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Sufficient conditions for unique stable sets in three agent pillage games[☆]

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HIGHLIGHTS

- Pillage games allow externalities between coalitions and allow power to depend on resources.
- They are richer than games in characteristic or partition function form.
- When $n = 3$, three axioms ensure stable sets' uniqueness, and restrict them to no more than 15 elements.
- We present an algorithm for deciding existence and computing stable sets under these conditions.

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ABSTRACT

Pillage games (Jordan, 2006a) have two features that make them richer than cooperative games in either characteristic or partition function form: they allow power externalities between coalitions; they allow resources to contribute to coalitions' power as well as to their utility. Extending von Neumann and Morgenstern's analysis of three agent games in characteristic function form to anonymous pillage games, we characterise the core for any number of agents; for three agents, all anonymous pillage games with an empty core represent the same dominance relation. When a stable set exists, and the game also satisfies a continuity and a responsiveness axiom, it is unique and contains no more than 15 elements, a tight bound. By contrast, stable sets in three agent games in characteristic or partition function form may not be unique, and may contain continua. Finally, we provide an algorithm for computing the stable set, and can easily decide non-existence. Thus, in addition to offering attractive modelling possibilities, pillage games seem well behaved and analytically tractable, overcoming a difficulty that has long impeded use of cooperative game theory's flexibility.

1. Introduction

Opponents of a newly elected government nonetheless pay taxes, knowing its supporters are more powerful than they are; international alliances are formed and allies supported to maintain both their allegiance and their effectiveness; firms transfer resources (such as patents) to other firms to help them compete against mutual rivals (Kwong, 2011); junior staff often tolerate abusive behaviour from their seniors without protest, knowing that a challenge would become a power contest which they would lose; merchant ships surrender rather than fight pirates or privateers (Kontorovich, 2004).

All the above are examples of power contests in which power is derived both from agents' inalienable attributes as well as their transferable resources. Further, while resources may be transferred, none are actually consumed by the contest.[1]

Their study presents two problems for typical analyses. First, Aumann (2008) has argued that, in general, "procedures are not

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[1] This latter feature seems most consistent with common knowledge of power. If so, the use of terms like 'pillage' and 'jungle' in the costless transfer literature (Jordan, 2006a; Piccione and Rubinstein, 2007) may be misleading, calling to mind exceptional interactions between relatively unknown parties.

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really all that relevant; that it is the possibilities for coalition forming, promising and threatening that are decisive, rather than whose turn it is to speak". This argument is especially strong when, in the absence of strong institutions, any proposed game forms would themselves be contested. Second, while cooperative game theory works directly with a dominance relation defined on outcomes, abstracting from game forms, its two most common formulations treat power and utility dichotomously: the contested goods yield utility, but cannot contribute to the contest of power.

Games in *characteristic function form*[2] (CF) allow power to depend only on coalitions' absolute inalienable attributes; consequently, analyses of such games tend to predict that the grand coalition of all agents forms, ruling out the very real possibility of conflict between coalitions "from the beginning" (Maskin, 2003).[3]

Games in *partition function form* (PF) generalise CF games, allowing power to depend on coalitions' relative inalienable attributes, and therefore for externalities across coalitions (Thrall and Lucas, 1963; Maskin, 2003; de Clippel and Serrano, 2008). However, they remain unable to model the possibility that transferring resources may transfer power as well as utility.

Jordan (2006a) introduced *pillage games*, a class of cooperative games whose dominance relations are represented by power functions, increasing in both coalitional membership and members' resource holdings. Thus, such games allow power to depend on both inalienable attributes and holdings of transferable resources without the imposition of restrictive game forms. As a result, they offer an alternative and possibly fruitful tool for the study of power contests.[4] To emphasise their relation to the better known classes of cooperative games, we also refer to pillage games as games in *power function form*.

To compare more concretely, consider a CF game that allows any sufficiently large coalition to split a pie. In its PF counterpart, the *majority game*, absolute size is insufficient: instead, the coalition must be larger than any other to split the pie. In the extension to power function form (the *majority pillage game* of Jordan and Obadia, 2004), larger coalitions again dominate, but ties in coalitional size are decided in favour of that with more resources.

Although games in power function form seem richer than PF games in allowing power to depend on resource holdings, they are nevertheless disjoint, replacing a constancy condition in PF games with a strict monotonicity condition. Further, the original theory of PF games (Thrall and Lucas, 1963) also made the "bloodthirsty" assumption (Ray and Vohra, 1997) that the residual agents opposed any coalition; by contrast, in power function form, a coalition is only opposed by those agents with opposed interests.[5]

Section 2 of this paper formally introduces games in power function form. While their independence from game forms and rich ability to model power are attractive, their behaviour is correspondingly sensitive to the choice of power function. Jordan (2006a) presented results for von Neumann and Morgenstern stable sets for three particular power functions: in the first, a unique stable set was derived; in the second, a (possibly non-unique) stable set was derived for odd numbers of agents; in the third, a non-existence result was proven. Jordan also proved that a set is stable in a pillage game if and only if there is a consistent expectation for which it is the (farsighted) core in expectation.[6] Thus, to the extent that the property of being undominated is an appealing one, and forward looking agents are seen as natural, stable sets are a compelling solution concept for pillage games. This is particularly fortunate given the problematic nature of specifying an extensive form non-cooperative game to describe a contest in which agents might seek to challenge such a game form: as noted by Harsanyi (1974), elements in stable sets of games in characteristic function form need not satisfy this dynamic consistency property, spawning – via Selten (1981) – a rich literature that seeks non-cooperative underpinnings to cooperative solutions.

As even a finite agent set generates an infinite number of possible power functions, a number of questions arise naturally. How are Jordan's three results related? How many other classes of stable sets are possible? What determines uniqueness and existence? Are there equivalence classes among power functions?

This paper answers these questions for three agents, when the power function satisfies three additional axioms. Analysis is restricted to three agents as stable sets are famously intractable: even basic questions like existence in the simple environment of CF games remained open for a quarter century (Lucas, 1968).[7] Consequently, exhaustive analyses of stable sets for three agents have been theoretically important[8]: von Neumann and Morgenstern (1953) provided these for CF games; Thrall and Lucas (1963) for PF games.

First, though, Section 3 presents results on the core, the set of undominated allocations. For anonymous n-agent pillage games, we characterise the core: if it is non-empty, it must contain all allocations giving the whole endowment to each single agent (one-way splits); it may also contain the set of all equal p-way splits, for as many p between 2 and n as one wishes. When $n = 3$, there are four possible non-empty cores; adding a responsiveness axiom reduces this to two.

Section 4 turns to stable sets. Its first result is that the core characterisation result does not generalise to stable sets — unsurprising given stable sets' greater complexity. Strikingly however, Section 4.1 proves that all anonymous, $n = 3$ power functions yielding empty cores represent the same dominance relation – that of the majority pillage game – and therefore yield the same stable set.

Section 4.2 – which treats the case of the non-empty core – is the heart of the paper, responsible for most of its notation and lemmas, and for its most involved reasoning. For three agent games, it presents an algorithm for deciding stable sets and, when they exist, computing them.[9] This uses three arguments. First, allocations in the core, a unique object, must belong to any stable set, a general result in cooperative game theory. Second, when power is continuous in resources, there may exist a balance of power locus, along which the most powerful agent is just as powerful as the other two combined; as dominance is transitive along this locus, external stability not only requires inclusion of allocations from the locus, but

2 While a small subset of von Neumann and Morgenstern's abstract games, the dominance of these games within cooperative game theory is so complete that they are often equated to cooperative game theory.

3 Rosenthal (1972) criticised characteristic functions as unable to model either externalities or strategic conditionalities; Ray and Vohra (1997) noted that games in partition function form share this latter failing.

4 Models of contest success functions (q.v. Skaperdas, 1992, 1996), a popular non-cooperative approach, differ in two ways from those based on cooperative games: game forms must be specified, and the power contest is costly, eliminating the possibility of efficiency from the outset.

5 Kóczy (2007) reviewed more recent approaches to the residual agents in PF games; its own model imposed commitment assumptions on agents, allowing cores to be defined recursively from those of singleton agents.

6 See Anesi (2006) and the references therein for more motivations of stable sets based on farsightedness.

7 For abstract cooperative games, we are grateful to Oleg Itskhoki for noting that Condorcet cycles produce non-existence immediately: e.g. $\{x, y, z\}$ and the following dominance relations, $x \succeq y \succeq z \succeq x$. See also Lucas (1992, p. 547).

8 Lucas (1971) notes that stable sets have only been "described for a few very particular games with arbitrary n". See Maskin (2003) and de Clippel and Serrano (2008) for an example of the difficulties of generalising beyond three agents.

9 An implementation of the algorithm in *Theorema* (Windsteiger et al., 2006), an automated theorem prover plug-in for *Mathematica* is available at www.cs.bham.ac.uk/~mmk/economics/theorema. Kerber and Rowat (2009) presented results on applying the algorithm to classes of power functions, and on deciding and computing stable sets for power functions violating these three additional axioms.

uniquely determines the allocations. Third, dominance over the allocations that are neither members of the unique set built by the first two steps, nor dominated by members of that set, is equivalent to that in the majority pillage game; this has a unique stable set of three allocations (Jordan and Obadia, 2004), producing a unique stable set on this final domain.

When power satisfies anonymity, continuity and responsiveness axioms, this procedure both yields a unique stable set, when one exists, and sets a tight upper bound of 15 allocations on it. This bound is much tighter than the finite bound of Jordan (2006a), the Ramsey bound of Kerber and Rowat (2011) or the doubly-exponential one of Saxton (2011). It also identifies the source of non-existence: stable sets do not exist when the balance of power locus lacks a maximal element, a new application of an old consequence of Zorn's lemma (von Neumann and Morgenstern, 1953, Section 65.4.2).[10] Similar arguments have been made previously to demonstrate the existence and uniqueness of stable sets (Gillies, 1959; Chang and Chang, 1991). Here, the power function's monotonicity allows the actual computation of stable sets.

Section 5 contains two examples to illustrate and expand upon points made in the preceding analysis.

Taken together, the paper identifies a well-behaved and tractable class of pillage games.[11] Furthermore, the stable sets in this class of pillage games behave differently from those in three agent CF or PF games. In the former, "stable sets are typically not unique", but are guaranteed to exist (Lucas, 1992, pp. 562–563). For "most" of the latter, they are unique, but are "often" larger than those in CF games (Lucas, 1971, p. 511). Thus, this class of pillage games allows richer modelling possibilities than either CF or PF games for three agents, and allows tighter predictions—but subject to the caveat that the stable set may not exist.

2. Pillage games

Let $I = \{1, \ldots, n\}$ be a finite set of *agents*; when indexed by i, j and k, these refer to distinct agents. An *allocation* divides a unit resource among them, so that the feasible set of allocations is a compact, continuous $(n-1)$-dimensional unit simplex:

$$X \equiv \left\{ \{x_i\}_{i \in I} \middle| x_i \geq 0, \sum_{i \in I} x_i = 1 \right\}.$$

Let $\subset$ denote a proper subset, and use $\subseteq$ to allow the possibility of equality. Jordan (2006a) defined a *power function* over subsets of agents and allocations, so that $\pi : 2^I \times X \to \mathbb{R}$ satisfies:

(WC) if $C \subset C' \subseteq I$ then $\pi\left(C', \boldsymbol{x}\right) \geq \pi\left(C, \boldsymbol{x}\right)\ \forall \boldsymbol{x} \in X$;
(WR) if $y_i \geq x_i\ \forall i \in C \subseteq I$ then $\pi\left(C, \boldsymbol{y}\right) \geq \pi\left(C, \boldsymbol{x}\right)$; and
(SR) if $\emptyset \neq C \subseteq I$ and $y_i > x_i\ \forall i \in C$ then $\pi\left(C, \boldsymbol{y}\right) > \pi\left(C, \boldsymbol{x}\right)$.

Axiom (WC) requires weak monotonicity in coalitional inclusion; (WR) requires weak monotonicity in resources; (SR) requires strong monotonicity in resources. These axioms imply the following representation[12]:

Lemma 1. *Any power function, $\pi\left(C, \boldsymbol{x}\right)$, can be represented by another, $\pi'\left(C, \{x_i\}_{i\in C}\right)$, which depends only on the resource holdings of its coalition members.*

Proof. Consider arbitrary $\boldsymbol{x}, \boldsymbol{y}$ such that $x_i = y_i\ \forall i \in C \subseteq I$. Then $y_i \geq x_i$ and $x_i \geq y_i$ so that axiom (WR) requires $\pi\left(C, \boldsymbol{y}\right) \geq \pi\left(C, \boldsymbol{x}\right) \geq \pi\left(C, \boldsymbol{y}\right)$. For this to hold, $\pi\left(C, \boldsymbol{x}\right)$ cannot depend on x_j for any $j \notin C$. □

The axioms also imply that $\pi\left(\emptyset, \boldsymbol{x}\right)$ is the smallest value that π can take, and is independent of $\boldsymbol{x}$.[13] Without loss of generality, we normalise $\pi\left(\emptyset, \boldsymbol{x}\right) = 0$.

The following additional axioms will be used in establishing our main result, on uniqueness conditions for stable sets:

(AN) let $\sigma : I \to I$ be a bijective function permuting the agent set; if $i \in C \Leftrightarrow \sigma(i) \in C'$ and $x_i = x'_{\sigma(i)}$, then $\pi\left(C, \boldsymbol{x}\right) = \pi\left(C', \boldsymbol{x}'\right)$.
(CX) $\pi\left(C, \boldsymbol{x}\right)$ is continuous in $\boldsymbol{x}$.
(RE) if $i \notin C$ and $\pi\left(\{i\}, \boldsymbol{x}\right) > 0$ then $\pi\left(C \cup \{i\}, \boldsymbol{x}\right) > \pi\left(C, \boldsymbol{x}\right)$.

Anonymity axiom (AN) means that power does not depend on the identity of agents, merely their cardinality and resources.[14] It therefore plays the same simplifying role that identical agent assumptions play in other economic models, and is as restrictive. One consequence (see the Discussion) is to restrict the datasets on which the current theory can be tested. Continuity axiom (CX) plays a standard technical role. Economically, its role is less clear: as dominance will be seen to be discontinuous on its own, is a second source of discontinuity needed? Example 2, below, will show that the axiom does have bite. Finally, responsiveness axiom (RE) (q.v. Jordan, 2009) states that the addition of an agent which has positive power even as a singleton strictly increases the power of its new coalition. Again, this will be seen to be technically useful. Economically, it does rule out apparently plausible cases, such as the champion power function, defined below.

A *pillage game* is then a triple, (n, X, π).

The three power functions defined by Jordan (2006a) are *wealth is power* (WIP), *strength in numbers* (SIN) and *Cobb–Douglas* (CD):

$$\pi_w\left(C, \boldsymbol{x}\right) = \sum_{i \in C} x_i; \tag{1}$$

$$\pi_v\left(C, \boldsymbol{x}\right) = \sum_{i \in C} \left(x_i + v\right); \tag{2}$$

$$\pi_c\left(C, \boldsymbol{x}\right) = \|C\|^\alpha \cdot \left(\sum_{i \in C} x_i\right)^{1-\alpha}; \tag{3}$$

where $v \geq 0$, $\|C\|$ denotes the cardinality of C and $\alpha \in (0, 1)$. When $v > 1$, SIN is the majority pillage game of Jordan and Obadia (2004). All three power functions satisfy additional axioms (AN), (CX) and (RE).

The following sinusoidal class of power functions appears less intuitively motivated, but will be useful in exemplifying particular cases and establishing the tightness of the upper bound on stable sets that will be derived later in the paper:

$$\pi_k\left(C, \boldsymbol{x}\right) \equiv \sum_{i \in C} \left[x_i + \frac{\sin\left(k\varpi x_i\right)}{k\varpi}\right]; \tag{4}$$

for $k \in \mathbb{Z}$, where – to avoid confusion with the power function – ϖ denotes the constant pi.

An allocation $\boldsymbol{y}$ *dominates* an allocation $\boldsymbol{x}$, written $\boldsymbol{y} \succ \boldsymbol{x}$, iff

$$\pi\left(W, \boldsymbol{x}\right) > \pi\left(L, \boldsymbol{x}\right);$$

where $W \equiv \{i \,|y_i > x_i\}$ and $L \equiv \{i \,|x_i > y_i\}$. By the strict inequality, domination is irreflexive; by axiom (SR), it is asymmetric.

[10] In the examples considered by Jordan (2006a), this occurred when the element that would be maximal was dominated by a tyrannical allocation. Kerber and Rowat (2009) showed that it may also occur when the power function is discontinuous in resource holdings.

[11] MacKenzie et al. (2013) are able to generate multiple stable sets in pillage games with four or more agents by relaxing the anonymity axiom.

[12] Kerber et al. (2011) presented automated proofs of this lemma, as well as Lemma 2, implemented in *Theorema* (Windsteiger et al., 2006).

[13] See Beardon and Rowat (2013) for a slightly longer discussion.

[14] Jordan and Obadia (2004) called this axiom 'symmetry', following the terminology of CF games (Lucas, 1971). We regard 'anonymity' as more precise, as it restricts the symmetry to agents' identity: intrinsically identical agents may differ in power when their resources differ.

The following result will be used later to prove Theorem 4:

Lemma 2. *Let $\boldsymbol{x}, \boldsymbol{y} \in X$ such that $W = \{i \,|y_i > x_i\} = \{1\}$ and $L = \{i \,|x_i > y_i\} = \{2\}$. Then, for any power function satisfying axiom (AN), $\boldsymbol{y} \succ \boldsymbol{x} \Leftrightarrow x_1 > x_2$.*

Proof. 'If': $x_1 > x_2 \Rightarrow \pi'(\{1\}, x_1) > \pi'(\{1\}, x_2) = \pi'(\{2\}, x_1)$, where the first implication and its inequality owe to axiom (SR) and the representation established by Lemma 1, and the equality owes to axiom (AN). By transitivity, Lemma 1 and the definition of dominance, $\pi'(\{1\}, x_1) > \pi'(\{2\}, x_1) \Leftrightarrow \boldsymbol{y} \succ \boldsymbol{x}$, completing the proof in this direction.

The 'only if' direction only requires reversing the direction of the 'if' step's first implication, to show that $\pi'(\{1\}, x_1) > \pi'(\{1\}, x_2) \Rightarrow x_1 > x_2$. Suppose otherwise, so that $x_2 = x_1 + \Delta x$, with $\Delta x \geq 0$. Then, by axiom (WR) (*resp.* (SR)), $\pi'(\{1\}, x_2) \geq \pi'(\{1\}, x_1)$ (*resp.* $\pi'(\{1\}, x_2) > \pi'(\{1\}, x_1)$), a contradiction, completing the proof. □

For $Y \subset X$, let

$$D(Y) \equiv \{\boldsymbol{x} \in X \,|\exists \boldsymbol{y} \in Y \text{ s.t. } \boldsymbol{y} \succ \boldsymbol{x}\} \tag{5}$$

be the *dominion* of Y, the set of allocations dominated by an allocation in Y. Similarly, $U(Y) = X \setminus D(Y)$, the set of allocations undominated by any allocation in Y.

As pillage games are examples of von Neumann and Morgenstern abstract games, the relationship between power functions and the underlying dominance relation is the same as that between utility functions and the underlying preference relation in utility theory: in both cases, more than one function may represent the same primitive.[15] Monotonic transformations of power functions clearly represent the same dominance relations (Jordan and Obadia, 2004); we would like to know how much broader these equivalence classes are.

3. The core

The *core*, K, is the set of undominated allocations, $U(X) = X \setminus D(X)$. Following Jordan (2006a), let $\boldsymbol{t}^i \in X$ be a *tyrannical* allocation such that $t^i_i = 1$ and $t^i_j = 0$ for all $j \neq i \in I$. Then:

Theorem 1 (*Jordan, 2006a, 2.6*)**.** *The core is the set*

$$K = \{\boldsymbol{x} \in X \,|\{i : x_i > 0\} = \{i : \pi(\{i\}, \boldsymbol{x}) \geq \pi(I \setminus \{i\}, \boldsymbol{x})\}\}.$$

In particular,

(a) *for each $i \in I$, $\boldsymbol{t}^i \in K$ iff $\pi\left(\{i\}, \boldsymbol{t}^i\right) \geq \pi\left(I \setminus \{i\}, \boldsymbol{t}^i\right)$; and*
(b) *if $\pi\left(\{i\}, \boldsymbol{t}^i\right) < \pi\left(I \setminus \{i\}, \boldsymbol{t}^i\right)$ $\forall i \in I$ then $K = \emptyset$.*

The inequality in item (b) was called the *no-tyranny* condition by Jordan (2006b).[16] The main result of this section characterises the core for anonymous pillage games with three or more agents. Doing so requires two preliminary lemmas:

Lemma 3. *Let K be the core for some n-agent pillage game with anonymous power function π. Consider a core allocation $\boldsymbol{x}$ which sets $x_i > 0$ for all $i \in C \subseteq N$, where $\|C\| \geq 2$. Then, for those $i \in C$, $x_i = \frac{1}{\|C\|}$.*

Proof. For $\boldsymbol{x}$ to belong to the core it must be undominated. This requires, for any $i, j \in C$, that $\pi(\{i\}, \boldsymbol{x}) = \pi(\{j\}, \boldsymbol{x})$. By Lemma 1, this may be represented as $\pi(\{i\}, x_i) = \pi\left(\{j\}, x_j\right)$ which, by anonymity, requires that $x_i = x_j$. □

Let $\mathcal{E}^c$ be the set of *egalitarian allocations* that divide the endowment equally among all coalitions of size c. Then:

Lemma 4. *There is an n-agent pillage game with an anonymous power function, whose core is $\bigcup_{c=1}^{n} \mathcal{E}^c$; the core of any other anonymous n-agent pillage game is a subset of this.*

Proof. Example 2.8 in Jordan (2006a) showed that this core is achieved by the *champion power function*

$$\bar{\pi}(C, \boldsymbol{x}) \equiv \max_{i \in C} x_i. \tag{6}$$

The subset property is an immediate consequence of Lemma 3. □

Jordan (2006a) referred to the *champion power function* as $\pi_{\max}$.[17]

Proposition 2.7 of Jordan (2006a) completely characterised the core in two agent pillage games. We now extend this result to characterise the core for anonymous pillage games with three or more agents:

Theorem 2. *Let P be the union of $\{1\}$ and any non-empty subset of $\{2, \ldots, n\}$. Then, for any pillage game with $n \geq 3$ agents and an anonymous power function, the core is either empty or is equal to $\bigcup_{p \in P} \mathcal{E}^p$.*

The proof is based around the recognition that, for an equal split of the endowment to be in the core, it must survive a number of different pillage attempts: if three or more agents split the resource, any agent holding resources must be as powerful as any two agents holding resources; if only two agents split the resource, then each must be as powerful as the other plus the agents without resources; finally, if a single agent holds the whole resource, it must be as powerful as all other agents combined. The proof therefore defines a power function that is tailored to either allow or prevent these sort of pillage operations, depending on whether the equal split in question is to included in the core or not.

Proof. As preliminaries, note that: the case of an empty core is possible, as exemplified by the SIN power function with $\upsilon > 1$; by Theorem 1, the tyrannical allocations must belong to any non-empty core; by Lemma 4, the core must be a subset of $\cup_{c=1}^{n} E^c$.

Thus, it remains to show that arbitrary sets of equal splits among three or more agents are possible. To do so, consider first the following intervals, indexed by $c = n, \ldots, 2$:

$$J_n \equiv \left[0, \tfrac{2}{2n-1}\right), \ldots, J_c \equiv \left[\tfrac{2}{2c+1}, \tfrac{2}{2c-1}\right), \ldots, J_2 \equiv \left[\tfrac{2}{5}, 1\right].$$

Now define the functions

$$m(C, \boldsymbol{x}, c) \equiv n^{n-c} \max_{i \in C} x_i; \quad \text{and}$$

$$s(C, \boldsymbol{x}, c) \equiv \left\{ \begin{array}{ll} n^{n-c} \sum_{i \in C} x_i & \text{if } c > 2 \\ n^{n-c} \sum_{i \in C} (x_i + \varepsilon) & \text{otherwise} \end{array} \right\}$$

where $\varepsilon > 0$ is small. Finally, let $B \equiv \times_{c=n}^{2} \{b_c\}$ be an ordered $(n-1)$-tuple, such that $b_c \in \{m, s\}$, and define the piecewise function:

$$\pi_B(C, \boldsymbol{x}) \equiv \left\{ \begin{array}{ll} m(C, \boldsymbol{x}, c) & \text{if } \max_{i \in C} x_i \in J_c \wedge b_c = m \\ s(C, \boldsymbol{x}, c) & \text{if } \max_{i \in C} x_i \in J_c \wedge b_c = s \end{array} \right\}.$$

We now prove that $\pi_B(C, \boldsymbol{x})$ is a power function. Start with axiom (WR), for which there are two cases:

[15] The crucial distinction is that a rational preference ordering is transitive, a property not required of dominance relations.

[16] The Bondareva–Shapley theorem provides a parallel result for CF games.

[17] In addition to proving that its core was the set of all equal divisions of the unit endowment among all non-empty coalitions, Jordan also proved that this was its unique stable set.

1. The increase from $\max_{i\in C} x_i$ to $\max_{i\in C} y_i$ *preserves the interval type* (i.e. either m or s). The following three facts then hold: neither the sum nor the max is decreasing; the c index's weak decrease increases n^{n-c}; the argument of $s(C, \boldsymbol{x}, c)$ increases by $\|C\|\varepsilon$ when c falls to two. Thus, $\pi_B(C, \boldsymbol{y}) \geq \pi_B(C, \boldsymbol{x})$, so that the result holds in this case.
2. The increase from $\max_{i\in C} x_i$ to $\max_{i\in C} y_i$ *changes the interval type*:
   (a) From m to s, changing power from $n^{n-c}\max_{i\in C} x_i$ to at least $n^{n-c'}\sum_{i\in C} y_i$, with $c' < c$. As $y_i \geq x_i\, \forall i \in C \Rightarrow n^{n-c'}\sum_{i\in C} y_i \geq n^{n-c}\max_{i\in C} x_i$, with $n - c' > n - c$, the result holds in this case.
   (b) From s to m, changing power from $n^{n-c}\sum_{i\in C} x_i$ to $n^{n-c'}\max_{i\in C} y_i$, with $c' < c$. As $y_i \geq x_i \Rightarrow n\max_{i\in C} y_i \geq \sum_{i\in C} x_i$, the result also holds in this case.

The same calculations with the strict inequality $y_i > x_i$ establish that axiom (SR) holds as well. Similar calculations also establish axiom (WC). Now, though, the allocation remains $\boldsymbol{x}$ but $C \subset C' \Rightarrow \max_{i\in C} x_i \leq \max_{i\in C'} x_i$, so that the interval index, c, weakly decreases with the addition of agents, weakly increasing n^{n-c}.

We now turn to the attempted pillage operations that core allocations must survive. When the endowment is equally split between any subset of three of more agents, π_B only grants power to coalitions holding resources. Under these circumstances, if $b_c = m$, then a singleton coalition holding resources is as powerful as any other coalition: for each $p \in P\setminus\{1,2\}$, setting $b_p = m$ ensures that all allocations splitting the resource equally between p agents are undominated. If, though, $b_c = s$ under these circumstances, then larger coalitions are more powerful than smaller ones: for each $p \notin P \cup \{1,2\}$, setting $b_p = s$ allows any two-agent coalition to pillage any singleton coalition, ensuring all p-way equal splits are dominated when $p \geq 3$.

When the endowment is equally split between only two agents, $b_2 = m$ only grants power to coalitions holding resources; by axiom (AN), these are equally powerful, and the corresponding allocations undominated. When, on the other hand, $b_2 = s$, a coalition of two agents (one with resources, one without) has power $\frac{1}{2} + 2\varepsilon > \frac{1}{2} + \varepsilon$, allowing it to pillage the other resource-holding agent (whose power is the latter term).

Finally, at a tyrannical allocation, $\pi_B\left(\{i\}, \boldsymbol{t}^i\right) = 1 + \varepsilon > (n-1)\varepsilon \geq \pi_B\left(I\setminus\{i\}, \boldsymbol{t}^i\right)$, so that the $\boldsymbol{t}^i$ belong to the core. □

Thus, all allocations in the core, $\boldsymbol{x} \in K$, lie on symmetry axes, so that $x_j = x_k$ for some $j \neq k$. We shall later see that this need not be true of allocations in stable sets.

As the focus of this paper are stable sets in $n = 3$ pillage games, we shall present two corollaries of the theorem under that restriction. They make use of some further notation, introduced now.

Let $\boldsymbol{s}^{jk} \in X$ be a *split* allocation with $s_i^{jk} = 0$ and $s_j^{jk} = s_k^{jk} = \frac{1}{2}$. For the $n = 3$ SIN power function (as defined in Eq. (2)) with $v \in (0, 1)$, Fig. 1 illustrates the tyrannical and split allocations, as well as their dominions. These latter are derived from Eq. (5):

$$\begin{aligned} D\left(\boldsymbol{t}^1\right) &= \left\{\boldsymbol{x} \in X \,\middle|\, \boldsymbol{t}^1 \succ \boldsymbol{x}\right\} \\ &= \{\boldsymbol{x} \in X \,|\pi(\{1\}, \boldsymbol{x}) > \pi(\{2,3\}, \boldsymbol{x})\} \\ &= \{\boldsymbol{x} \in X \,|x_1 + v > (1 - x_1) + 2v\} \\ &= \left\{\boldsymbol{x} \in X \,\middle|\, x_1 > \frac{1+v}{2}\right\}. \end{aligned} \tag{7}$$

$$\begin{aligned} D\left(\boldsymbol{s}^{23}\right) = &\left\{\boldsymbol{x} \in X| \left(\frac{1}{2} > x_2, x_3; x_2 + x_3 > \frac{1-v}{2}\right)\right. \\ &\left.\cup \left(\frac{1}{2} = \max\{x_2, x_3\} > \min\{x_2, x_3\} > x_1\right)\right\}. \end{aligned} \tag{8}$$

Finally, denote the simplex' centroid by $\boldsymbol{c} \equiv \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

We now present the corollaries of Theorem 2, characterising the core when $n = 3$:

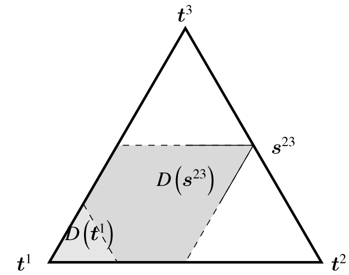

Fig. 1. Strength in numbers (SIN) when $v \in (0, 1)$.

Corollary 1. *When $n = 3$ and axiom* (AN) *holds, the core is one of:* (1) $\emptyset$; (2) $\mathcal{E}^1$; (3) $\mathcal{E}^1 \cup \mathcal{E}^2$; (4) $\mathcal{E}^1 \cup \mathcal{E}^3$; *and* (5) $\mathcal{E}^1 \cup \mathcal{E}^2 \cup \mathcal{E}^3$.

Proof. By Theorem 2, the core may be empty. If it is non-empty, it must include the $\boldsymbol{t}^i$ (which form $\mathcal{E}^1$), and may also include none, either or both the two-way ($\mathcal{E}^2$) and three-way splits (the centroid, $\mathcal{E}^3$). It may not include anything else. □

When axiom (RE) holds as well, the centroid may no longer belong to the core:

Corollary 2. *When $n = 3$ and axioms* (AN) *and* (RE) *hold, the core is one of:* (1) $\emptyset$; (2) $\mathcal{E}^1$; *and* (3) $\mathcal{E}^1 \cup \mathcal{E}^2$.

Proof. Axiom (SR) and the normalisation $\pi(\emptyset, \boldsymbol{c}) = 0$ ensure that $\pi(\{i\}, \boldsymbol{c}) > 0$. As axiom (AN) guarantees that $\pi(\{j\}, \boldsymbol{c}) = \pi(\{k\}, \boldsymbol{c})$, axiom (RE) then ensures that $\pi(\{i,j\}, \boldsymbol{c}) > \pi(\{k\}, \boldsymbol{c})$. Thus, $\boldsymbol{c}$ is dominated by any allocation that transfers resources from k to i and j; as $\boldsymbol{c} > \boldsymbol{0}$, such allocations exist, so that $\boldsymbol{c}$ cannot belong to any core. □

4. Stable sets

A set of allocations, $S \subseteq X$, is a *stable set* iff it satisfies *internal stability*,

$$S \cap D(S) = \emptyset; \tag{IS}$$

and *external stability*,

$$S \cup D(S) = X. \tag{ES}$$

The conditions combine to yield $S = X \setminus D(S)$. While stable sets may not exist, or may be non-unique in general cooperative games, the core necessarily belongs to any stable set; when the core also satisfies external stability, it is the unique stable set.[18]

As stable sets are more complex objects than the core, Jordan (2006a) derived results only for the WIP, SIN and CD power functions defined in Eqs. (1)–(3), above: WIP has a unique stable set consisting of the $\cup_{k=0,1,\ldots,\lfloor\log_2 n\rfloor}\mathcal{E}^{2^k}$; for SIN, a stable set for odd n is derived; no stable set exists for CD. This greater complexity prevents even a weaker version of Theorem 2 holding for stable sets:

Theorem 3. *For $n \geq 2m$, there exists no power function satisfying axiom* (AN) *such that, for any $P \subseteq \{1, \ldots, m\}$, $\cup_{p\in P}\mathcal{E}^p$ is stable.*

Proof. Let $\bar{p}$ denote the maximal element of P and consider the allocation

$$\boldsymbol{x} \equiv \left(\frac{1}{2\bar{p}}, \ldots, \frac{1}{2\bar{p}}, 0, \ldots, 0\right) \notin \cup_{p\in P}\mathcal{E}^p.$$

We establish that there is no $\boldsymbol{y} \in \cup_{p\in P}\mathcal{E}^p$ such that $\boldsymbol{y} \succ \boldsymbol{x}$; thus, $\cup_{p\in P}\mathcal{E}^p$ fails external stability, and cannot be a stable set. The coalition which could bring about a $\boldsymbol{y} \in \cup_{p\in P}\mathcal{E}^p$ that is most challenging to $\boldsymbol{x}$ consists of $\bar{p}$ agents, each holding $\frac{1}{2\bar{p}}$ in $\boldsymbol{x}$: a larger coalition could not reward all of its members with an allocation in

[18] Asilis and Kahn (1992) expressed this result in terms of *transparency*.

$\cup_{p \in P} \mathcal{E}^p$; these members are more powerful than those for which $x_i = 0$. Letting $W \equiv \{i \in I : y_i > x_i\}$ and $L \equiv \{i \in I : x_i > y_i\}$ sets $\pi(W, \boldsymbol{x}) = \pi(L, \boldsymbol{x})$ by axiom (AN). □

Smaller sets satisfy external stability less easily; thus, relative to Jordan's Example 2.8, this result does not allow the candidate stable set to be large enough.

The introduction mentioned that existing results either do not apply to the present environment, or require further work to apply. The existence and uniqueness theorems of Berge (1962, Chapter 5) are in the former category: these either require symmetry of domination (so that $\boldsymbol{y} \succeq \boldsymbol{x} \succeq \boldsymbol{y}$, disallowed by axiom (SR)), or that any allocation be dominated by only finitely many others, which is impossible by the following result:

Lemma 5 (*Convex Transitivity*). *If $\boldsymbol{x} \succeq \boldsymbol{y}$ then $\boldsymbol{x} \succeq \alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} \succeq \boldsymbol{y}\ \forall\alpha \in (0, 1)$.*

Proof. $\boldsymbol{x} \succeq \boldsymbol{y} \Leftrightarrow \pi(W_1, \boldsymbol{y}) > \pi(L_1, \boldsymbol{y})$, where $W_1 \equiv \{i \,|x_i > y_i\}$ and $L_1 \equiv \{i\,|y_i > x_i\}$. Now define

$$W_\alpha \equiv \{i\,|\alpha x_i + (1-\alpha) y_i > y_i\};$$

and L_α similarly for $\alpha \in (0, 1)$. It is then immediate that $W_\alpha = W_1$ (*resp.* $L_\alpha = L_1$) so that

$$\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} \succeq \boldsymbol{y}.$$

The second dominance operation follows similarly. □

The existing algorithm for constructing stable sets in pillage games owes to Roth (1976) via Jordan and Obadia (2004).[19] It uses a weaker condition than external stability, namely *self-protection*:

$$S \subseteq U^2(S); \tag{SP}$$

where $U^2(S) \equiv U(U(S))$. Whereas external stability requires that all allocations outside the set in question be dominated by at least one inside it, self-protection only requires this of those external allocations that dominate allocations within the set.

Algorithm 1 The Roth–Jordan algorithm

$S_0 \equiv K$
$i = 1$
repeat
  $S_i \equiv U^2(S_{i-1})$
  $i = i + 1$
until $S_i = S_{i-1}$
if S_i satisfies (ES) **then**
  $S = S_i$
end if

Algorithm 1 takes as input a non-empty set satisfying conditions (IS) and (SP); if the core is non-empty, this is the natural candidate to use. Each iteration of the algorithm generates weakly larger sets $S_0 \subseteq S_1 \subseteq \cdots$ satisfying both (IS) and (SP). As pillage games' stable sets are finite (Jordan, 2006a), the algorithm will terminate after finitely many iterations. If its final iterate also satisfies condition (ES), then it is the unique stable set. The algorithm may, however, terminate before this point without finding a stable set.[20]

This general algorithm is therefore incomplete in four respects. First, it provides no means for calculating the core. Second, when the core is empty, it provides no clue for finding an internally stable and self-protecting initial iterate, S_0. Indeed, for some $n > 2$ pillage games the only such sets are themselves stable (Jordan and Obadia, 2004), in which case the algorithm cannot help find them. Third, it does not specify an efficient way to compute $U^2(S_{i-1})$. Finally, it is not clear whether terminating at an S_i which is not externally stable means that no stable set exists, or merely that further steps must be taken independently of the algorithm.

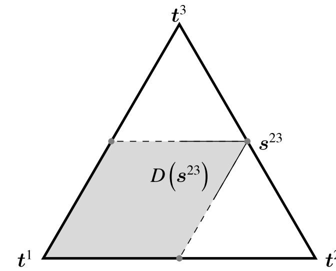

Fig. 2. Strength in numbers (SIN) when $v > 1 : S = \left\{\boldsymbol{s}^{12}, \boldsymbol{s}^{13}, \boldsymbol{s}^{23}\right\}$.

The next subsection proves that all anonymous, $n = 3$ pillage games with an empty core represent the same dominance relation. The subsequent subsection then proves that all such games with a non-empty core, also satisfying the (CX) and (RE) axioms, have either no stable set or a unique one containing no more than 15 elements. It does so by presenting a new algorithm that both decides the stable set question for the class of pillage games considered, and allows computation of their stable sets.

4.1. Empty core

When the core is empty, the Roth–Jordan algorithm can obviously not use the core as its first iterate. This subsection demonstrates that these cases do not actually pose a problem: any anonymous three agent pillage game with an empty core yields a unique stable set; further, as already noted in the discussion of Theorem 1, the core's emptiness is easily determined. The argument proceeds in two steps: an example of the preceding is found; Theorem 4 then shows that all other anonymous three agent pillage games with an empty core represent the same dominance ordering. Intuitively, the empty core leaves $D\left(\boldsymbol{s}^{jk}\right)$ defined only by the coalitional membership inequalities $x_j, x_k < \frac{1}{2}$, which are independent of $\pi()$.

Our example shall be the majority pillage game. This sets $v > 1$ in the SIN power function defined in Eq. (2). Propositions 4.2 and 4.3 of Jordan (2006a) proved general results which, in the $n = 3$ case, imply that: when $v > 1$ the core is empty, and $\mathcal{E}^2$ is stable. This may be confirmed by deriving the dominion of a split allocation, such as $D\left(\boldsymbol{s}^{23}\right)$ (q.v. Eq. (8)), which is illustrated in Fig. 2.

Theorem 4 (*SIN Equivalence*). *When $n = 3$, all power functions satisfying axiom* (AN) *and yielding an empty core represent the same dominance ordering.*

Proof. Fix an $\boldsymbol{x}$ and a $\boldsymbol{y}$, in turn fixing W and L. By axiom (AN) and Lemma 2, these power functions do not discriminate between dominance orderings when the induced W and L are singletons.

For the non-singleton W and L, make use of $K = \emptyset$. First consider $W = \{1\}$ and $L = \{2, 3\}$. There are no $\boldsymbol{y}$ and $\boldsymbol{x}$ yielding these W and L for which $\boldsymbol{y} \succeq \boldsymbol{x}$ as

$$\pi(\{1\}, \boldsymbol{x}) \leq \pi\left(\{1\}, \boldsymbol{t}^1\right) < \pi\left(\{2, 3\}, \boldsymbol{t}^1\right) \leq \pi(\{2, 3\}, \boldsymbol{x}).$$

The strict inequality above follows, by Theorem 1, from $K = \emptyset$.

As π satisfies axiom (AN), the only remaining case to consider is that in which $W = \{2, 3\}$ and $L = \{1\}$. Reading the above inequalities from right to left reveals that any $\boldsymbol{y}$ and $\boldsymbol{x}$ generating these W and L set $\boldsymbol{y} \succeq \boldsymbol{x}$ without further conditions, just as for SIN. □

Corollary 3. *When $n = 3$, all power functions satisfying axiom* (AN) *and yielding an empty core have $\mathcal{E}^2$ as their unique stable set.*

[19] See also Asilis and Kahn (1992) and Jordan (2006b, Proposition 3.9).

[20] If $S_0 = \emptyset$, which trivially satisfies (IS) and (SP), then the algorithm terminates immediately.

Proof. Theorem 3.4 of Jordan and Obadia (2004) proved that $\mathcal{E}^2$ is the unique stable set of SIN when $\upsilon > 1$ and $n = 3$. As this pillage game has an empty core, the result follows from the theorem. □

4.2. Non-empty core

When the core is non-empty, $D\left(\boldsymbol{s}^{jk}\right)$ will be defined not just by the coalitional membership inequalities, but by particular features of the power function. Thus, unlike the previous case, power functions admitting non-empty cores cannot generally be expected to represent the same dominance orderings, or even to yield the same stable sets. To illustrate, we assert that both the WIP power function and $\pi_{\sqrt{}}(C, \boldsymbol{x}) \equiv \sum_{i \in C} \sqrt{x_i}$ have the same core – $K = \mathcal{E}^1 \cup \mathcal{E}^2$ – but different stable sets (for the former, add $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ and its permutations to the core; for the latter, add $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$ and its permutations).

In this subsection, we show that all anonymous $n = 3$ pillage games with non-empty cores whose power functions also satisfy the continuity and responsiveness axioms either have no stable set, or have a unique stable set. In the former case, we isolate the source of non-existence; analysis of both cases also yields an algorithm for computing as well as deciding the stable set question in these games. This algorithm allows derivation of the previous paragraph's assertion about the stable sets of WIP and $\pi_{\sqrt{}}$.

This subsection – the paper's most involved – first introduces some preliminary notation and results, then presents the algorithm that decides and computes stable sets, and – finally – presents and derives the results undergirding the algorithm.

The most important analytical object is the *balance of power locus*, the set of allocations such that one agent is just as powerful as the other two. Understanding dominance relationships within these loci turns out to be the most involved aspect of this analysis. Whenever possible, we illustrate the concepts required by it in Fig. 3, which is derived from Eq. (6)'s $n = 3$ champion power function.

First, the balance of power locus of allocations lying between $D\left(\boldsymbol{t}^i\right)$ and $D\left(\boldsymbol{s}^{jk}\right)$ is:

$$B^i \equiv \{\boldsymbol{x} \in X \,|\, \pi(\{i\}, \boldsymbol{x}) = \pi(\{j, k\}, \boldsymbol{x})\}.$$

In Fig. 3, B^i is comprised of two line segments, each running from the simplex' centroid, $\boldsymbol{c}$, to one of the split allocations, $\boldsymbol{s}^{jk}$, inclusive. For other power functions, B^i may be empty. When it is non-empty, a number of objects are often of interest. The first is

$$\bar{\boldsymbol{b}}^i \in B^i \quad \text{s.t. } \bar{b}^i_j = \bar{b}^i_k,$$

the midpoint of B^i. In Fig. 3, $\bar{\boldsymbol{b}}^i = \boldsymbol{c}$ for all agents. The second is the set of *maximal* elements from agent i's point of view:

$$M^i \equiv \left\{ \boldsymbol{x} \in B^i \setminus D\left(\boldsymbol{t}^i\right) \,\middle|\, x_i = \max_{\boldsymbol{y} \in B^i \setminus D\left(\boldsymbol{t}^i\right)} y_i \right\}$$

with elements $\boldsymbol{m}^i \in M^i$, with $M \equiv \left\{M^i\right\}_{i=1}^3$.[21] In Fig. 3, these are the split allocations, $\boldsymbol{s}^{jk}$. Finally, there are the *extremal* elements of the maximal set most favourable to each of the other two agents,

$$E^i \equiv \left\{\boldsymbol{e}^i_j, \boldsymbol{e}^i_k \in M^i \,\middle|\, e^i_{jj} \geq x_j, e^i_{kk} \geq y_k \,\forall \boldsymbol{x}, \boldsymbol{y} \in M^i \right\};$$

where e^i_{jj} is the jth coordinate of $\boldsymbol{e}^i_j$, with $E \equiv \left\{E^i\right\}_{i=1}^3$. In Fig. 3, $E^i = M^i$.

Clearly, when B^i is empty, M^i and E^i are as well, and $\bar{\boldsymbol{b}}^i$ does not exist. These objects may also be empty or fail to exist if B^i is open in particular ways. Finally, it may also be the case that $\bar{\boldsymbol{b}}^i = \boldsymbol{e}^i_j = \boldsymbol{e}^i_k$.

The next lemma will become useful in establishing dominance along the balance of power locus, when it exists:

Lemma 6. *$\forall \boldsymbol{x}, \boldsymbol{y} \in B^i$, if $y_i > x_i$ then either $y_j > x_j$ or $y_k > x_k$.*

Proof. Suppose otherwise, so that $y_j \leq x_j$ and $y_k \leq x_k$. Then, by definition of B^i and axioms (WR) and (SR):

$$\pi(\{i\}, \boldsymbol{y}) > \pi(\{i\}, \boldsymbol{x}) = \pi(\{j, k\}, \boldsymbol{x}) \geq \pi(\{j, k\}, \boldsymbol{y});$$

contradicting $\boldsymbol{y} \in B^i$. □

An implication of the lemma is that, when $n = 3$, a ray from $\boldsymbol{t}^i$ through the simplex cuts B^i at most once.[22] This assists in depicting these pillage games in the simplex (q.v. Figs. 1, 3 and 4):

The subset of a balance of power in which the lone agent has strictly more power than do either of the other two on their own is also analytically important as, by anonymity, agent i is able to dominate either of the other two agents for allocations within it. Defining and analysing this subset is key to establishing dominance relations within B^i:

$$B^i_+ \equiv \left\{ \boldsymbol{x} \in B^i \,\middle|\, x_i > \max\left\{x_j, x_k\right\} \right\},$$

For some power functions, the restriction to B^i_+ has no consequence: in SIN with $\upsilon \in (0, 1)$, for example, $B^i_+ = B^i$. For other power functions, it is prohibitive: in Fig. 3's champion power function, B^i_+ is empty although B^i is not.

With these preliminaries established, we present Algorithm 2 to outline the structure of the coming arguments about deciding and computing stable sets in three agent pillage games satisfying the three additional axioms. As the algorithm cites notation that has yet to be defined, we ask that the reader regard it as a roadmap at this point, identifying the cases that must be addressed to establish our results. One case has already been addressed: line 2's stable set return for the empty core (line 1) was established in Corollary 3; it is exemplified by SIN with $\upsilon > 1$, as illustrated in Fig. 2.

Algorithm 2 Stable sets in $n = 3$ pillage games satisfying the additional axioms

1: **if** $\pi\left(\{i\}, \boldsymbol{t}^i\right) < \pi\left(\{j, k\}, \boldsymbol{t}^i\right)$ **then**
2: $\quad S = \mathcal{E}^2$ — Section 4.1: Corollary 3
3: **else**
4: $\quad$ **if** $E^i = \emptyset$ **then**
5: $\quad\quad$ **return** "no stable set exists" — Section 4.2.1: Theorem 6
6: $\quad$ **else**
7: $\quad\quad$ **if** $\pi\left(\{j\}, \boldsymbol{s}^{jk}\right) \geq \pi\left(\{i, k\}, \boldsymbol{s}^{jk}\right)$ **then**
8: $\quad\quad\quad S = \mathcal{E}^1 \cup \mathcal{E}^2 \cup \left\{S^i\right\}_{i=1}^3$ — Section 4.2.2: Theorem 7
9: $\quad\quad$ **else**
10: $\quad\quad\quad S = \mathcal{E}^1 \cup \left\{S^i\right\}_{i=1}^3 \cup A$ — Section 4.2.3: Theorem 9
11: $\quad\quad$ **end if**
12: $\quad$ **end if**
13: **end if**
14: **return** S

Theorems 6, 7 and 9 each establish one of the algorithm's remaining cases. Each of these cases is also non-empty: line 5's non-existence return (Theorem 6) is exemplified by CD or SIN with $\upsilon \in (0, 1)$ (q.v. Fig. 1); line 8's stable set return (Theorem 7) is exemplified by WIP (q.v. Fig. 4) or π_9 (q.v. Eq. (4)); line 10's stable set return (Theorem 9) is exemplified by the power function in Example 1. Two further theorems play a role in establishing the algorithm: Theorem 5 addresses existence and uniqueness of stable sets on the balance of power loci; Theorem 8 extends a uniqueness result of Jordan and Obadia (2004), allowing its use in Theorem 9.

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21 Fig. 1 illustrates how $B^i \setminus D\left(\boldsymbol{t}^i\right)$ can differ from the simpler B^i: when $\upsilon = \frac{1}{3}$, $\boldsymbol{x} = \left(\frac{2}{3}, 0, \frac{1}{3}\right)$ belongs to both B^i and $D\left(\boldsymbol{t}^i\right)$; thus, it does not belong to $B^i \setminus D\left(\boldsymbol{t}^i\right)$. Subsequent proofs will build transitive chains; by Zorn's lemma, maximal elements guarantee unique stable sets, while their absence guarantees non-existence. Defining M^i over B^i alone would complicate this by forcing consideration of dominance by $\boldsymbol{t}^i$.

22 If $\boldsymbol{x}$ and $\boldsymbol{y}$ lie on such a ray then $y_i > x_i$ implies $y_j < x_j$ and $y_k < x_k$, contradicting Lemma 6.

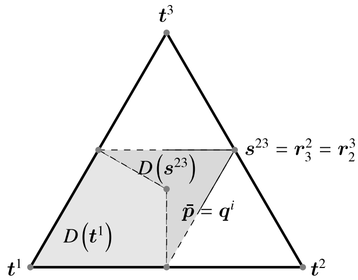

Fig. 3. The champion power function, $\bar{\pi}(C, \boldsymbol{x})$.

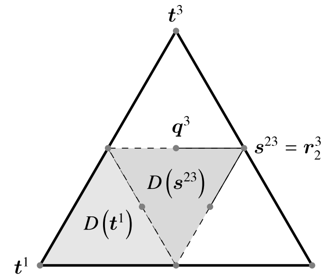

Fig. 4. Wealth is power (WIP).

The additional structure of these games therefore allows Algorithm 2 to overcome the limitations of the more general Roth–Jordan algorithm. First, the core need never be calculated; further, even if it must be, this is easily done. Second, the case of the empty core is dealt with. Third, the problem of calculating $U^2(S_{i-1})$ is replaced by easier calculations; the most difficult of these is the determination of whether the extremal set, E^i, is empty.[23] Fourth, the algorithm decides and, when a stable set exists, computes it in all cases.

The subsections that follow each address one of the cases arising from algorithm.[24] Before beginning, we establish some further notation and results. The most important of these latter is Theorem 5, which addresses existence and uniqueness of a stable set within the balance of power loci. A set of allocations $K^i \subseteq Y \subseteq X$ that satisfies $K^i = Y \setminus D(Y)$ is the *core on* Y. Similarly, a set of allocations $S^i \subseteq Y \subseteq X$ satisfying $S^i \cap D(S^i) = \emptyset$ and $S^i \cup D(S^i) = Y$, is *stable on* Y.[25]

Theorem 5 requires a standard definition and a lemma:

Definition 1. The dominance operator, $\succ$, is a *strict total order* on a set of allocations, X, iff, for any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$ the following hold:

1. trichotomy: exactly one of $\boldsymbol{x} = \boldsymbol{y}$, $\boldsymbol{x} \succ \boldsymbol{y}$ and $\boldsymbol{y} \succ \boldsymbol{x}$ holds; and
2. transitivity: $\boldsymbol{x} \succ \boldsymbol{y} \succ \boldsymbol{z}$ implies that $\boldsymbol{x} \succ \boldsymbol{z}$.

This is also called a *complete ordering* (von Neumann and Morgenstern, 1953, Section 65.3.1). For concision's sake, we adopt their convention.

The lemma will allow use of Zorn's lemma, via von Neumann and Morgenstern (1953, Section 65.4.2), to obtain either uniqueness or non-existence of a stable set on $(B^i_+ \cup E^i)$:

Lemma 7. *Let* $n = 3$ *and* π *satisfy axioms* (AN) *and* (RE). *Then dominance completely orders the set of allocations,* $\boldsymbol{x} \in (B^i_+ \cup E^i)$ *which satisfy* $x_j > x_k$.

The proof of Lemma 7 requires two further lemmas which we state and prove here:

Lemma 8. *When* $n = 3$ *and* π *satisfies axiom* (AN), *if either*

1. $\boldsymbol{x}, \boldsymbol{y} \in B^i$ *are such that* $y_i > x_i \geq x_j > x_k$ *and* $y_j > x_j$; *or*
2. $\boldsymbol{x} \in B^i_+$ *and* $\boldsymbol{y} \in B^i$ *are such that* $y_i > x_i$;

then $\boldsymbol{y} \succ \boldsymbol{x}$.

Proof. In the first case, $W \equiv \{l \,|y_l > x_l\} = \{i, j\}$ and $L \equiv \{l \,|x_l > y_l\} = \{k\}$ by assumption. In the second case, the same W and L can be assumed without loss of generality by Lemma 6.

In the first case, $x_i > x_k$ is assumed; in the second, it follows from $\boldsymbol{x} \in B^i_+$. In both cases, axioms (AN) and (SR) then ensure that $\pi(\{i\}, \boldsymbol{x}) > \pi(\{k\}, \boldsymbol{x})$; by axiom (WC), it follows that $\pi(W, \boldsymbol{x}) > \pi(L, \boldsymbol{x})$, hence $\boldsymbol{y} \succ \boldsymbol{x}$. □

Lemma 9. *When* $n = 3$ *and* π *satisfies axioms (AN) and (RE), if the extremal set is non-empty and* $\boldsymbol{x} > \boldsymbol{0}$ *lies in* $B^i \setminus (E^i \cup \{\bar{\boldsymbol{b}}^i\})$, *then* $\boldsymbol{x} \in D(E^i)$.

Proof. First consider the case in which $\boldsymbol{x}$ lies in $M^i \setminus (E^i \cup \{\bar{\boldsymbol{b}}^i\})$, so that agent i is indifferent between allocation $\boldsymbol{x}$ and those in E^i. As $\boldsymbol{x} \neq \bar{\boldsymbol{b}}^i$ let, without loss of generality, $x_j > x_k$, so that, by axiom (AN), $\pi(\{j\}, \boldsymbol{x}) > \pi(\{k\}, \boldsymbol{x})$. This, in turn, implies $\boldsymbol{e}^i_j \succ \boldsymbol{x}$, which establishes the result.

The remaining case to consider is that in which $\boldsymbol{x}$ lies in $B^i \setminus (M^i \cup \{\bar{\boldsymbol{b}}^i\})$. Consider this in two subcases:

1. $\boldsymbol{x} \in B^i_+ \setminus (M^i \cup \{\bar{\boldsymbol{b}}^i\})$. The set of agents preferring $\boldsymbol{e}^i_j$ to $\boldsymbol{x}$ therefore includes i (by definition of M^i) and at least one other (by Lemma 6). As $\boldsymbol{x} \in B^i_+, x_i > \max\{x_j, x_k\}$ so that $\pi(\{i\}, \boldsymbol{x}) > \pi(\{\arg\max\{x_j, x_k\}\}, \boldsymbol{x})$. By axiom (WC), $\boldsymbol{e}^i_j \succ \boldsymbol{x}$.
2. $\boldsymbol{x} \in B^i \setminus (B^i_+ \cup M^i \cup \{\bar{\boldsymbol{b}}^i\})$. Without loss of generality, this implies that

   $e^i_{ji} > x_i = x_j > x_k > 0,$

   with the first inequality owing to $\boldsymbol{e}^i_j \in M^i \not\ni \boldsymbol{x}$, the equality (without loss of generality) to $\boldsymbol{x} \notin B^i_+$, the second inequality to $\boldsymbol{x} \neq \bar{\boldsymbol{b}}^i$ and the final to $\boldsymbol{x} > \boldsymbol{0}$. Then the set of agents preferring $\boldsymbol{e}^i_j$ to $\boldsymbol{x}$ includes i and, by Lemma 6, at least one other; in the worst case, this is k. As $x_k > 0$ and axiom (RE) holds, $\pi(\{i, k\}, \boldsymbol{x}) > \pi(\{i\}, \boldsymbol{x}) = \pi(\{j\}, \boldsymbol{x})$ so that $\boldsymbol{e}^i_j \succ \boldsymbol{x}$. □

Proof of Lemma 7. Without loss of generality, let $x_i \geq y_i$. Then, under the stated conditions, for any two $\boldsymbol{x}$ and $\boldsymbol{y}$:

1. $x_i > y_i$, implies $\boldsymbol{x} \succ \boldsymbol{y}$ (by Lemma 8); and
2. $x_i = y_i$ and $x_j > y_j$, implies $\boldsymbol{x} \succ \boldsymbol{y}$ (by axiom (AN), applied as in Lemma 9).
3. $x_i = y_i$ and $x_j = y_j$, implies $\boldsymbol{x} = \boldsymbol{y}$.

Thus, $\succ$ completely orders allocations in the stated domain. □

Finally, define

$$S^i \equiv E^i \cup \left(\{\bar{\boldsymbol{b}}^i\} \cap M^i\right). \tag{9}$$

Theorem 5. *Let* $n = 3$, *and* π *satisfy axioms* (AN) *and* (RE). *When the extremal set is non-empty, the unique stable set to exist on* $(B^i_+ \cup E^i)$ *is* S^i; *otherwise, if the extremal set is empty, there is no stable set on* B^i_+.

By contrast to the other conditions in Algorithm 2, we do not have a computable condition that characterises the extremal set's non-emptiness, $E^i \neq \emptyset$. Such a condition would depend on the properties of M^i which, in turn, depends on the global properties of B^i. As even basic operations (e.g. computing the set of zeros) rapidly fail to have closed form solutions in even well-behaved

[23] In the *Theorema* implementation, we typically supply this information.

[24] Whereas the conditions in the algorithm are designed to be computable, the section headings are chosen for expositional clarity.

[25] Gillies (1959) called this Y*-stable* while Thrall and Lucas (1963) called this *stable for* Y. Our notation uses the index i rather than the more natural Y as we will only be interested in stability on B^i; using the natural index would require the double-index S^{B^i}.

functions (e.g. polynomials), we suspect that a computable characterisation is impossible, and that even sufficient conditions for non-emptiness will be extremely restrictive. In practice, this may not be much of an impediment: computing stable sets requires computing B^i, which then allows determination of whether $E^i = \emptyset$ on a case-by-case basis.

Proof of Theorem 5. If $\boldsymbol{e}^i_j \neq \bar{\boldsymbol{b}}^i$, then it belongs to $\left(B^i_+ \cup E^i\right)$ such that $x_j > x_k$. As $\boldsymbol{e}^i_j \in M^i$, it must also be that $e^i_{ji} \geq x_i$ for all other $\boldsymbol{x}$ in this set. When this inequality is strict, $\boldsymbol{e}^i_j$ is the set's maximal element by the first case of Lemma 7; otherwise, as $\boldsymbol{e}^i_j \in E^i$, the second case makes $\boldsymbol{e}^i_j$ the maximal element. In either case, it is the unique stable set on the set; otherwise, the set has no stable set on it (von Neumann and Morgenstern, 1953, Section 65.4.2).

When $E^i = \left\{\bar{\boldsymbol{b}}^i\right\}$, the same reasoning applies to $\boldsymbol{e}^i_j$ over $\left(B^i_+ \cup E^i\right)$ such that $x_j \geq x_k$. Otherwise, if $E^i \neq \left\{\bar{\boldsymbol{b}}^i\right\}$, consider the restriction of $\left(B^i_+ \cup E^i\right)$ to $x_j = x_k$, the singleton, $\bar{\boldsymbol{b}}^i$. When $\bar{\boldsymbol{b}}^i$ exists, it is trivially the unique stable set on itself.

Finally, as neither $\boldsymbol{e}^i_j \succeq \boldsymbol{e}^i_k$ nor the reverse, both must belong to any stable set on $\left(B^i_+ \cup E^i\right)$; $\bar{\boldsymbol{b}}^i$ will also belong to any stable set on $\left(B^i_+ \cup E^i\right)$ iff $\bar{\boldsymbol{b}}^i \in M^i$ as it will, otherwise, belong to $D\left(E^i\right)$. □

The proof may be compared to that by Jordan (2006a, Proposition 4.6), which established non-existence of a stable set for the Cobb–Douglas power function, π_c, and the SIN power function when $v \in (0, 1)$.[26] In that case, the proof demonstrated that any stable set that might exist was infinite, a contradiction; the proof here establishes a complete ordering over $B^i_+ \cup E^i$.

4.2.1. Stable sets when $K \neq \emptyset$, $E^i = \emptyset$

The first case we consider is that of a non-empty core, but empty extremal set. The main result of this section is Theorem 6, which proves that no stable set exists in this case, justifying Algorithm 2's **if** $E^i = \emptyset$ **then** block.

First, though:

Lemma 10. *Let S^i be the unique stable set on $\left(B^i_+ \cup E^i\right)$. If the core is non-empty, $\boldsymbol{x} \in S^i$, π satisfies axioms* (AN) *and* (CX), *and $\boldsymbol{y} \succeq \boldsymbol{x}$ then $\boldsymbol{y} \in D\left(\boldsymbol{t}^i\right)$.*

Proof. As $\boldsymbol{x} \in S^i$ it is also in B^i. If $\boldsymbol{y} \succeq \boldsymbol{x}$ then one of the following three disjoint cases applies:

1. $y_i > x_i$ so that, by axiom (SR), $\pi\left(\{i\}, \boldsymbol{y}\right) > \pi\left(\{i\}, \boldsymbol{x}\right)$. As $\boldsymbol{x} \in S^i$ it is also in M^i. Therefore, it cannot be that $\pi\left(\{i\}, \boldsymbol{y}\right) = \pi(\{j, k\}, \boldsymbol{y})$ as then $\boldsymbol{y} \in B^i$ with $y_i > x_i$, so that $\boldsymbol{x} \notin M^i$, a contradiction. If $\pi\left(\{i\}, \boldsymbol{y}\right) > \pi\left(\{j, k\}, \boldsymbol{y}\right)$ then $\boldsymbol{y} \in D\left(\boldsymbol{t}^i\right)$, the desired result. Finally, suppose that $\pi\left(\{i\}, \boldsymbol{y}\right) < \pi\left(\{j, k\}, \boldsymbol{y}\right)$; then, as $K \neq \emptyset$, $\pi\left(\{i\}, \boldsymbol{t}^i\right) \geq \pi\left(\{j, k\}, \boldsymbol{t}^i\right)$ so that axioms (SR) and (CX) guarantee that there exists a $\boldsymbol{z}$ such that $1 \leq z_i > y_i$ and $\pi\left(\{i\}, \boldsymbol{z}\right) = \pi\left(\{j, k\}, \boldsymbol{z}\right)$, again contradicting $\boldsymbol{x} \in M^i$.
2. $y_i = x_i$ and, permuting indices j and k if necessary, $y_j > x_j > x_k > y_k$. (Without $x_j > x_k$, $\boldsymbol{y}$ would not dominate $\boldsymbol{x}$ under axiom (AN).) As this precludes $\boldsymbol{x} = \bar{\boldsymbol{b}}^i$, it follows from Theorem 5 that $\boldsymbol{x} = \boldsymbol{e}^i_j$. It cannot be that $\pi\left(\{i\}, \boldsymbol{y}\right) = \pi\left(\{j, k\}, \boldsymbol{y}\right)$ as then $\boldsymbol{y}$ is an element of M^i with $y_j > x_j$, contradicting the assumption that $\boldsymbol{x} = \boldsymbol{e}^i_j$. Finally, $\pi\left(\{i\}, \boldsymbol{y}\right) < \pi\left(\{j, k\}, \boldsymbol{y}\right)$ is ruled out by implying the existence of the same $\boldsymbol{z}$ that yielded the contradiction in the previous case.
3. $y_i < x_i$. As $\boldsymbol{x} \in B^i$, $\pi\left(\{i\}, \boldsymbol{x}\right) = \pi\left(\{j, k\}, \boldsymbol{x}\right)$, so that $\boldsymbol{y} \not\succeq \boldsymbol{x}$, a contradiction. □

We now have the basis for Algorithm 2's **if** $E^i = \emptyset$ **then** block:

Theorem 6. *If $n = 3$, π satisfies axioms* (AN), (CX) *and* (RE), *and the core and extremal set are non-empty, then any stable set, S, must contain S^i, as defined in Eq.* (9). *If the extremal set is empty, no stable set exists on X.*

Proof. By assumption, $B^i_+ \cup E^i$ is non-empty. By Theorem 5, S^i is the unique stable set on it. Finally, as $\boldsymbol{t}^i$ must belong to S (as $\mathcal{E}^1 \subseteq K \subseteq S$), no allocation in $D\left(\boldsymbol{t}^i\right)$ can belong to S. Lemma 10 then ensures that including S^i is necessary for external stability. If it does not exist, no stable set can exist on X. This establishes the result. □

4.2.2. Stable sets when $K = \mathcal{E}^1 \cup \mathcal{E}^2$ and $E^i \neq \emptyset$

If $K, E^i \neq \emptyset$, so that the core and extremal set are non-empty, then the preceding theorem and general cooperative game theory imply that both the S^i and the core must belong to any stable set. The set of tyrannical allocations, $\mathcal{E}^1$, necessarily belong to a non-empty core. The set of split allocations, $\mathcal{E}^2$, need not though. Each of the next two subsections treats one of these two cases.

This section contains two results: the next lemma bounds the difference between B^i and B^i_+; the main result uses the lemma to compute the unique stable set under the conditions of this section (corresponding to lines 7 and 8 of the algorithm).

Lemma 11. *When $n = 3$ and π satisfies axioms* (AN) *and* (RE), *$B^i \setminus B^i_+ \subseteq \mathcal{E}^2$.*

Proof. By definition,

$$B^i \setminus B^i_+ = \left\{\boldsymbol{x} \in X \,\middle|\, x_i = \max\left\{x_j, x_k\right\}, \pi\left(\{i\}, \boldsymbol{x}\right) = \pi\left(\{j, k\}, \boldsymbol{x}\right)\right\}.$$

Without loss of generality, let $x_i = x_j \geq x_k$. It is immediate that $x_i = 0$ yields the infeasible $\boldsymbol{x} = \boldsymbol{0}$. However, it must be that $x_k = 0$: otherwise, $\pi\left(\{k\}, \boldsymbol{x}\right) > 0$ by axiom (SR) and the non-negativity normalisation so that, by axiom (RE), $\pi\left(\{j, k\}, \boldsymbol{x}\right) > \pi\left(\{j\}, \boldsymbol{x}\right)$; as axiom (AN) sets $\pi\left(\{i\}, \boldsymbol{x}\right) = \pi\left(\{j\}, \boldsymbol{x}\right)$, it would follow that $\pi\left(\{i\}, \boldsymbol{x}\right) < \pi\left(\{j, k\}, \boldsymbol{x}\right)$, preventing $\boldsymbol{x} \in B^i \setminus B^i_+$, a contradiction.

With $x_i = x_j$ and $x_k = 0$, the result follows from feasibility. □

Theorem 7. *If $n = 3$, π satisfies axioms* (AN), (CX) *and* (RE), *the core is $\mathcal{E}^1 \cup \mathcal{E}^2$, and the extremal set is non-empty, then there exists a unique stable set, $S = \mathcal{E}^1 \cup \mathcal{E}^2 \cup \left\{S^i\right\}_{i=1}^3$, where S^i is defined in Eq.* (9).

Proof. By Lemma 11, $B^i \setminus B^i_+ \subseteq \mathcal{E}^2$ under the conditions of the theorem.

As any stable set must contain the core, it must include the tyrannical allocations; thus, it cannot include $D\left(\mathcal{E}^1\right)$, which is comprised of the three

$$D\left(\boldsymbol{t}^i\right) = \left\{\boldsymbol{x} \in \left(X \setminus \mathcal{E}^1\right) | \pi\left(\{i\}, \boldsymbol{x}\right) > \pi\left(\{j, k\}, \boldsymbol{x}\right)\right\}.$$

Further, as the core includes the split $\mathcal{E}^2$, no stable set can include $D\left(\mathcal{E}^2\right)$, comprised of the three

$$D\left(\boldsymbol{s}^{jk}\right) = \left\{\boldsymbol{x} \in \left(X \setminus \left(\mathcal{E}^1 \cup \mathcal{E}^2\right)\right) | \pi\left(\{i\}, \boldsymbol{x}\right) < \pi\left(\{j, k\}, \boldsymbol{x}\right)\right\}.$$

Finally, any $\boldsymbol{x}$ such that $\pi\left(\{i\}, \boldsymbol{x}\right) = \pi\left(\{j, k\}, \boldsymbol{x}\right)$, belong to B^i. As, by Theorem 5, the S^i are the unique stable sets on $B^i \setminus \left(B^i_+ \cup E^i\right) \subseteq B^i \setminus B^i_+$, which, by Lemma 11, are subsets of $\mathcal{E}^2$, the remaining allocations in X are forced to either belong to S or are necessarily excluded. □

4.2.3. Stable sets when $K = \mathcal{E}^1$ and $E^i \neq \emptyset$

The final case to consider is that of the small core, $K = \mathcal{E}^1$ and the non-empty extremal set. As the $\boldsymbol{s}^{jk}$ do not belong to the core, this corresponds to the **else** clause in line 9 of Algorithm 2. The section's main result is Theorem 9, which presents the unique stable set under these conditions—corresponding to line 10 in the algorithm.

[26] There, the allocations indexed by λ defined a generalised version of B^i. The present lemma is obviously more restrictive insofar as it only applies to the $n = 3$ case, but more general in applying to a broader class of power functions.

The preliminary results used to establish the main theorem are Theorem 8 (which modifies the three agent version of the uniqueness result of Jordan and Obadia, 2004), Lemma 12 (which establishes that, when $\mathcal{E}^2$ is not in the core, it dominated by allocations in $\left\{S^i\right\}_{i=1}^3$), and Lemma 13 (which establishes that certain allocations are undominated by the core and the allocations stable on the S^i).

Theorem 8. *For the anonymous, $n = 3$ SIN pillage game with $v = 1$, the unique stable set is $S = \mathcal{E}^1 \cup \mathcal{E}^2$.*

Proof. It is immediate that the core consists of the tyrannical allocations, $K = \mathcal{E}^1$, so that

$$U(K) = \left\{\boldsymbol{x} \in X \left| \boldsymbol{x} > \boldsymbol{0} \cup \boldsymbol{x} \in \mathcal{E}^2\right.\right\}.$$

Adding $\mathcal{E}^2$ to the core produces the stable set in the theorem's statement.

To establish uniqueness, recall that, for $v > 1$, Jordan and Obadia (2004, Theorem 3.4) proved that $\mathcal{E}^2$ is the unique stable set on X. Two steps are required to extend their result to the case of $v = 1$. First dominance on X when $v > 1$ is equivalent to dominance on $U\left(\mathcal{E}^1\right)$ when $v = 1$: for any $\boldsymbol{y}, \boldsymbol{x}$, $W \equiv \{i \,| y_i > x_i\}$ and $L\left\{j \left| y_j < x_j\right.\right\}$, $\boldsymbol{y} \succ \boldsymbol{x}$ iff either:

1. $\|W\| = 2 > \|L\| = 1$, or
2. $\|W\| = \|L\| = 1$ such that $x_i > x_j$.

Second, the geometry of $U\left(\mathcal{E}^1\right)$ must allow the arguments used on X by Jordan and Obadia (2004). This is ensured by replacing Jordan and Obadia's $\left(\frac{3}{4}, 0, \frac{1}{4}\right)$ with $\left(\frac{3}{4}, \varepsilon, \frac{1}{4} - \varepsilon\right)$ for small $\varepsilon > 0$. Thus, $\mathcal{E}^2$ is the unique candidate for addition to the core, $\mathcal{E}^1$, when $v = 1$; as $\mathcal{E}^1 \cup \mathcal{E}^2$ is stable, the result follows. □

The previous theorem's reasoning is used to establish the algorithm's line 10, albeit on a restricted domain within X. Under Theorem 9's conditions, dominance over this restricted domain is similar to that over the full domain of the three agent majority pillage game considered by Jordan and Obadia (2004), again establishing uniqueness. Whereas the original domain, X, is a closed triangle, the restricted domain is open:

$$X_- \equiv \left\{\boldsymbol{x} \in X \left| x_1 > e_{11}^2, x_2 > e_{22}^3, x_3 > e_{33}^1\right.\right\}.$$

Define also the averages,

$$\boldsymbol{a}^{jk} \equiv \frac{1}{2}\left(\boldsymbol{e}_i^j + \boldsymbol{e}_i^k\right), \quad \text{and}$$

$$A \equiv \left\{\boldsymbol{a}^{12}, \boldsymbol{a}^{13}, \boldsymbol{a}^{23}\right\}.$$

Thus, if E is empty, so is A. Equally, X_- may be empty. As the $\boldsymbol{a}^{jk}$ are the midpoints of the edges of the closure of X_-, they are undominated by allocations in E^i.

The next two lemmas establish some dominance relationships that will be used to establish Theorem 9:

Lemma 12. *If $n = 3, \pi$ satisfies axioms (AN), (CX) and (RE), and the core and extremal set are non-empty, then $\mathcal{E}^2 \subset K \cup D\left(\left\{S^i\right\}_{i=1}^3\right)$.*

Proof. If $\mathcal{E}^2$ belongs to the core (which, by Corollary 2, it may), the result holds directly.

Now assume that $\mathcal{E}^2 \not\subseteq K$, but is dominated by some allocation, $\boldsymbol{x}$. By axiom (AN), refer, without loss of generality, to the dominated allocation as $\boldsymbol{s}^{jk}$. Axiom (AN) prevents $\boldsymbol{x} \succ \boldsymbol{s}^{jk}$ where $W = \{j\}$ and $L = \{k\}$. Axioms (WC) and (AN) together prevent $\boldsymbol{x} \succ \boldsymbol{s}^{jk}$ where $W = \{j\}$ and $L = \{i, k\}$. Similar logic rules out any domination by an $\boldsymbol{x}$ setting $W = \{i\}$.

Thus, if $\boldsymbol{x} \succ \boldsymbol{s}^{jk}$, it must be that $W = \{i, j\}$ and $L = \{k\}$. As dominance depends on coalitions and resources at the original allocation, any $\boldsymbol{x}$ generating these coalitions will dominate a $\boldsymbol{s}^{jk} \notin K$.

A non-core $\boldsymbol{s}^{jk}$ cannot belong to S^k: as the only possible dominating allocations set $W = \{i, j\}$ and $L = \{k\}$, domination would imply that $\boldsymbol{s}^{jk} \notin B^k$, a pre-requisite for membership of S^k.

If $\boldsymbol{s}^{jk} \notin S^k$, then it must also not belong to M^k: if it did, it would belong to E^k, as it is extremal, and therefore to S^k. Thus, by not belonging to M^k, it is dominated by any $\boldsymbol{x} \in S^k$, which each set $W = \{i, j\}$ and $L = \{k\}$. □

Lemma 13. *If $n = 3, \pi$ satisfies axioms (AN), (CX) and (RE), and the core and extremal sets are both non-empty, then $A \cup X_- \subset U\left(\mathcal{E}^1 \cup \left\{S^i\right\}_{i=1}^3\right)$, where S^i is defined as in Eq. (9).*

Proof. First show that $A \subset U\left(\mathcal{E}^1\right)$: define $W \equiv \left\{h \left| e_{jh}^k > a_h^{ik}\right.\right\}$ and $L \equiv \left\{h \left| a_h^{jk} > e_{jh}^k\right.\right\}$ so that $W = \{k\}$ and $L = \{i\}$. By Lemma 6 and axiom (WR) it follows that $\pi\left(W, \boldsymbol{a}^{ik}\right) < \pi\left(I \setminus W, \boldsymbol{a}^{ik}\right)$, so that $\boldsymbol{a}^{ik} \notin D\left(\boldsymbol{e}_j^k\right)$; comparing a_h^{ik} to t_h^k expands L to $\{i, j\}$ so that, by axiom (WC), $\boldsymbol{a}^{ik} \notin D\left(\boldsymbol{t}^k\right)$ either. Repeating this calculation with permuted indices establishes the result.

Proving that $A \subset U\left(\left\{S^i\right\}_{i=1}^3\right)$ is similar: repeat the above, now comparing a_h^{ik} to the remaining possible elements of S^i, namely $\bar{b}_h^k$ and $\boldsymbol{e}_{ih}^k$. In both cases, $W = \{k\}$ and $L = \{i, j\}$. Thus, as above, $\boldsymbol{a}^{ik} \notin D\left(S^k\right)$. Again, permuting the indices yields the result.

Finally, the above may be repeated to show that $X_- \subset U\left(\mathcal{E}^1\right)$ and $X_- \subset U\left(\left\{S^i\right\}_{i=1}^3\right)$. In all cases, the leading cases produce $W = \{k\}$ and $L = \{i, j\}$. □

Our final result is therefore:

Theorem 9. *If $n = 3, \pi$ satisfies axioms (AN), (CX) and (RE), the core is $\mathcal{E}^1$, and the extremal set E^i is non-empty, then there exists a unique stable set, $S = \mathcal{E}^1 \cup \left\{S^i\right\}_{i=1}^3 \cup A$, where S^i is defined in Eq. (9).*

Proof. The core must belong to any stable set; by Theorem 5, so must $\left\{S^i\right\}_{i=1}^3$. This excludes $D\left(\mathcal{E}^1\right)$ and $D\left(\left\{S^i\right\}_{i=1}^3\right)$. The remaining allocations to consider are:

1. $\mathcal{E}^2$: by Lemma 12, these belong to $D\left(\left\{S^i\right\}_{i=1}^3\right)$, so are excluded.
2. $A \cup X_-$: by Lemma 13, these are undominated by $\mathcal{E}^1 \cup \left\{S^i\right\}_{i=1}^3$. Applying the steps used in the proof of Theorem 8 to extend Jordan and Obadia (2004, Theorem 3.4) to X_i establishes that P is the unique stable set on $X_- \cup A$. □

4.2.4. A tight bound on the cardinality of stable sets

Collectively, the results presented above bound the number of elements in any stable set for a three agent pillage game satisfying the additional axioms:

Theorem 10. *When $n = 3$, and π satisfies axioms (AN), (CX) and (RE), if a stable set, S, exists, then $\|S\| \leq 15$.*

Proof. The preceding establishes that Algorithm 2 provides a complete classification of stable sets for $n = 3$ pillage games satisfying additional axioms (AN), (CX) and (RE). The largest stable set for such games may be found directly by counting the allocations arising from each of the cases. The case of an empty core yields a stable set containing only three allocations. When the core is non-empty, the three tyrannical allocation belong to S; each S^i contributes up to three more; finally, either $\mathcal{E}^2$ or A can contribute up to three more allocations. □

This bound improves enormously on the finite bound by Jordan (2006a), Kerber and Rowat's Ramsey bound, and the doubly exponential bound by Saxton (2011). The first two of those bounds derived implications of axiom (SR) alone; the last also took axiom (WC) into account. Further, the bound is tight. Setting $k = 9$ in the sinusoidal power functions defined by Eq. (4), yields – by Theorem 9 – $\mathcal{E}^1 \cup \mathcal{E}^2 \cup \left\{S^i\right\}_{i=1}^3$, as its unique stable set; as each S^i contains distinct $\boldsymbol{e}_j^i$ and $\boldsymbol{e}_k^i$ as well as a $\bar{\boldsymbol{b}}^i$, $\|S\| = 15$.

5. Examples

This section presents two examples that help to illustrate or expand upon points made in the body of the paper's analysis, above.

First we analyse the power function cited as exemplifying line 10 of Algorithm 2:

Example 1. Let

$$\pi(C, \boldsymbol{x}) = \sum_{i \in C} \left(\sqrt{x_i} + v\right)$$

where $v \in (0, 1)$. Now the conditions in both lines 1 and 7 of the algorithm are failed, so that line 10 returns the unique stable set, $S = \mathcal{E}^1 \cup \left\{\bar{\boldsymbol{b}}^1, \bar{\boldsymbol{b}}^2, \bar{\boldsymbol{b}}^3\right\} \cup A$, where each $\bar{\boldsymbol{b}}^i = S^i \in \left\{\bar{\boldsymbol{b}}^1, \bar{\boldsymbol{b}}^2, \bar{\boldsymbol{b}}^3\right\}$ implicitly solves $\sqrt{\bar{b}_i^i} = \sqrt{2\left(1 - \bar{b}_i^i\right)} + v$ and $\bar{b}_j^i = \bar{b}_k^i$, and A is $\left(\frac{1+q_i^i}{4}, \frac{1+q_i^i}{4}, \frac{1-q_i^i}{2}\right)$ and its permutations.

The next example shows that axiom (CX) is necessary for the proof of Lemma 10: when it is violated, a $\boldsymbol{y} \notin D\left(\boldsymbol{t}^i\right)$ may be found to dominate $\boldsymbol{x} \in B^i$:

Example 2. Consider a discontinuous version of SIN, so that

$$\pi(C, x) \equiv \sum_{i \in C} \left\{\begin{array}{ll} \frac{1}{2}x_i + v & \text{if } x_i \leq \dfrac{1-v}{2} \\ x_i + v & \text{otherwise} \end{array}\right\}; \tag{10}$$

where $v \in \left(\frac{1}{7}, 1\right)$. The existence of B^i requires that $x_i > \frac{1-v}{2} \geq x_j, x_k$:

$$B^i = \left\{\boldsymbol{x} \in X \,\middle|\, x_i = \frac{1+2v}{3}, \max\left\{x_j, x_k\right\} \leq \frac{1-v}{2}\right\}.$$

Thus, if depicted in the simplex, B^i is a line that stops short of the edges. As x_i remains constant over B^i, $M^i = B^i$; $\bar{\boldsymbol{b}}^i$ sets $\bar{b}_i^i = \frac{1+2v}{3}$ and $\bar{b}_j^i = \bar{b}_k^i = \frac{1-v}{3}$; finally, $E^i = \left\{\boldsymbol{e}_j^i, \boldsymbol{e}_k^i\right\}$, where $\boldsymbol{e}_j^i$ sets $e_{ji}^i = \frac{1+2v}{3}$, $e_{jj}^i = \frac{1-v}{2}$ and $e_{jk}^i = \frac{1-v}{6}$. There is therefore a $\boldsymbol{y} \in D\left(\boldsymbol{s}^{jk}\right)$ with $y_i = e_{ji}^i$ and $y_j > e_{jj}^i$ such that $\boldsymbol{y} \succ \boldsymbol{x}$ for any $\boldsymbol{x} \in B^i$ with $x_i = e_{ji}^i$ and $x_j > x_k$.

While Lemma 10 therefore does not apply to this example,

$$\mathcal{E}^1 \cup \mathcal{E}^2 \cup \left\{\bar{\boldsymbol{b}}^1, \bar{\boldsymbol{b}}^2, \bar{\boldsymbol{b}}^3\right\} \cup E$$

is, nonetheless, stable.

6. Discussion

The paper's main result is that, when a stable set exists in an $n = 3$ pillage game satisfying axioms (AN), (CX) and (RE), it is unique and contains no more than 15 allocations, a tight bound. Furthermore, the paper presents an algorithm for deciding and deriving stable sets which is fully computational in all but one step, namely the determination of whether $E^i = \emptyset$. We think that progress on converting this into a computational step is likely to require more structure than that which we impose. We also note that the algorithm implies that any two pillage games satisfying its conditions and sharing a B^i share a stable set, if they have one: the common B^i ensures that they both pass or fail the conditions in lines 1, 4 and 7 in the same way; the elements of the ensuing stable sets, if they exist, depend entirely on the geometry of the shared B^i.

By contrast with the cardinality bound derived for pillage games here, for $n = 3$ CF games, the best known class of cooperative games, "stable sets are typically not unique", but existence is guaranteed (Lucas, 1992, pp. 562–563).

To understand reasons for this difference, consider some $\boldsymbol{x}$ in a CF game. An infinite subset of the allocations near $\boldsymbol{x}$ may be incomparable, in the sense that neither $\boldsymbol{x} \succ \boldsymbol{y}$ nor $\boldsymbol{y} \succ \boldsymbol{x}$ (q.v. Lucas, 1992, Examples 3, 4), allowing construction of infinite stable sets. In pillage games, where power depends monotonically on resource holdings, the domain of this incomparability is reduced. This reduced domain also seems responsible for the finitude of stable sets in pillage games. Indeed, as Lucas (1992) notes, only one $n = 3$ CF game has a finite stable set, namely the majority game in which $\boldsymbol{y} \succ \boldsymbol{x}$ iff two agents prefer $\boldsymbol{y}$ to $\boldsymbol{x}$. In this case, the regions of incomparability are 1-dimensional curves in the simplex, which still admit a family of infinite stable sets.[27]

The paper provides guidance on handling the $n > 3$ case as well: there may be a $D\left(\boldsymbol{t}^i\right)$ region around each $\boldsymbol{t}^i$, separated from the rest of X by a B^i. Analysis of $D\left(\boldsymbol{t}^i\right)$ and B^i likely proceeds as in the $n = 3$ case. On the other side of the B^i, the game will again be equivalent to the SIN game with $v > 1$, for which few results are known (Jordan and Obadia, 2004). Thus, progress in analysing $n > 3$ pillage games likely depends on progress in analysing these SIN games.

Relaxing the anonymity axiom is likely an easier extension of the current analysis.[28] Success in doing so will ease empirical applications of the theory: natural applications include the three-player, empty-core version (Straffin, 1993) of the ten-player problem of how to launch and share communications satellites (McDonald, 1977), or the Hellman and Wasserman (2011) dataset on equity divisions in over 500 new ventures, many of them with three founders.

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27 See Jordan and Obadia (2004) for a discussion of the majority pillage game's use of resource holdings to break ties in the majority game.

28 See MacKenzie et al. (2013) for initial steps.

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