A_t$ (this is also true if $N_{A_z}(O_2(C_{A_z}(R)))$ is solvable, as then $H_z = Q_R$). Similarly we see $H_t \leq A_z$ and then we have that $O_2(N_{A_t}(R)) \leq O_2(C_{A_z}(z)(R))$ and $Q_R \leq O_2(C_{A_t}(t)(R))$. This shows
that $Q_R \leq O_2(C_{A_t}(R)) \leq Q_R$ and so $Q_R = O_2(C_{A_t}(R))$. We further get $H_t \leq H_z \leq H_t$, the assertion. \hfill \Box

**Lemma 6.16.** We have $A_z \cong A_t$.

*Proof.* Let first $A_z$ be sporadic. By Proposition 5.2 we have that $A_z \cong J_2$ or $M(24)'$. In both cases $N_{A_z}(Q_R)$ is nonsolvable. By Lemma 6.15 we have that $H_z = H_t \leq A_t$ and $H_z \cong 2^{1+4}A_5$ or $2^{1+12}3U_4(3)$. If $A_t$ is sporadic too, then we have that $A_z \cong A_t$. So we may assume that $A_t$ is a group of Lie type over GF(2). As $3U_4(3)$ is not a group of Lie type in characteristic two, we get a contradiction. In the first case we have that $|Q_R| = 2^5$. Then by Lemma 2.17 we get that $A_t \cong L_4(2)$ or $U_4(2)$, which contradicts Proposition 5.2. So we have $A_z \cong A_t$.

Next we assume that both $A_z$ and $A_t$ are groups of Lie type. If $N_{A_z}(Q_R)$ is nonsolvable we may argue as before, i.e. we compare the orders of $Q_R$ and the Levi factors, as given by Lemma 2.17. Then we receive $A_z \cong A_t$ or $A_z \cong L_3(2)$, $L_4(2)$, $\Omega_8^+(2)$, $U_4(2)$, $U_5(2)$. By Proposition 5.2 $A_z \not\cong L_3(2)$, $L_4(2)$ or $U_4(2)$. Now we have symmetry and so also $A_t \cong \Omega_8^+(2)$ or $U_5(2)$. But these groups are determined just by the order of $Q_R$, which is $2^9$, $2^7$, respectively, so $A_t \cong A_z$ too. \hfill \Box

**Proposition 6.17.** The main theorem holds.

*Proof.* Suppose false. Then according to Lemma 6.11 and Lemma 6.12 we have some $t \in \Omega_1(Z(S))$, $t \neq z$, $t \in Z(N)$. By Lemma 6.16 $A_z \cong A_t$ and by Lemma 6.14 both groups are standard. We first show

$$\text{(1)} \qquad A_t \cong A_z \cong L_n(2) \text{ or } U_n(2).$$

Suppose false. By [MaStr, Lemma 2.33] we have that $N_{A_z}(Q_R)$ acts irreducibly on $Q_R/R$. Set $V = \Omega_1(Z_2(S \cap A_z))$. We get with [MaStr, Lemma 2.35] that $|V| = 4$. Set $P = N_{A_t}(V)$. Then $P$ is normalized by $S$ and $P/O_2(P) \cong \Sigma_3$. For a group of Lie type this is just a minimal parabolic not in $N_{A_t}(R)$. For the sporadic groups this follows with Lemma 2.14.

Hence $\Omega_1(Z(O_2(P))) = \Omega_1(Z_2(S \cap A_z))$. Then $V \leq Q_R$ and so $V = \Omega_1(Z_2(S \cap A_t))$ by Lemma 6.15. On $V$ both $N_{A_t}(V)$ and $N_{A_t}(V)$ induce $\Sigma_3$. Now $\langle N_{A_t}(V), N_{A_t}(V) \rangle$ acts on $\langle z, V \rangle = \langle t, V \rangle$. As $z^G \cap \Omega_1(Z(S)) = \{z\}$ by Lemma 6.9 we have $z^G \cap \langle z, V \rangle = \{z\}$. So $N_{A_t}(V) \leq C_G(z)$. This implies $A_t = \langle N_{A_t}(V), N_{A_t}(Q_R) \rangle \leq C_G(z)$, a contradiction. This proves (1).

By (1) $A_t \cong A_z \cong U_n(2)$, or $L_n(2)$. In the latter by Lemma 6.13 we
have some graph automorphism induced. As $[C_S(A_z), Q_R] = 1$, we get $C_S(A_z) \leq C_S(A_t) \times R$. This yields $\Omega_1(\Phi(C_S(A_z))) \leq \Omega_1(\Phi(C_S(A_t))) \leq \langle t \rangle$. As $z \neq t$ this shows that $C_S(A_z) = \langle z \rangle$. By the Thompson transfer lemma (Lemma 2.2) and $z^G \cap \Omega_1(Z(S)) = \{z\}$ by Lemma 6.9, we have that $z$ is a square of some $x \in C_G(z)$, which induces an outer automorphism on $A_z$. The same of course is true for $t$. In particular

(2)

All involutions of $C_G(z)$ are in $\langle z \rangle \times A_z$

and all involutions of $C_G(t)$ are in $\langle t \rangle \times A_t$. By Lemma 6.11 and Lemma 6.12 there is some parabolic $N$ in $C_G(t)$, $N \not\leq N_{C_G(t)}(R)$. This shows that $N/C_N(\Omega_1(Z(O_2(N)))) \cong O^+_4(2)$ in case of $A_t \cong L_n(2)$ and $\Omega^-_4(2)$ or $O^-_4(2)$ in case of $A_t \cong U_n(2)$. Set again $U = \Omega_1(Z(O_2(N)))$ and $V = U \cap A_t$. Then $V$ is the natural module for $N/C_N(U)$. Further we have that $V \cap Q_R = [V, Q_R]$ is of order eight. By (2) and Lemma 2.28 we have that $U$ is uniquely determined in $S$. Then also there is a corresponding subgroup $N_1$ of $C_G(z)$ such that $N_1$ induces $\Omega^+_4(2)$ on $U$. This now implies the following. The orbits of $N \leq N_{C_G(t)}(U)$ on $U^t$ are $1, 5, 5, 10, 10, 10$ or $1, 6, 6, 9, 9$ and $N_1 \leq N_{C_G(z)}(U)$ induces the same orbit sizes. As $|z^{N_G(U)}|$ is odd, we see that under $N_G(U)$ the orbit of $z$ must have length $11$ or $21$ and $7$ or $13$, respectively. By (2) and Lemma 2.28 we have that $U$ is uniquely determined in $S$. Then also there is a corresponding subgroup $N_1$ of $C_G(z)$ such that $N_1$ induces $\Omega^+_4(2)$ on $U$. This now implies the following. The orbits of $N \leq N_{C_G(t)}(U)$ on $U^t$ are $1, 5, 5, 10, 10, 10$ or $1, 6, 6, 9, 9$ and $N_1 \leq N_{C_G(z)}(U)$ induces the same orbit sizes. As $|z^{N_G(U)}|$ is odd, we see that under $N_G(U)$ the orbit of $z$ must have length $11$ or $21$ and $7$ or $13$, respectively. Recall that $z \not\sim t$ or $r$. But $|z^{N_G(U)}|$ has to divide the order of $GL_5(2)$, which implies that $|z^{N_G(U)}| = 21$ in the first case and $7$ in the second. The same applies for $t$, i.e. $|t^{N_G(U)}| = 21, 7$, respectively. But there is obviously just one possibility to make up an orbit of length $21$ or $7$, which implies that $z \sim t$, the final contradiction.

\[\square\]

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