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DOI:
10.1016/j.jalgebra.2014.10.030

Journal of Algebra 423 (2015) 798–811

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On fixed-point-free automorphisms

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ARTICLE INFO

Article history:
Received 10 June 2014
Available online xxxx
Communicated by E.I. Khukhro

MSC:
20D45

Keywords:
Automorphism
Fixed-point-free
Fitting height

ABSTRACT

Let R be a cyclic group of prime order which acts on the extraspecial group F in such a way that $F = [F, R]$. Suppose RF acts on a group G so that $C_G(F) = 1$ and $(|R|, |G|) = 1$. It is proved that $F(C_G(R)) \subseteq F(G)$. As corollaries to this, it is shown that the Fitting series of $C_G(R)$ coincides with the intersections of $C_G(R)$ with the Fitting series of G, and that when $|R|$ is not a Fermat prime, the Fitting heights of $C_G(R)$ and G are equal.

1. Introduction

If a group A acts on a group G in such a way that $C_G(A) = 1$, then one can often say something about the structure of G given properties of A. For example, due to a result of V. Belyaev and B. Hartley [1, Theorem 0.11], if A is nilpotent, then G is soluble. It was conjectured by J. Thompson [11] that the Fitting height of a soluble group G, denoted $f(G)$, is bounded by a function of the order of one of its Carter subgroups (a Carter subgroup is a self-normalising nilpotent subgroup, and in any soluble group there is a single conjugacy class of such subgroups). It is applicable here since if a nilpotent group A acts on a group G so that $C_G(A) = 1$, then AG is a soluble group and A is a

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http://dx.doi.org/10.1016/j.jalgebra.2014.10.030

Carter subgroup. J. Thompson proved his conjecture in the case where $(|A|, |G|) = 1$. The bounds he obtained were improved in numerous papers that followed, most notably, linear bounds were found by H. Kurzweil [9] and best-possible by A. Turull [13]. This conjecture is a special case of the more general Fitting height conjecture, which can be stated as follows:

> Let A be a group which acts on the soluble group G so that $C_G(A) = 1$. Then the Fitting height of G is bounded above by the length of the longest chain of subgroups in A.

This has been largely settled when A is soluble of coprime order to G; many of these results are collected in [12]. Much of the recent work towards settling the Fitting height conjecture when $|A|$ is not assumed to be coprime to $|G|$ has concerned when A is cyclic. For example, it has been proved when A is cyclic of order a product of two and three distinct primes by K. Cheng [2] and G. Ercan and İ. Güloğlu [5] respectively. Further work has been done by G. Ercan in [3], where A is cyclic of order $p^n q$ for primes p and q greater than 3 and $n \in \mathbb{N}$.

However, E. Khukhro has taken a slightly different approach, and has considered the case where A has a nilpotent subgroup B so that $C_G(B) = 1$ and has asked: Can we bound the Fitting height of G in terms of how elements outside of B act on G? In particular, he has considered the case where A is a Frobenius group and has proved the following:

Theorem 1.1 *(Khukhro). Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H so that $C_G(F) = 1$. Then:*

1. *$F_i(C_G(H)) = F_i(G) \cap C_G(H)$ for all i; and*
2. *$f(G) = f(C_G(H))$.*

Proof. See [7, Theorem 2.1]. □

Since Frobenius kernels are nilpotent, E. Khukhro is still considering the situation where a nilpotent group acts fixed-point-freely on a group G, but there is also an 'additional' action which comes from the complement H; and indeed it is in terms of the action of this complement that he obtains structural information about G, namely, that its Fitting height is equal to that of the fixed-point subgroup of H.

In what follows, we also consider the situation where a group A acts on a group G in such a way that for some nilpotent subgroup $B < A$, we have $C_G(B) = 1$, and we obtain structural information about G in terms of how elements outside of this nilpotent subgroup act on G. Namely, we prove the following:

Theorem 1.2. *Let $R \cong \mathbb{Z}_r$ for some prime r and F be extraspecial. Suppose that R acts on F in such a way that $F = [F, R]$, and that RF acts on a group G so that $C_G(F) = 1$ and $(r, |G|) = 1$. Then $F(C_G(R)) \leq F(G)$.*

From this we obtain the following corollaries which reflect more obviously the recent work of E. Khukhro.

Corollary 1.3. *Let $R \cong \mathbb{Z}_r$ for some prime r and F be extraspecial. Suppose that R acts on F in such a way that $F = [F, R]$, and that RF acts on a group G so that $C_G(F) = 1$ and $(r, |G|) = 1$. Then $F_i(C_G(R)) = F_i(G) \cap C_G(R)$ for all i.*

Corollary 1.4. *Let $R \cong \mathbb{Z}_r$ for some non-Fermat prime r and F be extraspecial. Suppose that R acts on F in such a way that $F = [F, R]$, and that RF acts on a group G so that $C_G(F) = 1$ and $(r, |G|) = 1$. Then $f(C_G(R)) = f(G)$.*

It should be mentioned that by a theorem of A. Turull [13], we already have that $f(G) \leq f(C_G(R)) + 2$, even without any F.

In Section 2 we set some notation and recall some results which will be needed in the proof of Theorem 1.2. We will then prove Theorem 1.2 and Corollaries 1.3 and 1.4 in Section 3. The proof of Theorem 1.2 proceeds by considering a counterexample with $|RFG|$ minimal. A series of reductions are made until we find that $G = QV$ where Q is an RF-invariant Sylow q-subgroup of G, and $V = F(G)$ is minimal normal in RFG. Hence, V is an irreducible $\mathbb{F}_p[RFQ]$-module on which Q acts faithfully. We then consider $\overline{V}$, which we take to be an irreducible $k[RFQ]$-submodule of $W = V \otimes_{\mathbb{F}_p} k$ where k is a splitting field for RFQ. This is also a module on which Q acts faithfully. We obtain a contradiction by finding that the nontrivial subgroup $1 \neq O_q(C_G(R)) \subseteq Q$ acts trivially on $\overline{V}$.

After the present paper was submitted, the authors were informed by the referee about a recent paper by G. Ercan and İ. Güloğlu [4]. Here they consider a soluble finite group G admitting a 'Frobenius-like' group of automorphisms FR of odd order such that $|F'|$ is of prime order, $C_G(F) = 1$, and $(|G|, |R|) = 1$. ('Frobenius-like' means that F is a nilpotent normal subgroup and FR/F' is a Frobenius group with Frobenius kernel F/F' and complement R.) Theorem A of that paper asserts that $F_i(C_G(R)) = F_i(G) \cap C_G(R)$ for all i and $f(C_G(R)) = f(G)$. These results are of course very similar to Corollaries 1.3 and 1.4 of the present paper. They are more general in the sense they do not require R to be of prime order, but less general in their stipulation that RF must be of odd order. Furthermore, the authors were also made aware that [4] contains Proposition C, which can be used to significantly shorten the proof of Theorem 1.2 of the present paper. This proposition is as follows.

Proposition 1.5 *(Ercan–Güloğlu). Let FH be a Frobenius-like group such that F' is of prime order and $[F', H] = 1$. Suppose that FH acts on a q-group Q for some prime q coprime to the order of H. Let V be a $kQFH$-module where k is a field with characteristic not dividing $|QH|$. Suppose further that F acts fixed-point freely on the semidirect product VQ. Then we have*

$$\mathrm{Ker}\big(C_Q(H) \text{ on } C_V(H)\big) = \mathrm{Ker}\big(C_Q(H) \text{ on } V\big).$$

We will make it clear later how Proposition 1.5 can be used to shorten the proof of Theorem 1.2.

2. Preliminaries

Let G be a group. Then the *Fitting subgroup*, denoted $F(G)$, is the largest normal nilpotent subgroup of G. If we set $F_0(G) = 1$ and $F_1(G) = F(G)$, then we define $F_i(G)$ to be the full inverse image of $F(G/F_{i-1}(G))$ in G, for $i \geq 1$. Note that if G is soluble, then there exists $n \in \mathbb{N} \cup \{0\}$ such that $F_n(G) = G$, and the smallest such n is called the *Fitting height* of G. We denote this by $f(G)$.

The *Frattini subgroup*, denoted $\Phi(G)$, is defined to be the intersection of all maximal subgroups of G. We note that $\Phi(G) \subseteq F(G)$, and if $G \neq 1$, then $\Phi(G) \neq F(G)$. Also, for $N \trianglelefteq G$, we have $\Phi(N) \subseteq \Phi(G)$.

The next couple results highlight some very useful properties of the Fitting and Frattini subgroups.

Lemma 2.1. *Let G be a p-group such that $Z(\Phi(G)) \leq Z(G)$. Then $\Phi(G) \leq Z(G)$.*

Proof. Let $\overline{G} = G/Z(G)$ and let N be the inverse image of $\Omega_1(Z(\overline{G}))$. We obtain that $[N, \Phi(G)] = 1$. In particular, $N \cap \Phi(G) \leq Z(\Phi(G))$, and so by hypothesis, $N \cap \Phi(G) \leq Z(G)$. Then $\Omega_1(Z(\overline{G})) \cap \overline{\Phi(G)} = 1$. As $\overline{\Phi(G)} \trianglelefteq \overline{G}$ and $\overline{G}$ is a p-group, this implies $\overline{\Phi(G)} = 1$. $\square$

The following is a well-known generalisation of a theorem of Gaschütz [10, Theorem 1.12].

Lemma 2.2. *Let X be a group and $G \trianglelefteq X$. Set*

$$V = F(G)/\big(\Phi(X) \cap G\big).$$

1. *$V = F(G/(\Phi(X) \cap G))$;*
2. *V is a completely reducible X-module, possibly of mixed characteristic (by which we mean $V = V_1 \oplus \cdots \oplus V_n$ where for each i there exists a field $\mathbb{F}_i$ such that V_i is an $\mathbb{F}_i[X]$-module).*

One of the hypotheses of Theorem 1.2 is that the extraspecial group F acts on the group G so that $C_G(F) = 1$. The nilpotence of F here not only tells us that G is soluble, but also gives very useful information about the action of F on the Sylow subgroups of G and G/N for some F-invariant normal subgroup $N \trianglelefteq G$.

Theorem 2.3 *(Belyaev–Hartley). Let A be a finite nilpotent group which acts on a finite group G so that $C_G(A) = 1$. Then G is soluble.*

Proof. See [1, Theorem 0.11]. $\square$

Lemma 2.4. *Suppose that a finite group G admits a group RF of automorphisms where RF is the split extension of the nilpotent group F by R. Suppose further that $C_G(F) = 1$. Then there is a unique RF-invariant Sylow p-subgroup of G for each prime $p \in \pi(G)$.*

Proof. See [8, Lemma 2.6]. □

Lemma 2.5. *Let G be a finite group admitting a nilpotent group F of automorphisms such that $C_G(F) = 1$. If N is a normal F-invariant subgroup of G, then $C_{G/N}(F) = 1$.*

Proof. See [8, Lemma 2.2]. □

Throughout the proof of Theorem 1.2, we will often encounter the action of RF on some direct product. We now set some notation and state some results which will be very useful to us when considering these actions.

Definition 2.6. Let G be a group which acts on the set Ω. Then we define:

1. $\mathrm{Mov}_\Omega(G) = \{\alpha \in \Omega \mid \alpha^g \neq \alpha \text{ for some } g \in G\}$; and
2. $\mathrm{Fix}_\Omega(G) = \{\alpha \in \Omega \mid \alpha^g = \alpha \text{ for all } g \in G\}$.

Lemma 2.7. *Let RG be a group and V an irreducible RG-module on which G acts faithfully. Suppose $V = V_0 \oplus \cdots \oplus V_n$ where each V_i is a G-submodule of V. Let $H \leq RG$ be such that $H \subseteq C_G(V_1 \oplus \cdots \oplus V_n)$ and $G = \langle H^{RG} \rangle$. Then $G = G_0 \times \cdots \times G_n$ where $G_i = C_G(V_0 \oplus \cdots \oplus V_{i-1} \oplus V_{i+1} \oplus \cdots \oplus V_n)$.*

Proof. First note that $H \subseteq G_0$. Let $x \in RG$, and suppose $V_0^x = V_i$. Let $h \in H$ and $v \in V_j \neq V_i$. Then $v^{x^{-1}} \in V_k \neq V_0$, and so $[v^{x^{-1}}, h] = 1$. Hence $v^{h^x} = v$. Therefore, $h^x \in G_i$, and so we obtain that $G = \langle H^{RG} \rangle \subseteq G_0 \cdots G_n$. Note that each G_i is normal in G as the kernel of an action. Suppose there exists an i such that $G_i \bigcap \prod_{j \neq i} G_j \neq 1$, and let $1 \neq g \in G_i \bigcap \prod_{j \neq i} G_j$. Then g centralises $V_0 \oplus \cdots \oplus V_{i-1} \oplus V_{i+1} \oplus \cdots \oplus V_n$ since $g \in G_i$, and centralises V_i since $g \in \prod_{j \neq i} G_j$. Thus g is a nontrivial element of G which centralises V. This is a contradiction since V is a faithful G-module. Thus $G = G_i \times \prod_{j \neq i} G_j$. By induction it follows that $G = G_0 \times \cdots \times G_n$. □

Lemma 2.8. *Let G be a group which acts on a group $H = H_0 \times \cdots \times H_n$ in such a way that $C_H(G) = 1$ and for each $H_i \in \{H_0, \ldots, H_n\}$ and $g \in G$ we have $H_i^g \in \{H_0, \ldots, H_n\}$. Let $G_0 = N_G(H_0)$. Then $C_{H_0}(G_0) = 1$.*

Proof. Note that by induction, we may assume that G is transitive on $\{H_0, \ldots, H_n\}$. Now let $T = \{g_0, g_1, \ldots, g_n\}$ be a set of representatives for the right cosets of G_0 in G. Suppose $C_{H_0}(G_0) \neq 1$, and choose $1 \neq h \in C_{H_0}(G_0)$. Let $\hat{h} = \prod h^{g_i}$. We claim that $\hat{h}$ is fixed by G.

First note that elements in a common coset of G_0 in G act in the same way on h. Let $g_i' \in G_0 g_i$, so $g_i' = g g_i$ for some $g \in G_0$. Then $h^{g_i'} = h^{g g_i} = h^{g_i}$.

Now notice that the h^{g_i} commute. This follows since for distinct g_i, $g_j \in T$, h^{g_i} and h^{g_j} lie in distinct H_k. For any $g \in G$, the set Tg is another set of representatives for G_0 in G. Therefore, $\{h^{g_i g}\} = \{h^{g_i}\}$. Hence, $\hat{h}^g = (\prod\{h^{g_i}\})^g = \prod\{h^{g_i g}\} = \prod\{h^{g_i}\} = \hat{h}$ since the h^{g_i} commute. □

We now finish this section by outlining some of the representation theoretic results which we will require throughout Section 3.

Lemma 2.9. *Let $A = \langle a \rangle$ be a cyclic group which acts semiregularly on the abelian group N. Let V be a faithful $\mathbb{F}[AN]$-module where AN is the split extension of N by A. Assume that* $\mathrm{char}(\mathbb{F})$ *and $|N|$ are coprime and $C_V(N) = 0$. Then V_A is free.*

Proof. This is a special case of [7, Lemma 1.3]. □

Theorem 2.10 *(Flavell). Let r be a prime, $R \cong \mathbb{Z}_r$ and P an r'-group on which R acts. Let V be a faithful irreducible RP-module over a field of characteristic p such that $C_V(R) = 0$. Then either:*

1. $[R, P] = 1$*; or*
2. $[R, P]$ *is a nonabelian special* 2*-group and $r = 2^n + 1$ for some $n \in \mathbb{N}$.*

Proof. See [6, Theorem A]. □

3. Proof of the main result

The main aim of this section is to prove Theorem 1.2.

Let $R \cong \mathbb{Z}_r$ for some prime r act on the extraspecial s-group F in such a way that $F = [F, R]$. Then, clearly, $r \neq s$ and $C_F(R) \subseteq \Phi(F)$. In what follows we will show that if RF acts on a group G so that $C_G(F) = 1$, then $F(C_G(R)) \subseteq F(G)$. The proof will proceed by considering a minimal counterexample RFG. Thus we must have $C_F(R) = Z(F)$. Otherwise $C_F(R) = 1$; but we know that a counterexample does not exist in this case by Theorem 1.1. Hence, we will establish Theorem 1.2 by proving the following:

Theorem 3.1. *Let $R \cong \mathbb{Z}_r$ for some prime r and F be an extraspecial s-group. Suppose that R acts on F in such a way that $[R, Z(F)] = 1$ and $RF/Z(F)$ is a Frobenius group. Suppose further that RF acts on a group G so that $C_G(F) = 1$ and $(r, |G|) = 1$. Then $F(C_G(R)) \leq F(G)$.*

Proof. Since F is nilpotent, the condition $C_G(F) = 1$ forces G to be soluble by Theorem 2.3. We begin by considering a counterexample with $|RFG|$ minimal. So $F(C_G(R)) \nsubseteq F(G)$. For notational purposes set $X = RFG$, so $G \trianglelefteq X$.

Lemma 3.2. *With G and X as above, we obtain that $F(G)$ is a completely reducible X-module.*

Proof. We know by Lemma 2.2 that $F(G)/(\Phi(X) \cap G)$ is a completely reducible module for X. We work to show that $\Phi(X) \cap G = 1$. Suppose that $\Phi(X) \cap G \neq 1$ and set $\overline{G} = G/(\Phi(X) \cap G)$. By minimality, we have $F(C_{\overline{G}}(R)) \leq F(\overline{G})$. We also have by Lemma 2.2 that $\overline{F(G)} = F(\overline{G})$. Now $\overline{F(C_G(R))} \leq F(C_{\overline{G}}(R))$, and so $\overline{F(C_G(R))} \leq \overline{F(G)}$. Hence

$$F\bigl(C_G(R)\bigr)\bigl(\Phi(X) \cap G\bigr) \leq F(G)\bigl(\Phi(X) \cap G\bigr) = F(G).$$

However, this is a contradiction since $F(C_G(R)) \nsubseteq F(G)$. $\square$

Lemma 3.3. *There exists a prime p such that $F(G) = O_p(G)$ is an irreducible X-module.*

Proof. We know from Lemma 3.2 that $F(G)$ is a completely reducible X-module. Suppose that $F(G)$ is not an irreducible X-module, and let U and V denote two distinct irreducible X-submodules. Clearly, $U \cap V = 1$. Therefore, G embeds into $G/U \times G/V$ by the injective map given by $\phi(g) = (gU, gV)$.

Now let $\overline{G} = G/U$. Then $\overline{F(C_G(R))} \leq F(C_{\overline{G}}(R)) \leq F(\overline{G})$ where the inclusion on the right follows by minimality. Thus it follows that $\overline{\langle F(C_G(R))^G \rangle} \leq F(\overline{G})$. Similarly, if we set $\overline{G} = G/V$, then $\overline{\langle F(C_G(R))^G \rangle} \leq F(\overline{G})$. So the image of $\langle F(C_G(R))^G \rangle$ under ϕ is nilpotent. However, since ϕ is injective, $\langle F(C_G(R))^G \rangle$ must also be nilpotent. So $\langle F(C_G(R))^G \rangle \subseteq F(G)$, since $\langle F(C_G(R))^G \rangle \trianglelefteq G$. This is a contradiction since $F(C_G(R)) \nsubseteq F(G)$. $\square$

For notational purposes set $F(G) = V$.

Lemma 3.4. *There exists a nontrivial RF-invariant Sylow q-subgroup Q of G such that $G = QV$ for some prime $q \neq p$.*

Proof. Set $\overline{G} = G/V$. By minimality we have $\overline{F(C_G(R))} \leq F(\overline{G})$. Now $F(C_G(R)) \nsubseteq F(G)$, and so there exists a prime $q \neq p$ so that $\overline{O_q(C_G(R))} \neq 1$. By the above, we obtain that $\overline{O_q(C_G(R))} \leq O_q(\overline{G})$. Let K denote the full inverse image of $O_q(\overline{G})$ in G. So $O_q(C_G(R)) \subseteq K \trianglelefteq RFG$. Now K is RF-invariant, hence $C_K(F) = 1$. By Lemma 2.4, there exists a unique RF-invariant Sylow q-subgroup Q of K. Thus $K = QV$. However, $F(K) = V$, and so by minimality it follows that $G = K$. $\square$

Lemma 3.5. *Let $1 \neq H \leq O_q(C_G(R))$. Then $Q = \langle H^F \rangle$.*

Proof. By Lemma 3.4, $G = QV$ where Q is an RF-invariant Sylow q-subgroup of G. By coprime action, we obtain that $O_q(C_G(R)) \leq Q$.

Set $Q_0 = \langle H^{RF} \rangle$. Then $Q_0 = \langle H^F \rangle$, since H is centralised by R. Suppose $Q_0 < Q$, and set $G_0 = Q_0V$. Now $C_G(V) = V$, and so $O_q(G_0) = 1$. By minimality, we obtain that

$F(C_{G_0}(R)) \leq F(G_0) = V$. However, $1 \neq H \subseteq F(C_{G_0}(R))$. This contradiction forces $Q_0 V = G_0 = G = QV$, and so $Q_0 = Q$. $\square$

We may consider V as an irreducible $\mathbb{F}_p[RFQ]$-module. We now extend the ground field to a splitting field k for RFQ, and consider $W = V \otimes_{\mathbb{F}_p} k$. Henceforth, let $\overline{V}$ be an irreducible $k[RFQ]$-submodule of W.

Lemma 3.6. *Q acts faithfully on $\overline{V}$ and $C_{\overline{V}}(F) = 0$.*

Proof. Suppose Q does not act faithfully on $\overline{V}$. Then there exists $1 \neq K \subseteq Q$ with $K \trianglelefteq RFQ$ so that $C_{\overline{V}}(K) = \overline{V}$. Now $C_{\overline{V}}(K) \subseteq C_W(K) = C_V(K) \otimes_{\mathbb{F}_p} k$, and so $C_V(K) \neq 0$. Since $K \trianglelefteq RFQ$, it follows that $C_V(K)$ is normalised by RFQ. By the irreducibility of V, we have $C_V(K) = V$. However, Q acts faithfully on V.

The second claim follows as $C_W(F) = C_V(F) \otimes_{\mathbb{F}_p} k = 0$ and $C_{\overline{V}}(F) \subseteq C_W(F)$. $\square$

Lemma 3.7. $[C_{\overline{V}}(R), O_q(C_G(R))] = 0$.

Proof. Note that $C_G(R) = C_V(R)C_Q(R)$ where $C_V(R) \trianglelefteq C_G(R)$. Thus

$$\left[C_V(R), O_q\left(C_G(R)\right)\right] = C_V(R) \cap O_q\left(C_G(R)\right) = 1.$$

By considering $C_V(R)$ as an $\mathbb{F}_p[O_q(C_G(R))]$-module, we obtain that $[C_W(R), O_q(C_G(R))] = 0$. Since $C_{\overline{V}}(R) \subseteq C_W(R)$, it follows that $[C_{\overline{V}}(R), O_q(C_G(R))] = 0$. $\square$

At this stage we note that we could invoke Proposition 1.5 to finish the proof of Theorem 1.2. As $O_q(C_G(R)) \subseteq C_Q(R)$, Proposition 1.5 together with Lemma 3.7 tells us that $O_q(C_G(R))$ acts trivially on $\overline{V}$. However, since Q is faithful on $\overline{V}$, we obtain that $O_q(C_G(R)) = 1$. It follows that $F(C_G(R)) \subseteq F(G)$, which is a contradiction. We will now continue the proof of Theorem 1.2 without an appeal to Proposition 1.5.

Lemma 3.8. *Suppose $\overline{V}$ is an imprimitive module for RFQ. Then $O_q(C_G(R))$ centralises any block which is not normalised by R.*

Proof. Let $\overline{V} = U_0 \oplus \cdots \oplus U_n$ where the U_i are blocks of imprimitivity in the action of RFQ on $\overline{V}$, and set $\Omega = \{U_0, \ldots, U_n\}$. Let $R = \langle a \rangle$. We want to show that $O_q(C_G(R))$ centralises

$$U = \bigoplus_{U_i \in \mathrm{Mov}_\Omega(R)} U_i.$$

Clearly, $O_q(C_G(R))$ acts on $\mathrm{Mov}_\Omega(R)$. Let $U_i \in \mathrm{Mov}_\Omega(R)$, and consider

$$U' = \bigoplus_{j=1}^{r} U_i^{a^j}.$$

Then for $u \in U_i$, $w = u + u^a + \cdots + u^{a^{r-1}}$ is centralised by R. Thus it is also centralised by $O_q(C_G(R))$. Hence, U' is normalised by $O_q(C_G(R))$. An r-cycle in $\mathrm{Sym}(r)$ is self-centralising, so as $O_q(C_G(R))$ is an r'-group, it follows that $O_q(C_G(R))$ normalises U_i. Since w is centralised by $O_q(C_G(R))$, and $O_q(C_G(R))$ normalises U_i, it follows that $O_q(C_G(R))$ must centralise u. □

Henceforth, we will write $\overline{V} = V_0 \oplus \cdots \oplus V_n$ where the V_i are the homogeneous components with respect to $Z(Q)$. Set $\Gamma = \{V_0, \ldots, V_n\}$.

Our next major goal is to prove that $[Z(Q), Z(F)] \neq 1$. We thus proceed with the assumption that this is not the case and work to obtain a contradiction. We first need a few lemmas.

Lemma 3.9. *Assume $[Z(Q), Z(F)] = 1$. Then R has only one fixed point on Γ.*

Proof. Let $R = \langle a \rangle$. By Lemma 3.8, we obtain that $O_q(C_G(R))$ centralises all of the subspaces $V_i \in \mathrm{Mov}_\Gamma(R)$. Now $C_Q(\overline{V}) = 1$ by Lemma 3.6, and so we have the strict inclusion $\mathrm{Mov}_\Gamma(R) \subset \Gamma$. Now $\mathrm{Fix}_\Gamma(R) \neq \emptyset$, hence R is in the stabiliser of a point in the action of RFQ on Γ, and is a Sylow r-subgroup of this stabiliser. Thus $N_{RFQ}(R)$ acts transitively on $\mathrm{Fix}_\Gamma(R)$. Now $N_{RFQ}(R) = RZ(F)C_Q(R)$. Clearly R acts trivially on $\mathrm{Fix}_\Gamma(R)$. Also $[C_Q(R), Z(Q)] = 1$, and by hypothesis we have $[Z(Q), Z(F)] = 1$. Therefore, $Z(F)C_Q(R) \subseteq C_{RFQ}(Z(Q))$. Thus by Clifford's Theorem, $Z(F)C_Q(R)$ acts trivially on Γ. In particular, $Z(F)C_Q(R)$ acts trivially on $\mathrm{Fix}_\Gamma(R)$, and so $|\mathrm{Fix}_\Gamma(R)| = 1$. □

In the following lemma, let $Q_i = C_Q(V_0 \oplus \cdots \oplus V_{i-1} \oplus V_{i+1} \oplus \cdots \oplus V_n)$.

Lemma 3.10. *If $[Z(Q), Z(F)] = 1$, then $Q = Q_0 \times \cdots \times Q_n$.*

Proof. We can assume without loss of generality that $\mathrm{Fix}_\Gamma(R) = \{V_0\}$. Since R has no fixed points on $\Gamma - \{V_0\}$, it follows by Lemma 3.8 that $V_1 \oplus \cdots \oplus V_n$ is centralised by $O_q(C_G(R))$. Thus $O_q(C_G(R)) \subseteq Q_0$. The result now follows from Lemma 2.7 with Q and $\overline{V}$ in place of G and V respectively. □

Let $F_0 = N_F(V_0)$. Then $F_0 \neq 1$ otherwise V_0 would be in a regular orbit under the action of F on Γ. Thus $\langle V_0^F \rangle$ would be a free F-module and so $C_{\overline{V}}(F) \neq 1$ contrary to Lemma 3.6.

Lemma 3.11. *If $[Z(Q), Z(F)] = 1$, then $Q_0 = \langle O_q(C_G(R))^{F_0} \rangle$.*

Proof. Let $f \in F$ and suppose $V_0^f = V_i$. Let $g \in O_q(C_G(R))$ and $v \in V_j \neq V_i$. Then $v^{f^{-1}} \in V_k \neq V_0$, and so $[v^{f^{-1}}, g] = 1$. Hence $v^{g^f} = v$. Therefore, $g^f \in Q_i$, and so $O_q(C_G(R))^f \subseteq Q_0$ if and only if $f \in F_0$.

Now $Q/(Q_1\times\cdots\times Q_n)\cong Q_0$ and $\langle O_q(C_G(R))^{F-F_0}\rangle\subseteq Q_1\times\cdots\times Q_n$. So if we consider the canonical epimorphism $\phi: Q\longrightarrow Q/(Q_1\times\cdots\times Q_n)$, it follows that $\langle O_q(C_G(R))^{F_0}\rangle$ maps onto $Q/(Q_1\times\cdots\times Q_n)$ under ϕ. So

$$Q=\langle O_q\bigl(C_G(R)\bigr)^{F_0}\rangle\times Q_1\times\cdots\times Q_n.$$

By considering orders it follows that $|Q_0| = |\langle O_q(C_G(R))^{F_0}\rangle|$. Hence $Q_0 = \langle O_q(C_G(R))^{F_0}\rangle$. □

Lemma 3.12. *If* $[Z(Q),Z(F)]=1$*, then* $C_{Q_0}(F_0)=1$.

Proof. By noting that $C_Q(F)=1$, this follows by Lemma 2.8 with F and Q in place of G and H respectively. □

Lemma 3.13. $[Z(Q),Z(F)]\neq 1$.

Proof. Assume that this is not the case so $[Z(Q),Z(F)]=1$. Then $Q\cong Q_0\times\cdots\times Q_n$ where the Q_i are defined as in Lemma 3.10, and $C_{Q_0}(F_0)=1$ by Lemma 3.12.

Now since the V_i are homogeneous components for $Z(Q)$, and k is a splitting field for $Z(Q)$, $Z(Q)$ acts on V_0 by scalars. However, $Z(Q)=Z(Q_0)\times\cdots\times Z(Q_n)$, and $Z(Q_1)\times\cdots\times Z(Q_n)$ acts trivially on V_0. So $Z(Q_0)$ acts on V_0 nontrivially by scalars, otherwise $C_Q(\overline{V})\neq 1$ as $Z(Q_0)\neq 1$. This follows since $1\neq O_q(C_G(R))\subseteq Q_0$. We know that $Z(Q_0)$ acts by scalars on V_0, and so every element in $[F_0,Z(Q_0)]$ acts trivially on V_0. However, since Q_0 acts faithfully on V_0, and $[F_0,Z(Q_0)]\subseteq Z(Q_0)$, it follows that F_0 must centralise $Z(Q_0)$, and thus $1\neq C_{Q_0}(F_0)$. This is a contradiction to Lemma 3.12. □

Corollary 3.14. $Z(F)$ *acts semiregularly on* Γ.

Proof. Suppose $Z(F)$ normalises some $V_j\in\Gamma$. Since RF is transitive on Γ, and $Z(F)=Z(RF)$, we find that $Z(F)$ acts trivially on Γ. Now $Z(Q)$ acts on each $V_i\in\Gamma$ by scalars, and so $[Z(F),Z(Q)]$ must act trivially on each $V_i\in\Gamma$. This forces $[Z(F),Z(Q)]=1$ since $C_Q(\overline{V})=1$, which is a contradiction to Lemma 3.13. □

Lemma 3.15. Q *acts trivially on any system of imprimitivity in the action of* RFQ *on* $\overline{V}$.

Proof. Let $\overline{V}=U_0\oplus\cdots\oplus U_n$ where the U_i are blocks of imprimitivity in the action of RFQ on $\overline{V}$, and set $\Omega=\{U_0,\dots,U_n\}$. We work to show that $O_q(C_G(R))$ acts trivially on Ω. Then the normal closure of $O_q(C_G(R))$ in RFQ will also act trivially on Ω. Since $\langle O_q(C_G(R))^{RFQ}\rangle=Q$, the claim will follow.

Let $R=\langle a\rangle$. Then $O_q(C_G(R))$ centralises any $U_i\in \mathrm{Mov}_\Omega(R)$ by Lemma 3.8. Also, as in the proof of Lemma 3.9, we get that $\mathrm{Fix}_\Omega(R)\neq\emptyset$ and $N_{RFQ}(R)$ is transitive on $\mathrm{Fix}_\Omega(R)$. Now $N_{RFQ}(R)=RZ(F)C_Q(R)$. Clearly, R acts trivially on $\mathrm{Fix}_\Omega(R)$. Let $U_j\in\mathrm{Fix}_\Omega(R)$, and suppose $Z(F)\nsubseteq N_F(U_j)$. Then $N_F(U_j)\cap Z(F)=1$ since

$Z(F)$ is cyclic of prime order. In particular, R acts semiregularly on $N_F(U_j)$ because $C_F(R) = Z(F)$. Note that $N_F(U_j) \neq 1$, otherwise F would have a regular orbit on Ω and thus a nontrivial fixed point on $\overline{V}$, contrary to Lemma 3.6. Therefore, $N_F(U_j)$ must be elementary abelian since it is isomorphic to its image under the canonical epimorphism $\varphi : F \longrightarrow F/Z(F)$. Also $C_{U_j}(N_F(U_j)) = 0$ by Lemma 2.8. Hence $C_{U_j}(R) \neq 0$ by Lemma 2.9. Thus by Lemma 3.7, $O_q(C_G(R))$ normalises U_j.

Suppose that $O_q(C_G(R))$ does not normalise $U_j \in \mathrm{Fix}_\Omega(R)$. Then reasoning as above we must have $Z(F) \subseteq N_F(U_j)$ and $C_{U_j}(R) = 0$. Thus $C_Q(R)$ can only map U_j to a subspace $U_i \in \mathrm{Fix}_\Omega(R)$ which itself is normalised by $Z(F)$. So $Z(F)$ must act trivially on $\mathrm{Fix}_\Omega(R)$, otherwise we get two distinct orbits in the action of $N_{RFQ}(R)$ on $\mathrm{Fix}_\Omega(R)$. Since $Z(F)$ is trivial on $\mathrm{Fix}_\Omega(R)$, we obtain that $Z = [O_q(C_G(R)), Z(F)]$ is also trivial on $\mathrm{Fix}_\Omega(R)$. Also, Z centralises each subspace $U_j \in \mathrm{Mov}_\Omega(R)$ since $Z \subseteq O_q(C_G(R))$, and so Z acts trivially on Ω. Note that $Z \neq 1$, since $[Z(Q), Z(F)] \neq 1$, and so by Lemma 3.5 we have $Q = \langle Z^F \rangle$. Thus it follows that Q also acts trivially on Ω. This is a contradiction since U_j is not normalised by $O_q(C_G(R))$. □

Corollary 3.16. *Every characteristic abelian subgroup of Q is contained in $Z(Q)$.*

Proof. Let A be a characteristic abelian subgroup of Q. Let $\overline{V} = U_0 \oplus \cdots \oplus U_n$ where the U_i are homogeneous components with respect to A. Then $\Omega = \{U_0, \dots, U_n\}$ is a system of imprimitivity for RFQ on $\overline{V}$, and so Q is trivial on Ω. Since A acts by scalars on any given $U_i \in \Omega$, $[Q, A]$ centralises $\overline{V}$. This forces $[Q, A] = 1$, and thus $A \subseteq Z(Q)$. □

Corollary 3.17. *Q has nilpotence class at most two.*

Proof. Since every characteristic abelian subgroup of Q is contained in $Z(Q)$, it follows that $Z(\Phi(Q)) \subseteq Z(Q)$. Thus $\Phi(Q) \subseteq Z(Q)$ by Lemma 2.1, and so $Q/Z(Q)$ is abelian. □

Recall that Γ is the set of $Z(Q)$-homogeneous components in $\overline{V}$. We know that the subset of components in Γ which are normalised by R is nonempty, and that $N_{RF}(R) = R \times Z(F)$ acts transitively on this set. We also know, since $Z(F) \trianglelefteq RF$, that the orbits of the action of $Z(F)$ on Γ forms a system of imprimitivity $\overline{V} = W_0 \oplus \cdots \oplus W_m$ for the action of RF on Γ. We can assume without loss of generality that V_0 is normalised by R and that W_0 is the direct sum of components in the orbit of V_0 under the action of $Z(F)$ on Γ. Henceforth, we set $Q_i = C_{\overline{V}}(W_0 \oplus \cdots \oplus W_{i-1} \oplus W_{i+1} \oplus \cdots \oplus W_m)$, and find that $Q = Q_0 \times \cdots \times Q_m$, which follows exactly as in Lemma 3.10.

Lemma 3.18. $Q_0 = \langle O_q(C_G(R))^{N_F(V_0)} \rangle$.

Proof. We first show that $N_F(W_0) = Z(F) \times N_F(V_0)$. We can assume without loss of generality that $W_0 = V_0 \oplus \cdots \oplus V_{s-1}$. Set $\Delta = \{V_0, \dots, V_{s-1}\}$. By definition $Z(F) \subseteq N_F(W_0)$ and is transitive on Δ. In particular, since $|\Delta| = s$, $N_F(W_0)$ is primitive on Δ. Since $N_F(W_0)$ is an s-group and $|\Delta| = s$, $N_F(V_0)$ must be the full kernel in the action of $N_F(W_0)$ on Δ. We find that $N_F(W_0)/N_F(V_0)$ is regular on Δ, and so $N_F(W_0)/N_F(V_0) \cong \mathbb{Z}_s$. Thus it follows that $N_F(W_0) = Z(F) \times N_F(V_0)$. Arguing exactly as in the proof of Lemma 3.11, we find that $Q_0 = \langle O_q(C_G(R))^{N_F(W_0)}\rangle$. Now $[R, Z(F)] = 1$, and so $O_q(C_G(R))^{Z(F)} = O_q(C_G(R))$. Thus

$$Q_0 = \langle O_q\big(C_G(R)\big)^{Z(F)\times N_F(V_0)}\rangle = \langle O_q\big(C_G(R)\big)^{N_F(V_0)}\rangle. \qquad \square$$

Lemma 3.19. $[Z(Q_0), R] = 1$.

Proof. Since the subspaces $V_i \subseteq \overline{V}$ are homogeneous components for $Z(Q)$, $Z(Q_0)$ acts on them by scalars. Now W_0 is the direct sum of components which are normalised by R. Since $Z(Q_0)$ acts by scalars on any given $V_i \subseteq W_0$, it follows that $[Z(Q_0), R]$ acts trivially on W_0. However, Q is faithful on $\overline{V}$, and since Q_0 centralises $W_1 \oplus \cdots \oplus W_m$, this forces $[Z(Q_0), R] = 1$. $\square$

Lemma 3.20. *Q is abelian.*

Proof. Note that $Q' \cap O_q(C_G(R)) = 1$. Hence

$$\big[O_q\big(C_G(R)\big), C_Q(R)\big] \subseteq Q' \cap O_q\big(C_G(R)\big) = 1,$$

and so $O_q(C_G(R)) \subseteq Z(C_Q(R))$. By Lemma 3.19, we have $[Z(Q_0), R] = 1$, and so since Q has nilpotence class at most two, $Q_0 = [Q_0, R] * C_{Q_0}(R)$. However, since $O_q(C_G(R)) \subseteq Z(C_Q(R))$, it follows that $O_q(C_G(R)) \subseteq Z(C_{Q_0}(R))$, and so $O_q(C_G(R)) \subseteq Z(Q_0) \subseteq Z(Q)$. Set $G_0 = Z(Q)V$. If $G_0 < G$, then $F(C_{G_0}(R)) \subseteq F(G_0) = V$ by induction. However, since $O_q(C_G(R)) \subseteq Z(Q)$, there are clearly q-elements in $F(C_{G_0}(R))$. Thus $G_0 = G$, and so $Z(Q) = Q$. $\square$

We now complete the proof of Theorem 3.1.

It follows from Corollary 3.14 that $Z(F) \nsubseteq N_F(V_i)$ for any $V_i \in \Gamma$. Thus $Z(F) \cap N_F(V_0) = 1$ since $Z(F)$ is cyclic of prime order. Hence $N_F(V_0) = [R, N_F(V_0)]$. Since Q is abelian, Lemma 3.19 now says that $[Q_0, R] = 1$, and so $[Q_0, N_F(V_0)] = 1$. Thus $Q_0 = \langle O_q(C_G(R))^{N_F(V_0)}\rangle = O_q(C_G(R))$. Now $N_F(V_0)$ is abelian, and $C_{V_0}(N_F(V_0)) = 0$ by Lemma 2.8. Hence $C_{V_0}(R) \neq 0$ by Lemma 2.9. Since $[O_q(C_G(R)), C_V(R)] = 1$, we must have that Q_0 acts trivially on $C_{V_0}(R)$. However, V_0 is a homogeneous component for Q_0, and so Q_0 must act trivially on V_0. It follows that Q_0 acts trivially on W_0 and thus Q_0 acts trivially on $\overline{V}$. However, this is a contradiction since $C_Q(\overline{V}) = 1$. $\square$

Corollary 3.21. *Let $R \cong \mathbb{Z}_r$ for some prime r and F be an extraspecial s-group. Suppose that R acts on F in such a way that $[R, Z(F)] = 1$ and $RF/Z(F)$ is a Frobenius group. Suppose further that RF acts on a group G so that $C_G(F) = 1$ and $(r, |G|) = 1$. Then $F_i(C_G(R)) = F_i(G) \cap C_G(R)$ for all i.*

Proof. Let $i \in \mathbb{N}$ be the least such that $F_i(C_G(R)) \nsubseteq F_i(G)$. We know that $F(C_G(R)) \leq F(G)$ by Theorem 3.1, and so $i > 1$. Let $\overline{G} = G/F_{i-1}(G)$, and ϕ be the canonical epimorphism from G onto $\overline{G}$. Now $\overline{F_i(C_G(R))} \trianglelefteq \overline{C_G(R)}$ and $\overline{F_i(C_G(R))}$ is nilpotent since $F_{i-1}(C_G(R)) \subseteq \ker(\phi)$. Now

$$\overline{F_i\big(C_G(R)\big)} \leq F\big(\overline{C_G(R)}\big) = F\big(C_{\overline{G}}(R)\big) \leq F(\overline{G}).$$

By the definition of $F_i(G)$, we have $F(\overline{G}) = \overline{F_i(G)}$. Therefore, $\overline{F_i(C_G(R))} \subseteq \overline{F_i(G)}$. However, this is a contradiction since $F_i(C_G(R)) \nsubseteq F_i(G)$. □

Corollary 3.22. *Let $R \cong \mathbb{Z}_r$ for some non-Fermat prime r and F be an extraspecial s-group. Suppose that R acts on F in such a way that $[R, Z(F)] = 1$ and $RF/Z(F)$ is a Frobenius group. Suppose further that RF acts on a group G so that $C_G(F) = 1$ and $(r, |G|) = 1$. Then $f(C_G(R)) = f(G)$.*

Proof. Let $n \in \mathbb{N}$ be the Fitting height of $C_G(R)$. Now we know that $F_n(C_G(R)) = F_n(G) \cap C_G(R)$, and so $C_G(R) \leq F_n(G)$. We work to show that $F_n(G) = G$. Suppose this is not the case, so $F_n(G) < G$. Let S be an RF-invariant section of $G/F_n(G)$ which has no proper RF-invariant subgroups. By coprime action, R acts fixed point freely on $G/F_n(G)$. Also, F acts nontrivially on S since $C_{G/F_n(G)}(F) = 1$. By Theorem 2.10, it follows that either $[R, F/C_F(S)] = 1$ or r is a Fermat prime. By hypothesis, the former must hold. Since $C_F(R) = \Phi(F)$, we obtain that $F = C_F(S)\Phi(F)$. However, this implies $F = C_F(S)$, which is a contradiction. □

Note that if G is a minimal counterexample to Corollary 3.22, we obtain that $f(G) \leq f(C_G(R)) + 1$ in any case, since $G/F_n(G)$ admits a fixed-point-free automorphism of prime order.

Corollaries 3.21 and 3.22 together with Theorem 1.1 confirm Corollaries 1.3 and 1.4 stated in the introduction. Note that we cannot drop the condition that r be a non-Fermat prime in Corollary 3.22. In particular, if $R \cong \mathbb{Z}_r$ where r is a prime of the form $r = 2^n + 1$, then there exists an extraspecial group F on which R acts such that $F = [F, R]$, and a group G on which RF acts such that $C_G(F) = 1$, $f(G) = 1$ and $f(C_G(R)) = 0$.

Acknowledgments

This work forms part of the PhD thesis of the first author who was supported by an EPSRC doctoral training grant. The authors would also like to thank the referee for his thorough reading of this manuscript and his numerous suggestions for improvement.

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