# Predicative aspects of order theory in univalent foundations

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# Predicative Aspects of Order Theory in Univalent Foundations

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## Abstract

We investigate predicative aspects of order theory in constructive univalent foundations. By predicative and constructive, we respectively mean that we do not assume Voevodsky's propositional resizing axioms or excluded middle. Our work complements existing work on predicative mathematics by exploring what *cannot* be done predicatively in univalent foundations. Our first main result is that nontrivial (directed or bounded) complete posets are necessarily large. That is, if such a nontrivial poset is small, then weak propositional resizing holds. It is possible to derive full propositional resizing if we strengthen nontriviality to positivity. The distinction between nontriviality and positivity is analogous to the distinction between nonemptiness and inhabitedness. We prove our results for a general class of posets, which includes directed complete posets, bounded complete posets and sup-lattices, using a technical notion of a $\delta_{\mathcal{V}}$-complete poset. We also show that nontrivial locally small $\delta_{\mathcal{V}}$-complete posets necessarily lack decidable equality. Specifically, we derive weak excluded middle from assuming a nontrivial locally small $\delta_{\mathcal{V}}$-complete poset with decidable equality. Moreover, if we assume positivity instead of nontriviality, then we can derive full excluded middle. Secondly, we show that each of Zorn's lemma, Tarski's greatest fixed point theorem and Pataraia's lemma implies propositional resizing. Hence, these principles are inherently impredicative and a predicative development of order theory must therefore do without them. Finally, we clarify, in our predicative setting, the relation between the traditional definition of sup-lattice that requires suprema for all subsets and our definition that asks for suprema of all small families.

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## 1 Introduction

We investigate predicative aspects of order theory in constructive univalent foundations. By predicative and constructive, we respectively mean that we do not assume Voevodsky's propositional resizing axioms [26, 27] or excluded middle. Our work is situated in our larger programme of developing domain theory constructively and predicatively in univalent foundations. In previous work [12], we showed how to give a constructive and predicative account of many familiar constructions and notions in domain theory, such as Scott's $D_\infty$ model of untyped $\lambda$-calculus and the theory of continuous dcpos. The present work complements this and other existing work on predicative mathematics (e.g. [2, 21, 6]) by exploring what *cannot* be done predicatively, as in [7, 8, 9, 10, 11]. We do so by showing that certain statements crucially rely on resizing axioms in the sense that they are equivalent to them. Such arguments are important in constructive mathematics. For example, the constructive failure of trichotomy on the real numbers is shown [4] by reducing it to a nonconstructive instance of excluded middle.


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Our first main result is that nontrivial (directed or bounded) complete posets are necessarily large. In [12] we observed that all our examples of directed complete posets have large carriers. We show here that this is no coincidence, but rather a necessity, in the sense that if such a nontrivial poset is small, then weak propositional resizing holds. It is possible to derive full propositional resizing if we strengthen nontriviality to positivity in the sense of [19]. The distinction between nontriviality and positivity is analogous to the distinction between nonemptiness and inhabitedness. We prove our results for a general class of posets, which includes directed complete posets, bounded complete posets and sup-lattices, using a technical notion of a $\delta_{\mathcal{V}}$-complete poset. We also show that nontrivial locally small $\delta_{\mathcal{V}}$-complete posets necessarily lack decidable equality. Specifically, we can derive weak excluded middle from assuming the existence of a nontrivial locally small $\delta_{\mathcal{V}}$-complete poset with decidable equality. Moreover, if we assume positivity instead of nontriviality, then we can derive full excluded middle.

Secondly, we prove that each of Zorn's lemma, Tarski's greatest fixed point theorem and Pataraia's lemma implies propositional resizing. Hence, these principles are inherently impredicative and a predicative development of order theory in univalent foundations must thus forgo them.

Finally, we clarify, in our predicative setting, the relation between the traditional definition of sup-lattice that requires suprema for all subsets and our definition that asks for suprema of all small families. This is important in practice in order to obtain workable definitions of dcpo, sup-lattice, etc. in the context of predicative univalent mathematics.

Our foundational setup is the same as in [12], meaning that our work takes places in intensional Martin-Löf Type Theory and adopts the univalent point of view [24]. This means that we work with the stratification of types as singletons, propositions (or subsingletons or truth values), sets, 1-groupoids, etc., and that we work with univalence. At present, higher inductive types other than propositional truncation are not needed. Often the only consequences of univalence needed here are functional and propositional extensionality. An exception is Section 2.3. Full details of our univalent type theory are given at the start of Section 2.

## Related work

Curi investigated the limits of predicative mathematics in CZF [2] in a series of papers [7, 8, 9, 10, 11]. In particular, Curi shows (see [7, Theorem 4.4 and Corollary 4.11], [8, Lemma 1.1] and [9, Theorem 2.5]) that CZF cannot prove that various nontrivial posets, including sup-lattices, dcpos and frames, are small. This result is obtained by exploiting that CZF is consistent with the anti-classical generalized uniformity principle GUP [25, Theorem 4.3.5]. Our related Theorem 35 is of a different nature in two ways. Firstly, our theorem is in the spirit of reverse constructive mathematics [18]: Instead of showing that GUP implies that there are no non-trivial small dcpos, we show that the existence of a non-trivial small dcpo is *equivalent* to weak propositional resizing, and that the existence of a positive small dcpo is *equivalent* to full propositional resizing. Thus, if we wish to work with small dcpos, we are forced to assume resizing axioms. Secondly, we work in univalent foundations rather than CZF. This may seem a superficial difference, but a number of arguments in Curi's papers [9, 10] crucially rely on set-theoretical notions and principles such as transitive set, set-induction, weak regular extension axiom wREA, which cannot even be formulated in the underlying type theory of univalent foundations. Moreover, although Curi claims that the arguments of [7, 8] can be adapted to some version of Martin-Löf Type Theory, it is presently not known whether there is any model of univalent foundations which validates GUP.

### Organization

*Section 2*: Foundations and size matters, including impredicativity, relation to excluded middle, univalence and closure under embedded retracts. *Section 3*: Nontrivial and positive $\delta_{\mathcal{V}}$-complete posets and reductions to impredicativity and excluded middle. *Section 4*: Predicative invalidity of Zorn's lemma, Tarski's fixed point theorem and Pataraia's lemma. *Section 5*: Comparison of completeness w.r.t. families and w.r.t. subsets. *Section 6*: Conclusion and future work.

## 2 Foundations and Size Matters

We work with a subset of the type theory described in [24] and we mostly adopt the terminological and notational conventions of [24]. We include $+$ (binary sum), $\Pi$ (dependent products), $\Sigma$ (dependent sum), Id (identity type), and inductive types, including $\mathsf{0}$ (empty type), $\mathsf{1}$ (type with exactly one element $\star : \mathsf{1}$), $\mathsf{N}$ (natural numbers). We assume a universe $\mathcal{U}_0$ and two operations: for every universe $\mathcal{U}$ a successor universe $\mathcal{U}^+$ with $\mathcal{U} : \mathcal{U}^+$, and for every two universes $\mathcal{U}$ and $\mathcal{V}$ another universe $\mathcal{U} \sqcup \mathcal{V}$ such that for any universe $\mathcal{U}$, we have $\mathcal{U}_0 \sqcup \mathcal{U} \equiv \mathcal{U}$ and $\mathcal{U} \sqcup \mathcal{U}^+ \equiv \mathcal{U}^+$. Moreover, $(-) \sqcup (-)$ is idempotent, commutative, associative, and $(-)^+$ distributes over $(-) \sqcup (-)$. We write $\mathcal{U}_1 :\equiv \mathcal{U}_0^+$, $\mathcal{U}_2 :\equiv \mathcal{U}_1^+, \ldots$ and so on. If $X : \mathcal{U}$ and $Y : \mathcal{V}$, then $X + Y : \mathcal{U} \sqcup \mathcal{V}$ and if $X : \mathcal{U}$ and $Y : X \to \mathcal{V}$, then the types $\Sigma_{x:X} Y(x)$ and $\Pi_{x:X} Y(x)$ live in the universe $\mathcal{U} \sqcup \mathcal{V}$; finally, if $X : \mathcal{U}$ and $x, y : X$, then $\mathrm{Id}_X(x, y) : \mathcal{U}$. The type of natural numbers $\mathsf{N}$ is assumed to be in $\mathcal{U}_0$ and we postulate that we have copies $\mathsf{0}_{\mathcal{U}}$ and $\mathsf{1}_{\mathcal{U}}$ in every universe $\mathcal{U}$. We assume function extensionality and propositional extensionality tacitly, and univalence explicitly when needed. Finally, we use a single higher inductive type: the propositional truncation of a type $X$ is denoted by $\|X\|$ and we write $\exists_{x:X} Y(x)$ for $\|\sum_{x:X} Y(x)\|$.

### 2.1 The Notion of Size

We introduce the fundamental notion of a type having a certain size and specify the impredicativity axioms under consideration (Section 2.2). We also note the relation to excluded middle (Section 2.2) and univalence (Section 2.3). Finally in Section 2.4 we review embeddings and sections and establish our main technical result on size, namely that having a certain size is closed under retracts whose sections are embeddings.

▶ **Definition 1** (Size, `UF-Slice.html` in [16]). *A type $X$ in a universe $\mathcal{U}$ is said to* have size $\mathcal{V}$ *if it is equivalent to a type in the universe $\mathcal{V}$. That is, $X$* has-size $\mathcal{V} :\equiv \sum_{Y:\mathcal{V}} (Y \simeq X)$.

### 2.2 Impredicativity and Excluded Middle

We consider various impredicativity axioms and their relation to (weak) excluded middle. The definitions and propositions below may be found in [15, Section 3.36], so proofs are omitted here.

▶ **Definition 2** (Impredicativity axioms).

**(i)** *By* Propositional-Resizing$_{\mathcal{U},\mathcal{V}}$ *we mean the assertion that every proposition $P$ in a universe $\mathcal{U}$ has size $\mathcal{V}$.*

**(ii)** *The type of all propositions in a universe $\mathcal{U}$ is denoted by $\Omega_{\mathcal{U}}$. Observe that $\Omega_{\mathcal{U}} : \mathcal{U}^+$. We write* $\Omega$-Resizing$_{\mathcal{U},\mathcal{V}}$ *for the assertion that the type $\Omega_{\mathcal{U}}$ has size $\mathcal{V}$.*

**(iii)** *The type of all $\neg\neg$-stable propositions in a universe $\mathcal{U}$ is denoted by $\Omega_{\mathcal{U}}^{\neg\neg}$, where a proposition $P$ is* $\neg\neg$-stable *if $\neg\neg P$ implies $P$. By $\Omega_{\neg\neg}$-Resizing$_{\mathcal{U},\mathcal{V}}$ we mean the assertion that the type $\Omega_{\mathcal{U}}^{\neg\neg}$ has size $\mathcal{V}$.*

**(iv)** *For the particular case of a single universe, we write $\Omega$-Resizing$_{\mathcal{U}}$ and $\Omega_{\neg\neg}$-Resizing$_{\mathcal{U}}$ for the respective assertions that $\Omega_{\mathcal{U}}$ has size $\mathcal{U}$ and $\Omega_{\mathcal{U}}^{\neg\neg}$ has size $\mathcal{U}$.*

▶ **Proposition 3.**

**(i)** *The principle $\Omega$-Resizing$_{\mathcal{U},\mathcal{V}}$ implies* Propositional-Resizing$_{\mathcal{U},\mathcal{V}}$ *for every two universes $\mathcal{U}$ and $\mathcal{V}$.*

**(ii)** *The conjunction of* Propositional-Resizing$_{\mathcal{U},\mathcal{V}}$ *and* Propositional-Resizing$_{\mathcal{V},\mathcal{U}}$ *implies $\Omega$-Resizing$_{\mathcal{U},\mathcal{V}^+}$ for every two universes $\mathcal{U}$ and $\mathcal{V}$.*

It is possible to define a weaker variation of propositional resizing for $\neg\neg$-stable propositions only (and derive similar connections), but we don't have any use for it in this paper.

▶ **Definition 4** ((Weak) excluded middle)**.**

**(i)** Excluded middle *in a universe $\mathcal{U}$ asserts that for every proposition $P$ in $\mathcal{U}$ either $P$ or $\neg P$ holds.*

**(ii)** Weak excluded middle *in a universe $\mathcal{U}$ asserts that for every proposition $P$ in $\mathcal{U}$ either $\neg P$ or $\neg\neg P$ holds.*

We note that weak excluded middle says precisely that $\neg\neg$-stable propositions are decidable and is equivalent to de Morgan's Law.

▶ **Proposition 5.** *Excluded middle implies impredicativity. Specifically,*

**(i)** *Excluded middle in $\mathcal{U}$ implies $\Omega$-Resizing$_{\mathcal{U},\mathcal{U}_0}$.*

**(ii)** *Weak excluded middle in $\mathcal{U}$ implies $\Omega_{\neg\neg}$-Resizing$_{\mathcal{U},\mathcal{U}_0}$.*

## 2.3 Size and Univalence

Assuming univalence we can prove that Propositional-Resizing$_{\mathcal{U},\mathcal{V}}$ and $\Omega$-Resizing$_{\mathcal{U},\mathcal{V}}$ are subsingletons. More generally, univalence allows us to prove that the statement that $X$ has size $\mathcal{V}$ is a proposition, which is needed in Section 3.5.

▶ **Proposition 6** (cf. `has-size-is-subsingleton` in [15])**.** *If $\mathcal{V}$ and $\mathcal{U} \sqcup \mathcal{V}$ are univalent universes, then $X$* has-size *$\mathcal{V}$ is a proposition for every $X : \mathcal{U}$.*

The converse also holds in the following form.

▶ **Proposition 7.** *The type $X$* has-size *$\mathcal{U}$ is a proposition for every $X : \mathcal{U}$ if and only if $\mathcal{U}$ is a univalent universe.*

**Proof.** Note that $X$ has-size $\mathcal{U}$ is $\sum_{Y:\mathcal{U}} Y \simeq X$, so this can be found in [15, Section 3.14]. ◀

## 2.4 Size and Retracts

We show our main technical result on size here, namely that having a size is closed under retracts whose sections are embeddings.

▶ **Definition 8** (Sections, retractions and embeddings)**.**

**(i)** *A* section *is a map $s : X \to Y$ together with a left inverse $r : Y \to X$, i.e. the maps satisfy $r \circ s \sim \mathrm{id}$. We call $r$ the* retraction *and say that $X$ is a* retract *of $Y$.*

**(ii)** *A function $f : X \to Y$ is an embedding if the map $\mathrm{ap}_f : (x = y) \to (f(x) = f(y))$ is an equivalence for every $x, y : X$. (See [24, Definition 4.6.1(ii)].)*

**(iii)** *A* section-embedding *is a section $s : X \to Y$ that moreover is an embedding. We also say that $X$ is an* embedded retract *of $Y$.*

We recall the following facts about embeddings and sections.

▶ **Lemma 9.**
- **(i)** *A function $f : X \to Y$ is an* embedding *if and only if all its fibres are subsingletons, i.e.* $\prod_{y:Y}$ is-subsingleton$(\mathrm{fib}_f(y))$. *(See [24, Proof of Theorem 4.6.3].)*
- **(ii)** *If every section is an embedding, then every type is a set. (See [22, Remark 3.11(2)].)*
- **(iii)** *Sections to sets are embeddings. (See [15, `lc-maps-into-sets-are-embeddings`].)*

In phrasing our results it is helpful to extend the notion of size from types to functions.

▶ **Definition 10** (Size (for functions), `UF-Slice.html` in [16]). *A function $f : X \to Y$ is said to* have size $\mathcal{V}$ *if every fibre has size $\mathcal{V}$.*

▶ **Lemma 11** (cf. `UF-Slice.html` in [16]).
- **(i)** *A type $X$ has size $\mathcal{V}$ if and only if the unique map $X \to \mathbf{1}_{\mathcal{U}_0}$ has size $\mathcal{V}$.*
- **(ii)** *If $f : X \to Y$ has size $\mathcal{V}$ and $Y$ has size $\mathcal{V}$, then so does $X$.*
- **(iii)** *If $s : X \to Y$ is a section-embedding and $Y$ has size $\mathcal{V}$, then $s$ has size $\mathcal{V}$ too, regardless of the size of $X$.*

**Proof.** The first two claims follow from the fact that for any map $f : X \to Y$ we have an equivalence $X \simeq \sum_{y:Y} \mathrm{fib}_f(y)$ (see [24, Lemma 4.8.2]). For the third claim, suppose that $s : X \to Y$ an embedding with retraction $r : Y \to X$. By the second part of the proof of Theorem 3.10 in [22], we have $\mathrm{fib}_s(y) \simeq \|s(r(y)) = y\|$, from which the claim follows. ◀

▶ **Lemma 12.**
- **(i)** *If $X$ is an embedded retract of $Y$ and $Y$ has size $\mathcal{V}$, then so does $X$.*
- **(ii)** *If $X$ is a retract of a set $Y$ and $Y$ has size $\mathcal{V}$, then so does $X$.*

**Proof.** The first statement follows from (ii) and (iii) of Lemma 11. The second follows from the first and item (iii) of Lemma 9. ◀

## 3 Large Posets Without Decidable Equality

We show that constructively and predicatively many structures from order theory (directed complete posets, bounded complete posets, sup-lattices) are necessarily large and necessarily lack decidable equality. We capture these structures by a technical notion of a $\delta_{\mathcal{V}}$-complete poset in Section 3.1. In Section 3.2 we define when such structures are nontrivial and introduce the constructively stronger notion of positivity. Section 3.3 and Section 3.4 contain the two fundamental technical lemmas and the main theorems, respectively. Finally, Section 3.5 considers alternative formulations of being nontrivial and positive that ensure that these notions are properties, as opposed to data and shows how the main theorems remain valid, assuming univalence.

### 3.1 $\delta_{\mathcal{V}}$-complete Posets

We start by introducing a class of weakly complete posets that we call $\delta_{\mathcal{V}}$-complete posets. The notion of a $\delta_{\mathcal{V}}$-complete poset is a technical and auxiliary notion sufficient to make our main theorems go through. The important point is that many familiar structures (dcpos, bounded complete posets, sup-lattices) are $\delta_{\mathcal{V}}$-complete posets (see Examples 15).

▶ **Definition 13** ($\delta_{\mathcal{V}}$-complete poset, $\delta_{x,y,P}$, $\bigvee \delta_{x,y,P}$)**.** *A* poset *is a type $X$ with a subsingleton-valued binary relation $\sqsubseteq$ on $X$ that is reflexive, transitive and antisymmetric. It is not necessary to require $X$ to be a set, as this follows from the other requirements. A poset $(X, \sqsubseteq)$ is* $\delta_{\mathcal{V}}$-complete *for a universe $\mathcal{V}$ if for every pair of elements $x, y : X$ with $x \sqsubseteq y$ and every subsingleton $P$ in $\mathcal{V}$, the family*

$$\begin{aligned} \delta_{x,y,P} : 1 + P &\to X \\ \mathrm{inl}(\star) &\mapsto x; \\ \mathrm{inr}(p) &\mapsto y; \end{aligned}$$

*has a supremum $\bigvee \delta_{x,y,P}$ in $X$.*

▶ Remark 14 (Every poset is $\delta_{\mathcal{V}}$-complete, classically). Consider a poset $(X, \sqsubseteq)$ and a pair of elements $x \sqsubseteq y$. If $P : \mathcal{V}$ is a decidable proposition, then we can define the supremum of $\delta_{x,y,P}$ by case analysis on whether $P$ holds or not. For if it holds, then the supremum is $y$, and if it does not, then the supremum is $x$. Hence, if excluded middle holds in $\mathcal{V}$, then the family $\delta_{x,y,P}$ has a supremum for every $P : \mathcal{V}$. Thus, if excluded middle holds in $\mathcal{V}$, then every poset (in any universe) is $\delta_{\mathcal{V}}$-complete.

The above remark naturally leads us to ask whether the converse also holds, i.e. if every poset is $\delta_{\mathcal{V}}$-complete, does excluded middle in $\mathcal{V}$ hold? As far as we know, we can only get weak excluded middle in $\mathcal{V}$, as we will later see in Proposition 18. This proposition also shows that in the absence of excluded middle, the notion of $\delta_{\mathcal{V}}$-completeness isn't trivial. For now, we focus on the fact that, also constructively and predicatively, there are many examples of $\delta_{\mathcal{V}}$-complete posets.

▶ **Examples 15.**

**(i)** *Every $\mathcal{V}$-sup-lattices is $\delta_{\mathcal{V}}$-complete. That is, if a poset $X$ has suprema for all families $I \to X$ with $I$ in the universe $\mathcal{V}$, then $X$ is $\delta_{\mathcal{V}}$-complete.*

**(ii)** *The $\mathcal{V}$-sup-lattice $\Omega_{\mathcal{V}}$ is $\delta_{\mathcal{V}}$-complete. The type $\Omega_{\mathcal{V}}$ of propositions in $\mathcal{V}$ is a $\mathcal{V}$-sup-lattice with the order given by implication and suprema by existential quantification. Hence, $\Omega_{\mathcal{V}}$ is $\delta_{\mathcal{V}}$-complete. Specifically, given propositions $Q$, $R$ and $P$, the supremum of $\delta_{Q,R,P}$ is given by $Q \vee (R \times P)$.*

**(iii)** *The $\mathcal{V}$-powerset $\mathcal{P}_{\mathcal{V}}(X) :\equiv X \to \Omega_{\mathcal{V}}$ of a type $X$ is $\delta_{\mathcal{V}}$-complete. Note that $\mathcal{P}_{\mathcal{V}}(X)$ is another example of a $\mathcal{V}$-sup-lattice (ordered by subset inclusion and with suprema given by unions) and hence $\delta_{\mathcal{V}}$-complete.*

**(iv)** *Every $\mathcal{V}$-bounded complete posets is $\delta_{\mathcal{V}}$-complete. That is, if $(X, \sqsubseteq)$ is a poset with suprema for all bounded families $I \to X$ with $I$ in the universe $\mathcal{V}$, then $(X, \sqsubseteq)$ is $\delta_{\mathcal{V}}$-complete. A family $\alpha : I \to X$ is bounded if there exists some $x : X$ with $\alpha(i) \sqsubseteq x$ for every $i : I$. For example, the family $\delta_{x,y,P}$ is bounded by $y$.*

**(v)** *Every $\mathcal{V}$-directed complete poset (dcpo) is $\delta_{\mathcal{V}}$-complete, since the family $\delta_{x,y,P}$ is directed. We note that [12] provides a host of examples of $\mathcal{V}$-dcpos.*

## 3.2 Nontrivial and Positive Posets

In Remark 14 we saw that if we can decide a proposition $P$, then we can define $\bigvee \delta_{x,y,P}$ by case analysis. What about the converse? That is, if $\delta_{x,y,P}$ has a supremum and we know that it equals $x$ or $y$, can we then decide $P$? Of course, if $x = y$, then $\bigvee \delta_{x,y,P} = x = y$, so we don't learn anything about $P$. But what if add the assumption that $x \neq y$? It turns out that constructively we can only expect to derive decidability of $\neg P$ in that case. This is due to the fact that $x \neq y$ is a negated proposition, which is rather weak constructively, leading us to later define (see Definition 20) a constructively stronger notion for elements of $\delta_{\mathcal{V}}$-complete posets.

▶ **Definition 16** (Nontrivial). *A poset $(X, \sqsubseteq)$ is* nontrivial *if we have designated $x, y : X$ with $x \sqsubseteq y$ and $x \neq y$.*

▶ **Lemma 17.** *Let $(X, \sqsubseteq, x, y)$ be a nontrivial poset. We have the following implications for every proposition $P : \mathcal{V}$:*

**(i)** *if the supremum of $\delta_{x,y,P}$ exists and $x = \bigvee \delta_{x,y,P}$, then $\neg P$ is the case.*
**(ii)** *if the supremum of $\delta_{x,y,P}$ exists and $y = \bigvee \delta_{x,y,P}$, then $\neg\neg P$ is the case.*

**Proof.** Let $P : \mathcal{V}$ be an arbitrary proposition. For (i), suppose that $x = \bigvee \delta_{x,y,P}$ and assume for a contradiction that we have $p : P$. Then $y \equiv \delta_{x,y,P}(\mathrm{inr}(p)) \sqsubseteq \bigvee \delta_{x,y,P} = x$, which is impossible by antisymmetry and our assumptions that $x \sqsubseteq y$ and $x \neq y$. For (ii), suppose that $y = \bigvee \delta_{x,y,P}$ and assume for a contradiction that $\neg P$ holds. Then $x = \bigvee \delta_{x,y,P} = y$, contradicting our assumption that $x \neq y$. ◀

▶ **Proposition 18** (cf. Section 4 of [12]). *Let $\mathsf{2}$ be the poset with exactly two elements $0 \sqsubseteq 1$. If $\mathsf{2}$ is $\delta_{\mathcal{V}}$-complete, then weak excluded middle in $\mathcal{V}$ holds.*

**Proof.** Suppose that $\mathsf{2}$ were $\delta_{\mathcal{V}}$-complete and let $P : \mathcal{V}$ be an arbitrary subsingleton. We must show that $\neg P$ is decidable. Since $\mathsf{2}$ has exactly two elements, the supremum $\bigvee \delta_{0,1,P}$ must be 0 or 1. But then we apply Lemma 17 to get decidability of $\neg P$. ◀

That the conclusion of the implication in Lemma 17(ii) cannot be strengthened to say that $P$ is the case is shown by the following observation.

▶ **Proposition 19.** *Recall Examples 15, which show that $\Omega_{\mathcal{V}}$ is $\delta_{\mathcal{V}}$-complete. If for every two propositions $Q$ and $R$ with $Q \sqsubseteq R$ and $Q \neq R$ we have that the equality $R = \bigvee \delta_{Q,R,P}$ in $\Omega_{\mathcal{V}}$ implies $P$ for every proposition $P : \mathcal{V}$, then excluded middle in $\mathcal{V}$ follows.*

**Proof.** Assume the hypothesis in the proposition. We are going to show that $\neg\neg P \to P$ for every proposition $P : \mathcal{V}$, from which excluded middle in $\mathcal{V}$ holds. Let $P$ be a proposition in $\mathcal{V}$ and assume that $\neg\neg P$. This yields $\mathsf{0} \neq P$, so by assumption the equality $P = \bigvee \delta_{0,P,P}$ implies $P$. But, recalling item (ii) of Examples 15, we have exactly this equality $\bigvee \delta_{0,P,P} = P$. ◀

We have seen that having a pair of elements $x, y$ with $x \sqsubseteq y$ and $x \neq y$ is very weak constructively. As promised in the introduction of this section, we now introduce a constructively stronger notion.

▶ **Definition 20** (Strictly below, $x \sqsubset y$). *Let $(X, \sqsubseteq)$ be a $\delta_{\mathcal{V}}$-complete poset and $x, y : X$. We say that $x$ is* strictly below *$y$ if $x \sqsubseteq y$ and, moreover, for every $z \sqsupseteq y$ and every proposition $P : \mathcal{V}$, the equality $z = \bigvee \delta_{x,z,P}$ implies $P$.*

Note that with excluded middle, $x \sqsubset y$ is equivalent to the conjunction of $x \sqsubseteq y$ and $x \neq y$. But constructively, the former is much stronger, as the following example and proposition illustrate.

▶ **Example 21** (Strictly below in $\Omega_{\mathcal{V}}$). Recall from Examples 15 that $\Omega_{\mathcal{V}}$ is $\delta_{\mathcal{V}}$-complete. Let $P : \mathcal{V}$ be an arbitrary proposition. Observe that $\mathsf{0}_{\mathcal{V}} \neq P$ precisely when $\neg\neg P$ holds. However, $\mathsf{0}_{\mathcal{V}}$ is strictly below $P$ if and only if $P$ holds.

▶ **Proposition 22.** *For a $\delta_{\mathcal{V}}$-complete poset $(X, \sqsubseteq)$ and $x, y : X$, we have that $x \sqsubset y$ implies both $x \sqsubseteq y$ and $x \neq y$. However, if the conjunction of $x \sqsubseteq y$ and $x \neq y$ implies $x \sqsubset y$ for every $x, y : \Omega_{\mathcal{V}}$, then excluded middle in $\mathcal{V}$ holds.*

**Proof.** Note that $x \sqsubset y$ implies $x \sqsubseteq y$ by definition. Now suppose that $x \sqsubset y$ and assume $x = y$ for a contradiction. Since we assumed $x \sqsubset y$, the equality $y = \bigvee \delta_{x,y,0_\mathcal{V}}$ implies that $0_\mathcal{V}$ holds. But this equality holds since $x = y$ by our other assumption, so $x \neq y$, as desired.

For $P : \Omega_\mathcal{V}$ we observed that $0_\mathcal{V} \neq P$ is equivalent to $\neg\neg P$ and that $0_\mathcal{V} \sqsubset P$ is equivalent to $P$, so if we had $((x \sqsubseteq y) \times (x \neq y)) \to x \sqsubset y$ in general, then we would have $\neg\neg P \to P$ for every proposition $P$ in $\mathcal{V}$, which is equivalent to excluded middle in $\mathcal{V}$. ◀

▶ **Lemma 23.** *Let $(X, \sqsubseteq)$ be a $\delta_\mathcal{V}$-complete poset and $x, y, z : X$. The following hold:*
- **(i)** *If $x \sqsubseteq y \sqsubset z$, then $x \sqsubset z$.*
- **(ii)** *If $x \sqsubset y \sqsubseteq z$, then $x \sqsubset z$.*

**Proof.** For (i), assume $x \sqsubseteq y \sqsubset z$, let $P$ be an arbitrary proposition in $\mathcal{V}$ and suppose that $z \sqsubseteq w$. We must show that $w = \bigvee \delta_{x,w,P}$ implies $P$. But $y \sqsubset z$, so we know that the equality $w = \bigvee \delta_{y,w,P}$ implies $P$. Now observe that $\bigvee \delta_{x,w,P} \sqsubseteq \bigvee \delta_{y,w,P}$, so if $w = \bigvee \delta_{x,w,P}$, then $w = \bigvee \delta_{y,w,P}$, finishing the proof. For (ii), assume $x \sqsubset y \sqsubseteq z$, let $P$ be an arbitrary proposition in $\mathcal{V}$ and suppose that $z \sqsubseteq w$. We must show that $w = \bigvee \delta_{x,w,P}$ implies $P$. But $x \sqsubset y$ and $y \sqsubseteq w$, so this follows immediately. ◀

▶ **Proposition 24.** *Let $(X, \sqsubseteq)$ be a $\mathcal{V}$-sup-lattice and let $y : X$. The following are equivalent:*
- **(i)** *the least element of $X$ is strictly below $y$;*
- **(ii)** *for every family $\alpha : I \to X$ with $I : \mathcal{V}$ and $y \sqsubseteq \bigvee \alpha$, there exists some element $i : I$.*
- **(iii)** *there exists some $x : X$ with $x \sqsubset y$.*

**Proof.** Write $\bot$ for the least element of $X$. By Lemma 23 we have:

$$\bot \sqsubset y \iff \exists_{x:X}(\bot \sqsubseteq x \sqsubset y) \iff \exists_{x:X}(x \sqsubset y),$$

which proves the equivalence of (i) and (iii). It remains to prove that (i) and (ii) are equivalent. Suppose that $\bot \sqsubset y$ and let $\alpha : I \to X$ with $y \sqsubseteq \bigvee \alpha$. Using $\bot \sqsubset y \sqsubseteq \bigvee \alpha$ and Lemma 23, we have $\bot \sqsubset \bigvee \alpha$. Hence, we only need to prove $\bigvee \alpha \sqsubseteq \bigvee \delta_{\bot, \bigvee \alpha, \exists i:I}$, but $\alpha_j \sqsubseteq \bigvee \delta_{\bot, \bigvee \alpha, \exists i:I}$ for every $j : I$, so this is true indeed. For the converse, assume that $y$ satisfies (ii), suppose $z \sqsupseteq y$ and let $P : \mathcal{V}$ be a proposition such that $z = \bigvee \delta_{\bot,z,P}$. We must show that $P$ holds. But notice that $y \sqsubseteq z = \bigvee \delta_{\bot,z,P} = \bigvee((p : P) \mapsto z)$, so $P$ must be inhabited as $y$ satisfies (ii). ◀

Item (ii) in Proposition 24 says exactly that $y$ is a positive element in the sense of [19, p. 98]. We note that item (iii) in Proposition 24 makes sense even when $(X, \sqsubseteq)$ is not a $\mathcal{V}$-sup-lattice, but just a $\delta_\mathcal{V}$-complete poset. Accordingly, we make the following definition.

▶ **Definition 25** (Positive element)**.** *An element of a $\delta_\mathcal{V}$-complete poset is* positive *if it satisfies item (iii) in Proposition 24.*

An element of a $\mathcal{V}$-dcpo is called *compact* if it is inaccessible by directed joins of families indexed by types in $\mathcal{V}$ [12, Definition 44].

▶ **Proposition 26.** *A compact element $x$ of a $\mathcal{V}$-dcpo with least element $\bot$ is positive if and only if $x \neq \bot$.*

**Proof.** One implication is taken care of by Proposition 22. For the converse, suppose that $x \neq \bot$. We show that $\bot$ is strictly below $x$. For if $x \sqsubseteq y = \bigvee \delta_{\bot,y,P}$, then by compactness of $x$, there must exist $i : 1 + P$ such that $x \sqsubseteq \delta_{\bot,y,P}(i)$ already. But $i$ can't be equal to $\mathrm{inl}(\star)$, since $x$ is assumed to be different from $\bot$. Hence, $i = \mathrm{inr}(p)$ and $P$ must hold. ◀

Looking to strengthen the notion of a nontrivial poset, we make the following definition, whose terminology is inspired by Definition 25.

▶ **Definition 27** (Positive poset). *A $\delta_{\mathcal{V}}$-complete poset $X$ is* positive *if we have designated $x, y : X$ with $x$ strictly below $y$.*

▶ **Examples 28.**

**(i)** *Consider an element $P$ of the $\delta_{\mathcal{V}}$-complete poset $\Omega_{\mathcal{V}}$. The pair $(0_{\mathcal{V}}, P)$ witnesses nontriviality of $\Omega_{\mathcal{V}}$ if and only if $\neg\neg P$ holds, while it witnesses positivity if and only if $P$ holds.*

**(ii)** *Consider the $\mathcal{V}$-powerset $\mathcal{P}_{\mathcal{V}}(X)$ on a type $X$ as a $\delta_{\mathcal{V}}$-complete poset (recall Examples 15). We write $\emptyset : \mathcal{P}_{\mathcal{V}}(X)$ for the map $x \mapsto 0_{\mathcal{V}}$. Say that a subset $A : \mathcal{P}_{\mathcal{V}}(X)$ is nonempty if $A \neq \emptyset$ and inhabited if there exists some $x : X$ such that $A(x)$ holds. The pair $(\emptyset, A)$ witnesses nontriviality of $\mathcal{P}_{\mathcal{V}}(X)$ if and only if $A$ is nonempty, while it witnesses positivity if and only if $A$ is inhabited. In particular, $\mathcal{P}_{\mathcal{V}}(X)$ is positive if and only if $X$ is an inhabited type.*

## 3.3 Retract Lemmas

We show that the type of propositions in $\mathcal{V}$ is a retract of any positive $\delta_{\mathcal{V}}$-complete poset and that the type of $\neg\neg$-stable propositions in $\mathcal{V}$ is a retract of any nontrivial $\delta_{\mathcal{V}}$-complete poset.

▶ **Definition 29** ($\Delta_{x,y} : \Omega_{\mathcal{V}} \to X$). *Suppose that $(X, \sqsubseteq, x, y)$ is a nontrivial $\delta_{\mathcal{V}}$-complete poset. We define $\Delta_{x,y} : \Omega_{\mathcal{V}} \to X$ by the assignment $P \mapsto \bigvee \delta_{x,y,P}$.*

We will often omit the subscripts in $\Delta_{x,y}$ when it is clear from the context.

▶ **Definition 30** (Locally small). *A $\delta_{\mathcal{V}}$-complete poset $(X, \sqsubseteq)$ is* locally small *if its order has values of size $\mathcal{V}$, i.e. we have $\sqsubseteq_{\mathcal{V}} : X \to X \to \mathcal{V}$ with $(x \sqsubseteq y) \simeq (x \sqsubseteq_{\mathcal{V}} y)$ for every $x, y : X$.*

▶ **Examples 31.**

**(i)** *The $\mathcal{V}$-sup-lattices $\Omega_{\mathcal{V}}$ and $\mathcal{P}_{\mathcal{V}}(X)$ (for $X : \mathcal{V}$) are locally small.*

**(ii)** *All examples of $\mathcal{V}$-dcpos in [12] are locally small.*

▶ **Lemma 32.** *A locally small $\delta_{\mathcal{V}}$-complete poset $(X, \sqsubseteq)$ is nontrivial, witnessed by elements $x \sqsubseteq y$, if and only if the composite $\Omega_{\mathcal{V}}^{\neg\neg} \hookrightarrow \Omega_{\mathcal{V}} \xrightarrow{\Delta_{x,y}} X$ is a section.*

**Proof.** Suppose first that $(X, \sqsubseteq, x, y)$ is nontrivial and locally small. We define

$$\begin{aligned} r : X &\to \Omega_{\mathcal{V}}^{\neg\neg} \\ z &\mapsto z \not\sqsubseteq_{\mathcal{V}} x. \end{aligned}$$

Note that negated propositions are $\neg\neg$-stable, so $r$ is well-defined. Let $P : \mathcal{V}$ be an arbitrary $\neg\neg$-stable proposition. We want to show that $r(\Delta_{x,y}(P)) = P$. By propositional extensionality, establishing logical equivalence suffices. Suppose first that $P$ holds. Then $\Delta_{x,y}(P) \equiv \bigvee \delta_{x,y,P} = y$, so $r(\Delta_{x,y}(P)) = r(y) \equiv (y \not\sqsubseteq_{\mathcal{V}} x)$ holds by antisymmetry and our assumptions that $x \sqsubseteq y$ and $x \neq y$. Conversely, assume that $r(\Delta_{x,y}(P))$ holds, i.e. that we have $\bigvee \delta_{x,y,P} \not\sqsubseteq_{\mathcal{V}} x$. Since $P$ is $\neg\neg$-stable, it suffices to derive a contradiction from $\neg P$. So assume $\neg P$. Then $x = \bigvee \delta_{x,y,P}$, so $r(\Delta_{x,y}(P)) = r(x) \equiv x \not\sqsubseteq_{\mathcal{V}} x$, which is false by reflexivity.

For the converse, assume that $\Omega_{\mathcal{V}}^{\neg\neg} \hookrightarrow \Omega_{\mathcal{V}} \xrightarrow{\Delta_{x,y}} X$ has a retraction $r : \Omega_{\mathcal{V}}^{\neg\neg} \to X$. Then $0_{\mathcal{V}} = r(\Delta_{x,y}(0_{\mathcal{V}})) = r(x)$ and $1_{\mathcal{V}} = r(\Delta_{x,y}(1_{\mathcal{V}})) = r(y)$, where we used that $0_{\mathcal{V}}$ and $1_{\mathcal{V}}$ are $\neg\neg$-stable. Since $0_{\mathcal{V}} \neq 1_{\mathcal{V}}$, we get $x \neq y$, so $(X, \sqsubseteq, x, y)$ is nontrivial, as desired. ◀

The appearance of the double negation in the above lemma is due to the definition of nontriviality. If we instead assume a positive poset $X$, then we can exhibit all of $\Omega_{\mathcal{V}}$ as a retract of $X$.

▶ **Lemma 33.** *A locally small $\delta_{\mathcal{V}}$-complete poset $(X, \sqsubseteq)$ is positive, witnessed by elements $x \sqsubseteq y$, if and only if for every $z \sqsupseteq y$, the map $\Delta_{x,z} : \Omega_{\mathcal{V}} \to X$ is a section.*

**Proof.** Suppose first that $(X, \sqsubseteq, x, y)$ is positive and locally small and let $z \sqsupseteq y$ be arbitrary. We define

$$\begin{aligned} r_z : X &\mapsto \Omega_{\mathcal{V}} \\ w &\mapsto z \sqsubseteq_{\mathcal{V}} w. \end{aligned}$$

Let $P : \mathcal{V}$ be arbitrary proposition. We want to show that $r_z(\Delta_{x,z}(P)) = P$. Because of propositional extensionality, it suffices to establish a logical equivalence between $P$ and $r_z(\Delta_{x,z}(P))$. Suppose first that $P$ holds. Then $\Delta_{x,z}(P) = z$, so $r_z(\Delta_{x,z}(P)) = r_z(z) \equiv (z \sqsubseteq_{\mathcal{V}} z)$ holds as well by reflexivity. Conversely, assume that $r_z(\Delta_{x,z}(P))$ holds, i.e. that we have $z \sqsubseteq_{\mathcal{V}} \bigvee \delta_{x,z,P}$. Since $\bigvee \delta_{x,z,P} \sqsubseteq z$ always holds, we get $z = \bigvee \delta_{x,z,P}$ by antisymmetry. But by assumption and Lemma 23, the element $x$ is strictly below $z$, so $P$ must hold.

For the converse, assume that for every $z \sqsupseteq y$, the map $\Delta_{x,z} : \Omega_{\mathcal{V}} \to X$ has a retraction $r_z : X \to \Omega_{\mathcal{V}}$. We must show that the equality $z = \Delta_{x,z}(P)$ implies $P$ for every $z \sqsupseteq y$ and proposition $P : \mathcal{V}$. Assuming $z = \Delta_{x,z}(P)$, we have $1_{\mathcal{V}} = r_z(\Delta_{x,z}(1_{\mathcal{V}})) = r_z(z) = r_z(\Delta_{x,z}(P)) = P$, so $P$ must hold indeed. Hence, $(X, \sqsubseteq, x, y)$ is positive, as desired. ◀

## 3.4 Reductions to Impredicativity and Excluded Middle

We present our main theorems here, which show that, constructively and predicatively, nontrivial $\delta_{\mathcal{V}}$-complete posets are necessarily large and necessarily lack decidable equality.

▶ **Definition 34** (Small)**.** *A $\delta_{\mathcal{V}}$-complete poset is* small *if it is locally small and its carrier has size $\mathcal{V}$.*

▶ **Theorem 35.**
- **(i)** *There is a nontrivial small $\delta_{\mathcal{V}}$-complete poset if and only if $\Omega_{\neg\neg}$-Resizing$_{\mathcal{V}}$ holds.*
- **(ii)** *There is a positive small $\delta_{\mathcal{V}}$-complete poset if and only if $\Omega$-Resizing$_{\mathcal{V}}$ holds.*

**Proof.** (i) Suppose that $(X, \sqsubseteq, x, y)$ is a nontrivial small $\delta_{\mathcal{V}}$-complete poset. By Lemma 32, we can exhibit $\Omega_{\mathcal{V}}^{\neg\neg}$ as a retract of $X$. But $X$ has size $\mathcal{V}$ by assumption, so by Lemma 12 and the fact that $\Omega_{\mathcal{V}}^{\neg\neg}$ is a set, the type $\Omega_{\mathcal{V}}^{\neg\neg}$ has size $\mathcal{V}$ as well. For the converse, note that $(\Omega_{\mathcal{V}}^{\neg\neg}, \to, 0_{\mathcal{V}}, 1_{\mathcal{V}})$ is a nontrivial $\mathcal{V}$-sup-lattice with $\bigvee \alpha$ given by $\neg\neg\exists_{i:I}\alpha_i$. And if we assume $\Omega_{\neg\neg}$-Resizing$_{\mathcal{V}}$, then it is small.

(ii) Suppose that $(X, \sqsubseteq, x, y)$ is a positive small poset. By Lemma 33, we can exhibit $\Omega_{\mathcal{V}}$ as a retract of $X$. But $X$ has size $\mathcal{V}$ by assumption, so by Lemma 12 and the fact that $\Omega_{\mathcal{V}}$ is a set, the type $\Omega_{\mathcal{V}}$ has size $\mathcal{V}$ as well. For the converse, note that $(\Omega_{\mathcal{V}}, \to, 0_{\mathcal{V}}, 1_{\mathcal{V}})$ is a positive $\mathcal{V}$-sup-lattice. And if we assume $\Omega$-Resizing$_{\mathcal{V}}$, then it is small. ◀

▶ **Lemma 36** (`retract-is-discrete` and `subtype-is-¬¬-separated` in [16])**.**
- **(i)** *Types with decidable equality are closed under retracts.*
- **(ii)** *Types with ¬¬-stable equality are closed under retracts.*

▶ **Theorem 37.** *There is a nontrivial locally small $\delta_{\mathcal{V}}$-complete poset with decidable equality if and only if weak excluded middle in $\mathcal{V}$ holds.*

**Proof.** Suppose that $(X, \sqsubseteq, x, y)$ is a nontrivial locally small $\delta_{\mathcal{V}}$-complete poset with decidable equality. Then by Lemmas 32 and 36, the type $\Omega_{\mathcal{V}}^{\neg\neg}$ must have decidable equality too. But negated propositions are ¬¬-stable, so this yields weak excluded middle in $\mathcal{V}$. For the converse, note that $(\Omega_{\mathcal{V}}^{\neg\neg}, \to, 0_{\mathcal{V}}, 1_{\mathcal{V}})$ is a nontrivial $\mathcal{V}$-sup-lattice that has decidable equality if and only if weak excluded middle in $\mathcal{V}$ holds. ◀

▶ **Theorem 38.** *The following are equivalent:*
- **(i)** *There is a positive locally small $\delta_{\mathcal{V}}$-complete poset with $\neg\neg$-stable equality.*
- **(ii)** *There is a positive locally small $\delta_{\mathcal{V}}$-complete poset with decidable equality.*
- **(iii)** *Excluded middle in $\mathcal{V}$ holds.*

**Proof.** Note that (ii) $\Rightarrow$ (i), so we are left to show that (iii) $\Rightarrow$ (ii) and that (i) $\Rightarrow$ (iii). For the first implication, note that $(\Omega_{\mathcal{V}}, \to, \mathsf{0}_{\mathcal{V}}, \mathsf{1}_{\mathcal{V}})$ is a positive $\mathcal{V}$-sup-lattice that has decidable equality if and only if excluded middle in $\mathcal{V}$ holds. To see that (i) implies (iii), suppose that $(X, \sqsubseteq, x, y)$ is a positive locally small $\delta_{\mathcal{V}}$-complete poset with $\neg\neg$-stable equality. Then by Lemmas 33 and 36 the type $\Omega_{\mathcal{V}}$ must have $\neg\neg$-stable equality. But this implies that $\neg\neg P \to P$ for every proposition $P$ in $\mathcal{V}$ which is equivalent to excluded middle in $\mathcal{V}$. ◀

Lattices, bounded complete posets and dcpos are necessarily large and necessarily lack decidable equality in our predicative constructive setting. More precisely,

▶ **Corollary 39.**
- **(i)** *There is a nontrivial small $\mathcal{V}$-sup-lattice (or $\mathcal{V}$-bounded complete poset or $\mathcal{V}$-dcpo) if and only if $\Omega_{\neg\neg}$-Resizing$_{\mathcal{V}}$ holds.*
- **(ii)** *There is a positive small $\mathcal{V}$-sup-lattice (or $\mathcal{V}$-bounded complete poset or $\mathcal{V}$-dcpo) if and only if $\Omega$-Resizing$_{\mathcal{V}}$ holds.*
- **(iii)** *There is a nontrivial locally small $\mathcal{V}$-sup-lattice (or $\mathcal{V}$-bounded complete poset or $\mathcal{V}$-dcpo) with decidable equality if and only if weak excluded middle in $\mathcal{V}$ holds.*
- **(iv)** *There is a positive locally small $\mathcal{V}$-sup-lattice (or $\mathcal{V}$-bounded complete poset or $\mathcal{V}$-dcpo) with decidable equality if and only if excluded middle in $\mathcal{V}$ holds.*

## 3.5 Unspecified Nontriviality and Positivity

The above notions of non-triviality and positivity are data rather than property. Indeed, a nontrivial poset $(X, \sqsubseteq)$ is (by definition) equipped with two designated points $x, y : X$ such that $x \sqsubseteq y$ and $x \neq y$. It is natural to wonder if the propositionally truncated versions of these two notions yield the same conclusions. In this section we show that this is indeed the case if we assume univalence. The need for the univalence assumption comes from the fact that the notion of having a given size is property precisely if univalence holds, as shown in Propositions 6 and 7.

▶ **Definition 40** (Nontrivial/positive in an unspecified way)**.** *A poset $(X, \sqsubseteq)$ is* nontrivial in an unspecified way *if there exist some elements $x, y : X$ such that $x \sqsubseteq y$ and $x \neq y$, i.e. $\exists_{x:X}\exists_{y:X}((x \sqsubseteq y) \times (x \neq y))$. Similarly, we can define when a poset is* positive in an unspecified way *by truncating the notion of positivity.*

▶ **Theorem 41.** *Suppose that the universes $\mathcal{V}$ and $\mathcal{V}^+$ are univalent.*
- **(i)** *There is a small $\delta_{\mathcal{V}}$-complete poset that is nontrivial in an unspecified way if and only if $\Omega_{\neg\neg}$-Resizing$_{\mathcal{V}}$ holds.*
- **(ii)** *There is a small $\delta_{\mathcal{V}}$-complete poset that is positive in an unspecified way if and only if $\Omega$-Resizing$_{\mathcal{V}}$ holds.*

**Proof.** (i) Suppose that $(X, \sqsubseteq)$ is a $\delta_{\mathcal{V}}$-complete poset that is nontrivial in an unspecified way. By Proposition 6 and univalence of $\mathcal{V}$ and $\mathcal{V}^+$, type $\Omega_{\mathcal{V}}^{\neg\neg}$ has-size $\mathcal{V}$ is a proposition. By the universal property of the propositional truncation, in proving that $\Omega_{\mathcal{V}}^{\neg\neg}$ has-size $\mathcal{V}$ we can therefore assume that are given points $x, y : X$ with $x \sqsubseteq y$ and $x \neq y$. The result then follows from Theorem 35. (ii) By reduction to item (ii) of Theorem 35. ◀

Similarly, we can prove the following theorems by reduction to Theorems 37 and 38.

▶ **Theorem 42.**
**(i)** *There is a locally small $\delta_{\mathcal{V}}$-complete poset with decidable equality that is nontrivial in an unspecified way if and only if weak excluded middle in $\mathcal{V}$ holds.*
**(ii)** *There is a locally small $\delta_{\mathcal{V}}$-complete poset with decidable equality that is positive in an unspecified way if and only if excluded middle in $\mathcal{V}$ holds.*

## 4 Maximal Points and Fixed Points

In this section we construct a particular example of a $\mathcal{V}$-sup-lattice that will prove very useful in studying the predicative validity of some well-known principles in order theory.

▶ **Definition 43** (Lifting, cf. [14]). *Fix a proposition $P_{\mathcal{U}}$ in a universe $\mathcal{U}$. Lifting $P_{\mathcal{U}}$ with respect to a universe $\mathcal{V}$ is defined by*

$$\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}}) :\equiv \sum_{Q:\Omega_{\mathcal{V}}} (Q \to P_{\mathcal{U}}).$$

This is a subtype of $\Omega_{\mathcal{V}}$ and it is closed under $\mathcal{V}$-suprema (in particular, it contains the least element).

▶ **Examples 44.**
**(i)** *If $P_{\mathcal{U}} :\equiv 0_{\mathcal{U}}$, then $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}}) \simeq \left(\sum_{Q:\Omega_{\mathcal{V}}} \neg Q\right) \simeq \left(\sum_{Q:\Omega_{\mathcal{V}}} Q = 0_{\mathcal{V}}\right) \simeq 1$.*
**(ii)** *If $P_{\mathcal{U}} :\equiv 1_{\mathcal{U}}$, then $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}}) \equiv \left(\sum_{Q:\Omega_{\mathcal{V}}} (Q \to 1_{\mathcal{U}})\right) \simeq \Omega_{\mathcal{V}}$.*

What makes $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ useful is the following observation.

▶ **Lemma 45.** *Suppose that the poset $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ has a maximal element $Q : \Omega_{\mathcal{V}}$. Then $P_{\mathcal{U}}$ is equivalent to $Q$, which is the greatest element of $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$. In particular, $P_{\mathcal{U}}$ has size $\mathcal{V}$. Conversely, if $P_{\mathcal{U}}$ is equivalent to a proposition $Q : \Omega_{\mathcal{V}}$, then $Q$ is the greatest element of $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$.*

**Proof.** Suppose that $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ has a maximal element $Q : \Omega_{\mathcal{V}}$. We wish to show that $Q \simeq P_{\mathcal{U}}$. By definition of $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$, we already have that $Q \to P_{\mathcal{U}}$. So only the converse remains. Therefore suppose that $P_{\mathcal{U}}$ holds. Then, $1_{\mathcal{V}}$ is an element of $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$. Obviously $Q \to 1_{\mathcal{V}}$, but $Q$ is maximal, so actually $Q = 1_{\mathcal{V}}$, that is, $Q$ holds, as desired. Thus, $Q \simeq P_{\mathcal{U}}$. It is then straightforward to see that $Q$ is actually the greatest element of $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$, since $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}}) \simeq \sum_{Q':\Omega_{\mathcal{V}}} (Q' \to Q)$. For the converse, assume that $P_{\mathcal{U}}$ is equivalent to a proposition $Q : \Omega_{\mathcal{V}}$. Then, as before, $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}}) \simeq \sum_{Q':\Omega_{\mathcal{V}}} (Q' \to Q)$, which shows that $Q$ is indeed the greatest element of $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$. ◀

▶ **Corollary 46.** *Let $P_{\mathcal{U}}$ be a proposition in $\mathcal{U}$. The $\mathcal{V}$-sup-lattice $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ has all $\mathcal{V}$-infima if and only if $P_{\mathcal{U}}$ has size $\mathcal{V}$.*

**Proof.** Suppose first that $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ has all $\mathcal{V}$-infima. Then it must have a infimum for the empty family $0_{\mathcal{V}} \to \mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$. But this infimum must be the greatest element of $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$. So by Lemma 45 the proposition $P_{\mathcal{U}}$ must have size $\mathcal{V}$.

Conversely, suppose that $P_{\mathcal{U}}$ is equivalent to a proposition $Q : \mathcal{V}$. Then the infimum of a family $\alpha : I \to \mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ with $I : \mathcal{V}$ is given by $(Q \times \Pi_{i:I}\alpha_i) : \mathcal{V}$. ◀

▶ **Definition 47** (Zorn's-Lemma$_{\mathcal{V},\mathcal{U},\mathcal{T}}$). *Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{T}$ be universes.* Zorn's-Lemma$_{\mathcal{V},\mathcal{U},\mathcal{T}}$ *asserts that every pointed $\mathcal{V}$-dcpo with carrier in $\mathcal{U}$ and order taking values in $\mathcal{T}$ (cf. [12]) has a maximal element.*

It important to note that Zorn's lemma does *not* imply the Axiom of Choice in the absence of excluded middle [3]. If it did, then the following would be useless, since the Axiom of Choice implies excluded middle, which in turn implies propositional resizing.

▶ **Theorem 48.** Zorn's-Lemma$_{\mathcal{V},\mathcal{V}^+\sqcup\mathcal{U},\mathcal{V}}$ *implies* Propositional-Resizing$_{\mathcal{U},\mathcal{V}}$.

In particular, Zorn's-Lemma$_{\mathcal{V},\mathcal{V}^+,\mathcal{V}}$ implies Propositional-Resizing$_{\mathcal{V}^+,\mathcal{V}}$.

**Proof.** Suppose that Zorn's-Lemma$_{\mathcal{V},\mathcal{V}^+\sqcup\mathcal{U},\mathcal{V}}$ were true. Then $\mathcal{L}_{\mathcal{V}}(P) : \mathcal{V}^+\sqcup\mathcal{U}$ has a maximal element for every $P : \Omega_{\mathcal{U}}$. Hence, by Lemma 45, every $P : \Omega_{\mathcal{U}}$ has size $\mathcal{V}$. ◀

We can also use Lemma 45 to show that the following version of Tarski's fixed point theorem [23] is not available predicatively.

▶ **Definition 49** (Tarski's-Theorem$_{\mathcal{V},\mathcal{U},\mathcal{T}}$)**.** *The assertion* Tarski's-Theorem$_{\mathcal{V},\mathcal{U},\mathcal{T}}$ *says that every monotone endofunction on a $\mathcal{V}$-sup-lattice with carrier in a universe $\mathcal{U}$ and order taking values in a universe $\mathcal{T}$ has a greatest fixed point.*

▶ **Theorem 50.** Tarski's-Theorem$_{\mathcal{V},\mathcal{V}^+\sqcup\mathcal{U},\mathcal{V}}$ *implies* Propositional-Resizing$_{\mathcal{U},\mathcal{V}}$.

In particular, Tarski's-Theorem$_{\mathcal{V},\mathcal{V}^+,\mathcal{V}}$ implies Propositional-Resizing$_{\mathcal{V}^+,\mathcal{V}}$.

**Proof.** Suppose that Tarski's-Theorem$_{\mathcal{V},\mathcal{V}^+\sqcup\mathcal{U},\mathcal{V}}$ were true and let $P : \Omega_{\mathcal{U}}$ be arbitrary. Consider the $\mathcal{V}$-sup-lattice $\mathcal{L}_{\mathcal{V}}(P) : \mathcal{V}^+ \sqcup \mathcal{U}$. By assumption, the identity map on this poset has a greatest fixed point, but this must be the greatest element of $\mathcal{L}_{\mathcal{V}}(P)$, which implies that $P$ has size $\mathcal{V}$ by Lemma 45. ◀

Another famous fixed point theorem, for dcpos this time, is due to Pataraia [20, 13] which says that every monotone endofunction on a pointed dcpo has a least fixed point. (A dcpo is called pointed if it has a least element.) A crucial step in proving Pataraia's theorem is the observation that every dcpo has a greatest monotone inflationary endofunction. (An endomap $f : X \to X$ is inflationary when $x \sqsubseteq f(x)$ for every $x : X$.) We refer to this intermediate result as Pataraia's lemma.

▶ **Definition 51** (Pataraia's-Lemma$_{\mathcal{V},\mathcal{U},\mathcal{T}}$, Pataraia's-Theorem$_{\mathcal{V},\mathcal{U},\mathcal{T}}$)**.**

**(i)** Pataraia's-Theorem$_{\mathcal{V},\mathcal{U},\mathcal{T}}$ *says that every monotone endofunction on a pointed $\mathcal{V}$-dcpo with carrier in a universe $\mathcal{U}$ and order taking values in a universe $\mathcal{T}$ has a least fixed point.*

**(ii)** Pataraia's-Lemma$_{\mathcal{V},\mathcal{U},\mathcal{T}}$ *says that every $\mathcal{V}$-dcpo with carrier in a universe $\mathcal{U}$ and order taking values in a universe $\mathcal{T}$ has a greatest monotone inflationary endofunction.*

A careful analysis of the proof in [13, Section 2] shows that in our predicative setting we can still prove that Pataraia's-Lemma$_{\mathcal{V},\mathcal{U}\sqcup\mathcal{T},\mathcal{U}\sqcup\mathcal{T}}$ implies Pataraia's-Theorem$_{\mathcal{V},\mathcal{U},\mathcal{T}}$. However, Pataraia's lemma is not available predicatively.

▶ **Theorem 52.** Pataraia's-Lemma$_{\mathcal{V},\mathcal{V}^+\sqcup\mathcal{U},\mathcal{V}}$ *implies* Propositional-Resizing$_{\mathcal{U},\mathcal{V}}$.

In particular, Pataraia's-Lemma$_{\mathcal{V},\mathcal{V}^+,\mathcal{V}}$ implies Propositional-Resizing$_{\mathcal{V}^+,\mathcal{V}}$.

**Proof.** Suppose that Pataraia's-Lemma$_{\mathcal{V},\mathcal{V}^+\sqcup\mathcal{U},\mathcal{V}}$ were true and let $P : \Omega_{\mathcal{U}}$ be arbitrary. Consider the $\mathcal{V}$-dcpo $\mathcal{L}_{\mathcal{V}}(P) : \mathcal{V}^+ \sqcup \mathcal{U}$. By assumption, it has a greatest monotone inflationary endomap $g : \mathcal{L}_{\mathcal{V}}(P) \to \mathcal{L}_{\mathcal{V}}(P)$. We claim that $g(\mathsf{0}_{\mathcal{V}})$ is a maximal element of $\mathcal{L}_{\mathcal{V}}(P)$, which would finish the proof by Lemma 45. So suppose that we have $Q : \mathcal{L}_{\mathcal{V}}(P)$ with $g(\mathsf{0}_{\mathcal{V}}) \sqsubseteq Q$. Then we must show that $Q \sqsubseteq g(\mathsf{0}_{\mathcal{V}})$. Define $f_Q : \mathcal{L}_{\mathcal{V}}(P) \to \mathcal{L}_{\mathcal{V}}(P)$ by $Q' \mapsto Q' \vee Q$. Note that $f_Q$ is monotone and inflationary, so that $f_Q \sqsubseteq g$. Hence, $Q = f_Q(\mathsf{0}_{\mathcal{V}}) \sqsubseteq g(\mathsf{0}_{\mathcal{V}})$, as desired. ◀

▶ Remark 53. For a *single* universe $\mathcal{U}$, the usual proofs (see resp. [23] and [13, Section 2]) of Tarski's-Theorem$_{\mathcal{U},\mathcal{U},\mathcal{U}}$, Pataraia's-Lemma$_{\mathcal{U},\mathcal{U},\mathcal{U}}$ and (hence) Pataraia's-Theorem$_{\mathcal{U},\mathcal{U},\mathcal{U}}$ are also valid in our predicative setting. However, in light of Theorem 35, these statements are not useful predicatively, because one would never be able to find interesting examples of posets to apply the statements to.

Finally, we note that Zorn's lemma implies Pataraia's lemma with the following universe parameters. Together with Theorem 52 this yields another proof that Zorn's-Lemma$_{\mathcal{V},\mathcal{V}^+,\mathcal{V}}$ implies Propositional-Resizing$_{\mathcal{V}^+,\mathcal{V}}$.

▶ **Lemma 54.** Zorn's-Lemma$_{\mathcal{V},\mathcal{U}\sqcup\mathcal{T},\mathcal{U}\sqcup\mathcal{T}}$ *implies* Pataraia's-Lemma$_{\mathcal{V},\mathcal{U},\mathcal{T}}$.

**Proof.** Assume Zorn's-Lemma$_{\mathcal{V},\mathcal{U}\sqcup\mathcal{T},\mathcal{U}\sqcup\mathcal{T}}$ and let $D : \mathcal{U}$ be $\mathcal{V}$-dcpo with order taking values in $\mathcal{T}$. Consider the type $\mathrm{MI}_D$ of monotone and inflationary endomaps on $D$. We can order these maps pointwise to get a $\mathcal{V}$-dcpo with carrier and order taking values in $\mathcal{U} \sqcup \mathcal{T}$. Finally, $\mathrm{MI}_D$ has a least element: the identity map. Hence, by our assumption, it has a maximal element $g : D \to D$. It remains to show that $g$ is in fact the greatest element. To this end, let $f : D \to D$ be an arbitrary monotone inflationary endomap on $D$. We must show that $f \sqsubseteq g$. Since $f$ is inflationary, we have $g \sqsubseteq f \circ g$. So by maximality of $g$, we get $g = f \circ g$. But $f$ is monotone and $g$ is inflationary, so $f \sqsubseteq f \circ g = g$, finishing the proof. ◀

The answer to the question whether Pataraia's theorem (or similarly, a least fixed point theorem version of Tarki's theorem) is inherently impredicative or (by contrast) does admit a predicative proof has eluded us thus far.

## 5 Families and Subsets

In traditional impredicative foundations, completeness of posets is usually formulated using subsets. For instance, dcpos are defined as posets $D$ such that every directed subset $D$ has a supremum in $D$. Examples 15 are all formulated using small families instead of subsets. While subsets are primitive in set theory, families are primitive in type theory, so this could be an argument for using families above. However, that still leaves the natural question of how the family-based definitions compare to the usual subset-based definitions, especially in our predicative setting, unanswered. This section aims to answer this question. We first study the relation between subsets and families predicatively and then clarify our definitions in the presence of impredicativity. In our answers we will consider sup-lattices, but similar arguments could be made for posets with other sorts of completeness, such as dcpos.

### All Subsets

We first show that simply asking for completeness w.r.t. all subsets is not satisfactory from a predicative viewpoint. In fact, we will now see that even asking for all subsets $X \to \Omega_{\mathcal{T}}$ for some fixed universe $\mathcal{T}$ is problematic from a predicative standpoint.

▶ **Theorem 55.** *Let $\mathcal{U}$ and $\mathcal{V}$ be universes and fix a proposition $P_{\mathcal{U}} : \mathcal{U}$. Recall $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ from Definition 43, which has $\mathcal{V}$-suprema. Let $\mathcal{T}$ be any type universe. If $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ has suprema for all subsets $\mathcal{L}_{\mathcal{V}}(P_U) \to \Omega_{\mathcal{T}}$, then $P_{\mathcal{U}}$ has size $\mathcal{V}$ independently of $\mathcal{T}$.*

**Proof.** Let $\mathcal{T}$ be a type universe and consider the subset $S$ of $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ given by $Q \mapsto 1_{\mathcal{T}}$. Note that $S$ has a supremum in $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ if and only if $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ has a greatest element, but by Lemma 45, the latter is equivalent to $P_{\mathcal{U}}$ having size $\mathcal{V}$. ◀

### All Subsets Whose Total Spaces Have Size $\mathcal{V}$

The proof above illustrates that if we have a subset $S : X \to \Omega_{\mathcal{T}}$, then there is no reason why the total space $\sum_{x:X} x \in S :\equiv \sum_{x:X}(S(x) \text{ holds})$ should have size $\mathcal{T}$. In fact, for $S(x) :\equiv 1_{\mathcal{T}}$ as above, the latter is equivalent to asking that $X$ has size $\mathcal{T}$.

▶ **Definition 56** (Total space of a subset, $\mathbb{T}$). *Let $\mathcal{T}$ be a universe, $X$ a type and $S : X \to \Omega_{\mathcal{T}}$ a subset of $X$. The* total space *of $S$ is defined as $\mathbb{T}(S) :\equiv \sum_{x:X} x \in S$.*

A naive attempt to solve the problem described in Theorem 55 would be to stipulate that a $\mathcal{V}$-sup-lattice $X$ should have suprema for all subsets $S : X \to \Omega_{\mathcal{V}}$ for which $\mathbb{T}(S)$ has size $\mathcal{V}$. Somewhat less naively, we might be more liberal and ask for suprema of subsets $S : X \to \Omega_{\mathcal{U}\sqcup\mathcal{V}}$ for which $\mathbb{T}(S)$ has size $\mathcal{V}$. Here the carrier of $X$ is in a universe $\mathcal{U}$. Perhaps surprisingly, even this more liberal definition is too weak to be useful as the following example shows.

▶ **Example 57** (Naturally occurring subsets whose total spaces are not necessarily small). Let $X$ be a poset with carrier in $\mathcal{U}$ and suppose that it has suprema for all (directed) subsets $S : X \to \Omega_{\mathcal{U}\sqcup\mathcal{V}}$ for which $\mathbb{T}(S)$ has size $\mathcal{V}$. Now let $f : X \to X$ be a Scott continuous endofunction on $X$. We would want to construct the least fixed point of $f$ as the supremum of the directed subset $S :\equiv \{\bot, f(\bot), f^2(\bot), \dots\}$. Now, how do we show that its total space $\mathbb{T}(S) \equiv \sum_{x:X}(\exists_{n:\mathsf{N}}\, x = f^n(\bot))$ has size $\mathcal{V}$? A first guess might be that $\mathsf{N} \simeq \mathbb{T}(S)$, which would do the job. However, it's possible that $f^m(\bot) = f^{m+1}(\bot)$ for some natural number $m$, which would mean that $\mathbb{T}(S) \simeq \mathrm{Fin}(m)$ for the least such $m$. The problem is that in the absence of decidable equality on $X$ we might not be able to decide which is the case. But $X$ seldom has decidable equality, as we saw in Theorems 37 and 38.

▶ Remark 58. The example above also makes clear that it is undesirable to impose an injectivity condition on families, as the family $\mathsf{N} \to X, n \mapsto f^n(\bot)$ is not necessarily injective. In fact, for every type $X : \mathcal{U}$ there is an equivalence between embeddings $I \hookrightarrow X$ with $I : \mathcal{V}$ and subsets of $X$ whose total spaces have size $\mathcal{V}$, cf. [16, `Slice.html`].

### All $\mathcal{V}$-covered Subsets

The point of Example 57 is analogous to the difference between Bishop finiteness and Kuratowski finiteness. Inspired by this, we make the following definition.

▶ **Definition 59** ($\mathcal{V}$-covered subset). *Let $X$ be a type, $\mathcal{T}$ a universe and $S : X \to \Omega_{\mathcal{T}}$ a subset of $X$. We say that $S$ is $\mathcal{V}$-*covered *for a universe $\mathcal{V}$ if we have a type $I : \mathcal{V}$ with a surjection $e : I \twoheadrightarrow \mathbb{T}(S)$.*

In the example above, the subset $S :\equiv \{\bot, f(\bot), f^2(\bot), \dots\}$ is $\mathcal{U}_0$-covered, because $\mathsf{N} \twoheadrightarrow \mathbb{T}(S)$.

▶ **Theorem 60.** *For $X : \mathcal{U}$ and any universe $\mathcal{V}$ we have an equivalence between $\mathcal{V}$-covered subsets $X \to \Omega_{\mathcal{U}\sqcup\mathcal{V}}$ and families $I \to X$ with $I : \mathcal{V}$.*

**Proof.** The forward map $\varphi$ is given by $(S, I, e) \mapsto (I, \mathrm{pr}_1 \circ e)$. In the other direction, we define $\psi$ by mapping $(I, \alpha)$ to the triple $(S, I, e)$ where $S$ is the subset of $X$ given by $S(x) :\equiv \exists_{i:I}\, x = \alpha(i)$ and $e : I \twoheadrightarrow \mathbb{T}(S)$ is defined as $e(i) :\equiv (\alpha(i), |(i, \mathrm{refl})|)$. The composite $\varphi \circ \psi$ is easily seen to be equal to the identity. To show that $\psi \circ \varphi$ equals the identity, we need the following intermediate result, which is proved using function extensionality and path induction.

▷ Claim. Let $S, S' : X \to \Omega_{\mathcal{U} \sqcup \mathcal{V}}$, $e : I \to \mathbb{T}(S)$ and $e' : I \to \mathbb{T}(S')$. If $S = S'$ and $\mathrm{pr}_1 \circ e \sim \mathrm{pr}_1 \circ e'$, then $(S, e) = (S', e')$.

The result then follows from the claim using function extensionality and propositional extensionality. ◀

▶ **Corollary 61.** *Let $X$ be a poset with carrier in $\mathcal{U}$ and let $\mathcal{V}$ be any universe. Then $X$ has suprema for all $\mathcal{V}$-covered subsets $X \to \Omega_{\mathcal{U} \sqcup \mathcal{V}}$ if and only if $X$ has suprema for all families $I \to X$ with $I : \mathcal{V}$.*

### Families and Subsets in the Presence of Impredicativity

Finally, we compare our family-based approach to the subset-based approach in the presence of impredicativity.

▶ **Theorem 62.** *Assume $\Omega$-Resizing$_{\mathcal{T},\mathcal{U}_0}$ for every universe $\mathcal{T}$. Then the following are equivalent for a poset $X$ in a universe $\mathcal{U}$:*

- **(i)** *$X$ has suprema for all subsets;*
- **(ii)** *$X$ has suprema for all $\mathcal{U}$-covered subsets;*
- **(iii)** *$X$ has suprema for all subsets whose total spaces have size $\mathcal{U}$;*
- **(iv)** *$X$ has suprema for all families $I \to X$ with $I : \mathcal{U}$.*

**Proof.** Clearly (i) ⇒ (ii) ⇒ (iii). We show that (iii) implies (i), which proves the equivalence of (i)–(iii). Assume that $X$ has suprema for all subsets whose total spaces have size $\mathcal{U}$ and let $S : X \to \Omega_{\mathcal{T}}$ be any subset of $X$. Using $\Omega$-Resizing$_{\mathcal{T},\mathcal{U}_0}$, the total space $\mathbb{T}(S)$ has size $\mathcal{U}$. So $X$ has a supremum for $S$ by assumption, as desired. Finally, (ii) and (iv) are equivalent by Corollary 61. ◀

Notice that (iv) in Theorem 62 implies that $X$ has suprema for all families $I \to X$ with $I : \mathcal{V}$ and $\mathcal{V}$ such that $\mathcal{V} \sqcup \mathcal{U} \equiv \mathcal{U}$. Typically, in the examples of [12] for instance, $\mathcal{U} :\equiv \mathcal{U}_1$ and $\mathcal{V} :\equiv \mathcal{U}_0$, so that $\mathcal{V} \sqcup \mathcal{U} \equiv \mathcal{U}$ holds. Thus, our $\mathcal{V}$-families-based approach generalizes the traditional subset-based approach.

## 6 Conclusion

Firstly, we have shown, constructively and predicatively, that nontrivial dcpos, bounded complete posets and sup-lattices are all necessarily large and necessarily lack decidable equality. We did so by deriving a weak impredicativity axiom or weak excluded middle from the assumption that such nontrivial structures are small or have decidable equality, respectively. Strengthening nontriviality to the (classically equivalent) positivity condition, we derived a strong impredicativity axiom and full excluded middle.

Secondly, we proved that Zorn's lemma, Tarski's greatest fixed point theorem and Pataraia's lemma all imply impredicativity axioms. Hence, these principles are inherently impredicative and a predicative development of order theory (in univalent foundations) must thus do without them.

Thirdly, we clarified, in our predicative setting, the relation between the traditional definition of a lattice that requires completeness with respect to subsets and our definition that asks for completeness with respect to small families.

In future work, we wish to study the predicative validity of Pataraia's theorem and Tarski's *least* fixed point theorem. Curi [9, 10] develops predicative versions of Tarki's fixed point theorem in some extensions of CZF. It is not clear whether these arguments could be adapted

to univalent foundations, because they rely on the set-theoretical principles discussed in the introduction. The availability of such fixed-point theorems would be especially useful for application to inductive sets [1], which we might otherwise introduce in univalent foundations using higher inductive types [24]. In another direction, we have developed a notion of apartness [5] for continuous dcpos [12] that is related to the notion of being strictly below introduced in this paper. Namely, if $x \sqsubseteq y$ are elements of a continuous dcpo, then $x$ is strictly below $y$ if $x$ is apart from $y$. In upcoming work, we give a constructive analysis of the Scott topology [17] using this notion of apartness.

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