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FOUR-VALUED MODAL LOGIC: KRIPKE SEMANTICS AND DUALITY

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Abstract. We introduce a family of modal expansions of Belnap-Dunn four-valued logic and related systems, and interpret them in many-valued Kripke structures. Using algebraic logic techniques and topological duality for modal algebras, and generalizing the so-called twist-structure representation, we axiomatize by means of Hilbert-style calculi the least modal logic over the four-element Belnap lattice and some of its axiomatic extensions. We study the algebraic models of these systems, relating them to the algebraic semantics of classical multi-modal logic. This link allows us to prove that both local and global consequence of the least four-valued modal logic enjoy the finite model property and are therefore decidable.

1. Introduction

Combining many-valued and modal logics into a single system is a long-standing concern in mathematical logic and computer science, see for example [16, 17] and the literature cited there. The benefit of such an interaction is that it may allow us to deal with modal notions like belief, knowledge, obligations, in connection with other aspects of reasoning that can be best handled using many-valued logics, for instance vagueness and inconsistency. If our final aim is to provide a comprehensive model of human reasoning, it is obvious that all these aspects have to be dealt with at the same time, therefore such a study is especially interesting from the point of view of theoretical computer science, cognitive science and artificial intelligence.

Recent work in the tradition of mathematical fuzzy logic has provided a very general framework for studying modal expansions of fuzzy logic, whose truth values are usually linearly ordered: see for instance [11, 12, 7]. A parallel line of research has been developing modal versions of inconsistency-tolerant logical systems, such as Belnap-Dunn four-valued logic and paraconsistent Nelson logic: see [32, 33, 35, 34, 37]. These are also many-valued systems where truth values can be naturally ordered according to different criteria, none of which defines a linear order.

In this paper we make a first attempt at combining the two approaches mentioned above, investigating expansions of Belnap-Dunn logic and related paraconsistent systems from the point of view of general many-valued modal logic adopted in [7]. In this way we systematically lay out a framework for studying paraconsistent modal logic which extends and encompasses the work of [34, 31]. A preliminary version of the present work has appeared in [25]. While the approach employed here is essentially the same, we have simplified many proofs, and refined and extended most results. The last section of the present paper is entirely new.

Our starting point is a Kripke-style semantics whose models are four-valued in two different respects, both semantic valuations and the accessibility relation among worlds taking

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values into the four-element Belnap lattice. We axiomatize the *minimum* modal logic over this lattice in the sense of [7], i.e., the logic determined by the class of all four-valued Kripke frames. However, our completeness proofs follow an alternative strategy to both those of [7] and of [34]. We will then consider axiomatic extensions of our base logic and explore further possible generalizations.

We obtain what we consider particularly neat completeness proofs, for both the global and the local consequence relation, mainly relying on (i) an algebraic study of models of the logic, (ii) a convenient representation of these models as *twist-structures*, and (iii) relating Kripke semantics to the topological semantics for classical modal logic provided by the duality of Jónsson and Tarski for modal algebras. This strategy allows us to attack the problem of completeness for four-valued modal logic using analogous results for classical multi-modal logic. We show that axiomatic extensions of the minimum modal logic, corresponding to restrictions on the accessibility relation, can be easily axiomatized using the same methods. Taking advantage of the insight gained through our algebraic analysis of the logic, we also introduce and study a more general four-valued semantics that seems to us a natural modal expansion of Belnap-Dunn (and paraconsistent Nelson) logic, encompassing the above-mentioned existing work on modal expansions of these systems. We obtain axiomatizations and completeness results for the base logic and its extensions by an easy modification of the methods used in the previous case.

The paper is organized as follows. In Section 2 we introduce the non-modal core of our logics, which is essentially the logic of the four-element Belnap lattice, either viewed as a bilattice or as an N4-lattice, and recall some facts that will be used in the study of its modal expansions. In Section 3 we introduce the semantics of our logics, based on four-valued Kripke frames; it is essentially an instantiation of the definition proposed in [7] for the least modal logic over a residuated lattice. We associate two modal consequence relations to each class of frames, a *global* and a *local* one. Section 4 introduces Hilbert-style calculi that we prove to be complete with respect to our semantically defined modal consequences. In Section 5 we determine and study the algebraic models of our calculi. The findings, besides their intrinsic mathematical interest, are key for the developments in the remainder of the paper. They also provide additional semantic insight into four-valued modal logic. In Section 6 we develop a topological duality theory for the algebraic models of our logic, which turns out to be a straightforward application of Jónsson-Tarski duality for modal algebras. This allows us to prove completeness of the logic with respect to Kripke-style semantics, and also to axiomatize certain interesting axiomatic extensions of the base logic. We also see that the semantics introduced in Section 3 can be generalized by replacing the four-element Belnap bilattice with any complete algebra in the same variety. In Section 7 we introduce an even more general semantics inspired by our algebraic analysis of four-valued modal logic, and we sketch out how to axiomatize the resulting logic and its extensions. The final Section 8 discusses open problems and directions for future research.

2. The non-modal core of the logic

Our non-modal starting point is the logic determined by the four-element Belnap lattice $\mathcal{FOUR}$ (Figure 1) together with the subset of *designated elements* $\{\mathtt{t}, \top\}$. $\mathcal{FOUR}$ has two (bounded) lattice structures, namely the t-lattice $\langle \mathcal{FOUR}, \leq_t, \wedge, \vee, \mathtt{f}, \mathtt{t} \rangle$ and the k-lattice $\langle \mathcal{FOUR}, \leq_k, \otimes, \oplus, \bot, \top \rangle$. The four lattice operations are determined by the two Hasse diagrams shown in Figure 1. Moreover, we will consider a negation and an implication operator. *Negation* $\neg$ is a unary operator that swaps $\mathtt{t}$ and $\mathtt{f}$ while having both $\bot$ and $\top$ as fixed points. *Weak implication* $\supset$ (later on we will introduce a *strong implication*) is

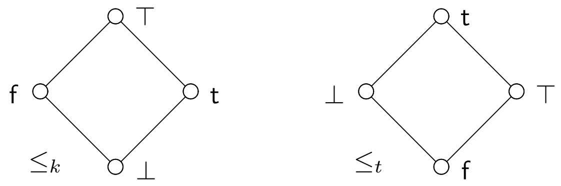

FIGURE 1. The four-element Belnap bilattice $\mathcal{FOUR}$ in its two orders

defined in $\mathcal{FOUR}$ by the following prescription:

$$x \supset y = \begin{cases} y & \text{if } x \in \{\mathsf{t}, \top\} \\ \mathsf{t} & \text{if } x \notin \{\mathsf{t}, \top\}. \end{cases}$$

The (non-modal) logical language we are mainly interested in is $\langle \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$, but it will sometimes be convenient to focus on more restricted languages, both for the sake of generality and in order to relate our study to known results on other non-classical logics.

The logical matrix $\langle \mathcal{FOUR}, \{\mathsf{t}, \top\} \rangle$ determines Belnap-Dunn logic [3, 4] in the following way. One considers the formula algebra $\mathbf{Fm}$ freely generated by a countable set of propositional variables over the language $\mathcal{L} = \langle \wedge, \vee, \neg \rangle$, whose connectives correspond to t-lattice meet, t-lattice join and negation, respectively. Given formulas $\Gamma \cup \{\varphi\} \subseteq Fm$, one sets $\Gamma \models \varphi$ if and only if, for all $\mathcal{L}$-homomorphisms $h \colon \mathbf{Fm} \to \mathcal{FOUR}$, we have $h(\varphi) \in \{\mathsf{t}, \top\}$ whenever $h[\Gamma] \subseteq \{\mathsf{t}, \top\}$.

Different choices of the propositional language $\mathcal{L}$, keeping the underlying set of truth values and the designated elements fixed, give rise to different logics:

1. $\mathcal{L} = \langle \wedge, \vee, \neg, \mathsf{f}, \mathsf{t} \rangle$ gives us Belnap-Dunn logic with propositional constants f (falsity) and t (truth).
2. $\mathcal{L} = \langle \wedge, \vee, \otimes, \oplus, \neg \rangle$ defines the implicationless bilattice logic of Arieli and Avron [2], to which one may add constants to obtain $\langle \wedge, \vee, \otimes, \oplus, \neg, \mathsf{f}, \mathsf{t} \rangle$. As we will see, the latter is in fact equivalent to $\langle \wedge, \vee, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$, in the sense that both constants $\bot$ and $\top$ can be obtained as terms in the language $\langle \wedge, \vee, \otimes, \oplus, \neg, \mathsf{f}, \mathsf{t} \rangle$ and, conversely, the connectives $\otimes$ and $\oplus$ are term-definable in $\langle \wedge, \vee, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$.
3. $\mathcal{L} = \langle \wedge, \vee, \supset, \neg \rangle$ gives us *four-valued paraconsistent Nelson logic*, which is an extension of *paraconsistent Nelson logic* [1, 28] obtained by adding the following axiom (Peirce's law): $((p \supset q) \supset p) \supset p$. The language with truth constants t and f is considered, for instance, in [30].
4. $\mathcal{L} = \langle \wedge, \vee, \otimes, \oplus, \supset, \neg \rangle$ gives us the full bilattice logic of Arieli and Avron [2]. As before, the language with truth constants $\langle \wedge, \vee, \otimes, \oplus, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$, which is considered in [25], is equivalent to $\langle \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$.

The equivalences stated in 2. and 4. depend on the fact that the following identities hold in $\mathcal{FOUR}$ [24, Lemma 1.5]:

$$x \otimes y = (x \wedge \bot) \vee (y \wedge \bot) \vee (x \wedge y)$$
$$x \oplus y = (x \wedge \top) \vee (y \wedge \top) \vee (x \wedge y).$$

This means that, in the presence of the constants $\bot$ and $\top$ in the language, the k-lattice operations can be simply introduced as derived connectives. Conversely, one can define

$$\bot := \mathsf{f} \otimes \mathsf{t} \qquad \top := \mathsf{f} \oplus \mathsf{t}.$$

We notice that all the above-mentioned logics can be finitely axiomatized, for instance, through Hilbert- and Gentzen-style syntactic calculi. We will introduce one of these in Section 4.

Unless otherwise stated, the language of our non-modal base logic is the one mentioned in the last item above, that is, $\mathcal{L} = \langle \wedge, \vee, \otimes, \oplus, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$. That is, we will be dealing with Arieli-Avron bilattice logic [2]. We will use the following abbreviations:

$$\begin{aligned} x \to y &:= (x \supset y) \wedge (\neg y \supset \neg x) \\ x * y &:= \neg(y \to \neg x) \\ x \equiv y &:= (x \supset y) \wedge (y \supset x) \\ x \leftrightarrow y &:= (x \to y) \wedge (y \to x). \end{aligned}$$

We use the same symbol for the algebraic operation and the corresponding propositional connective. The first two derived operations, that we call *strong implication* ($\to$) and *fusion* ($*$), play a particularly important role in this paper. The reason is that they together form a *residuated pair*: a fact, as we will see in the next section, that will allow us to relate our treatment of four-valued modal logic to existing literature on the modal logic of residuated lattices.

In our setting, being a residuated pair means that the following property holds for arbitrary elements x, y, z of $\mathcal{FOUR}$:

$$x * y \leq_t z \quad \text{iff} \quad y \leq_t x \to z.$$

This, together with the fact that $\langle \mathcal{FOUR}, *, \top \rangle$ is a monoid, entails that we can view $\mathcal{FOUR}$ as a *residuated lattice* [36, Proposition 5.4.1]. Residuated lattices are well-known in algebraic logic, for they provide algebraic semantics for a wide class of multi-valued logics, including the so-called fuzzy logics [20].

$*$	f	$\bot$	$\top$	t
f	f	f	f	f
$\bot$	f	f	$\bot$	$\bot$
$\top$	f	$\bot$	$\top$	t
t	f	$\bot$	t	t

$\to$	f	$\bot$	$\top$	t
f	t	t	t	t
$\bot$	$\bot$	t	$\bot$	t
$\top$	f	$\bot$	$\top$	t
t	f	$\bot$	f	t

Table 1. The residuated pair in $\mathcal{FOUR}$.

Table 1 shows the behaviour of the two operations in $\mathcal{FOUR}$. Some important points that we would like to highlight are the following:

- As suggested by the terminology, strong implication $\to$ has some logical features of classical implication. For instance, it satisfies the contraposition law ($\varphi \to \psi$ is semantically equivalent to $\neg\psi \to \neg\varphi$) and determines the t-lattice order of $\mathcal{FOUR}$ in the following way:

$$\begin{aligned} x \leq_t y \quad &\text{iff} \quad x \to y \in \{\mathsf{t}, \top\} \\ &\text{iff} \quad x \to y = (x \to y) \to (x \to y). \end{aligned}$$

  On the other hand, other good properties of classical implication are enjoyed by weak implication $\supset$ but not by the strong one, the most prominent example being the deduction theorem.

- A remarkable feature that distinguishes strong implication from the classical one, and that will have important consequences for our study, is the following. Given $x \in \mathcal{FOUR}$ and $y \in \{\mathsf{t}, \top\}$, it can happen that $x \to y \notin \{\mathsf{t}, \top\}$. The reason is that $\mathsf{t} \to \top = \mathsf{f}$. Logically, this means that, even if ψ is valid, $\varphi \to \psi$ might not be valid.
- Being an adjoint to strong implication, fusion $*$ has the logical role of a multiplicative conjunction. In fact, one can see that the formula that defines fusion from strong implication is the same as the one that defines classical conjunction from classical implication. As an algebraic operation, fusion is associative and commutative, but not idempotent, because $\bot * \bot = \mathsf{f}$ (this is in fact the only exception to idempotency). Notice also that the neutral element of the monoid $\langle \mathcal{FOUR}, *, \top \rangle$ is not the top element of the lattice order $\leq_t$. In the standard terminology of residuated lattices this is expressed by saying that $\mathcal{FOUR}$ is a *commutative non-integral residuated lattice.*

3. Relational semantics of the modal logic

For a modal expansion of our logic we initially focus on the necessity operator $\Box$ only. Semantically, we seek to interpret it in suitable *Kripke structures.* For motivation, let us consider first a classical Kripke model $\langle W, R, v \rangle$, where W is a non-empty set of "worlds", R an accessibility relation among them and v a valuation. Now view R as the characteristic function associated with the accessibility relation, i.e., as a map $R\colon W \times W \to \{\mathsf{t}, \mathsf{f}\}$. Similarly, view $v\colon \mathbf{Fm} \times W \to \{\mathsf{t}, \mathsf{f}\}$ as a map assigning to each formula $\varphi \in \mathbf{Fm}$ at each point $w \in W$ a truth value in $\{\mathsf{t}, \mathsf{f}\}$. By the so-called *standard translation* of modal logic into first-order logic, we obtain the following definition for the semantics of the necessity operator

$$v(\Box\varphi, w) := \bigwedge \{R(w, w') \to v(\varphi, w') : w' \in W\} \tag{1}$$

where $\bigwedge$ denotes the infinitary meet and $\to$ is Boolean implication. Note that conjunction is taken in the *complete* lattice of truth values, so there is no problem with applying it to an infinite set.

This definition can now easily be adapted to our four-valued setting. We consider Kripke models $\langle W, R, v \rangle$ where both R and v are four-valued, that is, we define $R\colon W \times W \to \mathcal{FOUR}$ and $v\colon \mathbf{Fm} \times W \to \mathcal{FOUR}$. As before, valuations are required to be homomorphisms in their first argument. We stress, as this will be important for our axiomatization, that we have included the constants $\mathsf{t}, \mathsf{f}, \top, \bot$ in the propositional language, so valuations must interpret each of them as the corresponding element of $\mathcal{FOUR}$.

Since $\mathcal{FOUR}$ carries three distinct conjunctions and two implications, there are six candidates for the translations of (1) to the four-valued setting. We reject the monoid operation $*$ because it is not idempotent and hence would require us to replace the set $\{R(w, w') \to v(\varphi, w') : w' \in W\}$ by a multi-set. The choice between $\wedge$ and $\otimes$ is more subtle as it relates to the intended interpretation of the necessity operator. Our choice is for the "logical" connective rather than the knowledge order one as it is here that there are useful interactions with the two implications. This leaves the pairs $\langle \wedge, \to \rangle$ and $\langle \wedge, \supset \rangle$.

The latter choice has, in our opinion, the disadvantage that the accessibility relation R, although formally introduced as four-valued, turns out to have a two-valued behaviour when interacting with weak implication. This is so because in $\mathcal{FOUR}$ the value of (1) (with $\to$ replaced by $\supset$) is the same as the following one:

$$\bigwedge \{v(\varphi, w') : R(w, w') \in \{\mathsf{t}, \top\}\}.$$

In fact, the choice $\langle \wedge, \supset \rangle$ has already been considered in [34] for a modal expansion of Belnap-Dunn logic. It turns out, however, that the resulting operator is strictly less expressive than the one defined by the pair $\langle \wedge, \to \rangle$. Denoting the two choices by $\Box_\supset$ and $\Box_\to$, we get:

Proposition 3.1. *For all formulas $\varphi \in Fm$, all four-valued Kripke models $\langle W, R, v\rangle$, and all $w \in W$:*

$$v(\Box_\supset \varphi,\, w) = v(\Box_\to(\varphi \vee \bot) \oplus (\Box_\to \varphi \wedge \bot),\, w).$$

Proof. Given that v is fixed, we abbreviate $v(\varphi,\, w)$ as $w(\varphi)$. Note that $x \to \bot \geq_t \bot$ for all $x \in \mathcal{FOUR}$, and $x \to \bot = \bot$ precisely when $x \geq_t \top$. This implies $w(\Box_\to(\varphi \vee \bot)) \geq_t \bot$ for all $w \in W$, and obviously we also have $w(\Box_\to \varphi \wedge \bot) \leq_t \bot$. Let us also notice that the definition $x \to y := (x \supset y) \wedge (\neg y \supset \neg x)$ immediately implies $w(\Box_\supset \varphi) \geq_t w(\Box_\to \varphi)$. Reasoning by cases, assume $w(\Box_\supset \varphi) = \mathsf{t}$. This means that $w'(\varphi) = \mathsf{t}$ for all w' s.t. $R(w, w') \geq_t \top$. Then $w(\Box_\to(\varphi \vee \bot)) = \mathsf{t}$. To prove that $\mathsf{t} \oplus (w(\Box_\to \varphi) \wedge \bot) = \mathsf{t}$, it remains to show that $w(\Box_\to \varphi) \wedge \bot \neq \mathsf{f}$, i.e., $w(\Box_\to \varphi) \geq_t \bot$. If we had $w(\Box_\to \varphi) = \top$, then there would be $w' \in W$ s.t. $R(w, w') \to w'(\varphi) = \top \to \top = \top$. But our assumption implies $w'(\varphi) = \mathsf{t}$, a contradiction. Suppose then $w(\Box_\to \varphi) = \mathsf{f}$. Under the assumptions, this means that there must be $w' \in W$ s.t. $R(w, w') \to w'(\varphi) = \mathsf{f}$. This can only happen if $R(w, w') \geq_t \top$, but then the assumptions imply $w'(\varphi) = \mathsf{t}$ and $x \to \mathsf{t} = \mathsf{t}$ for all $x \in \mathcal{FOUR}$. We conclude $w(\Box_\to \varphi) \geq_t \bot$ as required. Now assume $w(\Box_\supset \varphi) = \top$. This implies that, for all $w' \in W$, we have $w'(\varphi) \geq_t \top$ whenever $R(w, w') \geq_t \top$. Since $w(\Box_\to \varphi) \leq_t w(\Box_\supset \varphi) = \top$, we have $w(\Box_\to \varphi \wedge \bot) = \mathsf{f}$. We thus need to show that $w(\Box_\to(\varphi \vee \bot)) \oplus \mathsf{f} = \top$, i.e., $w(\Box_\to(\varphi \vee \bot)) = \mathsf{t}$. This happens when $R(w, w') \geq_t \top$ implies $w'(\varphi) \geq_t \top$ for all $w' \in W$, which is precisely our assumption. Now assume $w(\Box_\supset \varphi) = \bot$. This means that (i) there is $w' \in W$ s.t. $R(w, w') \geq_t \top$ and $w'(\varphi) = \bot$, and (ii) for all $w'' \in W$, we have $w''(\varphi) \geq_t \bot$ whenever $R(w, w'') \geq_t \top$. From (i) we obtain $w(\Box_\to(\varphi \vee \bot)) = \bot$. It remains to show that $w(\Box_\to \varphi \wedge \bot) \neq \mathsf{f}$, i.e., $w(\Box_\to \varphi) \geq_t \bot$. Now $w(\Box_\to \varphi) \not\geq_t \bot$ would mean that there is $w''' \in W$ s.t. $R(w, w''') \geq_t \top$ and $w'''(\varphi) \leq_t \top$, but this is forbidden by (ii). We conclude $w(\Box_\to \varphi) \geq_t \bot$ as required. Finally, assume $w(\Box_\supset \varphi) = \mathsf{f}$. This implies that there is $w' \in W$ s.t. $R(w, w') \in \{\mathsf{t}, \top\}$ and $w'(\varphi) \leq_t \bot$. Hence, $w(\Box_\to(\varphi \vee \bot)) \leq_t R(w, w') \to w'(\varphi \vee \bot) = R(w, w') \to \bot = \bot$. On the other hand, $w(\Box_\to \varphi) \leq_t w(\Box_\supset \varphi)$, implies $w(\Box_\to \varphi) = \mathsf{f}$. Thus we have $w(\Box_\to(\varphi \vee \bot) \oplus (\Box_\to \varphi \wedge \bot)) = \bot \oplus \mathsf{f} = \mathsf{f}$ as required, and this concludes our proof. $\Box$

One may wonder whether, conversely, it is possible to define $\Box_\to$ from $\Box_\supset$. This is already unlikely given the two-valued nature of the latter, and our algebraic analysis (Subsection 5.2) will indeed confirm this intuition.

To summarize, our choice for the semantics of the necessity operator is based on the pair $\langle \wedge, \to \rangle$, that is, in the four-valued context *we replace classical conjunction with the truth lattice meet and classical implication with the strong implication of Arieli-Avron logic.* From now on we will write simply $\Box$ in place of $\Box_\to$.

Let us point out a further pleasing feature of $\Box$. Given that $\mathcal{FOUR}$ is endowed with an involutive negation (in fact, since $\neg x = x \to \top$, we can view $\mathcal{FOUR}$ as an involutive residuated lattice in the sense of [21]), we can introduce a possibility operator $\Diamond$ which turns out to be dual to $\Box$ in the logic. Semantically, it is given by [7, p.746]:

$$v(\Diamond \varphi, w) := \bigvee \{R(w, w') * v(\varphi, w') : w' \in W\}. \tag{2}$$

This is again obviously a generalization of the classical definition with the monoid operation replacing classical conjunction (the fact that $*$ is not idempotent is not a problem here as it is applied to two terms, not to a set).

We are now ready to extend the semantic consequence relation of our base logic to the modal setting. We say that a point $w \in W$ of a four-valued model $M = \langle W, R, v\rangle$ satisfies a formula $\varphi \in \mathbf{Fm}$ if $v(\varphi, w) \in \{\mathsf{t}, \top\}$. In such a case we write $M, w \models \varphi$. For a set of formulas $\Gamma \subseteq \mathbf{Fm}$, we write $M, w \models \Gamma$ to mean that $M, w \models \gamma$ for each $\gamma \in \Gamma$. As is usual in modal logic, we consider two consequence relations. The *local* consequence $\Gamma \models_l \varphi$ holds if for every model $M = \langle W, R, v\rangle$ and every $w \in W$, it is the case that $M, w \models \Gamma$ implies $M, w \models \varphi$. The *global* consequence relation $\Gamma \models_g \varphi$ holds if, for every model M, if $M, w \models \Gamma$ for all $w \in W$, then $M, w \models \varphi$ for all $w \in W$.

We remind the reader that the above definitions imply that:

- if $\Gamma \models_l \varphi$, then $\Gamma \models_g \varphi$ (global consequence is a strengthening of the local one);
- $\emptyset \models_l \varphi$ if and only if $\emptyset \models_g \varphi$ (the two consequences have the same valid formulas).

Let us now explore the axioms and rules that are valid semantically. The following can be easily shown to follow from the definition of $\Box$ (see also [7]).

Proposition 3.2. *The following formulas are valid in all models:*

(i) $\Box\mathsf{t} \leftrightarrow \mathsf{t}$

(ii) $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$,

(iii) $\Box(\mathsf{c} \to \varphi) \leftrightarrow (\mathsf{c} \to \Box\varphi)$ *for all* $\mathsf{c} \in \{\mathsf{t}, \mathsf{f}, \top, \bot\}$.

As in [7], the last of these schemata will play a prominent role in the axiomatization of our logic, as will the following rule:

Proposition 3.3 (Monotonicity)**.** *The rule* $\varphi \to \psi \vdash \Box\varphi \to \Box\psi$ *is sound with respect to global consequence. In other words,* $\varphi \to \psi \models_g \Box\varphi \to \Box\psi$ *holds.*

Proof. We will use the following property, which holds in any residuated lattice. Let $x, y, z \in \mathcal{FOUR}$. If $x \leq_t y$, then $z \to x \leq_t z \to y$. From this the proposition easily follows. In fact, assume $\varphi \to \psi$ holds at every world w of a model $\langle M, R, v\rangle$. Then $v(\varphi \to \psi, w) \in \{\mathsf{t}, \top\}$, which means, as observed above, that $v(\varphi, w) \leq_t v(\psi, w)$. To compute $v(\Box\varphi, w)$ we take, according to (1), the t-meet of all expressions $R(w, w') \to v(\varphi, w')$. By the above property, each of those is smaller than $R(w, w') \to v(\psi, w')$, so the t-meets are comparable as well, that is, $v(\Box\varphi, w) \leq_t v(\Box\psi, w)$. And again this is equivalent to $v(\Box\varphi, w) \to v(\Box\psi, w) = v(\Box\varphi \to \Box\psi, w) \in \{\mathsf{t}, \top\}$. $\Box$

The following is an immediate consequence of monotonicity:

Corollary 3.4. *If* $\varphi \to \psi$ *is valid in all models then so is* $\Box\varphi \to \Box\psi$*.*

However, *necessitation* (from $\vdash \varphi$ derive $\vdash \Box\varphi$), which in classical modal logic is equivalent to monotonicity, is not sound, even with respect to global consequence. This is a consequence of what we observed in the previous section: $y \in \{\mathsf{t}, \top\}$ does not imply $x \to y \in \{\mathsf{t}, \top\}$. As a counter-example, consider the one-point Kripke model $M = \langle W, R, v\rangle$ where $W = \{w\}$, $R(w, w) = \mathsf{t}$ and $v(p, w) = \top$ for some variable $p \in Var$. Then $v(\Box p, w) = R(w, w) \to v(p, w) = \mathsf{t} \to \top = \mathsf{f}$. Hence $M \not\models \Box p$ but $M \models p$. The same model shows that the following monotonicity rule with respect to weak implication

$$\frac{\varphi \supset \psi}{\Box\varphi \supset \Box\psi}$$

is not globally sound. Let $q \in Var$ be such that $v(q, w) = \mathsf{t}$. Then $v(q \supset p, w) = \mathsf{t} \supset \top = \top$, which means that $M \models q \supset p$. However, $v(\Box q, w) = R(w, w) \to v(q, w) = \mathsf{t} \to \mathsf{t} = \mathsf{t}$ and

$v(\Box p, w) = R(w, w) \to v(p, w) = \mathsf{t} \to \top = \mathsf{f}$. This means that $v(\Box q \supset \Box p, w) = v(\Box q, w) \supset v(\Box p, w) = \mathsf{t} \to \mathsf{f} = \mathsf{f} \notin \{\mathsf{t}, \top\}$. That is, $M \not\models \Box q \supset \Box p$.

The *normality axiom*, $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, also fails. To see this, consider again a one-point model $M = \langle W, R, v\rangle$ where $W = \{w\}$ and $R(w, w) = \bot$. Let v be such that $v(p, w) = \bot$ and $v(q, w) = \mathsf{f}$ for $p, q \in Var$. Then we have that

$$v(p \to q, w) = \bot \to \mathsf{f} = \bot$$
$$v(\Box p, w) = \bot \to \bot = \mathsf{t}$$
$$v(\Box q, w) = \bot \to \mathsf{f} = \bot$$
$$v(\Box(p \to q), w) = \bot \to \bot = \mathsf{t}$$
$$v((\Box p \to \Box q), w) = \mathsf{t} \to \bot = \bot$$
$$v(\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\varphi), w) = \mathsf{t} \to \bot = \bot \notin \{\mathsf{t}, \top\}.$$

Thus $M \not\models \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$. The same model shows that similar normality axioms for the weak implication fail as well, that is, we have

$$\not\models \Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\varphi) \quad \text{and} \quad \not\models \Box(\varphi \supset \psi) \to (\Box\varphi \supset \Box\varphi).$$

The modal logic we are studying is thus non-normal: this constitutes one of the main difficulties in providing a complete axiomatization for it, as the standard canonical model construction cannot be applied to prove completeness.

4. Axiomatizations

In this section we introduce Hilbert-style calculi which we will prove to be complete with respect to the global and the local consequence relations, respectively. Our starting point is the axiomatization of the non-modal fragment of our logic, provided by Arieli and Avron [2, p. 47]. We present the axiom schemata in stages:

$(\supset 1)$	$p \supset (q \supset p)$	
$(\supset 2)$	$(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$	
$(\supset 3)$	$((p \supset q) \supset p) \supset p$	
$(\neg\,\neg)$	$p \supset \neg\neg p$	$\neg\neg p \supset p$

Note that the schema $(\neg p \supset \neg q) \supset (q \supset p)$, usually called *contraposition*, is absent but the classical nature of the calculus has been preserved by the inclusion of *Peirce's Law* $(\supset 3)$ and *double negation*. In fact, it is not difficult to check that the $\langle \wedge, \vee, \supset\rangle$-fragment of Arieli-Avron logic coincides with the negation-free fragment of classical logic [10, Remark 1.2].

The next set of schemata establishes the link with the truth lattice operations and is entirely standard:

$(\wedge \supset)$	$(p \wedge q) \supset p$	$(p \wedge q) \supset q$
$(\supset \wedge)$	$p \supset (q \supset (p \wedge q))$	
$(\supset \mathsf{t})$	$p \supset \mathsf{t}$	
$(\supset \vee)$	$p \supset (p \vee q)$	$q \supset (p \vee q)$
$(\vee \supset)$	$(p \supset r) \supset ((q \supset r) \supset ((p \vee q) \supset r))$	
$(\supset \mathsf{f})$	$\mathsf{f} \supset p$	

The analogous schemata for the information lattice operations are:

$(\otimes \supset)$	$(p \otimes q) \supset p$	$(p \otimes q) \supset q$
$(\supset \otimes)$	$p \supset (q \supset (p \otimes q))$	
$(\supset \top)$	$p \supset \top$	
$(\supset \oplus)$	$p \supset (p \oplus q)$	$q \supset (p \oplus q)$
$(\oplus \supset)$	$(p \supset r) \supset ((q \supset r) \supset ((p \oplus q) \supset r))$	
$(\supset \bot)$	$\bot \supset p$	

In the absence of contraposition one also has to stipulate how negation interacts with the other operations:

$(\neg \wedge)$	$\neg(p \wedge q) \equiv (\neg p \vee \neg q)$
$(\neg \vee)$	$\neg(p \vee q) \equiv (\neg p \wedge \neg q)$
$(\neg \otimes)$	$\neg(p \otimes q) \equiv (\neg p \otimes \neg q)$
$(\neg \oplus)$	$\neg(p \oplus q) \equiv (\neg p \oplus \neg q)$
$(\neg \supset)$	$\neg(p \supset q) \equiv (p \wedge \neg q)$
$(\neg\, \mathsf{t})$	$\neg \mathsf{t} \supset p$
$(\neg\, \mathsf{f})$	$p \supset \neg \mathsf{f}$
$(\neg \top)$	$p \supset \neg \top$
$(\neg \bot)$	$\neg \bot \supset p$

The only rule of the Arieli-Avron calculus is *modus ponens*:

(mp)	$p, p \supset q \vdash q$

As is shown in [2], this calculus is complete with respect to the semantics based on $\mathcal{FOUR}$ introduced in Section 2.

We now proceed to expand the Arieli-Avron calculus to accommodate the modal necessity operator, taking our cue from the semantic considerations in the previous subsection. We begin by adding the axiom schemata

$(\Box\, \mathsf{t})$	$\Box \mathsf{t} \leftrightarrow \mathsf{t}$
$(\Box \wedge)$	$\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$
$(\Box \bot)$	$\Box(\bot \to p) \leftrightarrow (\bot \to \Box p)$

Interestingly, the last of these covers only one of the four cases that make up Proposition 3.2 (iii), and indeed, one of the consequences of our completeness result is that the other three are not needed. In order to capture the closure property expressed in Corollary 3.4, we need to make sure that we first generate all valid instances of the shape $\varphi \to \psi$. The official definition of our logic is therefore slightly more involved than usual:

Definition 4.1. Let Fm be the set of formulas generated by a countable set of variables Var in the modal language $\langle \wedge, \vee, \otimes, \oplus, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top, \Box \rangle$. The set Σ of *axioms* of *modal bilattice logic* is the least subset of Fm containing all substitution instances of the schemata exhibited in this subsection, and closed under

(val-mp) if φ and $\varphi \supset \psi$ are in Σ, then so is ψ;
(val-mono) if $\varphi \to \psi$ is in Σ, then so is $\Box\varphi \to \Box\psi$.

The *rules* of *modal bilattice logic* are

$$\frac{\varphi, \varphi \supset \psi}{\psi} \text{ (mp)} \qquad\qquad \frac{\varphi \to \psi}{\Box\varphi \to \Box\psi} \text{ (mono)}$$

Local inference $\vdash_l$ employs only (mp), while *global inference* $\vdash_g$ is generated by (mp) and (mono).

Note that, although structurally similar, the rules (val-mp) and (val-mono) are only ever applied to *valid* formulas, while (modus ponens) and (monotonicity) can be applied to arbitrary assumptions.

5. Algebraic models of the logic

5.1. **Modal bilattices.** We start by looking at the algebraic models of the non-modal core of the logic. This will allow us to determine the models of the modal calculi which, as is to be expected, will turn out to be language expansions of the non-modal algebras.

The second author proved in [36, Theorem 4.2.4] that Arieli-Avron logic is algebraizable in the sense of [6]. This means in particular that the non-modal calculus introduced in the previous section enjoys strong algebraic completeness with respect to a class of algebras introduced in [36, Definition 4.3.1] under the name *implicative bilattices*[1].

Definition 5.1. A (bounded) *bilattice* is an algebra $\langle B, \wedge, \vee, \otimes, \oplus, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$ such that $\langle B, \wedge, \vee, \mathsf{f}, \mathsf{t} \rangle$ and $\langle B, \otimes, \oplus, \bot, \top \rangle$ are both (bounded) lattices. The order $\leq_t$ arising from $\wedge$ or $\vee$ is called the *truth order* (*t-order*), that arising from $\otimes$ or $\oplus$ the *knowledge order* (*k-order*) $\leq_k$. The *negation* operation $\neg$ is required to satisfy the properties

(i) $x \leq_t y$ iff $\neg y \leq_t \neg x$;
(ii) $x \leq_k y$ iff $\neg x \leq_k \neg y$;
(iii) $\neg\neg x = x$.

Conditions (i)-(iii) uniquely determine the behaviour of negation on the bounds: $\neg \mathsf{t} = \mathsf{f}$, $\neg \mathsf{f} = \mathsf{t}$, $\neg \top = \top$, and $\neg \bot = \bot$. We note that conditions (i)-(ii) can be expressed by equations (*De Morgan Laws*), which implies that bilattices form an equational class (a variety). Notice also that $\mathcal{FOUR}$ is an (in fact, the smallest non-trivial) algebra in this variety.

Definition 5.2. A (bounded) *implicative bilattice* is a (bounded) bilattice with an additional operation $\supset$ satisfying the following identities:

(IB1) $(x \supset x) \supset y = y$
(IB2) $x \supset (y \supset z) = (x \wedge y) \supset z = (x \otimes y) \supset z$
(IB3) $((x \supset y) \supset x) \supset x = x \supset x$
(IB4) $(x \vee y) \supset z = (x \supset z) \wedge (y \supset z) = (x \oplus y) \supset z$
(IB5) $x \wedge ((x \supset y) \supset (x \otimes y)) = x$
(IB6) $\neg(x \supset y) \supset z = (x \wedge \neg y) \supset z$.

Implicative bilattices obviously form a variety. Once again, $\mathcal{FOUR}$, viewed as an algebra in the language $\langle \wedge, \vee, \otimes, \oplus, \supset, \neg \rangle$ (possibly also including the bounds) is the smallest non-trivial implicative bilattice. We also notice that Definition 5.2 implies that each of the four lattice operations distributes over the other three [36, Proposition 4.3.4.]. This also follows from the following important fact [36, Theorem 5.2.1].

[1] These algebras are called *classical implicative bilattices* in [10, 24]

Theorem 5.3. *The variety of (bounded) implicative bilattices is generated by $\mathcal{FOUR}$.*

Algebraizability of the Arieli-Avron calculus introduced in the previous section means that the derivability relation of this calculus can be faithfully interpreted in the equational consequence of the variety of implicative bilattices, and vice versa, by mutually inverse interpretations. Consider a translation $\tau\colon Fm \to Eq$ from propositional formulas Fm into equations Eq over the same language, i.e., $\langle \wedge, \vee, \otimes, \oplus, \supset, \neg \rangle$, possibly enriched with the four constants. For $\varphi \in Fm$, we define

$$\tau\colon \varphi \;\longmapsto\; \varphi = \varphi \supset \varphi.$$

This is extended to sets of formulas in the usual way: $\tau(\Gamma) := \bigcup\{\tau(\gamma) : \gamma \in \Gamma\}$. Algebraizability of Arieli-Avron calculus $\vdash$ then implies the following.

Theorem 5.4. *$\Gamma \vdash \varphi$ if and only if $\tau(\Gamma) \models \tau(\varphi)$ in the equational consequence of the variety of (bounded) implicative bilattices.*

A translation $\rho\colon Eq \to Fm$ can be defined in order to obtain a "reverse completeness" theorem that may be seen as a converse to the above one. This is not central in our setting, but it will be useful to know that the translation can be defined as follows:

$$\rho(\varphi = \psi) \;\longmapsto\; \varphi \leftrightarrow \psi.$$

Theorem 5.3 tells us that $\Gamma \vdash \varphi$ is also equivalent to $\tau(\Gamma) \models \tau(\varphi)$ holding in $\mathcal{FOUR}$. Combining this result with what we already know from Section 2, we obtain the following equivalences.

Corollary 5.5. *Let $\Gamma \cup \{\varphi\} \subseteq Fm$. The following are equivalent:*

(i) $\Gamma \vdash \varphi$
(ii) $\tau(\Gamma) \models \tau(\varphi)$ *holds in* $\mathcal{FOUR}$
(iii) $\tau(\Gamma) \models \tau(\varphi)$ *holds in any (bounded) implicative bilattice*
(iv) $\Gamma \models \varphi$ *holds in the matrix* $\langle \mathcal{FOUR}, \{\mathsf{t}, \top\} \rangle$.

The last item of the preceding corollary can also be formulated in a more general way, replacing $\mathcal{FOUR}$ by an arbitrary implicative bilattice, and this will be particularly important for us. In the standard theory of logical matrices, one considers pairs $\langle \mathbf{A}, D \rangle$ where $\mathbf{A}$ is an algebra with carrier set A and $D \subseteq A$. One then defines a notion of consequence in the same way as we have done in Section 2 for the matrix $\langle \mathcal{FOUR}, \{\mathsf{t}, \top\} \rangle$. That is, we consider the formula algebra $\mathbf{Fm}$ freely generated by a countable set of propositional variables over the appropriate propositional language $\mathcal{L}$ and we set $\Gamma \models_{\langle \mathbf{A}, D \rangle} \varphi$ if and only if, for all $\mathcal{L}$-homomorphisms $h\colon \mathbf{Fm} \to \mathbf{A}$, we have $h(\varphi) \in D$ whenever $h[\Gamma] \subseteq D$. We can then add one more piece of information to the above-stated equivalences:

(v) $\Gamma \models \varphi$ *holds in any matrix* $\langle \mathbf{B}, F_0 \rangle$*, where* $\mathbf{B}$ *is a (bounded) implicative bilattice and* $F_0 := \{x \in B : x \supset x = x\}$.

This means that Arieli-Avron logic is complete with respect to the above-defined class of matrices. This is also a consequence of algebraizability, and one can see that the equation defining the elements in F_0 is determined by the translation τ. It is important for us to notice that item (v) can be restated in even more general terms:

(vi) $\Gamma \models \varphi$ *holds in any matrix* $\langle \mathbf{B}, F \rangle$*, where* $\mathbf{B}$ *is a (bounded) implicative bilattice and* F *is a bifilter of* $\mathbf{B}$.

By a *bifilter* of $\mathbf{B}$ we mean a subset $F \subseteq B$ that is a lattice filter with respect to both the t- and the k-lattice order (see [10, Proposition 2.11]). Using this terminology, it is easy to check that the above-defined set F_0 is the least bifilter of any implicative bilattice.

Algebraizability is an intrinsic property of a logical calculus that is preserved by extensions and, under certain conditions, by language expansions. These are determined by the shape of the translation ρ from equations into propositional formulas. In our case, when adding a modal operator $\Box$ to the Arieli-Avron calculus, the condition that we need in order to preserve algebraizability is that $\varphi \leftrightarrow \psi$ imply $\Box\varphi \leftrightarrow \Box\psi$. This is an easy consequence of the monotonicity rule (mono) introduced in the previous section, which is a rule of the global but not of the local calculus.

Theorem 5.6. *The global calculus $\vdash_g$ of modal bilattice logic is algebraizable with the same translations τ and ρ that ensure algebraizability of Arieli-Avron logic.*

It is easy to see that the local calculus $\vdash_l$ is not algebraizable. In fact, we will see that if it were algebraizable, then it would coincide with the global one. This situation mirrors classical modal logic, where local and global consequence share the same algebraic counterpart, the latter being algebraizable while the former is not.

The general theory of algebraizable logics [6] allows us to straightforwardly determine the algebraic models of the global calculus. These are algebras in the language

$$\langle \wedge, \vee, \otimes, \oplus, \supset, \neg, \Box, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$$

having an implicative bilattice reduct and satisfying identities and quasi-identities that are the τ-translations of the new axioms and rules that we have added to the non-modal calculus. Notice that we have now included the constants in the language, as they appear, crucially, in the new axioms. We are thus led to introduce the following structures.

Definition 5.7. A *modal bilattice* is a bounded implicative bilattice $\mathbf{B}$ having an extra unary operation $\Box$ that satisfies the following identities:

(i) $\Box\mathsf{t} = \mathsf{t}$

(ii) $\Box(x \wedge y) = \Box x \wedge \Box y$

(iii) $\Box(\bot \to x) = \bot \to \Box x$.

The reader may have noticed that the above equations are not prima facie the τ-translations of the axioms. For instance, the axiom $\Box\mathsf{t} \leftrightarrow \mathsf{t}$ translates as

$$\Box\mathsf{t} \leftrightarrow \mathsf{t} = (\Box\mathsf{t} \leftrightarrow \mathsf{t}) \supset (\Box\mathsf{t} \leftrightarrow \mathsf{t}).$$

It is however easy to show that, in an implicative bilattice, the equation $x \leftrightarrow y = (x \leftrightarrow y) \supset (x \leftrightarrow y)$ is equivalent to $x = y$. Notice also that we have not included the quasi-identity corresponding to the monotonicity rule because it holds just as a consequence of the second item (monotonicity of $\Box$ with respect to the t-lattice order). Returning to a comment we made above, we note that every modal bilattice satisfies the equation $\Box(\mathsf{c} \to \varphi) = \mathsf{c} \to \Box\varphi$ for each $\mathsf{c} \in \{\mathsf{f}, \mathsf{t}, \bot, \top\}$. This can be shown purely algebraically, but it will also follow from our completeness result. Finally, we notice that (iii) is equivalent, in any implicative bilattice, to the simpler one

(iii') $\Box(x \supset \bot) = \Diamond x \supset \bot$

where $\Diamond x := \neg\Box\neg x$.

The above considerations immediately imply the following results.

Theorem 5.8. *The global consequence relation $\vdash_g$ of modal bilattice logic is algebraizable with respect to the variety of modal bilattices.*

Theorem 5.9. *The global consequence relation $\vdash_g$ is complete with respect to the class of all matrices $\langle \mathbf{B}, F_0 \rangle$ such that $\mathbf{B}$ is a modal bilattice and F_0 is the least bifilter of $\mathbf{B}$.*

We now want to show that $\vdash_g$ and $\vdash_l$ indeed share the same algebraic counterpart, that is, that a similar result to Theorem 5.9 can be proved about the local calculus. For this we will need a few lemmas.

Following standard algebraic logic terminology [19], we say that a matrix $\langle \mathbf{A}, D \rangle$ is a *model* of a logic $\vdash$ when $\Gamma \vdash \varphi$ implies $\Gamma \models_{\langle \mathbf{A}, D \rangle} \varphi$ for all formulas $\Gamma \cup \{\varphi\} \subseteq Fm$. In such a case we call D a *logical filter* of $\vdash$.

Lemma 5.10. *For any modal bilattice $\mathbf{B}$, the matrix $\langle \mathbf{B}, F \rangle$ is a model of the local calculus $\vdash_l$ if and only if $F \subseteq B$ is a bifilter.*

Proof. Assume $\langle \mathbf{B}, F \rangle$ is such that $F \subseteq B$ is a bifilter of $\mathbf{B}$. In order to prove that $\langle \mathbf{B}, F \rangle$ is a model of $\vdash_l$ it is sufficient to prove that F contains the image of all axioms and is closed under the rules of the local consequence. The axioms are the same as those for the global consequence. Then Theorem 5.8 ensures that $\mathbf{B} \models h(\psi) = h(\psi) \supset h(\psi)$ for any axiom ψ and any homomorphism $h \colon \mathbf{Fm} \to \mathbf{B}$. This means that $h(\psi)$ belongs to the least non-empty bifilter of $\mathbf{B}$, namely $F_0 = \{a \in B : a = a \supset a\}$ [10, Theorem 2.12]. Since F_0 is contained in any non-empty bifilter, we easily obtain that $h(\psi) \in F$. As for rules, the only rule of $\vdash_l$ is modus ponens relative to $\supset$ and we know that bifilters are closed under modus ponens [10, Proposition 2.11]. Conversely, if $\langle \mathbf{B}, F \rangle$ is a model of $\vdash_l$ with $\mathbf{B}$ a modal bilattice and $F \subseteq B$, then F is non-empty because $h(\psi) \in F$ for any theorem ψ of $\vdash_l$ and any homomorphism $h \colon \mathbf{Fm} \to \mathbf{B}$. Moreover, F must be closed under modus ponens, which implies, again by [10, Proposition 2.11], that F is a bifilter. $\square$

We can already notice that the previous lemma indicates that, when considering models of the local consequence, it is necessary to consider *arbitrary* bifilters rather than just the minimal one.

Any logic is complete with respect to the class of all its matrix models. More interestingly, it is known that any logic is complete with respect to the class of all its *reduced* matrix models. We say that a matrix $\langle \mathbf{A}, D \rangle$ is *reduced* when identity is the only congruence θ of $\mathbf{A}$ which is *compatible* with D, by compatible meaning that, for all $a, b \in A$ such that $\langle a, b \rangle \in \theta$, it holds that $a \in D$ if and only if $b \in D$. It can be shown that, for any subset $D \subseteq A$, there is always a greatest congruence that is compatible with D. This is denoted by $\Omega(D)$ and is called the *Leibniz congruence* of the matrix $\langle \mathbf{A}, D \rangle$. Thus, a reduced matrix can be defined as one whose Leibniz congruence is the identity.

We are going to exploit the completeness result with respect to reduced models to characterize the algebraic counterpart of the local calculus. For an arbitrary logic $\vdash$ (not necessarily syntactically defined), we denote

$$\mathsf{Alg}^*(\vdash) := \{\mathbf{A} : \langle \mathbf{A}, D \rangle \text{ is a reduced matrix model of the logic} \vdash\}.$$

Algebraizability of $\vdash_g$ implies that a matrix $\langle \mathbf{A}, D \rangle$ is a reduced model of $\vdash_g$ if and only if $\mathbf{A}$ is a modal bilattice and D is the least bifilter of $\mathbf{A}$. It follows that $\mathsf{Alg}^*(\vdash_g)$ is exactly the variety of modal bilattices. This allows us to prove the next lemma that we need.

Lemma 5.11. $\mathsf{Alg}^*(\vdash_l)$ *is the variety of modal bilattices.*

Proof. Let us denote by ModBil the class of all modal bilattices. As mentioned above, $\mathsf{Alg}^*(\vdash_g) = \mathsf{ModBil}$. Moreover, $\mathsf{Alg}^*(\vdash_g) \subseteq \mathsf{Alg}^*(\vdash_l)$, because $\vdash_g$ is an extension of $\vdash_l$. Thus, $\mathsf{ModBil} \subseteq \mathsf{Alg}^*(\vdash_l)$. We also know from [19] that $V(\mathsf{Alg}^*(\vdash_l))$, the variety generated by $\mathsf{Alg}^*(\vdash_l)$, coincides with $V(\mathbf{Fm}/\Omega)$, where

$$\Omega := \{\langle \varphi, \psi \rangle \in \mathbf{Fm}\times\mathbf{Fm} : \ \emptyset \vdash_g \varphi \leftrightarrow \psi\}.$$

By Theorem 5.8, we have that $\mathbf{Fm}/\Omega$ is a modal bilattice. This implies that $V(\mathbf{Fm}/\Omega) = V(\mathsf{Alg}^*(\vdash_l)) \subseteq \mathsf{ModBil}$. Hence, $\mathsf{Alg}^*(\vdash_l) \subseteq \mathsf{ModBil}$, which implies that $\mathsf{Alg}^*(\vdash_g) = \mathsf{Alg}^*(\vdash_l) = \mathsf{ModBil}$. □

The previous lemmas do not yet give us necessary and sufficient conditions for a matrix to be a reduced model of $\vdash_l$ (this issue will be settled in Subsection 5.3, with the help of the construction introduced in Subsection 5.2). However, they allow us to prove the following completeness result.

Theorem 5.12. *The local consequence relation $\vdash_l$ is complete with respect to the class of all matrices $\langle \mathbf{B}, F\rangle$ such that $\mathbf{B}$ is a modal bilattice $\mathbf{B}$ and F is a bifilter of $\mathbf{B}$.*

Proof. Let K be the class of all matrix models $\langle \mathbf{B}, F\rangle$ such that $\mathbf{B}$ is a modal bilattice and $F \subseteq B$ a bifilter of $\mathbf{B}$. Denote by $\models_K$ the associated consequence relation, defined as follows: $\Gamma \models_K \varphi$ iff $\Gamma \models_{\langle \mathbf{B}, F\rangle} \varphi$ for any matrix $\langle \mathbf{B}, F\rangle \in K$. By Lemma 5.10, we have $\vdash_l \leq \models_K$ (i.e., $\models_K$ is an extension of $\vdash_l$). By Lemma 5.11 we know that $K^* \subseteq K$, where K^* denotes the class of all reduced matrix models of $\vdash_l$. Hence, $\models_K \leq \models_{K^*}$ and, as mentioned above, $\models_{K^*} = \vdash_l$ is an instance of a result that holds for any logic. Thus, we have that $\models_K \leq \models_{K^*} = \vdash_l$ which implies $\models_K = \vdash_l$. □

Given a finite set of formulas $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$, we abbreviate $\bigwedge \Gamma := \gamma_1 \wedge \ldots \wedge \gamma_n$.

Corollary 5.13. *Let $\Gamma \cup \{\varphi\} \subseteq Fm$. The following are equivalent:*

(i) $\Gamma \vdash_l \varphi$,
(ii) there exists a finite $\Gamma_0 \subseteq \Gamma$ *such that* $\Gamma_0 \vdash_l \varphi$,
(iii) there exists a finite $\Gamma_0 \subseteq \Gamma$ *such that the equation* $\bigwedge \Gamma_0 \wedge \top \leq_t \varphi$ *is valid in the variety of modal bilattices.*

Proof. The equivalence between (i) and (ii) follows immediately from the fact that all rules of the calculus $\vdash_l$ involve only finitely many premises. To show that (ii) implies (iii), assume $\Gamma_0 \vdash_l \varphi$ for a finite Γ_0. Then $\bigwedge \Gamma_0 \vdash_l \varphi$, as this already holds in the non-modal fragment of the calculus. By Theorem 5.12, this means that, for every matrix $\langle \mathbf{B}, F\rangle$ and every homomorphism $h\colon \mathbf{Fm} \to \mathbf{B}$, we have that $h(\bigwedge \Gamma_0) \in F$ implies $h(\varphi) \in F$. This implies that the element $h(\varphi)$ belongs to the bifilter generated by $h(\bigwedge \Gamma_0)$. By [9, p. 203] this means $h(\bigwedge \Gamma_0) \leq_t h(\bigwedge \Gamma_0) \otimes h(\varphi)$ or, equivalently, $h(\bigwedge \Gamma_0) \wedge \top \leq_t h(\varphi)$. Since this holds for any homomorphism h, we can conclude that $\mathbf{B}$ satisfies the equation $\bigwedge \Gamma_0 \wedge \top \leq_t \varphi$. Moreover, $\mathbf{B}$ itself being an arbitrary modal bilattice, we have that the equation holds in the variety. Conversely, assume (iii) holds. Then, if $h(\bigwedge \Gamma_0) \in F$ for some matrix $\langle \mathbf{B}, F\rangle$ and some homomorphism $h\colon \mathbf{Fm} \to \mathbf{B}$, the equation of (iii) tells us that $h(\varphi)$ belongs to the bifilter generated by $h(\bigwedge \Gamma_0)$, which is included in F. Hence, $h(\varphi) \in F$. □

We may ask ourselves what is the analogue of Lemma 5.10 for the global calculus $\vdash_g$, that is, given a matrix $\langle \mathbf{B}, F\rangle$ with $\mathbf{B}$ a modal bilattice, which properties must F satisfy in order for $\langle \mathbf{B}, F\rangle$ to be a model of the global calculus? Obviously F must be a bifilter, and the next proposition indicates that the only further requirement is that F be closed under rule (mono), that is, if $a \to b \in F$, then $\Box a \to \Box b \in F$.

Proposition 5.14. *For any modal bilattice* $\mathbf{B}$*, the matrix* $\langle \mathbf{B}, F \rangle$ *is a model of the global consequence relation* $\vdash_g$ *if and only if* $F \subseteq B$ *is a non-empty bifilter that is closed under rule (mono).*

Proof. Assume $\langle \mathbf{B}, F \rangle$ is such that $F \subseteq B$ is a non-empty bifilter of $\mathbf{B}$ closed under the monotonicity rule. Then we know that $\langle \mathbf{B}, F \rangle$ is a model of $\vdash_l$ by Proposition 5.10. Since monotonicity is the only rule that distinguishes $\vdash_g$ from $\vdash_l$, the assumption immediately implies that $\langle \mathbf{B}, F \rangle$ is a model of $\vdash_g$ as well. Conversely, if $\langle \mathbf{B}, F \rangle$ is a model of $\vdash_g$ with $\mathbf{B}$ a modal bilattice and $F \subseteq B$, then F is non-empty because $h(\psi) \in F$ for any theorem ψ of $\vdash_g$ and any homomorphism $h \colon \mathbf{Fm} \to \mathbf{B}$. Moreover, F must be closed under modus ponens, which implies, by [10, Proposition 2.11], that F is a bifilter, and it must also clearly be closed under rule (mono). ☐

Corollary 5.15. *The global consequence relation* $\vdash_g$ *is complete with respect to the class of all matrices* $\langle \mathbf{B}, F \rangle$ *such that* $\mathbf{B}$ *is a modal bilattice and* $F \subseteq B$ *a non-empty bifilter of* $\mathbf{B}$ *that is moreover closed under rule (mono).*

Proof. Let K be the class of all matrix models $\langle \mathbf{B}, F \rangle$ such that $\mathbf{B}$ a modal bilattice and $F \subseteq B$ a non-empty bifilter of $\mathbf{B}$ closed under rule (mono). By algebraizability of $\vdash_g$, we know that $\vdash_g$ is the logic determined by the class of matrices $\langle \mathbf{B}, F_0 \rangle$ where $\mathbf{B}$ is a modal bilattice and F_0 is the minimal (non-empty) bifilter. Since F_0 is closed under monotonicity, we immediately have $\models_K \leq \vdash_g$. On the other hand, Proposition 5.14 implies $\vdash_g \leq \models_K$, so we are finished. ☐

5.2. **Twist-structure representation of modal bilattices.** Several classes of bilattices can be conveniently represented through a construction called *twist-structure* [29, 8]. In this section we extend it to obtain a representation for modal bilattices. This will enhance our understanding of the necessity operator $\Box$ as well as clarify the connection between our logic and that of [34], and will eventually allow us to prove completeness of our modal calculi with respect to the four-valued Kripke semantics.

Definition 5.16. A *bimodal Boolean algebra* is a structure $\mathbf{A} = \langle A, \sqcap, \sqcup, \sim, 0, 1, \Box_+, \Box_- \rangle$ such that $\langle A, \sqcap, \sqcup, \sim, 0, 1 \rangle$ is a Boolean algebra and both $\Box_+$ and $\Box_-$ are unary operators that preserve finite (possibly empty) meets.

The above definition implies that both $\langle A, \sqcap, \sqcup, \sim, 0, 1, \Box_+ \rangle$ and $\langle A, \sqcap, \sqcup, \sim, 0, 1, \Box_- \rangle$ are modal Boolean algebras in the usual sense [13]. Given a bimodal Boolean algebra $\mathbf{A}$, we consider the *(full) twist-structure*

$$\mathbf{A}^{\bowtie} = \langle A \times A, \wedge, \vee, \otimes, \oplus, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top, \Box \rangle$$

whose operations are defined, for $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in A \times A$, as follows:

$$\begin{aligned}
\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle &:= \langle a_1 \sqcap b_1,\ a_2 \sqcup b_2\rangle \\
\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle &:= \langle a_1 \sqcup b_1,\ a_2 \sqcap b_2\rangle \\
\langle a_1, a_2\rangle \otimes \langle b_1, b_2\rangle &:= \langle a_1 \sqcap b_1,\ a_2 \sqcap b_2\rangle \\
\langle a_1, a_2\rangle \oplus \langle b_1, b_2\rangle &:= \langle a_1 \sqcup b_1,\ a_2 \sqcup b_2\rangle \\
\langle a_1, a_2\rangle \supset \langle b_1, b_2\rangle &:= \langle \sim a_1 \sqcup b_1, a_1 \sqcap b_2\rangle \\
\neg\langle a_1, a_2\rangle &:= \langle a_2,\ a_1\rangle \\
\mathsf{f} &:= \langle 0, 1\rangle \\
\mathsf{t} &:= \langle 1, 0\rangle \\
\bot &:= \langle 0, 0\rangle \\
\top &:= \langle 1, 1\rangle \\
\Box\langle a_1, a_2\rangle &:= \langle \Box_+ a_1 \sqcap \Box_- \sim a_2,\ \Diamond_+ a_2\rangle
\end{aligned}$$

where $\Diamond_+ a_2 := \sim \Box_+ \sim a_2$. This construction is obviously related to (and to some extent generalizes) those of [34, 37, 31]. The term *full* is meant to distinguish our twist-structures from those of, e.g., [31], whose underlying set can be a proper subset of the direct square $A \times A$ (see also the construction considered in Subsection 7.3). Notice that the k-order in $\mathbf{A}^{\bowtie}$ is the direct power of the lattice order of $\mathbf{A}$, i.e., $\leq_k = \leq \times \leq$, whereas the t-order is the direct product of $\leq$ and its dual: $\leq_t = \leq \times \geq$.

We are going to see that every twist-structure $\mathbf{A}^{\bowtie}$ is indeed a modal bilattice. With respect to the construction used in [34, 31] to represent so-called *BK-lattices*, we note that a twist-structure $\mathbf{A}^{\bowtie}$ is a BK-lattice precisely when the underlying bimodal Boolean algebra $\mathbf{A}$ satisfies the equation $\Box_- x = 1$, so that $\Box\langle a_1, a_2\rangle = \langle \Box_+ a_1,\ \Diamond_+ a_2\rangle$. It is also easy to check that

$$\Box(\langle a_1, a_2\rangle \vee \langle 0, 0\rangle) \oplus (\Box\langle a_1, a_2\rangle \wedge \langle 0, 0\rangle) = \langle \Box_+ a_1,\ \Diamond_+ a_2\rangle$$

which explains the relation between our modal operator and that of [31, 34] stated in Proposition 3.1. Obviously, our modal operator cannot be recovered as a term in the language of [34], because $\Box$ is defined using two independent operators $\Box_+$ and $\Box_-$ on the underlying Boolean algebra, while [34] only makes use of one operator (together with its dual).

Proposition 5.17. *Every twist-structure* $\mathbf{A}^{\bowtie}$ *is a modal bilattice.*

Proof. We do not need to worry about non-modal connectives, as the result has been proven, e.g., in [8, Proposition 4.11]. Let us check that $\mathbf{A}^{\bowtie}$ satisfies the axioms defining modal bilattices, namely:

(i) $\Box \mathsf{t} = \mathsf{t}$

(ii) $\Box(x \wedge y) = \Box x \wedge \Box y$

(iii) $\Box(\bot \to x) = \bot \to \Box x$.

(i) $\Box\langle 1,0\rangle = \langle \Box_+ 1 \wedge \Box_- \sim 0,\ \Diamond_+ 0\rangle = \langle \Box_+ 1 \wedge \Box_- 1,\ 0\rangle = \langle 1,0\rangle$.
(ii) Given $\langle a_1,a_2\rangle, \langle b_1,b_2\rangle \in A\times A$, we have

$$\begin{aligned}
\Box(\langle a_1,a_2\rangle \wedge \langle b_1,b_2\rangle) &= \Box\,\langle a_1\wedge b_1, a_2\vee b_2\rangle\\
&= \langle \Box_+(a_1\wedge b_1)\wedge \Box_- \sim(a_2\vee b_2),\ \Diamond_+(a_2\vee b_2)\rangle\\
&= \langle \Box_+ a_1 \wedge \Box_+ b_1 \wedge \Box_-(\sim a_2 \wedge \sim b_2),\ \Diamond_+ a_2 \vee \Diamond_+ b_2\rangle\\
&= \langle \Box_+ a_1 \wedge \Box_+ b_1 \wedge \Box_- \sim a_2 \wedge \Box_- \sim b_2,\ \Diamond_+ a_2 \vee \Diamond_+ b_2\rangle\\
&= \Box\langle a_1,a_2\rangle \wedge \Box\langle b_1,b_2\rangle.
\end{aligned}$$

(iii) In order to simplify our calculations, we will prove the equation (iii') $\Box(x\supset\bot) = \Diamond x \supset \bot$, which we have already noted to be equivalent, in any implicative bilattice, to (iii). Given $\langle a_1,a_2\rangle \in A\times A$, we have

$$\begin{aligned}
\Box(\langle a_1,a_2\rangle \supset \langle 0,0\rangle) &= \Box\langle \sim a_1, 0\rangle\\
&= \langle \Box_+ \sim a_1 \wedge \Box_- \sim 0,\ \Diamond_+ 0\rangle\\
&= \langle \Box_+ \sim a_1 \wedge \Box_- 1,\ 0\rangle\\
&= \langle \Box_+ \sim a_1 \wedge 1,\ 0\rangle\\
&= \langle \sim \Diamond_+ a_1,\ 0\rangle\\
&= \langle \Diamond_+ a_1,\ \Box_+ a_2 \wedge \Box_- \sim a_1\rangle \supset \langle 0,0\rangle\\
&= \neg\langle \Box_+ a_2 \wedge \Box_- \sim a_1,\ \Diamond_+ a_1\rangle \supset \langle 0,0\rangle\\
&= \neg\Box\neg\langle a_1,a_2\rangle \supset \langle 0,0\rangle\\
&= \Diamond\langle a_1,a_2\rangle \supset \langle 0,0\rangle. \qquad \Box
\end{aligned}$$

A first and most important example of a twist-structure is $\mathcal{FOUR}$ itself, which is isomorphic (if we ignore the modal operator) to $\mathbf{2}^{\bowtie}$, where $\mathbf{2}$ is the two-element Boolean algebra. Concerning the modal operator, given that there are two modal algebras whose non-modal reduct is the two-element Boolean algebra, we see that there are exactly four modal bilattices whose non-modal reduct is $\mathcal{FOUR}$.

Our next aim is to show that, as happens with (non-modal) bilattices, every modal bilattice is isomorphic to a twist-structure.

First of all, let us notice that, if we leave out the modal operator, then we know that *every bounded implicative bilattice is isomorphic to a twist-structure* $\mathbf{A}^{\bowtie}$*, where* $\mathbf{A}$ *is a Boolean algebra* [8, Theorem 4.13]. Given a (modal) bilattice $\mathbf{B}$, we can recover the associated Boolean algebra by defining an equivalence relation as follows: for $a,b\in B$, we let

$$a \approx b \;\text{ iff }\; a\supset \mathsf{f} = b \supset \mathsf{f}.$$

This relation, which can be defined in several alternative ways (cf. [9, Definition 3.7]), is not only an equivalence relation, but also a congruence with respect to all the algebraic operations of a bounded implicative bilattice except negation. This means that we can consider the quotient $\langle B,\wedge,\vee,\supset,\mathsf{f},\mathsf{t}\rangle/\approx$ which is a Boolean algebra. Notice that in the quotient the t-meet and the k-meet coincide, and likewise for the two joins. Also, for $a\in B$, the Boolean negation of its corresponding class $[a]\in B/\approx$ is defined as usual: $\sim[a] := [a]\supset[\mathsf{f}]$.

However, the relation $\approx$ need not be a congruence with respect to $\Box$. In order to define modal operators on the quotient $B/{\approx}$, we thus need slightly more involved definitions: for an equivalence class $[a] \in B/\!\approx$, we let

$$\begin{aligned}
\Box_+[a] &:= [\Diamond(a \supset \mathsf{f}) \supset \mathsf{f}] \\
\Box_-[a] &:= [\Box(\neg(a \supset \mathsf{f}) \vee \top)] \\
\Diamond_+[a] &:= [\Diamond a]
\end{aligned}$$

where $\Diamond$ abbreviates $\neg\Box\neg$. Notice that $\Box_+[a] = \sim\Diamond_+\sim[a]$. We can thus view $\Box_+$ as a defined operation.

Let us prove that our definitions are sound. Assume then $a \approx b$, and notice that this is equivalent to $a \supset \bot = b \supset \bot$ (this can be checked in any implicative bilattice, for instance using the twist-structure representation). Then, $\Box(a \supset \bot) = \Box(b \supset \bot)$. Now we can apply equation (iii) of Definition 5.7, in its equivalent form (iii'), to conclude $\Diamond a \supset \bot = \Diamond b \supset \bot$. Thus we have $\Diamond a \approx \Diamond b$. We omit the proofs of the other two cases as they are straightforward. It remains to prove that $\Box_-$ is indeed a meet-preserving operator and $\Diamond_+$ is a join preserving operator (from which it will follow that $\Box_+$ is a meet-preserving operator). It is immediate to see that $\Diamond_+[\mathsf{f}] = [\mathsf{f}]$. That $\Diamond_+$ preserves joins follows easily from De Morgan Laws. That $\Box_-[\mathsf{t}] = [\mathsf{t}]$ is also immediate. To see that $\Box_-$ preserves meets, we notice that any implicative bilattice satisfies the following equation:

$$\neg((x \wedge y) \supset \mathsf{f}) = \neg(x \supset \mathsf{f}) \wedge \neg(y \supset \mathsf{f}). \tag{3}$$

This can be checked in $\mathcal{FOUR}$ (relying on Theorem 5.3) or using the twist-structure representation of implicative bilattices. We can now easily check that

$$\begin{aligned}
\Box_-[a \wedge b] &= [\Box(\neg((a \wedge b) \supset \mathsf{f}) \vee \top)] \\
&= [\Box((\neg(a \supset \mathsf{f}) \wedge \neg(b \supset \mathsf{f})) \vee \top)] && \text{by (3)} \\
&= [\Box((\neg(a \supset \mathsf{f}) \vee \top) \wedge (\neg(b \supset \mathsf{f}) \vee \top))] && \text{by distributivity} \\
&= [\Box(\neg(a \supset \mathsf{f}) \vee \top) \wedge \Box(\neg(b \supset \mathsf{f}) \vee \top)] && \Box \text{ preserves meets} \\
&= [\Box(\neg(a \supset \mathsf{f}) \vee \top)] \wedge [\Box(\neg(b \supset \mathsf{f}) \vee \top)] \\
&= \Box_-[a] \wedge \Box_-[b].
\end{aligned}$$

We have thus shown that we can obtain a bimodal Boolean algebra $\mathbf{B}/\!\approx$ as a quotient of our modal bilattice $\mathbf{B}$. It remains to prove that $\mathbf{B}$ is isomorphic to the full twist-structure $(\mathbf{B}/\!\approx)^{\bowtie}$. The isomorphism is defined by the same map $j_{\mathbf{B}}$ as employed in the non-modal case (cf. [8, Theorem 4.13]): for all $a \in B$,

$$j_{\mathbf{B}}(a) := \langle [a], [\neg a] \rangle.$$

Building on the representation result for non-modal bilattices, we only need to check that $j_{\mathbf{B}}(\Box a) = \Box(j_{\mathbf{B}}(a))$. We will use the fact that the following equation holds in any bounded implicative bilattice:

$$x = ((x \supset \mathsf{f}) \supset \bot) \wedge (\neg((\neg x \supset \mathsf{f}) \supset \mathsf{f}) \vee \top). \tag{4}$$

This can be directly checked in $\mathcal{FOUR}$ or using the twist-structure representation of implicative bilattices.

We then have

$$\begin{aligned}
j_{\mathbf{B}}(\Box a) &= \langle [\Box a], [\neg\Box a]\rangle \\
&= \langle [\Box a], [\neg\Box\neg\neg a]\rangle && x = \neg\neg x \\
&= \langle [\Box(((a \supset \mathsf{f}) \supset \bot) \wedge (\neg((\neg a \supset \mathsf{f}) \supset \mathsf{f}) \vee \top))], [\Diamond\neg a]\rangle && \text{by (4)} \\
&= \langle [\Box((a \supset \mathsf{f}) \supset \bot) \wedge \Box(\neg((\neg a \supset \mathsf{f}) \supset \mathsf{f}) \vee \top)], [\Diamond\neg a]\rangle && \Box(x \wedge y) = \Box x \wedge \Box y \\
&= \langle [\Box((a \supset \mathsf{f}) \supset \bot)] \wedge [\Box(\neg((\neg a \supset \mathsf{f}) \supset \mathsf{f}) \vee \top)], [\Diamond\neg a]\rangle \\
&= \langle [\Diamond(a \supset \mathsf{f}) \supset \bot] \wedge [\Box(\neg((\neg a \supset \mathsf{f}) \supset \mathsf{f}) \vee \top)], [\Diamond\neg a]\rangle && \text{by Def. 5.7 (iii)} \\
&= \langle [\Diamond(a \supset \mathsf{f})] \supset [\bot] \wedge [\Box(\neg((\neg a \supset \mathsf{f}) \supset \mathsf{f}) \vee \top)], [\Diamond\neg a]\rangle \\
&= \langle [\Diamond(a \supset \mathsf{f})] \supset [\mathsf{f}] \wedge [\Box(\neg((\neg a \supset \mathsf{f}) \supset \mathsf{f}) \vee \top)], [\Diamond\neg a]\rangle \\
&= \langle [\Diamond(a \supset \mathsf{f}) \supset \mathsf{f}] \wedge \Box_-[\neg a \supset \mathsf{f}], [\Diamond\neg a]\rangle \\
&= \langle \Box_+[a] \wedge \Box_- \sim[\neg a], \Diamond_+[\neg a]\rangle \\
&= \Box\langle [a], [\neg a]\rangle \\
&= \Box(j_{\mathbf{B}}(a)).
\end{aligned}$$

Theorem 5.18. *Any modal bilattice* $\mathbf{B}$ *is isomorphic to the modal twist-structure* $(\mathbf{B}/{\equiv})^{\bowtie}$ *through the map* $j_{\mathbf{B}}$ *defined by* $j_{\mathbf{B}}(a) := \langle [a], [\neg a]\rangle$ *for all* $a \in B$.

Thanks to Theorem 5.18, from now on, when it is convenient to do so, we will be able to view a modal bilattice as a twist-structure. The correspondence between modal bilattices and twist-structures can be extended to an equivalence between two naturally associated categories, as was done for non-modal bilattices in [27, 8]. For what follows, it will be useful to recall a property of twist-structures that does not depend on the presence of modal operators [9, Proposition 3.18]:

Proposition 5.19. *Assume* $\mathbf{B} = \mathbf{A}^{\bowtie}$ *is a (modal) bilattice, viewed as a twist-structure over a (bimodal) Boolean algebra* $\mathbf{A}$, *and* $F \subseteq B$ *is a bifilter. Then* $F = \nabla \times A$, *where* ∇ *is a lattice filter of* $\mathbf{A}$.

A consequence of the twist-structure representation which is particularly important from a logical point of view is that it makes it possible to translate formulas from the language of modal bilattice logic into that of classical bimodal logic (studied, e.g., in [26]). This is quite straightforward. Let us consider the language of modal bilattice logic. Drawing inspiration from [22], we define a translation ν that maps the formulas of this language to pairs of formulas in the language of classical bimodal logic $\langle \sqcap, \sqcup, \sim, 0, 1, \Box_+, \Box_- \rangle$ as follows. First we assign to every propositional variable p a pair of different propositional variables $\langle p_1, p_2\rangle$ in such a way that if p is different from q, then p_1 is different from q_1 nad p_2 is different from q_2. Then we define ν inductively by:

$$\begin{aligned}
&\nu_1(p) := p_1 \quad\text{and}\quad \nu_2(p) := p_2 \\
&\nu_1(\mathsf{f}) := \nu_1(\bot) := \nu_2(\bot) := \nu_2(\mathsf{t}) := 0 \quad\text{and}\quad \nu_2(\mathsf{f}) := \nu_2(\top) := \nu_1(\top) := \nu_1(\mathsf{t}) := 1 \\
&\nu_1(\neg\varphi) := \nu_2(\varphi) \quad\text{and}\quad \nu_2(\neg\varphi) := \nu_1(\varphi) \\
&\nu_1(\Box\varphi) := \Box_+\nu_1(\varphi) \sqcap \Box_- \sim \nu_2(\varphi) \quad\text{and}\quad \nu_2(\Box\varphi) := \sim\Box_+ \sim \nu_2(\varphi) \\
&\nu_1(\varphi \wedge \psi) := \nu_1(\varphi \otimes \psi) := \nu_1(\varphi) \sqcap \nu_1(\psi) \quad\text{and}\quad \nu_2(\varphi \wedge \psi) := \nu_2(\varphi \oplus \psi) := \nu_2(\varphi) \sqcup \nu_2(\psi) \\
&\nu_1(\varphi \vee \psi) := \nu_1(\varphi \oplus \psi) := \nu_1(\varphi) \sqcup \nu_1(\psi) \quad\text{and}\quad \nu_2(\varphi \vee \psi) := \nu_2(\varphi \otimes \psi) := \nu_2(\varphi) \sqcap \nu_2(\psi) \\
&\nu_1(\varphi \supset \psi) := \sim\nu_1(\varphi) \sqcup \nu_1(\psi) \quad\text{and}\quad \nu_2(\varphi \supset \psi) := \nu_1(\varphi) \sqcap \nu_2(\psi).
\end{aligned}$$

Using the twist-structure representation we immediately obtain the following result (the proof is essentially the same as that of [22, Proposition 4.1]).

Proposition 5.20. *A modal bilattice* $\mathbf{B} = \mathbf{A}^{\bowtie}$ *satisfies an equation* $\varphi = \psi$ *if and only if the bimodal Boolean algebra* $\mathbf{A}$ *satisfies the equations* $\nu_1(\varphi) = \nu_1(\psi)$ *and* $\nu_2(\varphi) = \nu_2(\psi)$.

We will see a concrete application of the translation to particular equations in Subsection 6.3, when we look at extensions of the least modal bilattice logic. For now we would like to point out the following remarkable consequence. Let us denote the global and local consequence relations of classical bimodal logic by $\vdash_{cbg}$ and $\vdash_{cbl}$. We then have the following:

Proposition 5.21. *Let* $\Gamma \cup \{\varphi\} \subseteq Fm$ *be formulas of modal bilattice logic. Then*

(i) $\Gamma \vdash_g \varphi$ *if and only if* $\nu_1(\Gamma) \vdash_{cbg} \nu_1(\varphi)$,
(ii) $\Gamma \vdash_l \varphi$ *if and only if* $\nu_1(\Gamma) \vdash_{cbl} \nu_1(\varphi)$.

Proof. (i) Algebraizability of $\vdash_g$ (Theorem 5.6) implies that $\Gamma \vdash_g \varphi$ if and only if $\tau(\Gamma) \models \tau(\varphi)$ holds in the equational consequence of the class of modal bilattices, that is, $\{\gamma = \gamma \supset \gamma : \gamma \in \Gamma\} \models \varphi = \varphi \supset \varphi$. In any implicative bilattice, this is equivalent to $\{\gamma \wedge \top = \top : \gamma \in \Gamma\} \models \varphi \wedge \top = \top$. By Proposition 5.20, we then have that $\{\nu_1(\gamma \wedge \top) = \nu_1(\top) : \gamma \in \Gamma\} \models \nu_1(\varphi \wedge \top) = \nu_1(\top)$ and $\{\nu_2(\gamma \wedge \top) = \nu_2(\top) : \gamma \in \Gamma\} \models \nu_2(\varphi \wedge \top) = \nu_2(\top)$ hold in the class of bimodal Boolean algebras. The latter condition is vacuous, because $\nu_2(\psi \wedge \top) = \nu_2(\psi) \sqcup 1 = 1 = \nu_2(\top)$ for all $\psi \in Fm$. As to the former, since $\nu_1(\psi \wedge \top) = \nu_1(\psi) \sqcap 1 = \nu_1(\psi)$ for all $\psi \in Fm$, we can rewrite it as $\{\nu_1(\gamma) = 1 : \gamma \in \Gamma\} \models \nu_1(\varphi) = 1$. By algebraizability of the global consequence of classical bimodal logic [26, Corollary 4.2.12], this is exactly equivalent to $\nu_1(\Gamma) \vdash_{cbg} \nu_1(\varphi)$.
(ii) Assume $\Gamma \vdash_l \varphi$. By Corollary 5.13, this means that $\Gamma_0 \vdash_l \varphi$ for a finite $\Gamma_0 \subseteq \Gamma$ and that the equation $\bigwedge \Gamma_0 \wedge \top \leq_t \varphi$ holds in the variety of modal bilattices. This is a shorthand for $(\bigwedge \Gamma_0 \wedge \top) \vee \varphi = \varphi$. We apply ν to both sides and we obtain $\nu_1(\bigwedge \Gamma_0) \sqcup \nu_1(\varphi) = \nu_1(\varphi)$ and $(\nu_2(\bigwedge \Gamma_0) \sqcup 1) \sqcap \nu_2(\varphi) = \nu_2(\varphi)$. By Proposition 5.20, these equations are valid in any bimodal Boolean algebra (although the latter is obviously vacuous). By algebraic completeness of classical bimodal logic, this means that $\nu_1(\bigwedge \Gamma_0) \vdash_{cbl} \nu_1(\varphi)$, which clearly implies $\nu_1(\Gamma) \vdash_{cbl} \nu_1(\varphi)$. Conversely, assuming $\nu_1(\Gamma) \vdash_{cbl} \nu_1(\varphi)$, we invoke finiteness of classical bimodal logic to find a finite subset $\nu_1(\Gamma_0) = \{\nu_1(\gamma) : \gamma \in \Gamma_0 \subseteq \Gamma\} \subseteq \nu_1(\Gamma)$ such that $\nu_1(\Gamma_0) \vdash_{cbl} \nu_1(\varphi)$. By algebraic completeness of classical bimodal logic, this means that the equation $\bigsqcap \nu_1(\Gamma_0) \sqcup \nu_1(\varphi) = \nu_1(\varphi)$ is valid in the variety of bimodal Boolean algebras, where

$$\bigsqcap \nu_1(\Gamma_0) := \nu_1(\gamma_1) \sqcap \ldots \sqcap \nu_1(\gamma_n) = \nu_1(\gamma_1 \wedge \ldots \wedge \gamma_n) = \nu_1(\bigwedge \Gamma_0).$$

We can thus rewrite the left-hand side of the previous equation as follows:

$$\begin{aligned} \bigsqcap \nu_1(\Gamma_0) \sqcup \nu_1(\varphi) &= \nu_1(\bigwedge \Gamma_0) \sqcup \nu_1(\varphi) \\ &= (\nu_1(\bigwedge \Gamma_0) \sqcap 1) \sqcup \nu_1(\varphi) \\ &= (\nu_1(\bigwedge \Gamma_0) \sqcap \nu_1(\top)) \sqcup \nu_1(\varphi) \\ &= \nu_1((\bigwedge \Gamma_0 \wedge \top) \vee \varphi). \end{aligned}$$

Thus, we have $\nu_1((\bigwedge \Gamma_0 \wedge \top) \vee \varphi) = \nu_1(\varphi)$. As we have observed above, $\nu_2((\bigwedge \Gamma_0 \wedge \top) \vee \varphi) = \nu_2(\varphi)$ is trivially true, so we can apply Proposition 5.20 to conclude that $(\bigwedge \Gamma_0 \wedge \top) \vee \varphi = \varphi$ holds in any modal bilattice. Hence, by Corollary 5.13, we have $\bigwedge \Gamma_0 \vdash_l \varphi$, which implies $\Gamma \vdash_l \varphi$. □

Classical bimodal logic has both the local and the global *finite model property* [26, Theorems 2.7.9 and 3.1.10]. This means that if $\Gamma \not\vdash_{cbg} \varphi$ (or $\Gamma \not\vdash_{cbl} \varphi$), then this is witnessed by a

Kripke model whose underlying set of points is finite. This property, together with the fact that both logics are finitely axiomatizable, implies that they are decidable. Proposition 5.21 allows us to transfer these results to our logics:

Theorem 5.22. *Both calculi $\vdash_l$ and $\vdash_g$ of modal bilattice logic have the finite model property, and are therefore decidable.*

Proof. We only consider local consequence, as the proof for the global one is completely analogous. Assume $\Gamma \nvdash_l \chi$. By Proposition 5.21, we then have $\nu_1(\Gamma) \nvdash_{cbl} \nu_1(\chi)$. Since classical bimodal logic enjoys the finite model property, there is a finite Kripke model $M = \langle W, R_+, R_-, v\rangle$ witnessing this, where $R_+, R_- \subseteq W \times W$ are accessibility relations corresponding to the two operators $\Box_+, \Box_-$ and $v\colon Fm_{cb} \to P(W)$ is a valuation from the formula algebra $\mathbf{Fm}_{cb}$ over the language of classical bimodal logic $\langle \sqcap, \sqcup, \sim, 0, 1, \Box_+, \Box_-\rangle$. There is thus a point $w_0 \in W$ such that $M, w_0 \models \nu_1(\Gamma)$ but $M, w_0 \not\models \nu_1(\chi)$, that is, $w_0 \in v(\nu_1(\gamma))$ for each $\gamma \in \Gamma$ but $w_0 \notin v(\nu_1(\chi))$. This is enough to establish the claim that $\vdash_l$ (and similarly $\vdash_g$) is decidable, but notice that M is not a four-valued Kripke model of the kind introduced in Section 3. Thus, in order to conclude the proof, we need to turn M into a four-valued model. It is clear that we can combine the relations R_+ and R_- into a single four-valued relation $R_4\colon W \times W \to \mathcal{FOUR}$ by defining, for all $w, w' \in W$,

$$R_4(w,w') := \begin{cases} \mathsf{t} & \text{iff } \langle w,w'\rangle \in R_+ \text{ and } \langle w,w'\rangle \in R_- \\ \top & \text{iff } \langle w,w'\rangle \in R_+ \text{ and } \langle w,w'\rangle \notin R_- \\ \bot & \text{iff } \langle w,w'\rangle \notin R_+ \text{ and } \langle w,w'\rangle \in R_- \\ \mathsf{f} & \text{iff } \langle w,w'\rangle \notin R_+ \text{ and } \langle w,w'\rangle \notin R_-. \end{cases}$$

This gives us a four-valued Kripke frame $\langle W, R_4\rangle$. In order to define a four-valued valuation, we let, for each formula φ in the language of modal bilattice logic $\langle \wedge, \vee, \otimes, \oplus, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top, \Box\rangle$,

$$v_4(\varphi,w) = \begin{cases} \mathsf{t} & \text{iff } \ w \in v(\nu_1(\varphi)) \text{ and } \ w \notin v(\nu_2(\varphi)) \\ \top & \text{iff } \ w \in v(\nu_1(\varphi)) \text{ and } \ w \in v(\nu_2(\varphi)) \\ \bot & \text{iff } \ w \notin v(\nu_1(\varphi)) \text{ and } \ w \notin v(\nu_2(\varphi)) \\ \mathsf{f} & \text{iff } \ w \notin v(\nu_1(\varphi)) \text{ and } \ w \in v(\nu_2(\varphi)). \end{cases}$$

Checking that v_4 acts homomorphically on non-modal formulas is straightforward: we have, for instance,

$$\begin{aligned} v_4(\varphi \wedge \psi, w) = \mathsf{t} \ &\Leftrightarrow \ w \in v(\nu_1(\varphi \wedge \psi)) \ \text{ and } \ w \notin v(\nu_2(\varphi \wedge \psi)) \\ &\Leftrightarrow \ w \in v(\nu_1(\varphi) \sqcap \nu_1(\psi)) \ \text{ and } \ w \notin v(\nu_2(\varphi) \sqcup \nu_2(\psi)) \\ &\Leftrightarrow \ w \in v(\nu_1(\varphi)) \cap v(\nu_1(\psi)) \ \text{ and } \ w \notin v(\nu_2(\varphi)) \cup v(\nu_2(\psi)) \\ &\Leftrightarrow \ w \in v(\nu_1(\varphi)),\, w \in v(\nu_1(\psi)) \ \text{ and } \ w \notin v(\nu_2(\psi)),\, w \notin v(\nu_2(\psi)) \\ &\Leftrightarrow \ v_4(\varphi, w) = \mathsf{t} \ \text{ and } \ v_4(\psi, w) = \mathsf{t}. \end{aligned}$$

Concerning modal formulas, we need to check that

$$v_4(\Box\varphi, w) = \bigwedge\{R_4(w,w') \to v_4(\varphi,w') : w' \in W\}.$$

This amounts to showing that

(i) $\bigwedge\{R_4(w,w') \to v_4(\varphi,w') : w' \in W\} \in \{\mathsf{t}, \top\}$ iff
$w \in v(\nu_1(\Box\varphi)) = v(\Box_+\nu_1(\varphi) \sqcap \Box_- \sim \nu_2(\varphi)) = v(\Box_+\nu_1(\varphi)) \cap v(\Box_- \sim \nu_2(\varphi))$

(ii) $\bigwedge\{R_4(w,w') \to v_4(\varphi,w') : w' \in W\} \in \{\mathsf{f}, \top\}$ iff
$w \in v(\nu_2(\Box\varphi)) = v(\sim \Box_+ \sim \nu_2(\varphi)) = \sim v(\Box_+ \sim \nu_2(\varphi)) = (v(\Box_+ \sim \nu_2(\varphi)))^c$.

(i) Recall that $v(\Box_+\delta) := \{w \in W : R_+[w] \subseteq v(\delta)\}$ for all $\delta \in Fm_{cb}$ and the definition $v(\Box_-\delta)$ is similar. Now, on the one hand, $\bigwedge\{R_4(w,w') \to v_4(\varphi,w') : w' \in W\} \in \{\mathsf{t},\top\}$ means that $R_4(w,w') \to v_4(\varphi,w') \in \{\mathsf{t},\top\}$ for all $w' \in W$. By residuation, this is equivalent to $R_4(w,w') \leq_t v_4(\varphi,w')$. In $\mathcal{FOUR}$, this means that $R_4(w,w') \in \{\mathsf{t},\top\}$ implies $v_4(\varphi,w') \in \{\mathsf{t},\top\}$ and that $R_4(w,w') \in \{\mathsf{t},\bot\}$ implies $v_4(\varphi,w') \in \{\mathsf{t},\bot\}$. According to our definitions, these conditions translate as follows: $\langle w,w'\rangle \in R_+$ implies $w' \in v(\nu_1(\varphi))$ and $\langle w,w'\rangle \in R_-$ implies $w' \notin v(\nu_2(\varphi))$. On the other hand, $w \in v(\Box_+\nu_1(\varphi)) \cap v(\Box_- \sim\nu_2(\varphi))$ means that $R_+[w] \subseteq v(\nu_1(\varphi))$ and $R_-[w] \subseteq (v(\sim\nu_2(\varphi))) = (v(\nu_2(\varphi)))^c$. The former says that $\langle w,w'\rangle \in R_+$ implies $w' \in v(\nu_1(\varphi))$, while the latter says that $\langle w,w'\rangle \in R_-$ implies $w' \notin v(\nu_2(\varphi))$. We see then that the two conditions are equivalent.
(b). Notice that $\bigwedge\{R_4(w,w') \to v_4(\varphi,w') : w' \in W\} \in \{\mathsf{f},\top\}$ implies, in $\mathcal{FOUR}$, that there is $w'' \in W$ such that $R_4(w,w'') \to v_4(\varphi,w'') \in \{\mathsf{f},\top\}$. The table of strong implication in $\mathcal{FOUR}$ (Table 1) shows that, if $a \to b \in \{\mathsf{f},\top\}$, then $a \in \{\mathsf{t},\top\}$ and $b \in \{\mathsf{f},\top\}$. So, $R_4(w,w'') \in \{\mathsf{t},\top\}$ and $v_4(\varphi,w'') \in \{\mathsf{f},\top\}$. According to our definitions, these mean that $\langle w,w''\rangle \in R_+$ and $w'' \in v(\nu_2(\varphi))$. It is then sufficient to realize that

$$(v(\Box_+ \sim\nu_2(\varphi)))^c = \{w \in W : \exists w'' \in v(\nu_2(\varphi)) \text{ s.t. } \langle w,w''\rangle \in R_+\}$$

to see that the two conditions are equivalent.
We thus conclude that $M_4 = \langle W, R_4, v_4\rangle$ is indeed a four-valued Kripke model. To finish the proof, it is sufficient to check that $M_4, w_0 \models \Gamma$ but $M_4, w_0 \not\models \chi$, that is, $v_4(\gamma,w_0) \in \{\mathsf{t},\top\}$ for each $\gamma \in \Gamma$ but $v_4(\chi,w_0) \in \{\mathsf{t},\top\}$. According to our definitions, this amounts to proving that $w_0 \in v(\nu_1(\gamma))$ for all $\gamma \in \Gamma$ but $w_0 \notin v(\nu_1(\chi))$, which are exactly the assumptions we started from. $\Box$

The proof of Theorem 5.22 already suffices to establish completeness of our calculi with respect to the four-valued Kripke semantics introduced in Section 3, although only through completeness of classical bimodal logic. We are going to see in Subsection 6.2 that, thanks to duality, the same argument can be used to establish the result without relying on completeness of classical bimodal logic.

5.3. **Congruences and reduced models.** As we have anticipated, the twist-structure representation will allow us to characterize the reduced models of $\vdash_l$. In order to obtain this, we will use twist-structures to obtain further information on models of $\vdash_g$ also and on congruences of modal bilattices.

Firstly we are going to prove an analogue of Proposition 5.19 concerning those bifilters which are logical filters of $\vdash_g$, that is, those $F \subseteq B$ such that $\langle \mathbf{B}, F\rangle$ is a model of the global calculus. In order to do this, we introduce the following definition. Given a bimodal Boolean algebra $\langle \mathbf{A}, \Box_+, \Box_-\rangle$, we define an *open filter* (cf., e.g., [26, Definition 3.1.4]) to be a lattice filter $\nabla \subseteq A$ that satisfies: if $a \in \nabla$, then $\Box_+ a,\ \Box_- a \in \nabla$.

Proposition 5.23. *Let $\mathbf{B} = \mathbf{A}^{\bowtie}$ be a modal bilattice, viewed as a twist-structure over a bimodal Boolean algebra $\mathbf{A}$. Then $F \subseteq B$ is a logical filter of $\vdash_g$ if and only if $F = \nabla \times A$ and ∇ is an open filter of $\mathbf{A}$.*

Proof. Assume $F = \nabla \times A$, with ∇ an open filter of $\mathbf{A}$. We already know that F is a bifilter, so, by Proposition 5.14, we only need to check that it is closed under the rule (mono). Assume then $\langle a_1,a_2\rangle \to \langle b_1,b_2\rangle \in F$ for some $\langle a_1,a_2\rangle, \langle b_1,b_2\rangle \in B$, which means $a_1 \to b_1,\ b_2 \to a_2 \in \nabla$. We need to show that $\Box\langle a_1,a_2\rangle \to \Box\langle b_1,b_2\rangle \in F$, which amounts to showing that the first component of $\Box\langle a_1,a_2\rangle \to \Box\langle b_1,b_2\rangle$ belongs to ∇, i.e., $(\Box_+ a_1 \wedge \Box_- \sim a_2) \to (\Box_+ b_1 \wedge \Box_- \sim b_2),\ \Diamond_+ b_2 \to \Diamond_+ a_2 \in \nabla$. The assumption $b_2 \to a_2 \in \nabla$ allows us to conclude $\Box_+(b_2 \to a_2)$ because ∇ is open. Moreover, $\Box_+(b_2 \to a_2) \leq \Diamond_+ b_2 \to \Diamond_+ a_2$

holds in any (bi)modal Boolean algebra. So we have $\Diamond_+ b_2 \to \Diamond_+ a_2 \in \nabla$. Since $(\Box_+ a_1 \wedge \Box_- \sim a_2) \to (\Box_+ b_1 \wedge \Box_- \sim b_2) = (\Box_- \sim a_2 \to (\Box_+ a_1 \to \Box_+ b_1)) \wedge (\Box_+ a_1 \to (\Box_- \sim a_2 \to \Box_- \sim b_2))$, it remains to show that $\Box_- \sim a_2 \to (\Box_+ a_1 \to \Box_+ b_1)$, $\Box_+ a_1 \to (\Box_- \sim a_2 \to \Box_- \sim b_2) \in \nabla$. For this, it is sufficient to note that we have, on the one hand, $\Box_+(a_1 \to b_1) \in \nabla$ by assumption and $\Box_+(a_1 \to b_1) \leq \Box_+ a_1 \to \Box_+ b_1 \leq \Box_- \sim a_2 \to (\Box_+ a_1 \to \Box_+ b_1)$. Similarly, on the other hand, we have $\Box_-(b_2 \to a_2) = \Box_-(\sim a_2 \to \sim b_2) \leq \Box_- \sim a_2 \to \Box_- \sim b_2 \leq \Box_+ a_1 \to (\Box_- \sim a_2 \to \Box_- \sim b_2)$, so the result follows from the assumption that $b_2 \to a_2 \in \nabla$, which implies $\Box_-(b_2 \to a_2) \in \nabla$. Hence we conclude that F is a $\vdash_g$ filter. Conversely, assume F is a $\vdash_g$ filter, i.e., by Proposition 5.14, a non-empty bifilter that is closed under rule (mono). Then we know that $F = \nabla \times A$, with ∇ a lattice filter. We need to prove that ∇ is open. Assume $a \in \nabla$. This means that $\langle a, 1\rangle \in F$. Then we also have $\langle 1, 0\rangle \to (\neg(\langle a, 1\rangle \supset \langle 0, 1\rangle) \vee \langle 1, 1\rangle) \in F$. To see this, we only need to compute the first component of this expression, which is $(1 \to 1) \wedge ((a \to 0) \to 0) = a \in \nabla$. We can then apply (mono) to conclude $\Box\langle 1, 0\rangle \to \Box(\neg(\langle a, 1\rangle \supset \langle 0, 1\rangle) \vee \langle 1, 1\rangle) = \langle 1, 0\rangle \to \Box(\neg(\langle a, 1\rangle \supset \langle 0, 1\rangle) \vee \langle 1, 1\rangle) \in F$. We compute the first component of this expression, which is $(\Box_+ 1 \wedge \Box_- \sim\sim a) \wedge (\Diamond_+ \sim a \to 0) = \Box_- a \wedge \Box_+ a$. We have thus $\Box_- a \wedge \Box_+ a \in \nabla$, which shows that ∇ is open. □

In order to obtain more information on reduced models, we need to look at congruences of modal bilattices, in particular at those congruences that are compatible with a given logical filter. To this end, we will extend to the modal setting a known result of implicative bilattices, namely that the congruences of a twist-structure $\mathbf{A}^{\bowtie}$ are isomorphic, as a lattice, to the congruences of $\mathbf{A}$ [8, Theorem 4.13].

The existence of an isomorphism between congruences of a modal bilattice $\mathbf{B}$ (viewed as a twist-structure $\mathbf{A}^{\bowtie}$) and the underlying bimodal Boolean algebra $\mathbf{A}$ follows from the general theory of algebraizable logics. In fact, algebraizability of $\vdash_g$ with respect to the variety of modal bilattices implies that the congruences of any modal bilattice $\mathbf{B}$ are isomorphic to the lattice of logical filters of $\vdash_g$ on $\mathbf{B}$. Now, by Proposition 5.23, we have that $\mathrm{Con}(\mathbf{B})$ is isomorphic to the lattice of open filters of $\mathbf{A}$. Classical modal logic, even when two independent necessity operators are present in the language, is algebraizable (see [26]), and the logical filters of this logic are precisely the open filters. Thus we have that the lattice of open filters of $\mathbf{A}$ is isomorphic to $\mathrm{Con}(\mathbf{A})$. Putting these results together, we obtain $\mathrm{Con}(\mathbf{B}) \cong \mathrm{Con}(\mathbf{A})$.

In order to see how this isomorphism can be concretely established, consider a modal bilattice $\mathbf{B} = \mathbf{A}^{\bowtie}$ and define a map $H\colon \mathrm{Con}(\mathbf{A}^{\bowtie}) \to \mathrm{Con}(\mathbf{A})$, for all $\theta \in \mathrm{Con}(\mathbf{B})$, as follows:

$$H(\theta) := \{\langle x, y\rangle \in A^2 : \langle\langle x, 1\rangle, \langle y, 1\rangle\rangle \in \theta\}. \tag{5}$$

Let us check that this definition is sound.

Lemma 5.24. *Let $\theta \in \mathrm{Con}(\mathbf{A}^{\bowtie})$ and $\langle\langle x, x'\rangle, \langle y, y'\rangle\rangle \in \theta$. Then $\langle\langle x, z\rangle, \langle y, z\rangle\rangle \in \theta$ for any $z \in A$.*

Proof. From the assumption we obtain $\langle\langle x, x'\rangle \wedge \langle 1, 1\rangle, \langle y, y'\rangle \wedge \langle 1, 1\rangle\rangle = \langle\langle x, 1\rangle, \langle y, 1\rangle\rangle \in \theta$. Then, $\langle\langle x, 1\rangle \vee \langle 0, z\rangle, \langle y, 1\rangle \vee \langle 0, z\rangle\rangle = \langle\langle x, z\rangle, \langle y, z\rangle\rangle \in \theta$. □

Proposition 5.25. *For all $\theta \in \mathrm{Con}(\mathbf{A}^{\bowtie})$, $H(\theta)$ is a congruence of $\mathbf{A}$.*

Proof. Clearly $H(\theta)$ is an equivalence relation, and it is easy to check that it is compatible with the algebraic operations of $\mathbf{A}$. One needs, for instance, to show that $\langle x, y\rangle, \langle x', y'\rangle \in$

$H(\theta)$, that is, $\langle\langle x,1\rangle,\langle y,1\rangle\rangle,\langle\langle x',1\rangle,\langle y',1\rangle\rangle\in\theta$, implies $\langle x\sqcap x', y\sqcap y'\rangle\in H(\theta)$. By definition, the latter means that $\langle\langle x\sqcap x',1\rangle,\langle y\sqcap y',1\rangle\rangle\in\theta$, which easily follows from the assumptions. Concerning the modal operators, we need to check that $\langle\langle x,1\rangle,\langle y,1\rangle\rangle\in\theta$ implies $\langle\langle\Box_+ x,1\rangle,\langle\Box_+ y,1\rangle\rangle,\langle\langle\Box_- x,1\rangle,\langle\Box_- y,1\rangle\rangle\in\theta$. From the assumption, using Lemma 5.24, we obtain $\langle\langle x,0\rangle,\langle y,0\rangle\rangle\in\theta$. From this we have

$$\langle\Box\langle x,0\rangle,\Box\langle y,0\rangle\rangle=\langle\langle\Box_+x\sqcap\Box_-\sim 0,\Diamond 0\rangle,\langle\Box_+y\sqcap\Box_-\sim 0,\Diamond 0\rangle\rangle=\langle\langle\Box_+x,0\rangle,\langle\Box_+y,0\rangle\rangle\in\theta$$

and again by the Lemma we obtain $\langle\langle\Box_+x,1\rangle,\langle\Box_+y,1\rangle\rangle\in\theta$ as desired. The case of $\Box_-$ is analogous. $\Box$

The inverse map $H^{-1}\colon \mathrm{Con}(\mathbf{A})\to\mathrm{Con}(\mathbf{A}^{\bowtie})$ can be defined, for $\eta\in\mathrm{Con}(\mathbf{A})$, by

$$H^{-1}(\eta):=\{\langle\langle x_1,x_2\rangle,\langle y_1,y_2\rangle\rangle\in(A\times A)^2:\langle x_1,y_1\rangle,\langle x_2,y_2\rangle\in\eta\}. \tag{6}$$

Proposition 5.26. *For all $\eta\in\mathrm{Con}(\mathbf{A})$, $H^{-1}(\eta)$ is a congruence of $\mathbf{A}^{\bowtie}$.*

Proof. As before, it is clear that $H^{-1}(\eta)$ is an equivalence relation and compatibility with the algebraic operations of $\mathbf{A}^{\bowtie}$ is also easy to prove. Let us check the case of the modal operator. Assume $\langle\langle x_1,x_2\rangle,\langle y_1,y_2\rangle\rangle\in H^{-1}(\eta)$, i.e., $\langle x_1,y_1\rangle,\langle x_2,y_2\rangle\in\eta$. We need to prove that $\langle\Box\langle x_1,x_2\rangle,\Box\langle y_1,y_2\rangle\rangle\in H^{-1}(\eta)$, i.e., that $\langle\Box_+x_1\sqcap\Box_-\sim x_2,\Box_+y_1\sqcap\Box_-\sim y_2\rangle,\langle\Diamond_+x_2,\Diamond_+x_2y_2\rangle\in\eta$, and these follow easily from the fact that η is a congruence of $\mathbf{A}$. $\Box$

Theorem 5.27. *H and H^{-1} are mutually inverse order isomorphisms between the lattices $\langle\mathrm{Con}(\mathbf{A}^{\bowtie}),\subseteq\rangle$ and $\langle\mathrm{Con}(\mathbf{A}),\subseteq\rangle$.*

Proof. Let us check that $\theta=H^{-1}(H(\theta))$ for all $\theta\in\mathrm{Con}(\mathbf{A}^{\bowtie})$. Using (5) and (6), we have $\langle\langle x_1,x_2\rangle,\langle y_1,y_2\rangle\rangle\in H^{-1}(H(\theta))$ if and only if $\langle x_1,y_1\rangle,\langle x_2,y_2\rangle\in H(\theta)$ if and only if $\langle\langle x_1,1\rangle,\langle y_1,1\rangle\rangle,\langle\langle x_2,1\rangle,\langle y_2,1\rangle\rangle\in\theta$. Assume the latter condition holds. Note that $\langle x_1,1\rangle=\langle x_1,x_2\rangle\wedge\langle 1,1\rangle$, $\langle y_1,1\rangle=\langle y_1,y_2\rangle\wedge\langle 1,1\rangle$, $\langle x_2,1\rangle=\neg\langle x_1,x_2\rangle\wedge\langle 1,1\rangle$ and $\langle y_2,1\rangle=\neg\langle y_1,y_2\rangle\wedge\langle 1,1\rangle$. Then in the quotient $\mathbf{A}^{\bowtie}/\theta$, which is a modal bilattice, we have $[\langle x_1,x_2\rangle]_\theta\wedge[\langle 1,1\rangle]_\theta=[\langle x_1,x_2\rangle]_\theta\wedge\langle 1,1\rangle=[\langle y_1,y_2\rangle]_\theta\wedge\langle 1,1\rangle$ and, similarly, $\neg[\langle x_1,x_2\rangle]_\theta\wedge\langle 1,1\rangle=\neg[\langle y_1,y_2\rangle]_\theta\wedge\langle 1,1\rangle$. It is easy to show that, in any twist-structure, this implies $[\langle x_1,x_2\rangle]_\theta=[\langle y_1,y_2\rangle]_\theta$. Hence, $\langle\langle x_1,x_2\rangle,\langle y_1,y_2\rangle\rangle\in\theta$, which shows that $H^{-1}(H(\theta))\subseteq\theta$. The converse inclusion is easy, for $\langle\langle x_1,x_2\rangle,\langle y_1,y_2\rangle\rangle\in\theta$ implies $\langle\langle x_1,x_2\rangle\wedge\langle 1,1\rangle,\langle y_1,y_2\rangle\wedge\langle 1,1\rangle\rangle=\langle\langle x_1,1\rangle,\langle y_1,1\rangle\rangle\in\theta$ and $\langle\neg\langle x_1,x_2\rangle\wedge\langle 1,1\rangle,\neg\langle y_1,y_2\rangle\wedge\langle 1,1\rangle\rangle=\langle\langle x_2,1\rangle,\langle y_2,1\rangle\rangle\in\theta$. Thus, $\theta=H^{-1}(H(\theta))$.
It is obvious that $\eta=H(H^{-1}(\eta))$ for all $\eta\in\mathrm{Con}(\mathbf{A})$. In fact, applying the definitions, we have $\langle x,y\rangle\in H(H^{-1}(\eta))$ if and only if $\langle\langle x,1\rangle,\langle y,1\rangle\rangle\in H^{-1}(\eta)$ if and only if $\langle x,y\rangle,\langle 1,1\rangle\in\eta$. It is also clear that $\theta\subseteq\theta'$ implies $H(\theta)\subseteq H(\theta')$. Conversely, if $H(\theta)\subseteq H(\theta')$, then $H^{-1}(H(\theta))=\theta\subseteq H^{-1}(H(\theta'))=\theta'$. So $H\colon\mathrm{Con}(\mathbf{A}^{\bowtie})\cong\mathrm{Con}(\mathbf{A})$ is actually an order isomorphism. $\Box$

We are now going to use the insight given by the previous theorem to look at congruences of a modal bilattice that are *compatible* with logical filters.

Proposition 5.28. *Let $\langle\mathbf{B},F\rangle$ be a matrix such that $\mathbf{B}=\mathbf{A}^{\bowtie}$ is a modal bilattice and $F=\nabla\times A$ is a bifilter. Then a congruence $\theta\in Con(\mathbf{B})$ is compatible with F if and only if $H(\theta)\in Con(\mathbf{A})$ is compatible with ∇.*

Proof. Assume $\theta\in\mathrm{Con}(\mathbf{B})$ is compatible with F and let $x,y\in A$ be such that $\langle x,y\rangle\in H(\theta)$ and $x\in\nabla$. By (5), $\langle x,y\rangle\in H(\theta)$ iff $\langle\langle x,1\rangle,\langle y,1\rangle\rangle\in\theta$. Since $x\in\nabla$, we have $\langle x,1\rangle\in F$, which implies, by compatibility, $\langle y,1\rangle\in F$. This means that $y\in\nabla$, as required.
Conversely, assume $H(\theta)\in\mathrm{Con}(\mathbf{A})$ is compatible with ∇, $\langle x,y\rangle\in F$ and $\langle\langle x,y\rangle,\langle x',y'\rangle\rangle\in$

θ. From the last assumption, using Lemma 5.24, we obtain $\langle\langle x, 1\rangle, \langle x', 1\rangle\rangle \in \theta$. This means that $\langle x, x'\rangle \in H(\theta)$. Since $\langle x, y\rangle \in F$, we have $x \in \nabla$, hence $x' \in \nabla$. This implies $\langle x', y'\rangle \in F$. $\square$

Corollary 5.29. *Let $\langle \mathbf{B}, F\rangle$ be a matrix such that $\mathbf{B} = \mathbf{A}^{\bowtie}$ is a modal bilattice and $F = \nabla \times A$ is a bifilter. Then $\langle \mathbf{B}, F\rangle$ is reduced if and only if $\langle \mathbf{A}, \nabla\rangle$ is reduced.*

Corollary 5.30. *A matrix $\langle \mathbf{B}, F\rangle$ is a reduced model of $\vdash_l$ if and only if $\mathbf{B}$ is a modal bilattice and F is a bifilter such that $F_0 = \{a \in B : a = a \supset a\}$ is the only logical filter of $\vdash_g$ contained in F.*

Proof. Assume $\langle \mathbf{B}, F\rangle$ is a reduced model of $\vdash_l$. Then $\mathbf{B}$ is a modal bilattice (Proposition 5.11) and F is a bifilter (Proposition 5.10). Suppose G is a filter of $\vdash_g$ (hence, a fortiori, a filter of $\vdash_l$) such that $F_0 \subseteq G \subseteq F$. Since $\vdash_l$ is protoalgebraic, the Leibniz operator is monotone on filters of $\vdash_l$. This means that $\Omega(G) \subseteq \Omega(F) = Id_B$, therefore $\Omega(G) = \{\langle a, b\rangle \in B \times B : a \leftrightarrow b \in G\} = Id_B$. Assume $a \in G$. Since the equation $x \leftrightarrow (x \supset x) = x$ holds in any implicative bilattice, we have $a = a \leftrightarrow (a \supset a) \in G$. This means that $\langle a, a \supset a\rangle \in \Omega(G)$ and therefore $a = a \supset a$. Hence, $G = F_0$. Conversely, assume $\mathbf{B}$ is a modal bilattice and F is a bifilter such that F_0 is the only filter of $\vdash_g$ contained in F. By Proposition 5.10, $\langle \mathbf{B}, F\rangle$ is a model of $\vdash_l$. Let $G := \{a \in B : \langle a, a \supset a\rangle \in \Omega(F)\}$. We are going to prove that G is a filter of $\vdash_g$. It is clear that $b \in G$ for all $b \in F_0$, that is, $F_0 \subseteq G$. Moreover, G is closed under modus ponens, i.e., b, $b \supset c \in G$ imply $c \in G$. To see this, notice that in the quotient $\mathbf{B}/\Omega(F)$ we have $[b] = [b] \supset [b]$ and $[b] \supset [c] = ([b] \supset [c]) \supset ([b] \supset [c])$, which implies, in any implicative bilattice, $[c] = [c] \supset [c]$. Thus, G is a bifilter. Assume $b \to c \in G$, i.e., $\langle b \to c, (b \to c) \supset (b \to c)\rangle \in \Omega(F)$. Reasoning as before, in the quotient $\mathbf{B}/\Omega(F)$ we have $[b] \to [c] = ([b] \to [c]) \supset ([b] \to [c])$, which implies $[b] = [b] \wedge [c]$. Hence, $\langle b, b \wedge c\rangle \in \Omega(F)$. Since $\Omega(F)$ is a congruence of $\mathbf{B}$, we have $\langle \Box b, \Box(b \wedge c)\rangle \in \Omega(F)$, from which we can derive $\langle \Box b \to \Box c, \Box(b \wedge c) \to \Box c\rangle \in \Omega(F)$ and $\langle (\Box b \to \Box c) \supset (\Box b \to \Box c), (\Box(b \wedge c) \to \Box c) \supset (\Box b \to \Box c)\rangle \in \Omega(F)$. Since $(\Box(b \wedge c) \to \Box c) \supset (\Box b \to \Box c) = \Box b \to \Box c$, we obtain $\langle (\Box b \to \Box c) \supset (\Box b \to \Box c), \Box b \to \Box c\rangle \in \Omega(F)$. This means that $\Box b \to \Box c \in G$, so we conclude that G is a filter of $\vdash_g$. Then, by assumption, $G = F_0$. Now, if $\langle a, b\rangle \in \Omega(F)$, then $\langle (a \leftrightarrow b) \supset (a \leftrightarrow b), (b \leftrightarrow b) \supset (a \leftrightarrow b)\rangle = \langle (a \leftrightarrow b) \supset (a \leftrightarrow b), a \leftrightarrow b\rangle \in \Omega(F)$, which means that $a \leftrightarrow b \in G = F_0$. This implies $a = b$, so indeed $\Omega(F) = Id_B$ as required. $\square$

6. Duality and completeness

In this section we develop a topological duality for bimodal Boolean algebras, which will essentially turn out to be just an application of Jónsson-Tarski duality for modal algebras [5, Chapter 5]. Since, as mentioned earlier, bimodal Boolean algebras are equivalent (as a category) to modal bilattices, this will give us a duality for modal bilattices as well. More importantly, we will prove that the relational semantics obtained through duality is not only equivalent to the algebraic semantics for our calculi given in Section 5, but also to the four-valued Kripke semantics introduced in Section 3, and this will allow us to prove completeness of our modal calculi with respect to Kripke semantics.

6.1. **Duality.** As mentioned before, a bimodal Boolean algebra $\mathbf{A} = \langle A, \sqcap, \sqcup, \sim, 0, 1, \Box_+, \Box_-\rangle$ can be viewed as a pair of standard modal algebras $\langle A, \sqcap, \sqcup, \sim, 0, 1, \Box_+\rangle$ and $\langle A, \sqcap, \sqcup, \sim, 0, 1, \Box_-\rangle$. According to Jónsson-Tarski duality, to these algebras correspond modal spaces $\langle X(\mathbf{A}), \tau_{\mathbf{A}}, R_{\Box_+}\rangle$ and $\langle X(\mathbf{A}), \tau_{\mathbf{A}}, R_{\Box_-}\rangle$ constructed as follows:

- $X(\mathbf{A})$ is the set of ultrafilters of $\mathbf{A}$;

- $\tau_{\mathbf{A}}$ is the topology on $X(\mathbf{A})$ having as basis the family of sets $\Phi(a) := \{P \in X(\mathbf{A}) : a \in P\}$ for each $a \in A$;
- $R_{\Box_+} \subseteq X(\mathbf{A}) \times X(\mathbf{A})$ is defined, for all $P, Q \in X(\mathbf{A})$, as follows: $\langle P, Q \rangle \in R_+$ iff $\Box_+^{-1}[P] \subseteq Q$;
- $R_{\Box_-} \subseteq X(\mathbf{A}) \times X(\mathbf{A})$ is defined in the same way as R_+: $\langle P, Q \rangle \in R_-$ iff $\Box_-^{-1}[P] \subseteq Q$.

We remind the reader that a *modal space* is a structure $\langle X, \tau, R \rangle$, where $R \subseteq X \times X$, such that

- $\langle X, \tau \rangle$ is a Stone space, i.e., a compact Hausdorff space having a basis of clopen sets;
- $R[x]$ is a closed set for every $x \in X$;
- $R^{-1}[U]$ is clopen for every clopen set $U \subseteq X$.

The following definition seems thus to be the most natural one in our context.

Definition 6.1. A *bimodal space* is a structure $\mathcal{X} = \langle X, \tau, R_+, R_- \rangle$ such that both $\langle X, \tau, R_+ \rangle$ and $\langle X, \tau, R_- \rangle$ are modal spaces.

It is clear that to each bimodal space $\mathcal{X} = \langle X, \tau, R_+, R_- \rangle$ a bimodal Boolean algebra can be associated in the way prescribed by Jónsson-Tarski duality. We denote this algebra by

$$\mathbf{A}(\mathcal{X}) = \langle A(X), \cap, \cup, \sim, \Box_{R_+}, \Box_{R_-}, \emptyset, X \rangle$$

where $\langle A(\mathcal{X}), \cap, \cup, \sim, \emptyset, X \rangle$ is the Boolean algebra of clopens of the Stone space $\langle X, \tau \rangle$ and, for each $U \in A(\mathcal{X})$ and for $\bullet \in \{+, -\}$,

$$\Box_{R_\bullet} U := \{x \in X : R_\bullet[x] \subseteq U\} \tag{7}$$

The above correspondence between bimodal Boolean algebras and bimodal spaces can be extended to a dual equivalence of categories by defining suitable notions of morphisms. For the algebras, we adopt the obvious definition: morphisms of bimodal Boolean algebras are algebraic homomorphisms. On the topological side we follow once again Jónsson-Tarski duality.

Definition 6.2. A *bimodal function* $f \colon \mathcal{X} \to \mathcal{X}'$ between bimodal spaces $\mathcal{X} = \langle X, \tau, R_+, R_- \rangle$ and $\mathcal{X}' = \langle X', \tau', R'_+, R'_- \rangle$ is a continuous function such that, for $\bullet \in \{+, -\}$,

(i) $\langle x, y \rangle \in R_\bullet$ implies $\langle f(x), f(y) \rangle \in R'_\bullet$ for all $x, y \in X$;
(ii) for all $x \in X$ and $z \in X'$, if $\langle f(x), z \rangle \in R'_\bullet$, then there is $y \in X$ such that $f(y) = z$ and $\langle x, y \rangle \in R_\bullet$.

We thus have a category of bimodal Boolean algebras with algebraic homomorphisms on the one side, and a category of bimodal spaces with bimodal functions on the other. The same functors involved in Jónsson-Tarski duality will establish our duality. To an algebraic homomorphism of bimodal Boolean algebras $h \colon \mathbf{A} \to \mathbf{A}'$ corresponds a bimodal function $X(h) \colon X(\mathbf{A}') \to X(\mathbf{A})$ defined by $X(h)(P) := h^{-1}[P]$ for all $P \in X(\mathbf{A}')$. Similarly, to a bimodal function $f \colon \mathcal{X} \to \mathcal{X}'$ one associates a bimodal Boolean algebra homomorphism $A(f) \colon A(\mathcal{X}') \to A(\mathcal{X})$ defined by $A(f)(U) := f^{-1}[U]$ for all $U \in A(\mathcal{X}')$. Thus we have:

Theorem 6.3. *The category of bimodal spaces and bimodal functions is dually equivalent to the category of bimodal Boolean algebras and algebraic homomorphisms.*

As mentioned before, this result can be easily used to obtain a duality for modal bilattices (see also [8], where this strategy is applied to several classes of non-modal bilattices viewed as twist-structures).

Corollary 6.4. *The category of bimodal spaces and bimodal functions is dually equivalent to the category of modal bilattices and algebraic homomorphisms.*

6.2. **Completeness.** We are now going to use our duality, together with the algebraic results of Section 5, to prove completeness of our modal calculi with respect to the four-valued Kripke semantics introduced in Section 3.

We are going to expound the details of the completeness proof for the local calculus, which is essentially the same as that for the global calculus, and we will point out where the differences lie as we go along. The overall strategy is the following. Assuming $\Gamma \nvdash_l \varphi$, we first look for an algebraic counter-model, using the algebraic completeness results established in Section 5. Then, using duality, we turn the algebraic counter-model into a topological one. Finally, we will show how to view the topological model thus constructed as a four-valued Kripke model, and so our proof will be complete.

Theorem 6.5. *For all $\Gamma \cup \{\varphi\} \subseteq Fm$, the following are equivalent:*

(i) $\Gamma \vdash_l \varphi$*;*
(ii) for every four-valued Kripke model $M = \langle W, R, v\rangle$ and every $w \in W$, it holds that $M, w \models \Gamma$ implies $M, w \models \varphi$.

Theorem 6.6. *For all $\Gamma \cup \{\varphi\} \subseteq Fm$, the following are equivalent:*

(i) $\Gamma \vdash_g \varphi$*;*
(ii) for any four-valued Kripke model $M = \langle W, R, v\rangle$, if $M, w \models \Gamma$ for all $w \in W$, then $M, w \models \varphi$ for all $w \in W$.

Proof. Assume $\Gamma \nvdash_l \varphi$. Thanks to algebraic completeness (Theorem 5.12), we can find an algebraic counter-model, i.e., a matrix $\langle \mathbf{B}, F\rangle$, with $\mathbf{B}$ a modal bilattice and $F \subseteq B$ a bifilter of $\mathbf{B}$, and a homomorphism $h\colon \mathbf{Fm} \to \mathbf{B}$ such that $h[\Gamma] \subseteq F$ but $h(\varphi) \notin F$. In the case of $\vdash_g$, we moreover know that $F = F_0$ is the least bifilter of $\mathbf{B}$. Thanks to the twist-structure representation (Theorem 5.18), we may assume that $\mathbf{B} = \mathbf{A}^{\bowtie}$, with $\mathbf{A}$ a bimodal Boolean algebra. We also know, by Proposition 5.19, that $F = \nabla \times A$, with ∇ a lattice filter of $\mathbf{A}$. In the case of the global calculus, we have $F = F_0 = \{1\} \times A$, where 1 is the top element of $\mathbf{A}$. As before, we denote by π_1 the first projection mapping, so that $\pi_1[F] = \nabla$. Clearly, $\pi_1[h[\Gamma]] \subseteq \nabla$ but $\pi_1(h(\varphi)) \notin \nabla$. By the Ultrafilter Theorem, we can extend ∇ to an ultrafilter P such that $\pi_1(h(\varphi)) \notin P$. Then, by duality, we have $P \in \Phi(\pi_1(h(\gamma)))$ for all $\gamma \in \Gamma$ and $P \notin \Phi(\pi_1(h(\varphi)))$. We thus obtain a topological counter-model by considering the bimodal space $\langle X(\mathbf{A}), \tau_{\mathbf{A}}, R_{\Box_+}, R_{\Box_-}\rangle$ which is dual to $\mathbf{A}$. From this point on we follow the proof of Theorem 5.22, which showed that a model of classical bimodal logic can be turned into a four-valued Kripke model. We first define a four-valued relation $R_4\colon X(\mathbf{A}) \times X(\mathbf{A}) \to \mathcal{FOUR}$ as follows: for all $Q, Q' \in X(\mathbf{A})$,

$$R_4(Q,Q') := \begin{cases} \mathsf{t} & \text{iff } \langle Q,Q'\rangle \in R_{\Box_+} \text{ and } \langle Q,Q'\rangle \in R_{\Box_-} \\ \top & \text{iff } \langle Q,Q'\rangle \in R_{\Box_+} \text{ and } \langle Q,Q'\rangle \notin R_{\Box_-} \\ \bot & \text{iff } \langle Q,Q'\rangle \notin R_{\Box_+} \text{ and } \langle Q,Q'\rangle \in R_{\Box_-} \\ \mathsf{f} & \text{iff } \langle Q,Q'\rangle \notin R_{\Box_+} \text{ and } \langle Q,Q'\rangle \notin R_{\Box_-} \end{cases}$$

We thus have a four-valued Kripke frame $\langle X(\mathbf{A}), R_4\rangle$. Next, we need to define a four-valued valuation to obtain a Kripke model. We do this in two stages. We first define two standard (two-valued) valuations $v_+, v_-\colon Var \to P(X(\mathbf{A}))$ as follows: for all $p \in Var$,

$$v_+(p) := \{Q \in X(\mathbf{A}) : Q \in \Phi(\pi_1(h(p)))\}$$
$$v_-(p) := \{Q \in X(\mathbf{A}) : Q \in \Phi(\pi_1(\neg h(p)))\}.$$

These are extended to arbitrary formulas $\psi, \chi \in Fm$ as follows:

$v_+(\psi \wedge \chi) := v_+(\psi \otimes \chi) := v_+(\psi) \cap v_+(\chi)$ and $v_-(\psi \wedge \chi) := v_-(\psi \oplus \chi) := v_-(\psi) \cup v_-(\chi)$

$v_+(\psi \vee \chi) := v_+(\psi \oplus \chi) := v_+(\psi) \cup v_+(\chi)$ and $v_-(\psi \vee \chi) := v_-(\psi \otimes \chi) := v_-(\psi) \cap v_-(\chi)$

$v_+(\psi \supset \chi) := \sim v_+(\psi) \cup v_+(\chi)$ and $v_-(\psi \supset \chi) := v_+(\psi) \cap v_-(\chi)$

$v_+(\neg\psi) := v_-(\psi)$ and $v_-(\neg\psi) := v_+(\psi)$

$v_+(\Box\psi) := \Box_{R_{\Box_+}} v_+(\psi) \cap \Box_{R_{\Box_-}} \sim v_-(\psi)$ and $v_-(\Box\psi) := \sim \Box_{R_{\Box_+}} \sim v_-(\psi)$

$v_+(\mathsf{f}) := v_+(\bot) := v_-(\bot) := v_-(\mathsf{t}) := \emptyset$ and $v_-(\mathsf{f}) := v_-(\top) := v_+(\top) := v_+(\mathsf{t}) := X(\mathbf{A})$.

We then combine v_+ and v_- into one four-valued valuation $v_4 \colon \mathbf{Fm} \times X(\mathbf{A}) \to \mathcal{FOUR}$ as follows: for all $\psi \in Fm$ and $Q \in X(\mathbf{A})$,

$$v_4(\psi, Q) = \begin{cases} \mathsf{t} & \text{iff } Q \in v_+(\psi) \text{ and } Q \notin v_-(\psi) \\ \top & \text{iff } Q \in v_+(\psi) \text{ and } Q \in v_-(\psi) \\ \bot & \text{iff } Q \notin v_+(\psi) \text{ and } Q \notin v_-(\psi) \\ \mathsf{f} & \text{iff } Q \notin v_+(\psi) \text{ and } Q \in v_-(\psi). \end{cases}$$

The same argument used in the proof of Theorem 5.22 shows that v_4 acts homomorphically on both non-modal and modal formulas. We may thus conclude that $M_{\mathbf{A}} = \langle X(\mathbf{A}), R_4, v_4 \rangle$ is indeed a four-valued Kripke model. It only remains to show that $M_{\mathbf{A}}$ is a witness that Γ does not imply φ. This is easy, for $P \in v_+(\gamma)$ for all $\gamma \in \Gamma$ but $P \notin v_+(\varphi)$. According to our definition of v_4, this means that $v_4(\gamma, P) \in \{\mathsf{t}, \top\}$ for all $\gamma \in \Gamma$ and $v_4(\varphi, P) \notin \{\mathsf{t}, \top\}$. That is, $M_{\mathbf{A}}, P \models \Gamma$ but $M_{\mathbf{A}}, P \not\models \varphi$. Thus, $\Gamma \not\models_l \varphi$. Applying the same reasoning we can show that $\Gamma \not\models_g \varphi$, if we take into account that in this case $Q \in \Phi(\pi_1(h(\gamma)))$ for all $\gamma \in \Gamma$ and for all $Q \in X(\mathbf{A})$, which means that Γ holds globally in $M_{\mathbf{A}}$. $\Box$

We would now like to show that the completeness results of Theorems 6.5 and 6.6 apply to a more general semantics than the one introduced at the beginning of Section 3.

Consider a Kripke model $\langle W, R, v \rangle$ where both R and v are $\mathbf{B}$-valued instead of $\mathcal{FOUR}$-valued, where $\mathbf{B}$ is an implicative bilattice. That is, we define $R \colon W \times W \to \mathbf{B}$ and $v \colon \mathbf{Fm} \times W \to \mathbf{B}$. Notice that valuations are still required to preserve the four lattice bounds, which are included in the signature of implicative bilattices. As in Section 3, we can define the modal operator by

$$v(\Box\varphi, w) := \bigwedge \{R(w, w') \to v(\varphi, w') : w' \in W\}$$

where the algebraic operations are now those of $\mathbf{B}$. In order for this definition to make sense, we need to further require that the t-lattice reduct of $\mathbf{B}$ be complete in the usual lattice-theoretical sense. It is easy to show (using the twist-structure representation, for instance), that the k-lattice reduct of $\mathbf{B}$ will be complete as well.

We can now define (global and local) modal consequence relations determined by the class of $\mathbf{B}$-valued Kripke models as we did in Section 3. All definitions are the same, replacing $\mathcal{FOUR}$ with $\mathbf{B}$ and the set $\{\mathsf{t}, \top\}$ with F_0, the least bifilter of $\mathbf{B}$. The non-modal core of these logics will thus be the consequence determined by the matrix $\langle \mathbf{B}, F_0 \rangle$. It is an easy consequence of [36, Theorem 4.1.4, Proposition 4.3.14] that this logic coincides with that of the matrix $\langle \mathcal{FOUR}, \{\mathsf{t}, \top\} \rangle$. This result holds true even when we move to the modal setting.

Theorem 6.7. $\mathbf{B}$*-valued and* $\mathcal{FOUR}$*-valued modal logics coincide.*

Proof. To see that **B**-valued modal consequence is weaker than $\mathcal{FOUR}$-valued one, it is sufficient to notice that $\mathcal{FOUR}$ is a subalgebra of any implicative bilattice [36, Proposition 4.3.12]. Thus, any $\mathcal{FOUR}$-valued Kripke model can be viewed as a **B**-valued one, namely one where both R and v only take values in $\{\mathsf{f}, \mathsf{t}, \bot, \top\}$. The logics (both global and local) determined by the class of all **B**-valued Kripke models are obviously weaker than those of a subclass of it, hence the result follows. In order to prove that $\mathcal{FOUR}$-valued is weaker than the **B**-valued one, we will need completeness of our Hilbert calculi with respect to $\mathcal{FOUR}$-valued modal consequence. Let us notice that all axioms and rules of our calculi are sound with respect to **B**-valued modal consequence. This can be checked directly, and also easily follows from the considerations of [7, Section 3.1]. Soundness implies that the consequence determined by our calculus $\vdash_g$ (or $\vdash_l$) is weaker than **B**-valued modal consequence. By completeness (Theorems 6.5 and 6.6), $\vdash_g$ and $\vdash_l$ coincide with the corresponding semantically defined $\mathcal{FOUR}$-valued consequence, so we are done. $\square$

6.3. **Extensions.** We have mentioned in the proof of Proposition 6.7 the possibility of imposing restrictions on the values that the accessibility relation can take. These determine subclasses of frames and, therefore, extensions (strengthenings) of the four-valued modal logic that we have been considering throughout the paper. One may ask, as the authors of [7] do, whether it is possible to axiomatize the logic corresponding to these frames. In our case, this turns out to be quite straightforward, and we will see that in this respect too the splitting of the modal operator $\Box$ into two operators $\Box_+$ and $\Box_-$ is a great help. Taking inspiration from [7], we focus on:

1) *Idempotent frames*, i.e., those where the value of R is restricted to those elements of $\mathcal{FOUR}$ that are idempotent with respect to fusion, that is, $R(w, w') * R(w, w') = R(w, w')$ for for all $w, w' \in W$. As mentioned at the end of Section 2, in our case this amounts to the requirement that $R(w, w') \neq \bot$.
2) *Consistent frames*, where $R(w, w') \neq \top$.
3) *Classical frames*, where R is allowed to take only classical values: $R(w, w') \in \{\mathsf{f}, \mathsf{t}\}$. These are exactly the frames that are at the same time idempotent and consistent.

1) Idempotent frames. It is straightforward (if tedious) to check that in idempotent frames the normality axiom

$$\Box(p \to q) \to (\Box p \to \Box q)$$

is valid. In fact, as happens in [7], this axiom characterizes the class of idempotent frames. That is, it is possible to show that, if we add the axiom to our calculus $\vdash_l$ ($\vdash_g$), we obtain a sound and complete axiomatization for the local (global) consequence determined by the class of idempotent $\mathcal{FOUR}$-valued frames. We will not pursue this, instead we adopt a simpler axiomatization. We add to $\vdash_l$ (or $\vdash_g$) the following version of normality:

$$(\Box \supset) \qquad \Box(p \supset q) \supset (\Box p \supset \Box q)$$

Again, it is easy (and slightly less tedious) to check that $(\Box \supset)$ is valid in idempotent frames. Thus, our enriched calculi are sound with respect to the intended semantics. In order to prove completeness, it will be sufficient to show that, repeating the proof strategy of Theorems 6.5 and 6.6, we obtain as a counter-model a Kripke frame that is idempotent. In order to see this, we are going to look once more at algebraic models of our enriched calculi.

Any axiomatic extension of an algebraizable logic is itself algebraizable with the same translations. The corresponding algebraic semantics is a subvariety of the original one, axiomatized by adding the equations that result as the translation of the new logical axioms.

Thus, we see that the algebraic semantics of $\vdash_g + (\Box \supset)$ is precisely the class of modal bilattices that satisfy the equation

$$\Box(x \supset y) \supset (\Box x \supset \Box y) = (\Box(x \supset y) \supset (\Box x \supset \Box y)) \supset (\Box(x \supset y) \supset (\Box x \supset \Box y))$$

which can be equivalently and conveniently rewritten as

$$\Box(x \supset y) \supset (\Box x \supset \Box y) \geq_t \top.$$

This implies that reduced models of $\vdash_g + (\Box \supset)$ are matrices $\langle B, F_0 \rangle$ with $\mathbf{B}$ a modal bilattice satisfying the above equation and F_0 the minimal bifilter of $\mathbf{B}$. The same argument of Lemma 5.11 and Theorem 5.12 shows that the local calculus $\vdash_l + (\Box \supset)$ is complete with respect to the class of matrices $\langle B, F \rangle$ with $\mathbf{B}$ a modal bilattice satisfying the above equation and F a bifilter. In order to obtain more information on this class of modal bilattices, we once more exploit the twist-structure representation.

Proposition 6.8. *A modal bilattice* $\mathbf{B} = \mathbf{A}^{\bowtie}$ *satisfies* $\Box(x \supset y) \supset (\Box x \supset \Box y) \geq_t \top$ *if and only if the underlying bimodal Boolean algebra* $\mathbf{A}$ *satisfies the equation* $\Box_+ x \leq \Box_- x$.

Proof. Assume $\mathbf{A}^{\bowtie}$ satisfies $\Box(x \supset y) \supset (\Box x \supset \Box y) \geq_t \top$. According to the twist-structure construction, this means that the first component of

$$\Box(\langle a_1, a_2 \rangle \supset \langle b_1, b_2 \rangle) \supset (\Box\langle a_1, a_2 \rangle \supset \Box\langle b_1, b_2 \rangle)$$

is 1. We instantiate the above equation by taking $b_1 = {\sim} a_2 = 1$ and $b_2 = {\sim} a_1$, so that it becomes $\Box(\langle a_1, 0 \rangle \supset \langle 1, {\sim} a_1 \rangle) \supset (\Box\langle a_1, 0 \rangle \supset \Box\langle 1, {\sim} a_1 \rangle)$. We compute the first component, which is

$${\sim}(\Box_+({\sim} a_1 \sqcup 1) \sqcap \Box_-({\sim} a_1 \sqcup {\sim}{\sim} a_1)) \sqcup ({\sim}(\Box_+ a_1 \sqcap \Box_- {\sim} 0) \sqcup (\Box_+ 1 \sqcap \Box_- {\sim}{\sim} a_1)).$$

This simplifies as ${\sim}(1 \sqcap 1) \sqcup ({\sim} \Box_+ a_1 \sqcup \Box_- a_1) = {\sim} \Box_+ a_1 \sqcup \Box_- a_1$. Given that a_1 is an arbitrary element of A, we see that ${\sim} \Box_+ a_1 \sqcup \Box_- a_1 = 1$ means that $\mathbf{A}$ satisfies the equation ${\sim} \Box_+ x \sqcup \Box_- x = 1$, i.e., $\Box_+ x \leq \Box_- x$.
Conversely, assume $\mathbf{A}$ satisfies $\Box_+ x \leq \Box_- x$. We need to prove that

$${\sim}(\Box_+({\sim} a_1 \sqcup b_1) \sqcap \Box_-({\sim} a_1 \sqcup {\sim} b_2)) \sqcup ({\sim}(\Box_+ a_1 \sqcap \Box_- {\sim} a_2) \sqcup (\Box_+ b_1 \sqcap \Box_- {\sim} b_2)) = 1$$

which is equivalent, in any Boolean algebra, to

$$\Box_+({\sim} a_1 \sqcup b_1) \sqcap \Box_-({\sim} a_1 \sqcup {\sim} b_2) \sqcap \Box_+ a_1 \sqcap \Box_- {\sim} a_2 \leq \Box_+ b_1 \sqcap \Box_- {\sim} b_2.$$

Since $\Box_+$ and $\Box_-$ preserve meets, we can simplify the left-hand side of the inequality as follows: $\Box_+({\sim} a_1 \sqcup b_1) \sqcap \Box_-({\sim} a_1 \sqcup {\sim} b_2) \sqcap \Box_+ a_1 \sqcap \Box_- {\sim} a_2 = \Box_+(a_1 \sqcap ({\sim} a_1 \sqcup b_1)) \sqcap \Box_-({\sim} a_2 \sqcap ({\sim} a_1 \sqcup {\sim} b_2)) = \Box_+(a_1 \sqcap b_1) \sqcap \Box_-({\sim} a_2 \sqcap ({\sim} a_1 \sqcup {\sim} b_2))$. Obviously

$$\Box_+(a_1 \sqcap b_1) \sqcap \Box_-({\sim} a_2 \sqcap ({\sim} a_1 \sqcup {\sim} b_2)) \leq \Box_+ b_1$$

so it only remains to prove that

$$\Box_+(a_1 \sqcap b_1) \sqcap \Box_-({\sim} a_2 \sqcap ({\sim} a_1 \sqcup {\sim} b_2)) \leq \Box_- {\sim} b_2.$$

By our assumption that $\Box_+ x \leq \Box_- x$, we have

$$\Box_+(a_1 \sqcap b_1) \sqcap \Box_-({\sim} a_2 \sqcap ({\sim} a_1 \sqcup {\sim} b_2)) \leq \Box_-(a_1 \sqcap b_1) \sqcap \Box_-({\sim} a_2 \sqcap ({\sim} a_1 \sqcup {\sim} b_2)).$$

The right-hand side of the inequality can be rewritten as follows:

$$\begin{aligned} \Box_-(a_1 \sqcap b_1) \sqcap \Box_-({\sim} a_2 \sqcap ({\sim} a_1 \sqcup {\sim} b_2)) &= \Box_-(a_1 \sqcap b_1 \sqcap {\sim} a_2 \sqcap ({\sim} a_1 \sqcup {\sim} b_2)) \\ &= \Box_-(a_1 \sqcap {\sim} b_2 \sqcap b_1 \sqcap {\sim} a_2). \end{aligned}$$

We thus have

$$\Box_+(a_1 \sqcap b_1) \sqcap \Box_-({\sim} a_2 \sqcap ({\sim} a_1 \sqcup {\sim} b_2)) \leq \Box_-(a_1 \sqcap {\sim} b_2 \sqcap b_1 \sqcap {\sim} a_2) \leq \Box_- {\sim} b_2$$

which finishes our proof. $\Box$

We are now in a position to prove completeness of $\vdash_l + (\Box \supset)$ and of $\vdash_g + (\Box \supset)$ with respect to the consequence determined by idempotent Kripke frames. We assume that $\Gamma + (\Box \supset) \nvdash_l \varphi$. By algebraic completeness, we know that this is witnessed by a matrix $\langle \mathbf{B}, F \rangle$, with $\mathbf{B} = \mathbf{A}^{\bowtie}$ and F a bifilter of $\mathbf{B}$. By Proposition 6.8, we moreover know that $\mathbf{A}$ satisfies $\Box_+ x \leq \Box_- x$. This means that the bimodal space $\langle X(\mathbf{A}), \tau_{\mathbf{A}}, R_{\Box_+}, R_{\Box_-} \rangle$ will satisfy $R_{\Box_-} \subseteq R_{\Box_+}$. To see this, assume $\langle P, Q \rangle \in R_{\Box_-}$ for some $P, Q \in X(\mathbf{A})$. By definition, this means $\Box_-^{-1}[P] \subseteq Q$, that is, $\Box_- a \in P$ implies $a \in Q$ for all $a \in A$. Now, if $\Box_+ b \in P$ for some $b \in A$, then $\Box_- b \in P$ as well, because $\Box_+ b \leq \Box_- b$ and P is an up-set with respect to the lattice order of $\mathbf{A}$. Then, by assumption, $b \in Q$ and this means that $\langle P, Q \rangle \in R_{\Box_+}$ as claimed.

By looking at the proofs of Theorems 6.5 and 6.6, we see that the relation $R_4 \colon X(\mathbf{A}) \times X(\mathbf{A}) \to \mathcal{FOUR}$ that we constructed can only take value $\perp$ in case $\langle Q, Q' \rangle \notin R_{\Box_+}$ and $\langle Q, Q' \rangle \in R_{\Box_-}$ for some $Q, Q' \in X(\mathbf{A})$. In our case, as we have seen, this is impossible. We conclude that the model that we have constructed is actually idempotent. Hence, if φ is not derivable from Γ in $\vdash_l + (\Box \supset)$, then there is an idempotent model witnessing that Γ does not semantically entail φ. The same obviously holds for $\vdash_g + (\Box \supset)$. We have thus the following.

Theorem 6.9. *The calculus $\vdash_l + (\Box \supset)$ is sound and complete with respect to the local consequence determined by the class of idempotent Kripke models.*

Theorem 6.10. *The calculus $\vdash_g + (\Box \supset)$ is sound and complete with respect to the global consequence determined by the class of idempotent Kripke models.*

2) Consistent frames. The same strategy will allow us to axiomatize the consequence determined by the class of all consistent frames.

As before, we begin by conjecturing an axiomatization. By looking at the truth table of strong implication in $\mathcal{FOUR}$ (Table 1), one easily notices that, if the relation R is not allowed to take value $\top$, then no implication of the form $R(w, w') \to v(\varphi, w')$ can ever take value $\top$. By looking at the definition of $\Box$ given in Equation (1) of Section 3, we see that this implies that no modal formula can evaluate to $\top$. This means that a formula such as $\Box p \supset \Box p$, which is obviously valid in the logic, can only evaluate to t. This suggest that a sensible axiom to add to our calculi is the following:

$$(\to \Box) \qquad\qquad \mathsf{t} \to (\Box p \supset \Box p)$$

We are going to prove that this is in fact enough to axiomatize the consequence of consistent frames. As before, we look at the equation resulting from the translation of the new axiom, which is

$$\mathsf{t} \to (\Box x \supset \Box x) = (\mathsf{t} \to (\Box x \supset \Box x)) \supset (\mathsf{t} \to (\Box x \supset \Box x))$$

or, equivalently,

$$\Box x \supset \Box x = \mathsf{t}.$$

Proposition 6.11. *A modal bilattice $\mathbf{B} = \mathbf{A}^{\bowtie}$ satisfies $\Box x \supset \Box x = \mathsf{t}$ if and only if the underlying bimodal Boolean algebra $\mathbf{A}$ satisfies the equation $\Box_- x \leq \Box_+ x$.*

Proof. Assume $\mathbf{A}^{\bowtie}$ satisfies $\Box x \supset \Box x = \mathsf{t}$, which means that $\Box\langle a_1, a_2 \rangle \supset \Box\langle a_1, a_2 \rangle = \langle 1, 0 \rangle$ for all $a_1, a_2 \in A$. This only gives us information about the second component, given that the first component of any expression of the form $x \supset x$ is always 1 in an implicative bilattice. The part corresponding to the second components is $\Box_+ a_1 \sqcap \Box_- \sim a_2 \sqcap \sim \Box_+ \sim a_2 = 0$. We instantiate the equation by taking $a_1 = 1$, so that it becomes

$$\Box_+ 1 \sqcap \Box_- \sim a_2 \sqcap \sim \Box_+ \sim a_2 = \Box_- \sim a_2 \sqcap \sim \Box_+ \sim a_2 = 0.$$

This is equivalent, in any Boolean algebra, to $\Box_- \sim a_2 \leq \Box_+ \sim a_2$. Given that the element a_2 is arbitrary, we conclude that $\mathbf{A}$ satisfies $\Box_- x \leq \Box_+ x$.
Conversely, if $\mathbf{A}$ satisfies $\Box_- x \leq \Box_+ x$, then $\Box_- \sim a_2 \sqcap \sim \Box_+ \sim a_2 = 0$ for all $a_1, a_2 \in A$ and, a fortiori, $\Box_+ a_1 \sqcap \Box_- \sim a_2 \sqcap \sim \Box_+ \sim a_2 = 0$, which concludes our proof. □

From this point on the completeness proof for consistent frames is analogous to the one for idempotent frames. We just need to observe that, if a bimodal Boolean algebra satisfies $\Box_- x \leq \Box_+ x$, then in the dual bimodal space we will have $R_{\Box_+} \subseteq R_{\Box_-}$. This means that the relation $R_4 \colon X(\mathbf{A}) \times X(\mathbf{A}) \to \mathcal{FOUR}$ that we constructed in the proofs of Theorems 6.5 and 6.6 will never take value $\top$, as this corresponds to the case where $\langle Q, Q' \rangle \in R_{\Box_+}$ and $\langle Q, Q' \rangle \notin R_{\Box_-}$ for some $Q, Q' \in X(\mathbf{A})$. The following completeness results immediately follow.

Theorem 6.12. *The calculus* $\vdash_l + (\to \Box)$ *is sound and complete with respect to the local consequence determined by the class of consistent Kripke models.*

Theorem 6.13. *The calculus* $\vdash_g + (\to \Box)$ *is sound and complete with respect to the global consequence determined by the class of consistent Kripke models.*

3) *Classical frames.* This case is now an easy consequence of the previous ones. We just need to add both axioms $(\Box \supset)$ and $(\to \Box)$ to the logic to obtain a sound and complete axiomatization.

Proposition 6.14. *A modal bilattice* $\mathbf{B} = \mathbf{A}^{\bowtie}$ *satisfies both* $\Box(x \supset y) \supset (\Box x \supset \Box y) \geq_t \top$ *and* $\Box x \supset \Box x = \mathsf{t}$ *if and only if the underlying bimodal Boolean algebra* $\mathbf{A}$ *satisfies the equation* $\Box_+ x = \Box_- x$.

Theorem 6.15. *The calculus* $\vdash_l + (\Box \supset) + (\to \Box)$ *is sound and complete with respect to the global consequence determined by the class of classical Kripke models.*

Theorem 6.16. *The calculus* $\vdash_g + (\Box \supset) + (\to \Box)$ *is sound and complete with respect to the global consequence determined by the class of classical Kripke models.*

It is perhaps interesting to note that none of the restrictions considered above corresponds to the logic of [34], viewed as a particular case of ours (see Proposition 3.1). As we observed in Subsection 5.2, the requirement is in this case that the equation $\Box_- x = 1$ be satisfied in the underlying bimodal Boolean algebra. This corresponds to adding the axiom $\Box(p \supset p)$ to the logic, and this is in fact one of the axioms that appear in the calculus of [34]. If a bimodal Boolean algebra $\mathbf{A}$ satisfies $\Box_- x = 1$, then $R_{\Box_-} = \emptyset$ in the dual bimodal space $\mathcal{X}(\mathbf{A})$. As a consequence, the valuation $R_4 \colon X(\mathbf{A}) \times X(\mathbf{A}) \to \mathcal{FOUR}$ constructed in our completeness proof is only allowed to take values in $\{\mathsf{f}, \top\}$. This also shows that the class of frames corresponding to $\vdash_l + \Box(p \supset p)$ is a subclass of the idempotent frames, and hence the logic of [34] is normal, i.e., satisfies our axiom $(\Box \supset)$.

For completeness' sake, let us mention that the symmetric equation $\Box_+ x = 1$ entails that R_4 can only take values in $\{\mathsf{f}, \bot\}$, hence the corresponding frames are also consistent. The consequence determined by this class of frames can be axiomatized by adding the logical axiom $\Diamond p \supset \mathsf{f}$ to our base calculi.

We conclude the section with some considerations on the finite model property and decidability. By examining the proof of Theorem 5.22, it is easy to realize that all the lemmas involved are still true when we move from the base logics to their axiomatic extensions. This involves in particular restricting Corollary 5.13 and Proposition 5.20 to subvarieties of modal bilattices and of bimodal Boolean algebras corresponding to idempotent frames, classical frames etc. In fact, Proposition 5.20 can be used to show that there is an isomorphism between the lattices of subvarieties of modal bilattices and of bimodal Boolean algebras (cf. [22, Proposition 4.2]). This one-to-one correspondence extends to a correspondence between extensions of the least modal bilattice logic and extensions of classical bimodal logic. The only reason why analogues of Theorem 5.22 may fail is then that the extension of classical bimodal logic corresponding to a given modal bilattice logic may not itself enjoy the finite model property. This can happen, for the finite model property is not necessarily preserved under axiomatic extensions, as shown in [26], to which we also refer for several examples of logic which do enjoy it.

7. A more general approach

In this section we introduce an alternative and more general semantics for our four-valued modal logic, which makes a more explicit use of the insight gained from the twist-structure representation of modal bilattices. This alternative semantics is also closer to, and is a more direct generalization of, that of [34].

Let us first of all explain why we are interested in introducing an alternative semantics. This brings us back to the four-element algebra $\mathcal{FOUR}$. As mentioned in Section 2, we can view this structure as an algebra in different algebraic signatures, which correspond to different logics, each one being a conservative expansion of the previous one: Belnap-Dunn logic, paraconsistent Nelson logic, bilattice logic. The authors of [34], for instance, took four-valued paraconsistent Nelson logic as their non-modal core logic and axiomatized the least modal expansion of it, assuming that the accessibility relation of Kripke frames remains classical. We, instead, have worked throughout this paper with four-valued bilattice logic, and we managed to axiomatize the least modal expansion of it, allowing both valuations and the accessibility relation to be themselves four-valued (or, indeed, **B**-valued, for any complete implicative bilattice **B**). As we have seen, the restriction that the accessibility relation be classical corresponds to an extension of the least logic, which we have also axiomatized.

At this point we may ask ourselves if, analogously, it is possible to axiomatize the least modal expansion of four-valued paraconsistent Nelson logic, if we also allow the accessibility relation to be four-valued. This would be a logic in the language $\langle \wedge, \vee, \supset, \neg, \Box, \mathsf{f}, \mathsf{t} \rangle$ as opposed to the one we have been considering throughout the paper, namely $\langle \wedge, \vee, \otimes, \oplus, \supset, \neg, \Box, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$ or, equivalently, $\langle \wedge, \vee, \supset, \neg, \Box, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$. We see that the only difference lies in the presence of the constants $\bot$ and $\top$, and it is no coincidence that they play a quite crucial role in both our axiomatization for the logic and our twist-structure representation. Interestingly, the role of constants is also crucial in [7], the completeness proofs of which are based on a completely different strategy from ours. From a mathematical point of view, such a result would be desirable because it would be more general than the one we have proved in the previous section. In logic it is, as a rule, best to work with a language that is as restricted as possible, for results on language expansions will then follow just as special cases.

Unfortunately, we have not been able to achieve this. We do not know if four-valued modal logic can be axiomatized without including $\top$ as a logical constant; however, we do know that $\bot$ is not necessary. Indeed, we can show that axiom ($\Box\ \bot$) from Section 4 can be replaced in the logic by equivalent ones that only involve $\top$ and the t-constants, and similarly the twist-structure representation can be obtained without using $\bot$ as an algebraic constant. The reason why this is so can be best understood by looking at the proof of the twist-structure representation, which indeed appears to be the main hinge on which our whole completeness proofs rely.

As we have seen, the twist-structure representation tells us that the behaviour of the modal operator $\Box$ on a modal bilattice $\mathbf{B}$ is determined by a pair of operators $\Box_+$ and $\Box_-$ on the underlying Boolean algebra $\mathbf{B}/\!\approx$ which is a quotient of the $\langle \wedge, \vee, \supset, \mathsf{f}, \mathsf{t}\rangle$-reduct of $\mathbf{B}$. The two operators are obtained on $\mathbf{B}/\!\approx$ as

$$\Box_+[a] := [\Diamond(a \supset \mathsf{f}) \supset \mathsf{f}]$$
$$\Box_-[a] := [\Box(\neg(a \supset \mathsf{f}) \vee \top)]$$

where $\Diamond x$ is a shorthand for $\neg\Box\neg x$. This definition obviously relies on the existence of certain terms in the language of modal bilattices, namely $\Diamond(x \supset \mathsf{f}) \supset \mathsf{f}$ and $\Box(\neg(x \supset \mathsf{f}) \vee \top)$, and notice that neither of these involves $\bot$. Other properties are required, for instance that the terms respect the relation $\approx$ in the sense that $a \approx b$ entails $\Diamond(a \supset \mathsf{f}) \supset \mathsf{f} \approx \Diamond(b \supset \mathsf{f}) \supset \mathsf{f}$ and $\Box(\neg(a \supset \mathsf{f}) \vee \top) \approx \Box(\neg(b \supset \mathsf{f}) \vee \top)$. It is not difficult to check that, for the purpose of the twist-structure representation, other terms would have worked as well. In the same way as one could define $[a] \sqcap [b] := [a \otimes b]$ instead of $[a] \sqcap [b] := [a \wedge b]$ because $a \wedge b \approx a \otimes b$ for all $a, b \in B$, we could have defined, for instance,

$$\Box_+[a] := [\Box(a \vee \bot)]$$
$$\Box_-[a] := [\Box(\neg(a \supset \bot) \vee \top)].$$

While the definition of $\Box_+$ does not pose any problem, it seems that, in order to define a term that will allow us to recover to $\Box_-$ in the quotient $\mathbf{B}/\!\approx$, at least $\top$ is required.

The above analysis suggests, if not yet a solution to the problem, at least a possible way of approximating it. Indeed, if we cannot construct the algebraic terms that we need in the language $\langle \wedge, \vee, \supset, \neg, \Box, \mathsf{f}, \mathsf{t}\rangle$, we can assume that they already exist, i.e., introduce them as primitive algebraic operations. That is, we can augment the non-modal language $\langle \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}\rangle$ with two operators, which will be denoted by $\boxplus$ and $\boxminus$, that will allow us to simply define (in a suitably defined quotient algebra):

$$\Box_+[a] := [\boxplus a]$$
$$\Box_-[a] := [\boxminus a].$$

As mentioned earlier, the above requirement leaves us a certain freedom in the choice of the two operators. Our particular choice will depend on two criteria. On the one hand, we would like to relate our approach to that of [34], preserving in some way the property stated in Proposition 3.1. For this reason, we will choose $\boxplus$ to be defined on twist-structures exactly in the same way as the modal operator of [34]. A pleasing consequence of this is that $\boxplus$ is in this way guaranteed to be a normal and finite meet-preserving operator. In a similar spirit, we will define $\boxminus$ in such a way that (i) we obtain another finite meet-preserving operator and (ii) our original operator $\Box$ will not have to be included in the primitive language, because we will be able to define $\Box x := \boxplus x \wedge \boxminus x$.

We are now going to describe this approach in more detail, but we will allow ourselves to only sketch the parts which do not essentially differ from the constructions we have described in the previous sections.

7.1. **Axiomatization.** Let us begin by recalling that the non-modal core of the logic, which is the consequence determined by the matrix $\langle \mathcal{FOUR}, \{\mathsf{t}, \top\}\rangle$ in the language $\langle \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}\rangle$, is four-valued paraconsistent Nelson logic, which is an extension of paraconsistent Nelson logic [1, 28, 30] obtained by adding Peirce's axiom $((p \supset q) \supset p) \supset p$. A complete axiomatization of this logic can be obtained by taking all the axiom schemata introduced in Section 4 that only involve connectives in $\langle \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}\rangle$. The only rule is also the same, that is, modus ponens relative to weak implication. This calculus, which is going to supply the non-modal core of the Hilbert-style presentation for our new logic, is algebraizable. Its algebraic semantics is the variety of *bounded N4-lattices* [28] satisfying Peirce's equation $((x \supset y) \supset x) \supset x = x \supset x$. For us, the most straightforward way to introduce this class of algebras is to say that they are exactly the $\langle \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t} \rangle$-subreducts of implicative bilattices. This means, on the one hand, that this class is precisely the variety generated by $\mathcal{FOUR}$ viewed as an algebra in the language $\langle \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t} \rangle$. On the other hand, it entails that each algebra in the variety can be represented as a $\langle \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t} \rangle$-subalgebra of a twist-structure $\mathbf{A}^{\bowtie}$ over a Boolean algebra $\mathbf{A}$.

Then, when expanding this logic with a modal operator, we obtain a logic whose algebraic models (at least for the non-modal reduct) can be viewed as twist-structures. In fact, as mentioned in Subsection 5.2, Odintsov and Wansing [34] proved that their modal operator $\Box$ is represented, using our notation, as follows: $\Box\langle a_1, a_2 \rangle = \langle \Box_+ a_1,\ \Diamond_+ a_2 \rangle$. We have also seen that this operator can be defined as a term (that crucially uses the constant $\bot$ and the $\oplus$ operation) in four-valued bilattice modal logic. The idea is then to take this term as primitive in our new logic, that is, we are going to introduce an operator $\boxplus$ which will be represented, on twist-structures, as $\boxplus\langle a_1, a_2 \rangle = \langle \Box_+ a_1,\ \Diamond_+ a_2 \rangle$. Given that this operator coincides, on twist-structures, with the Odintsov-Wansing one, there is a natural candidate for its axiomatization in our logic, namely axiom schemata employed in [34]:

$(\boxplus \supset)$	$\boxplus\,(p \supset p)$
$(\boxplus \wedge)$	$\boxplus\,(p \wedge q) \supset (\boxplus p \wedge \boxplus q)$
$(\sim \boxplus)$	$\sim \boxplus p \equiv \neg \boxplus \neg \sim p$
$(\boxplus \sim)$	$\boxplus \sim p \equiv \sim \neg \boxplus \neg p$

where $\sim p := p \supset \mathsf{f}$. Alternatively, if we wanted to adopt an axiomatization that is closer to the one we introduced for modal bilattice logic, we could replace $(\boxplus \supset)$ and $(\boxplus \wedge)$ by:

$(\boxplus\, \mathsf{t})$	$\boxplus\, \mathsf{t} \leftrightarrow \mathsf{t}$
$(\boxplus \wedge)$	$\boxplus\,(p \wedge q) \leftrightarrow (\boxplus p \wedge \boxplus q)$.

In analogy with the previous case, in order to find suitable axioms for $\boxminus$ we are guided by the term which will correspond to it on the algebraic models. We adopt the following term from the language of modal bilattices:

$$\Box(x \vee \top)$$

which is particularly simple and meets the requirements explained above. In a twist-structure, this gives us

$$\boxminus\langle a_1, a_2 \rangle := \langle \Box_- \sim a_2, \Diamond_+ a_2 \rangle.$$

This operator can be captured through the following axioms:

$(\boxminus\, \mathsf{t})$	$\boxminus\, \mathsf{t} \leftrightarrow \mathsf{t}$
$(\boxminus \wedge)$	$\boxminus\,(p \wedge q) \leftrightarrow (\boxminus p \wedge \boxminus q)$
$(\boxplus \boxminus)$	$\neg \boxplus p \equiv \neg \boxminus p$
$(\boxminus \sim \neg)$	$\boxminus p \equiv \boxminus \sim \neg p$.

From the above axioms modal syntactic consequences $\vdash_l$ and $\vdash_g$ can be obtained as we did in Definition 4.1.

Definition 7.1. Let Fm be the set of formulas generated by a countable set of variables Var in the modal language $\langle \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}, \boxplus, \boxminus \rangle$. The set Σ of *axioms* of *modal N4-logic* is the least subset of Fm containing all substitution instances of the schemata exhibited in this subsection, and closed under

(val-mp) if φ and $\varphi \supset \psi$ are in Σ, then so is ψ;

(val-mono) if $\varphi \to \psi \in \Sigma$, then $\boxplus\varphi \to \boxplus\psi$, $\boxminus\varphi \to \boxminus\psi \in \Sigma$.

The *rules* of *modal bilattice logic* are

$$\frac{\varphi, \varphi \supset \psi}{\psi}\ (\mathrm{mp}) \qquad \frac{\varphi \to \psi}{\boxplus\varphi \to \boxplus\psi}\ (\mathrm{mono}\ \boxplus) \qquad \frac{\varphi \to \psi}{\boxminus\varphi \to \boxminus\psi}\ (\mathrm{mono}\ \boxminus)$$

Local inference $\vdash_l^*$ employs only (mp), while *global inference* $\vdash_g^*$ is generated by (mp), (mono $\boxplus$) and (mono $\boxminus$).

We note that for $\vdash_g$ we could alternatively use the rules of [34]

$$\frac{\varphi \supset \psi}{\boxplus\varphi \supset \boxplus\psi} \qquad \frac{\varphi \supset \psi}{\neg \boxplus \neg\varphi \supset \neg \boxplus \neg\psi}$$

but only for $\boxplus$, as they would not be sound with respect to the semantics of $\boxminus$.

One first pleasing feature of the axiomatization that we have proposed for $\boxplus$ and $\boxminus$ is that it allows us to recover the logic of [34] as an axiomatic extension of ours, and thus its algebraic counterpart [31] as a subclass of our algebraic models. The logic of Odintsov-Wansing is obtained by adding the following axiom:

($\boxminus\,\boxplus$) $\qquad\qquad \boxminus\, p \leftrightarrow {\sim} \neg \boxplus p$

which means, as was to be expected, that we can essentially ignore $\boxminus$ as it can be viewed as just a shorthand for ${\sim}\neg\boxplus$.

7.2. **Relational semantics.** We have seen that the algebraic (twist-structure) semantics has suggested us an axiomatization for the logic. Proceeding somehow in reverse order to what we have done in the previous sections, we are now going to see how the algebraic semantics also suggests a Kripke-style semantics. In fact, the latter will be essentially an adaptation of the semantics of [34] to the modal operators that we have chosen.

Consider a four-valued Kripke model $\langle W, R, v\rangle$ defined as in Section 3, i.e., such that $R\colon W \times W \to \mathcal{FOUR}$ and $v\colon \mathbf{Fm} \times W \to \mathcal{FOUR}$. As observed in Subsection 6.1, we can view R as a pair of two-valued relations $R_+, R_- \subseteq W \times W$ defined as follows: for all $w, w' \in W$,

$$\langle w, w'\rangle \in R_+ \ \text{ iff } \ R(w, w') \in \{\mathsf{t}, \top\}$$
$$\langle w, w'\rangle \in R_- \ \text{ iff } \ R(w, w') \in \{\mathsf{t}, \bot\}.$$

Similarly (but, as in Subsection 6.1, not symmetrically), we view v as a pair of valuations $v_+, v_-\colon Var \to P(W)$ defined, for all $w \in W$ and $p \in Var$, by

$$w \in v_+(p) \ \text{ iff } \ v(p, w) \in \{\mathsf{t}, \top\}$$
$$w \in v_-(p) \ \text{ iff } \ v(p, w) \in \{\mathsf{f}, \top\}.$$

Models will thus be structures $M = \langle W, R_+, R_-, v_+, v_-\rangle$. The valuations are extended to arbitrary formulas $\varphi, \psi \in Fm$ as follows:

$$v_+(\varphi \wedge \psi) := v_+(\varphi) \cap v_+(\psi) \ \text{ and } \ v_-(\varphi \wedge \psi) := v_-(\varphi) \cup v_-(\psi)$$
$$v_+(\varphi \vee \psi) := v_+(\varphi) \cup v_+(\psi) \ \text{ and } \ v_-(\varphi \vee \psi) := v_-(\varphi) \cap v_-(\psi)$$
$$v_+(\varphi \supset \psi) := {\sim} v_+(\varphi) \cup v_+(\psi) \ \text{ and } \ v_-(\varphi \supset \psi) := v_+(\varphi) \cap v_-(\psi)$$
$$v_+(\neg\varphi) := v_-(\varphi) \ \text{ and } \ v_-(\neg\varphi) := v_+(\varphi)$$
$$v_+(\mathsf{f}) := v_-(\mathsf{t}) := \emptyset \ \text{ and } \ v_-(\mathsf{f}) := v_+(\mathsf{t}) := W$$
$$v_+(\boxplus\varphi) := \{w \in W : R_+[w] \subseteq v_+(\varphi)\} \ \text{ and } \ v_-(\boxplus\varphi) := \{w \in W : R_+[w] \cap v_-(\varphi) \neq \emptyset\}$$

$$v_+(\boxminus\varphi) := \{w \in W : R_-[w] \cap v_-(\varphi) = \emptyset\} \ \text{ and } \ v_-(\boxminus\varphi) := v_-(\boxplus\varphi).$$

Let us point out that the somehow unusual semantics of $\boxminus$ simply reflects the algebraic definition introduced above, $\boxminus\langle a_1, a_2\rangle := \langle \Box_- \sim a_2, \Diamond_+ a_2\rangle$, which only considers the second component (hence only v_- appears) and in the first component operates on the Boolean complement of it.

Satisfaction, local and global consequence are defined in the way to be expected. We say that a point $w \in W$ of a model $M = \langle W, R_+, R_-, v_+, v_-\rangle$ satisfies a formula $\varphi \in Fm$ if $w \in v_+(\varphi)$ and we write $M, w \models \varphi$. For a set of formulas $\Gamma \subseteq Fm$, we write $M, w \models \Gamma$ to mean that $M, w \models \gamma$ for each $\gamma \in \Gamma$. Local and global consequence relation, denoted $\models_l^*$ and $\models_g^*$, are then defined as in Section 3.

Soundness of the axioms concerning $\boxplus$ with respect to this semantics follows from [34]. As to $\boxminus$, let us consider, for instance, the last axiom:

$$\boxminus p \equiv \boxminus \sim \neg p.$$

This is a shorthand for the two axioms $\boxminus p \supset \boxminus(\neg p \supset \mathsf{f})$ and $\boxminus(\neg p \supset \mathsf{f}) \supset \boxminus p$. We need to prove that

$$v_+(\boxminus p \supset \boxminus(\neg p \supset \mathsf{f})) = v_+(\boxminus(\neg p \supset \mathsf{f}) \supset \boxminus p) = W$$

for any model $M = \langle W, R_+, R_-, v_+, v_-\rangle$. According to the semantics of weak implication, this means $v_+(\boxminus(\neg p \supset \mathsf{f})) = v_+(\boxminus p)$. Applying the definitions, we see that the right-hand side of this equation is

$$\begin{aligned} v_+(\boxminus(\neg p \supset \mathsf{f})) &= \{w \in W : R_-[w] \cap v_-(\neg p \supset \mathsf{f}) = \emptyset\} \\ &= \{w \in W : R_-[w] \cap v_+(\neg p) \cap v_-(\mathsf{f}) = \emptyset\} \\ &= \{w \in W : R_-[w] \cap v_-(p) \cap W = \emptyset\} \\ &= \{w \in W : R_-[w] \cap v_-(p) = \emptyset\} \\ &= v_+(\boxminus p). \end{aligned}$$

7.3. **Algebraic models.** As in the case of bilattices, it is immediate to conclude that the global calculus $\vdash_g^*$ is algebraizable with the same translations that ensure algebraizability of paraconsistent Nelson logic (see, e.g., [37, Theorem 2.6]), which are those of Theorem 5.6. The equivalent algebraic semantics of $\vdash_g^*$ is a class of algebras in the language $\langle \wedge, \vee, \supset, \neg, \boxplus, \boxminus, \mathsf{f}, \mathsf{t}\rangle$, which we call *modal N4-lattices*. A (quasi)equational presentation of this class is given by the τ-translation of the axioms and rules introduced in the preceding Subsection. Clearly, the non-modal reduct of any modal N4-lattice is a bounded N4-lattice satisfying Peirce's equation, that is, a member of the variety generated by $\mathcal{FOUR}$ viewed as an algebra in the language $\langle \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}\rangle$. Instead of introducing modal N4-lattices through an abstract presentation, we will directly look at their concrete representation.

Let us first consider the non-modal reduct. Let $\mathbf{A} = \langle A, \sqcap, \sqcup, \sim, 0, 1\rangle$ be a Boolean algebra with associated (full) twist-structure $\mathbf{A}^{\bowtie} = \langle A \times A, \wedge, \vee, \otimes, \oplus, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top\rangle$, defined as in Subsection 5.2. We define a *Peirce N4-lattice* to be any $\langle \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}\rangle$-subalgebra of $\mathbf{A}^{\bowtie}$. We say that a Peirce N4-lattice $\mathbf{B}$ is a (non-full) twist-structure over $\mathbf{A}$, and we write $\mathbf{B} \leq \mathbf{A}^{\bowtie}$. An equational presentation for this class of algebras can be easily obtained by adding Peirce's equation and equations for the lattice bounds to the presentation of N4-lattices introduced in [28]. The following results from [28, 29] will also be useful:

Theorem 7.2. *Any Peirce N4-lattice* $\mathbf{B}$ *can be viewed as a twist-structure* $\mathbf{B} \leq \mathbf{A}^{\bowtie}$, *where* $\mathbf{A}$ *is a Boolean algebra, such that:*

(i) $\pi_1[B] := \{x \in A : \exists y \in A \text{ s.t. } \langle x, y\rangle \in B\} = A$,

(ii) $B = \{\langle x, y\rangle \in A \times A : \ x \sqcup y \in \nabla, \ x \sqcup y \in \Delta\}$, *where* $\nabla \subseteq A$ *is a lattice filter of* $\mathbf{A}$ *and* $\Delta \subseteq A$ *is a lattice ideal,*

(iii) $\nabla = \pi_1[A^*]$, *where* $A^* := \{a \vee \neg a : a \in B\}$,

(iv) $\Delta = \pi_2[A^*] := \{y \in A : \exists x \in A \ s.t. \ \langle x, y\rangle \in A^*\}$.

The non-modal reduct of any modal N4-lattice is a Peirce N4-lattice, which we can view as a twist-structure $\mathbf{B} \leq \mathbf{A}^{\bowtie}$ defined as above. On $\mathbf{B}$, the modal operators will be defined as explained in Subsection 7.1, that is, for $a_1, a_2 \in A \times A$,

$$\boxplus\langle a_1, a_2\rangle = \langle \Box_+ a_1, \ \Diamond_+ a_2\rangle \qquad\qquad \boxminus \langle a_1, a_2\rangle := \langle \Box_- \sim a_2, \Diamond_+ a_2\rangle$$

where $\Box_+$ and $\Box_-$ are finite meet-preserving operators that turn $\mathbf{A}$ into a bimodal Boolean algebra. The representation of $\boxplus$ simply follows from [34, 31]. For that of $\boxminus$, axioms ($\boxplus\,\boxminus$) and ($\boxminus \sim \neg$) are crucial. The former tells us that $\boxplus$ and $\boxminus$ agree on the second component, while the latter takes care of the first component. The remaining axioms ($\boxminus$ t) and ($\boxminus \wedge$) ensure that the operator $\Box_-$ actually preserves finite meets.

We can thus extend the twist-structure construction to obtain a representation of modal N4-lattices. It is also possible to obtain an analogue of Theorem 7.2: items (i), (iii) and (iv) are the same, whereas (ii) has to be adapted by imposing further restrictions on ∇ and Δ.

Theorem 7.3. *Any modal N4-lattice* $\mathbf{B}$ *can be viewed as a twist-structure* $\mathbf{B} \leq \mathbf{A}^{\bowtie}$, *where* $\mathbf{A}$ *is a bimodal Boolean algebra, such that:*

(i) $\pi_1[B] = A$,

(ii) $B = \{\langle x, y\rangle \in A \times A : \ x \sqcup y \in \nabla, \ x \sqcup y \in \Delta\}$, *where* $\nabla \subseteq A$ *is a lattice filter of* $\mathbf{A}$ *and* $\Delta \subseteq A$ *is a lattice ideal such that*
  (1) $x \in \nabla$ *implies* $\Box_+ x \in \nabla$
  (2) $x \in \Delta$ *implies* $\Diamond_+ x \in \Delta$
  (3) $\Box_- x \vee \sim \Box_+ x \in \nabla$ *and* $\Box_- x \wedge \sim \Box_+ x \in \Delta$ *for all* $x \in A$,

(iii) $\nabla = \pi_1[A^*]$, *where* $A^* := \{a \vee \neg a : a \in B\}$,

(iv) $\Delta = \pi_2[A^*]$.

Algebraic completeness theorems analogous to those of Subsection 5.1 can be obtained in the same way. In the case of global consequence, algebraizability immediately implies the following.

Theorem 7.4. *The global consequence relation* $\vdash^*_g$ *is complete with respect to the class of all matrices* $\langle \mathbf{B}, F_0\rangle$ *such that* $\mathbf{B}$ *is a modal N4-lattice and* $F_0 := \{a \in B : a \supset a = a\}$.

Similarly to the case of modal bilattices, the above theorem can be used to prove that $\mathsf{Alg}^*(\vdash^*_l) = \mathsf{Alg}^*(\vdash^*_g)$ is precisely the variety of modal N4-lattices. In order to obtain an analogue of Theorem 5.12, we need to replace bifilters by *special filters*, which can be defined as follows (see also [29]). A special filter of a (modal) N4-lattice $\mathbf{B}$ is a subset $F \subseteq B$ such that

(i) $F_0 := \{a \in B : a \supset a = a\} \subseteq F$,

(ii) F is closed under (mp), that is, $a, \ a \supset b \in F$ imply $b \in F$.

It is easy to check that the definition implies that any special filter is a non-empty lattice filter of $\langle B, \wedge, \vee\rangle$, and that F_0 is the least special filter of $\mathbf{B}$. A characterization of special filters, which generalizes that of bifilters of Proposition 5.19, can be obtained as well. In this case we have that any special filter $F \subseteq B$ of a (modal) N4-lattice $\mathbf{B} \leq \mathbf{A}^{\bowtie}$ is of the form $F = (\nabla \times A) \cap B$, where ∇ is a lattice filter of $\mathbf{A}$.

Theorem 7.5. *The local consequence relation $\vdash_l^*$ is complete with respect to the class of all matrices $\langle \mathbf{B}, F \rangle$ such that $\mathbf{B}$ is a modal N4-lattice and F is a special filter of $\mathbf{B}$.*

We will not pursue this here, but it is possible to combine the twist-structure representation of modal N4-lattices with the above results to obtain more information on reduced models of $\vdash_g^*$ and $\vdash_l^*$ as we have done in Subsection 5.3 with modal bilattice logic.

Similarly, one may also ask if $\vdash_g^*$ and $\vdash_l^*$ enjoy the finite model property and are therefore decidable. This can be shown following essentially the same proof as Theorem 5.22. The translation ν, restricted to formulas in the language of modal N4-lattices, is defined in the same way. However, some adjustments are needed, because Proposition 5.20 is no longer true for non-full twist-structures. This is essentially due to the restriction imposed by Theorem 7.3 on the elements of the direct product $A \times A$ that belong to the universe $B \subseteq A \times A$ of the twist-structure. Because of this, the only implication of Proposition 5.20 that still holds true is the leftward one. In fact, if a bimodal Boolean algebra $\mathbf{A}$ satisfies the equations $\nu_1(\varphi) = \nu_1(\psi)$ and $\nu_2(\varphi) = \nu_2(\psi)$, then the full twist-structure $\mathbf{A}^{\bowtie}$ satisfies the equation $\varphi = \psi$ and therefore every subalgebra $\mathbf{B} \leq \mathbf{A}^{\bowtie}$ will also satisfy $\varphi = \psi$. Fortunately, this direction of the implication (by contraposition) is the only one that is needed in the proof of Theorem 5.22. For the local consequence Corollary 5.13 also needs to be adapted, but this is not a problem. The equation appearing in the second item of the corollary can be replaced by an equivalent one in the language of modal N4-lattices, obtaining the equivalence of the following:

(i) $\mathit{\Gamma} \vdash_l^* \varphi$,
(ii) $\mathit{\Gamma}_0 \vdash_l^* \varphi$ for a finite $\mathit{\Gamma}_0 \subseteq \mathit{\Gamma}$,
(iii) the equation $\bigwedge \mathit{\Gamma}_0 \supset \varphi = (\bigwedge \mathit{\Gamma}_0 \supset \varphi) \supset (\bigwedge \mathit{\Gamma}_0 \supset \varphi)$ is valid in the variety of modal bilattices.

The rest of the proof of Theorem 5.22 can be straightforwardly adapted to the new logics. The relational semantics introduced in Subsection 7.2 no longer requires us to combine the two relations R_+, R_- of a classical model $\langle W, R_+, R_-, v \rangle$ into a single four-valued one, so this part is even easier than in the original proof. We do need to duplicate the classical valuation $v \colon Fm_{cb} \to P(W)$, which we can do by defining, for each formula φ in the language $\langle \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}, \boxplus, \boxminus \rangle$ and each $w \in W$,

$$w \in v_+(p) \text{ iff } w \in v(\nu_1(\varphi))$$
$$w \in v_-(p) \text{ iff } w \in v(\nu_2(\varphi)).$$

Checking that v_+ and v_- act homomorphically on both non-modal and modal formulas is straightforward (see the next section), as is to conclude that $\langle W, R_+, R_-, v_+, v_- \rangle$ is the counter-model we were looking for. We thus obtain both a method for proving completeness of $\vdash_g^*$ and $\vdash_l^*$, which again relies on completeness of classical bimodal logic, and the desired finite model property result.

7.4. **Duality and completeness.** As we did in Subsection 6.1, we can obtain a duality for modal N4-lattices through the duality for bimodal Boolean algebras. In fact, the correspondence between modal N4-lattices and twist-structures is still one-to-one, provided we associate to a given modal N4-lattice a triple $\langle \mathbf{A}, \nabla, \Delta \rangle$ with $\mathbf{A}$ a bimodal Boolean algebra and $\nabla, \Delta \subseteq A$ being respectively, a filter and an ideal satisfying the property stated in item (ii) of Theorem 7.3. Of course, the duality for bimodal Boolean algebras must itself be adjusted to account for the additional structure given by ∇ and Δ. This is rather straightforward and can be done along the lines of [23], where a duality of this type is developed for non-modal N4-lattices viewed as twist-structures.

A full duality is anyway not needed for proving completeness of $\vdash_l^*$ and $\vdash_g^*$ with respect to the relational semantics introduced in Subsection 7.2. The proof strategy is the same as that of the proofs of Theorems 6.5 and 6.6. Let us see the case of local consequence.

We assume $\Gamma \nvdash_l^* \varphi$. By Theorem 7.5, we can find a modal N4-lattice $\mathbf{B}$, a special filter $F \subseteq B$ and a homomorphism $h\colon \mathbf{Fm} \to \mathbf{B}$ such that $h[\Gamma] \subseteq F$ but $h(\varphi) \notin F$. By Theorem 7.3, we can assume that $\mathbf{B} \leq \mathbf{A}^{\bowtie}$ with $\mathbf{A}$ a bimodal Boolean algebra. In this case we also know that $F = (\nabla \times A) \cap B$, with ∇ a lattice filter of $\mathbf{A}$. Then $\pi_1[h[\Gamma]] \subseteq \nabla$ but $\pi_1(h(\varphi)) \notin \nabla$. By the Ultrafilter Theorem, there is an ultrafilter $P \supseteq \nabla$ such that $\pi_1(h(\varphi)) \notin P$. Then, the bimodal space $\mathcal{X}(\mathbf{A}) = \langle X(\mathbf{A}), \tau_{\mathbf{A}}, R_{\Box_+}, R_{\Box_-} \rangle$ gives us a topological counter-model, for $P \in \Phi(\pi_1(h(\gamma)))$ for all $\gamma \in \Gamma$ but $P \notin \Phi(\pi_1(h(\varphi)))$. In this case $\langle X(\mathbf{A}), R_{\Box_+}, R_{\Box_-} \rangle$ is already a Kripke frame of the kind defined in Subsection 7.2. We turn it into a model as we have done in Subsection 6.2, that is, defining two standard (two-valued) valuations $v_+, v_-\colon Var \to P(X(\mathbf{A}))$, for all $p \in Var$, as

$$v_+(p) := \{Q \in X(\mathbf{A}) : Q \in \Phi(\pi_1(h(p)))\}$$
$$v_-(p) := \{Q \in X(\mathbf{A}) : Q \in \Phi(\pi_1(\neg h(p)))\}.$$

These are extended to arbitrary formulas in the language $\langle \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t} \rangle$ in the same way as in Subsection 6.2. As for modal formulas, we let

$$v_+(\boxplus\varphi) := \Box_+ v_+(\varphi) \quad \text{and} \quad v_-(\boxplus\varphi) := {\sim}\Box_+ {\sim} v_-(\varphi)$$
$$v_+(\boxminus\varphi) := \Box_- {\sim} v_-(\varphi) \quad \text{and} \quad v_-(\boxminus\varphi) := v_-(\boxplus\varphi).$$

It is obvious that v_+ and v_- act homomorphically on both modal and non-modal formulas. That is, $M_{\mathbf{A}} = \langle X(\mathbf{A}), R_{\Box_+}, R_{\Box_-}, v_+, v_- \rangle$ is a Kripke model such that $M_{\mathbf{A}}, P \models \Gamma$ but $M_{\mathbf{A}}, P \not\models \varphi$. Hence, $\Gamma \not\models_l^* \varphi$ as desired.

Axiomatizing the extensions of $\vdash_l^*$ and $\vdash_g^*$ corresponding to restrictions on the accessibility relations considered in Subsection 6.3 is also straightforward. Consider, for example, idempotent frames, which are frames $\langle W, R_+, R_- \rangle$ such that $R_- \subseteq R_+$. As we have seen in Subsection 6.3, this corresponds to requiring that, for any algebraic model $\mathbf{B} \leq \mathbf{A}^{\bowtie}$, the bimodal Boolean algebra $\mathbf{A}$ satisfy the identity $\Box_+ x \leq \Box_- x$. It is easy to check that $\mathbf{A}$ satisfies $\Box_+ x \leq \Box_- x$ if and only if $\mathbf{B}$ satisfies ${\sim}\neg\boxplus x \supset \boxminus x = ({\sim}\neg\boxplus x \supset \boxminus x) \supset ({\sim}\neg\boxplus x \supset \boxminus x)$, which corresponds to the logical axiom ${\sim}\neg\boxplus p \supset \boxminus p$. Thus, if we add this axiom to our axiomatization of $\vdash_l^*$ (or $\vdash_g^*$), we obtain a sound and complete axiomatization for the local (global) consequence determined by the class of idempotent frames. Analogously, consistent frames are axiomatized by adding the axiom $\boxminus p \supset {\sim}\neg\boxplus p$ and classical frames correspond to $\boxminus p \equiv {\sim}\neg\boxplus p$. The latter is easily seen to be equivalent to $\boxminus p \leftrightarrow {\sim}\neg\boxplus p$, that is, to axiom ($\boxminus\,\boxplus$) of Subsection 7.1. This tells us that, as expected, the logic of [34] can now be viewed as the extension of our logic that corresponds to classical frames.

8. Further work

We list below a few open problems and what we believe might be interesting directions for future work.

- We have axiomatized extensions of the base logic that correspond to restrictions on the four-valued accessibility relation. We now know that the base logic can be equivalently defined starting from an arbitrary implicative bilattice. Thus, we might apply the restrictions considered in Subsection 6.3, suitably generalized, to an accessibility relation that is, instead of four-valued, $\mathbf{B}$-valued, $\mathbf{B}$ being any complete implicative bilattice. The definitions of idempotent and classical frames can be applied as they are, whereas it may make sense to consider a more liberal formulation for consistent frames. For instance, we could adopt the definition of *consistent element* of a bilattice given in [15, Definition 3.6] and say that a frame is consistent when the value of the accessibility relation is always a consistent element of the underlying bilattice. At this point we do not know whether and how it would be possible to axiomatize these extensions.

- We have dealt with the logic of the four-element bilattice or, equivalently, of any complete bilattice belonging to the variety generated by it. From a technical point of view, the semantic definition of the logic given in Section 3 could be recast replacing the four-element bilattice with any Brouwerian bilattice [8] or even one of the more general bilattices considered in [22]. One may thus wonder if it is possible to axiomatize these logics by the same methods as applied in this paper. This may not be straightforward, even in the case of Brouwerian bilattices, for these correspond to intuitionistic logic in the same way as implicative bilattices correspond to classical logic; and intuitionistic modal logic is at present far from being as well understood as the classical one.

- The semantics introduced in Section 7, unlike that of Section 3, does not require the presence of an implication in the logical language. This means that it might be possible to define a modal expansion of Belnap-Dunn logic whose non-modal core is a logic in the conjunction-disjunction-negation language, which is the one originally considered in [3, 4]. Algebraically, this would mean working with De Morgan lattices (see, e.g., [18]) instead of N4-lattices or bilattices. At this point it is not at all obvious whether the methods of this paper would be immediately applicable in this more general setting, because Belnap-Dunn logic in this language is not algebraizable [18, Theorem 2.11] and, moreover, a twist-structure representation is not available for De Morgan lattices.

- One problem that is left unsolved in [7] is whether it is possible to axiomatize the least modal logic over a finite residuated lattice in a language that does not include all the elements of the lattice as logical constants. As mentioned before, in the case of the four-element Belnap lattice we know that it is at least possible to dispense with one constant, namely $\perp$. Although we believe that the approach described in Section 7 is indeed an approximation to a solution of this question, we must point out that a fully general solution is yet to be found. As mentioned at the beginning of Section 7, the core of the problem seems to be that of devising a twist-structure representation that does not use any algebraic constant in an essential way.

- As mentioned before, our methods seem to be more powerful that those of [7] in the sense that the same strategy allowed us to prove completeness for both the global and the local consequence relation, whereas the proofs of [7] only work for local consequence. However, the scope of [7] is more general than ours as the authors were able to axiomatize the logic of an arbitrary finite residuated lattice (the four-element Belnap lattice being but one example, except the fact that it is not integral, which is however not essential). It is thus natural to ask ourselves if our methods could be applied to find an alternative and hopefully more satisfactory solution to the problems posed in [7]. The main obstacle in this respect seems to be that a topological duality theory for (non-modal) residuated lattices is not immediately available. However, if we restrict our attention to finitely-generated varieties of residuated lattices (which is the same setting as [7], too), then the theory of natural dualities [14] might provide a suitable basis to work on.

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References

[1] A. Almukdad and D. Nelson. Constructible falsity and inexact predicates. *The Journal of Symbolic Logic*, 49(1):231–233, 1984.

[2] O. Arieli and A. Avron. Reasoning with logical bilattices. *Journal of Logic, Language and Information*, 5(1):25–63, 1996.

[3] N. D. Belnap. How a computer should think. In G. Ryle, editor, *Contemporary Aspects of Philosophy*, pages 30–56. Oriel Press, Boston, 1976.

[4] N. D. Belnap, Jr. A useful four-valued logic. In J. M. Dunn and G. Epstein, editors, *Modern uses of multiple-valued logic (Fifth Internat. Sympos., Indiana Univ., Bloomington, Ind., 1975)*, pages 5–37. Episteme, Vol. 2. Reidel, Dordrecht, 1977.

[5] P. Blackburn, M. de Rijke, and Y. Venema. *Modal logic*. Number 53 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, Cambridge, 2001.

[6] W. J. Blok and D. Pigozzi. *Algebraizable logics*, volume 396 of *Mem. Amer. Math. Soc.* A.M.S., Providence, January 1989.

[7] F. Bou, F. Esteva, L. Godo, and R. Rodríguez. On the minimum many-valued modal logic over a finite residuated lattice. *Journal of Logic and Computation*, 21(5):739–790, 2011.

[8] F. Bou, R. Jansana, and U. Rivieccio. Varieties of interlaced bilattices. *Algebra Universalis*, 66(1):115–141, 2011.

[9] F. Bou and U. Rivieccio. The logic of distributive bilattices. *Logic Journal of the I.G.P.L.*, 19(1):183–216, 2011.

[10] F. Bou and U. Rivieccio. Bilattices with implications. *Studia Logica*, 2013. To appear.

[11] X. Caicedo and R. Oscar Rodríguez. Standard Gödel modal logics. *Studia Logica*, 94(2):189–214, 2010.

[12] X. Caicedo and R. Oscar Rodríguez. Bi-modal Gödel logic over $[0,1]$-valued Kripke frames, 2011.

[13] A. Chagrov and M. Zakharyaschev. *Modal Logic*, volume 35 of *Oxford Logic Guides*. Oxford University Press, 1997.

[14] D. M. Clark and B. A. Davey. *Natural dualities for the working algebraist*, volume 57 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1998.

[15] M. Fitting. Bilattices and the semantics of logic programming. *J. Logic Programming*, 11:91–116, 1991.

[16] M. Fitting. Many-valued modal logics. *Fundamenta Informaticae*, 15:235–254, 1991.

[17] M. Fitting. Many-valued modal logics, II. *Fundamenta Informaticae*, 17:55–73, 1992.

[18] J. M. Font. Belnap's four-valued logic and De Morgan lattices. *Logic Journal of the I.G.P.L.*, 5(3):413–440, 1997.

[19] J. M. Font and R. Jansana. *A general algebraic semantics for sentential logics*, volume 7 of *Lecture Notes in Logic*. Springer-Verlag, second edition, 2009. Electronic version freely available through Project Euclid at `projecteuclid.org/euclid.lnl/1235416965`.

[20] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: an algebraic glimpse at substructural logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2007.

[21] N. Galatos and J. G. Raftery. Adding involution to residuated structures. *Studia Logica*, 77(2):181–207, 2004.

[22] R. Jansana and U. Rivieccio. Residuated bilattices. *Soft Computing*, 16(3):493–504, 2012.

[23] R. Jansana and U. Rivieccio. Priestley duality for N4-lattices. *Proceedings of the 8th conference of the European Society for Fuzzy Logic and Technology*, to appear.

[24] A. Jung and U. Rivieccio. Priestley duality for bilattices. *Studia Logica*, 100(1-2):223–252, 2012.

[25] A. Jung and U. Rivieccio. Kripke semantics for modal bilattice logic. *Proceedings of the 28th Annual ACM/IEEE Symposium on Logic in Computer Science*, IEEE Computer Society Press, 438–447, 2013.

[26] M. Kracht. *Tools and techniques in modal logic*, volume 142 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1999.

[27] B. Mobasher, D. Pigozzi, G. Slutzki, and G. Voutsadakis. A duality theory for bilattices. *Algebra Universalis*, 43(2-3):109–125, 2000.

[28] S. P. Odintsov. Algebraic semantics for paraconsistent Nelson's logic. *Journal of Logic and Computation*, 13(4):453–468, 2003.

[29] S. P. Odintsov. On the representation of N4-lattices. *Studia Logica*, 76(3):385–405, 2004.

[30] S. P. Odintsov. The class of extensions of Nelson paraconsistent logic. *Studia Logica*, 80(2):291–320, 2005.

[31] S. P. Odintsov and E. I. Latkin. BK-lattices. Algebraic semantics for Belnapian modal logics, *Studia Logica*, 100(1-2): 319–338, 2012.
[32] S. P. Odintsov and H. Wansing. Inconsistency-tolerant description logic: motivation and basic systems. In *Trends in logic*, volume 21 of *Trends Log. Stud. Log. Libr.*, pages 301–335. Kluwer Acad. Publ., Dordrecht, 2003.
[33] S.P. Odintsov and H. Wansing. Constructive predicate logic and constructive modal logic. formal duality versus semantical duality. pages 269–286. Logos Verlag, 2004.
[34] S. P. Odintsov and H. Wansing. Modal logics with Belnapian truth values. *Journal of Applied Non-Classical Logics*, 20:279–301, 2010.
[35] G. Priest. Many-valued modal logics: a simple approach. *The Review of Symbolic Logic*, 1(2):190–203, 2008.
[36] U. Rivieccio. *An Algebraic Study of Bilattice-based Logics*. Ph. D. Dissertation, University of Barcelona, 2010.
[37] U. Rivieccio. Paraconsistent modal logics. *Electronic Notes in Theoretical Computer Science*, 278:173–186, 2011.

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