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The Evolution of Travelling Waves in a KPP Reaction-Diffusion Model with cut-off Reaction Rate. II. Evolution of Travelling Waves.

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Abstract

In Part II of this series of papers, we consider an initial-boundary value problem for the Kolmogorov–Petrovskii–Piscounov (KPP) type equation with a discontinuous cut-off in the reaction function at concentration $u = u_c$. For fixed cut-off value $u_c \in (0, 1)$, we apply the method of matched asymptotic coordinate expansions to obtain the complete large-time asymptotic form of the solution which exhibits the formation of a permanent form travelling wave structure. In particular, this approach allows the correction to the wave speed and the rate of convergence of the solution onto the permanent form travelling wave to be determined via a detailed analysis of the asymptotic structures in small-time and, subsequently, in large-space. The asymptotic results are confirmed against numerical results obtained for the particular case of a cut-off Fisher reaction function.

Keywords: reaction-diffusion equations, permanent form travelling waves, asymptotic expansions, singular perturbations

1 Introduction

Travelling waves arise as the long-time solution to many reaction-diffusion models and are relevant to a broad range of applications in chemistry, biology, ecology, epidemiology and genetics [9, 17]. The most celebrated model where such waves emerge is the KPP or Fisher-KPP model named after the pioneering work by Fisher [11] and Kolmogorov, Petrovskii, Piscounov [13]. In one spatial coordinate (x) this model describes the temporal (t) evolution of the concentration of a chemical or biological substance $u(x, t)$ as

$$u_t = u_{xx} + f(u), \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \tag{1a}$$

subject to an initial condition

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R} \tag{1b}$$

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and boundary conditions

$$u(x,t) \to \begin{cases} 1, & \text{as } x \to -\infty \\ 0, & \text{as } x \to \infty, \end{cases} \tag{1c}$$

with the limits being uniform for time $t \in [0,T]$ and any $T > 0$. Here, $u_0 : \mathbb{R} \to \mathbb{R}$ is taken to be piecewise continuous, non-negative and non-increasing with $\lim_{x\to\infty} u_0(x) = 0$ and $\lim_{x\to-\infty} u_0(x) = 1$. The function $f : \mathbb{R} \to \mathbb{R}$ is a normalised KPP-type reaction function which satisfies $f \in C^1(\mathbb{R})$ with

$$f(0) = f(1) = 0, \quad f'(0) = 1, \quad f'(1) < 0 \tag{2a}$$

and

$$0 < f(u) \le u \quad \text{for all} \quad u \in (0,1), \qquad f(u) < 0 \quad \text{for all} \quad u \in (1,\infty). \tag{2b}$$

A prototypical example of such a KPP reaction function is the Fisher reaction function [11] given by

$$f(u) = u(1-u). \tag{3}$$

The initial-boundary value problem (1) has a classical and global solution $u : \mathbb{R}\times[0,\infty) \to \mathbb{R}$. In addition, on using the classical maximum principle and comparison theorem (see, for example, [1] and [9]), $0 < u(x,t) < 1$ and $u_x(x,t) < 0$ for all $(x,t) \in \mathbb{R}\times\mathbb{R}^+$. The conditions (2) on f imply also that the initial-boundary value problem (1) admits a one-parameter family of permanent form travelling wave (PTW) solutions $u(x,t) = U_v(x-vt)$ that are strictly monotone decreasing, with $U_v \ge 0$, $U_v : \mathbb{R} : \mathbb{R}$ such that $U_v > 0$ with $\lim_{y\to-\infty} U_v(y) = 1$ and $\lim_{y\to\infty} U_v(y) = 0$. The parameterisation is through the propagation speed v, with a unique (up to translation) PTW for each v where v satisfies $v \ge v_m = 2$.

A central question is whether a PTW evolves in the solution to (1) at large times and if so what is its speed of propagation. It is well established [2, 10, 13] that for Heaviside initial conditions:

$$u_0 = \begin{cases} 1, & \text{for } x < 0 \\ 0, & \text{for } x \ge 0, \end{cases} \tag{4}$$

the solution to (1) converges onto the PTW solution with minimum propagation speed $v = v_m = 2$ in the sense that there exists a function $s_m(t)$ such that as $t \to \infty$, $s_m(t)/t \to 2$ and

$$u(z + s_m(t), t) \to U_2(z), \tag{5}$$

uniformly for $z \in \mathbb{R}$. A more detailed asymptotic description was provided by McKean [15, 16] and Bramson [4, 5] who, using a probabilistic approach, obtained that the rate of convergence of the solution to the initial-boundary value problem (1) to the PTW is algebraically small in t as $t \to \infty$, specifically $O(\dot{s}_m(t) - 2)$, where

$$\dot{s}_m(t) = 2 - \frac{3}{2}t^{-1} + o(t^{-1}) \quad \text{as } t \to \infty \tag{6}$$

with the dot denoting differentiation with respect to t. More recently, the same result has been established using a range of alternative approaches, based on a point patching procedure [6, 8], the theory of matched asymptotic expansions [3, 14] and rigorous bounds [12]. All of these approaches involve the solution to a linearized version of (1) that describes the behaviour at the leading edge of the front and is obtained by replacing $f(u)$ with $f'(0)u$. The common observation is that, with the appropriate boundary conditions, the linear version of (1) mainly determines the large-t structure of the solution to (1).

A linearized approach is not available to apply in the case of the cut-off KPP model that Brunet and Derrida [6] proposed and considered and was the focus of a companion paper [18] (hereafter referred to as Part I). In this model, the cut-off value $u_c \in (0,1)$ is introduced by replacing $f(u)$ in the initial-boundary value problem (1) with $f_c(u)$ where

$$f_c(u) = \begin{cases} f(u), & u \in (u_c, \infty) \\ 0, & u \in (-\infty, u_c] \end{cases} \tag{7}$$

and $f(u)$ continues to satisfy the KPP conditions (2). The discontinuity in $f_c(u)$ at $u = u_c$ suggests that the corresponding initial-boundary value problem is expressed as a moving boundary problem with the location of the moving boundary given by $s(t)$ where $s(t)$ satisfies $u(s(t),t) = u_c$ for $t > 0$ (see Part I). For Heaviside initial conditions (4), this boundary separates the domain D^L where $u > u_c$ from the domain D^R where $u < u_c$. A simple coordinate transformation $(x,t) \to (y,t)$ with $y = x - s(t)$ fixes the boundary at the origin and transforms the domains D^L and D^R into $Q^L = \mathbb{R}^- \times \mathbb{R}^+$ and $Q^R = \mathbb{R}^+ \times \mathbb{R}^+$ and the moving boundary problem becomes the following equivalent initial-boundary value problem that we refer to as QIVP (with a detailed derivation given in Part I):

$$u_t - \dot{s}(t) u_y = u_{yy} + f_c(u), \quad (y,t) \in Q^L \cup Q^R, \tag{8a}$$

$$u \geq u_c \text{ in } \bar{Q}^L, \quad u \leq u_c \text{ in } \bar{Q}^R, \tag{8b}$$

$$u(y,0) = \begin{cases} 1, & y < 0 \\ 0, & y \geq 0 \end{cases} \tag{8c}$$

$$u(y,t) \to \begin{cases} 1, & \text{as} \quad y \to -\infty \\ 0, & \text{as} \quad y \to \infty \end{cases} \tag{8d}$$

uniformly for $t \in [0,T]$ for all $T > 0$. At the boundary,

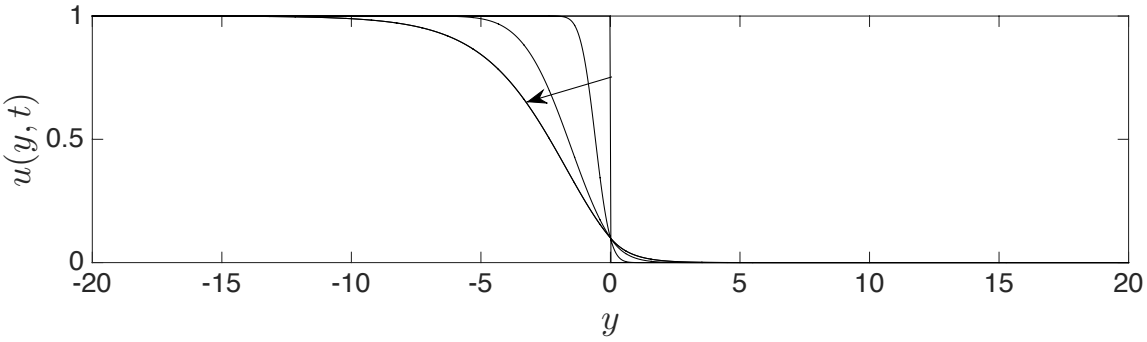
$$u(0,t) = u_c, \quad t \in (0,\infty), \tag{8e}$$

$$u_y(0^+,t) = u_y(0^-,t), \quad t \in (0,\infty). \tag{8f}$$

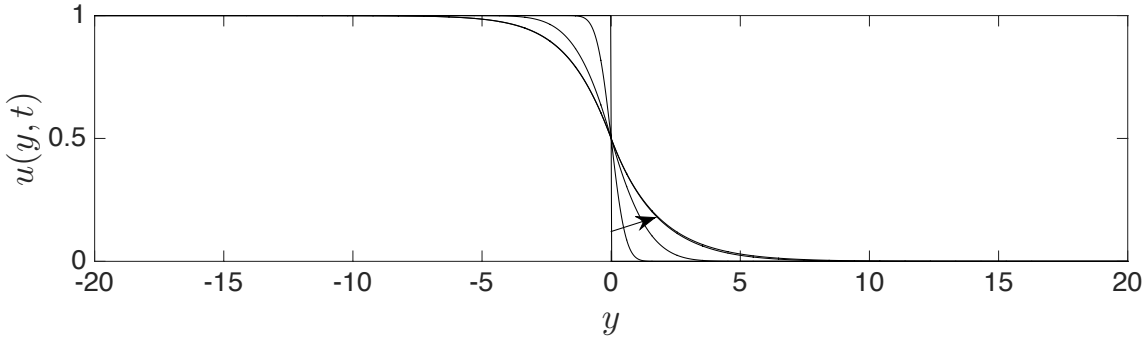
$$s(0^+) = 0. \tag{8g}$$

In Part I we stated regularity conditions (see equation (18)) for the solution $u(y,t)$ and $s(t)$ to be classical for all $t > 0$, and on using the classical maximum principle and comparison theorem (see, for example, [1] and [9]), obtained that $0 < u(y,t) < u_c$ for all $(y,t) \in Q^R$, $u_c < u(y,t) < 1$ for all $(y,t) \in Q^L$, and $u_y(y,t) < 0$ for all $t > 0$ and $y \in \mathbb{R}$ with $[u_{yy}(y,t)]_{y=0^-}^{y=0^+} = f_c^+$ for all $t \in \mathbb{R}^+$ with $f_c^+ = f_c(u_c^+)$. We then established that in the presence of a cut-off, the initial-boundary value problem (8) admits exactly one PTW solution (up to translation) $u(y,t) = U_T(y)$ that is strictly monotone decreasing and positive, with $\lim_{y\to-\infty} U_T(y) = 1$ and $\lim_{y\to\infty} U_T(y) = 0$ where the speed $v = v^*(u_c)$ is, for fixed $u_c \in (0,1)$, uniquely defined. An explicit expression of $v^*(u_c)$ is in general not known, it is however straightforward to establish that $v^*(u_c)$ is a continuous, monotone decreasing function of $u_c \in (0,1)$, with $v^*(u_c) \to 2^-$ as $u_c \to 0^+$ and $v^*(u_c) \to 0^+$ as $u_c \to 1^-$ [18]. Brunet and Derrida [6] predicted that the difference between $v^*(u_c)$ and $v_m = 2$ is strongly influenced at small values of u_c, being only logarithmically small in u_c as $u_c \to 0^+$. This behaviour was rigorously verified by Dumortier, Popovic and Kaper [7], with higher order corrections obtained in Part I. This behaviour is in contrast with the behaviour of $v^*(u_c)$ obtained as $u_c \to 1^-$ in which case it vanishes algebraically in $(1 - u_c)$ (see Part I).

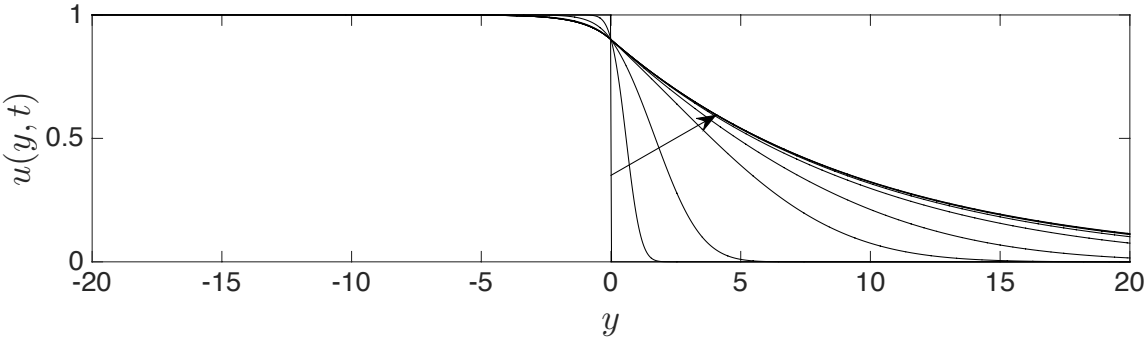
We may now once again enquire as to whether or not a PTW solution evolves in the solution to (8) for arbitrary cut-off $u_c \in (0,1)$ at large time, and, if this is the case, what is the



(a) $u_c = 0.1$



(b) $u_c = 0.5$



(c) $u_c = 0.9$

Figure 1: A graph of the solution $u(y,t)$ to QIVP as it evolves over time. Results are obtained numerically for (a) $u_c = 0.1$, (b) $u_c = 0.5$ and (c) $u_c = 0.9$ for $t = 0$, 0.1, 1, 10 and $t = 30$ with the arrow pointing in the direction of increasing t. Panel (c) includes additional graphs of solutions obtained at $t = 100$, 200, 300, 350 and $t = 400$.

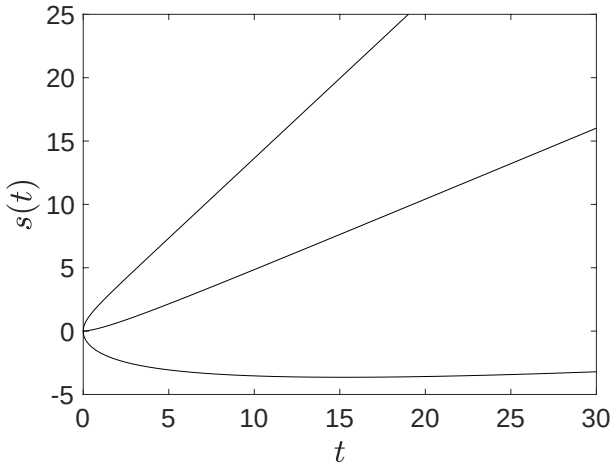


Figure 2: A graph of the solution $s(t)$ to QIVP obtained numerically for $u_c = 0.1$ (top), $u_c = 0.5$ (middle) and $u_c = 0.9$ (bottom).

rate of convergence onto the PTW solution. In this paper we observe that a PTW of speed $\lim_{t\to\infty} \dot{s}(t) = v^*(u_c)$ emerges in the solution of (8) for $t \to \infty$ via numerical simulations obtained for the specific case of f_c with f given by (3). We then adapt the approach introduced in [14], where $u_c = 0$, to obtain the detailed description of the large-t structure of the solution to (8). In particular, we use the theory of matched asymptotic coordinate expansions to establish that for each value of $u_c \in (0, 1)$, the solution to (8) converges to the PTW solution with propagation speed $v = v^*(u_c)$ at a rate that is linearly exponentially small in t as $t \to \infty$, specifically $O(\dot{s}(t) - v^*(u_c))$, where

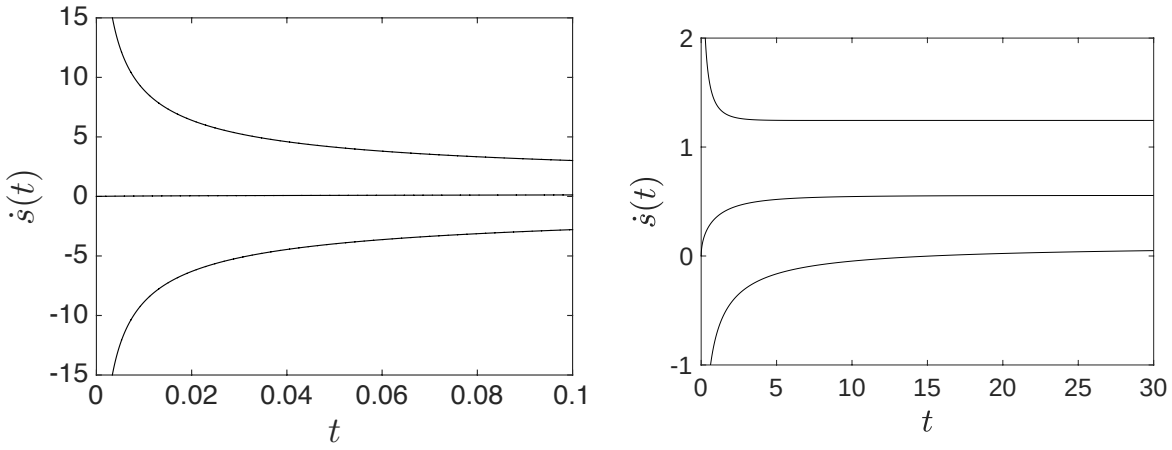
$$\dot{s}(t) = v^*(u_c) + O\left(t^\gamma \exp\left(-\frac{1}{4}v^*(u_c)^2 t\right)\right), \quad \text{as } t \to \infty, \tag{9}$$

(with $\gamma = -1/2$ or $-3/2$ depending on the structure of $f(u)$, specifically $f'(U_T)$, which determines the solution to (172) on which the choice in the value of γ depends) so that convergence slows down as u_c increases. Thus, introducing an arbitrary cut-off into the reaction function changes the rate of convergence of the large-time solution onto the PTW from algebraic to exponential. The paper is organised as follows: in section 2, we present numerical results for the specific case of the cut-off Fisher reaction function with f given by (3). Sections 3 and 4 are respectively devoted to the small-t ($y \in \mathbb{R}$) and large-$|y|$ ($t \geq O(1)$) structure of the solution to QIVP. These are used in section 5 to develop the complete asymptotic structure to QIVP as $t \to \infty$, uniformly in $y \in \mathbb{R}$. At the end of sections 3 and 5, we illustrate the theory for the specific case of the cut-off Fisher reaction function (for which $\gamma = -3/2$). The paper ends with the concluding section 6.

2 Numerical solution to QIVP

In this section we consider a numerical solution to QIVP to indicate whether the solution converges onto a PTW solution at large times. We present results for the particular case of the cut-off Fisher reaction function, namely,

$$f_c(u) = \begin{cases} u(1-u), & u \in (u_c, \infty), \\ 0, & u \in (-\infty, u_c], \end{cases} \tag{10}$$

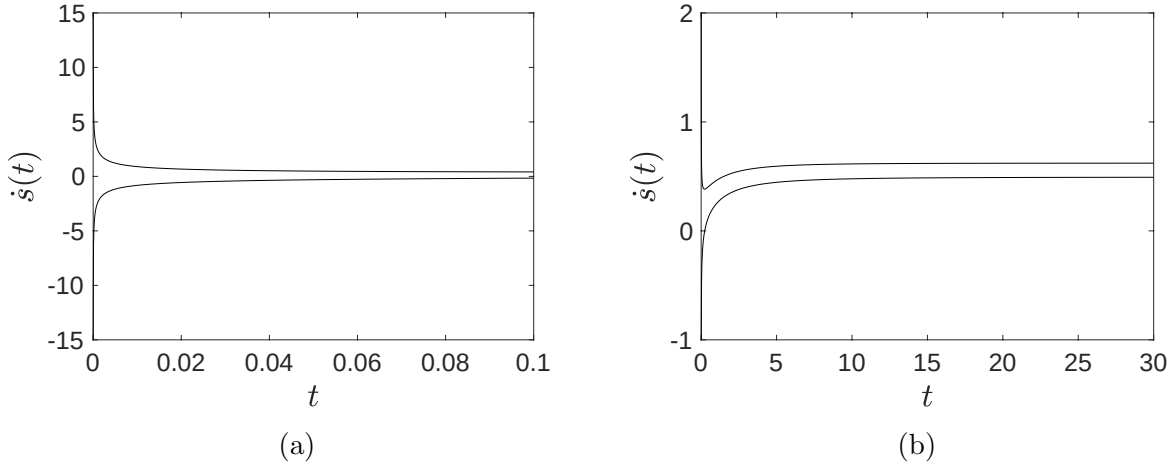


(a) (b)

Figure 3: A graph of $\dot{s}(t)$ to QIVP (solid lines) obtained numerically for cut-off value $u_c = 0.1$ (top), 0.5 (middle) and 0.9 (bottom) plotted for a (a) small and (b) large range of values of t.

for fixed cut-off value $u_c \in (0, 1)$. We adopt an explicit finite difference scheme, detailed in Appendix A. We choose this scheme over an implicit scheme despite the severe numerical stability restrictions on the time step. This is because an explicit scheme is very straightforward to use: at each time step, the associated numerical calculation requires the solution of a linear algebraic system (rather than a nonlinear algebraic system that would be required for an implicit scheme).

We examine the behaviour of $u(y, t)$, $s(t)$ and $\dot{s}(t)$, obtained numerically for illustrative values of $u_c \in (0, 1)$. Figures 1–3 respectively focus on the structure of $u(y, t)$, $s(t)$ and $\dot{s}(t)$ obtained for $u_c = 0.1$, 0.5 and 0.9. These confirm all of the qualitative properties obtained in Part I (see equation (20)) and described in section 1. Figure 1 indicates that a PTW



(a) (b)

Figure 4: Same as Figure 3 but this time $u_c = 0.45$ (top) and $u_c = 0.55$ (bottom).

develops in the large-time structure of the solution to QIVP, that is, as $t \to \infty$. Moreover, the rate of convergence of the solution to the PTW depends on the value of u_c (compare panel (a) with panel (c)). Figures 2 and 3 show that this PTW will have propagation speed given by

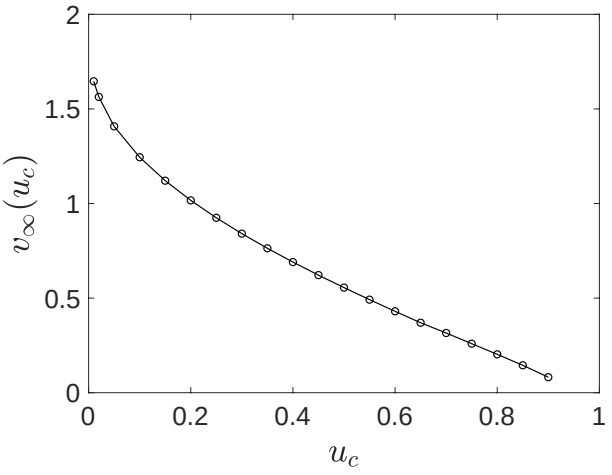


Figure 5: A graph of $\lim_{t\to\infty} \dot{s}(t) = v_\infty(u_c)$ obtained from the numerical solution to QIVP for selected values of $u_c \in (0,1)$.

$\lim_{t\to\infty} \dot{s}(t) = v_\infty(u_c)$ and in this case, this limit has

$$v_\infty(u_c) \simeq \begin{cases} 1.248, & \text{for } u_c = 0.1, \\ 0.558, & \text{for } u_c = 0.5, \\ 0.100, & \text{for } u_c = 0.9. \end{cases} \tag{11}$$

Figure 3 also illustrates that $\dot{s}(t)$ appears to have a (integrable) singularity at $t = 0^+$ when $u_c \neq 0.5$. This is further supported in Figure 4 which shows the behaviour of $\dot{s}(t)$ when $u_c = 0.45$ and $u_c = 0.55$. For $u_c = 0.5$, Figure 3 suggests that $\dot{s}(t)$ is regular in this limit, tending to 0 from above. Figures 3 and 4 show that the sign of $\dot{s}(t)$ as $t \to 0^+$ depends upon u_c, with $\dot{s}(t)$ initially positive when $0 < u_c < 0.5$ and initially negative when $0.5 < u_c < 1$. Moreover, when $0 < u_c \lesssim 0.2$, then $\dot{s}(t)$ is monotonic decreasing for all $t > 0$; when $0.2 \lesssim u_c < 0.5$, then $\dot{s}(t)$ decreases to a minimum value, before increasing to $v_\infty(u_c)$; and when $0.5 < u_c < 1$, then $\dot{s}(t)$ is monotonic increasing for all $t > 0$. Finally, the correction to $\dot{s}(t)$ as $t \to \infty$ appears to be exponentially small in t. These features are persistent for all considered values of $u_c \in (0,1)$.

We conclude that the numerical solution of QIVP involves the formation of a PTW as $t \to \infty$, which has propagation speed $v_\infty(u_c)$ for all values of $u_c \in (0,1)$. A graph of numerically calculated values $v_\infty(u_c)$ for $u_c \in (0,1)$ is given in Figure 5, which indicates that $v_\infty(u_c)$ is monotone decreasing with $u_c \in (0,1)$. The numerical cost increases drastically as $u_c \to 0^+$ and $u_c \to 1^-$. Nevertheless, we expect that $v_\infty(u_c) \to 2^-$ as $u_c \to 0^+$, whilst, $v_\infty(u_c) \to 0^+$ as $u_c \to 1^-$. Finally, it is instructive to compare the travelling wave speed obtained in the large-time limit of the numerical solution to QIVP, namely $v_\infty(u_c)$, with a permanent form travelling wave propagation speed, $v^*(u_c)$, obtained numerically in Part I. As anticipated, we find that, with a significant degree of accuracy (at least up to two decimal places), $v_\infty(u_c) \approx v^*(u_c)$.

3 Asymptotic solution to QIVP as $t \to 0^+$

We now develop the asymptotic structure to QIVP as $t \to 0^+$ via the method of matched asymptotic coordinate expansions. We anticipate that the structure of the solution to QIVP as $t \to 0^+$ will have two asymptotic regions in $y < 0$, and two asymptotic regions in $y > 0$. An

examination of the leading order balances in equation (8a), together with the initial condition (8c) and the connection conditions (8e), (8f) determine the asymptotic structure as:

$$\text{region } \mathbf{I_L}: \quad y = O(t^{\frac{1}{2}}) < 0 \text{ with } u = O(1) \text{ as } t \to 0^+, \tag{12a}$$

$$\text{region } \mathbf{I_R}: \quad y = O(t^{\frac{1}{2}}) > 0 \text{ with } u = O(1) \text{ as } t \to 0^+, \tag{12b}$$

$$\text{region } \mathbf{II_L}: \ y = O(1) < 0 \text{ with } u = 1 + o(1) \text{ as } t \to 0^+, \tag{12c}$$

$$\text{region } \mathbf{II_R}: \ y = O(1) > 0 \text{ with } u = o(1) \text{ as } t \to 0^+. \tag{12d}$$

The situation is illustrated in Figure 6 (for any variable λ, we will henceforth write $\lambda = O(1) > 0$ as $\lambda = O(1)^+$, and correspondingly, $\lambda = O(1) < 0$ as $\lambda = O(1)^-$). It follows from the small-time asymptotic structure (12) of QIVP that we anticipate an asymptotic expansion for $s(t)$ of the form

$$s(t) = s_0 t^\alpha + s_1 t^\beta + o(t^\beta) \quad \text{as} \quad t \to 0^+, \tag{13}$$

where the constants s_0, s_1, α and $\beta(> \alpha)$ are to be found. The initial condition (8g), together with a leading order balance in equation (8a) determines

$$\alpha = \frac{1}{2}. \tag{14}$$

3.1 Regions $\mathbf{I_L}$ and $\mathbf{I_R}$

We begin in region $\mathbf{I_L}$, following (12a), where we introduce the coordinate $\eta = yt^{-\frac{1}{2}} = O(1)^-$ as $t \to 0^+$ and where $u = u(\eta, t)$ satisfies, from (8a),

$$u_t - \frac{1}{t}\frac{\eta}{2}u_\eta - \frac{\dot{s}(t)}{t^{\frac{1}{2}}}u_\eta = \frac{1}{t}u_{\eta\eta} + f(u), \quad \eta < 0. \tag{15}$$

We expand $u(\eta, t)$ in the form,

$$u(\eta, t) = u_{L0}(\eta) + \phi_L(t)u_{L1}(\eta) + o(\phi_L(t)) \quad \text{as} \quad t \to 0^+, \tag{16}$$

with $\eta = O(1)^-$ and $\phi_L(t) = o(1)$ as $t \to 0^+$ to be determined. On substituting expansions (13) and (16) into equation (15), we obtain at leading order as $t \to 0^+$,

$$u''_{L0} + \frac{1}{2}(\eta + s_0)u'_{L0} = 0, \quad \eta < 0, \tag{17a}$$

which must be solved subject to the boundary condition (8e) at $\eta = 0$, together with the matching condition with region $\mathbf{II_L}$ as $\eta \to -\infty$. Using (12c) and (16), these conditions require,

$$u_{L0}(0) = u_c, \tag{17b}$$

$$u_{L0}(\eta) \to 1 \quad \text{as} \quad \eta \to -\infty. \tag{17c}$$

Due to the coupling condition (8f) across $y = 0$, it is necessary now to consider region $\mathbf{I_R}$, in which, via (12b), $\eta = O(1)^+$ and $u = O(1)$ as $t \to 0^+$ and where $u = u(\eta, t)$ satisfies, from (8a),

$$u_t - \frac{1}{t}\frac{\eta}{2}u_\eta - \frac{\dot{s}(t)}{t^{\frac{1}{2}}}u_\eta = \frac{1}{t}u_{\eta\eta}, \quad \eta > 0. \tag{18}$$

We expand $u(\eta, t)$ in the form,

$$u(\eta, t) = u_{R0}(\eta) + \phi_R(t)u_{R1}(\eta) + o(\phi_R(t)) \quad \text{as} \quad t \to 0^+, \tag{19}$$

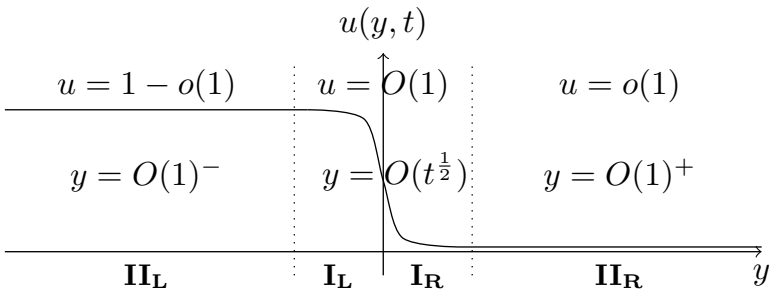


Figure 6: A sketch of the structure of the solution to QIVP as $t \to 0^+$.

with $\eta = O(1)^+$ as $t \to 0^+$. Here $\phi_R = o(1)$ as $t \to 0^+$, and is to be determined. Now, substituting expansions (13) and (19) into equation (18), we obtain at leading order as $t \to 0^+$,

$$u''_{R0} + \frac{1}{2}(\eta + s_0)u'_{R0} = 0, \quad \eta > 0, \tag{20a}$$

which must be solved subject to the boundary condition (8e) at $\eta = 0$, together with the matching condition with region $\mathbf{II_R}$ as $\eta \to \infty$, which requires,

$$u_{R0}(0) = u_c, \tag{20b}$$
$$u_{R0}(\eta) \to 0 \quad \text{as} \quad \eta \to \infty. \tag{20c}$$

Finally, the boundary value problems (17) and (20) must be solved subject to the coupling condition (8f) across $\eta = 0$, which requires

$$u'_{L0}(0) = u'_{R0}(0). \tag{21}$$

The solutions to (17) and (20) respectively, are readily obtained as

$$u_{L0}(\eta) = \frac{u_c\left(1 + \operatorname{erf}(\frac{\eta+s_0}{2})\right) - \operatorname{erf}(\frac{\eta+s_0}{2}) + \operatorname{erf}(\frac{s_0}{2})}{\left(1 + \operatorname{erf}(\frac{s_0}{2})\right)}, \quad \eta \le 0, \tag{22a}$$

$$u_{R0}(\eta) = u_c\frac{\left(1 - \operatorname{erf}(\frac{\eta+s_0}{2})\right)}{\left(1 - \operatorname{erf}(\frac{s_0}{2})\right)}, \quad \eta \ge 0. \tag{22b}$$

Finally, an application of condition (21) to (22) determines

$$s_0 = 2\operatorname{erf}^{-1}(1 - 2u_c), \tag{23}$$

and thus, the leading order terms in region $\mathbf{I_L}$ and region $\mathbf{I_R}$, respectively, are given by

$$u_{L0}(\eta) = \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{\eta}{2} + \operatorname{erf}^{-1}(1 - 2u_c)\right)\right], \quad \eta \le 0, \tag{24a}$$

$$u_{R0}(\eta) = \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{\eta}{2} + \operatorname{erf}^{-1}(1 - 2u_c)\right)\right], \quad \eta \ge 0. \tag{24b}$$

We now proceed to the correction terms in expansions (13), (16) and (19). A balancing of terms requires $\phi_L(t) = \phi_R(t) = O(t)$ as $t \to 0^+$ and $\beta = \frac{3}{2}$. Thus, we set $\phi_L(t) = \phi_R(t) = t$, without loss of generality. On substitution from expansions (13), (16) and (19) into equations

(15) and (18), we obtain the coupled problem for $u_{L1}(\eta)(\eta < 0)$, $u_{R1}(\eta)(\eta > 0)$ and s_1, namely,

$$u''_{L1} + \frac{1}{2}(\eta + s_0)u'_{L1} - u_{L1} = -\frac{3}{2}s_1 u'_{L0} - f(u_{L0}(\eta)), \quad \eta < 0, \tag{25a}$$

$$u''_{R1} + \frac{1}{2}(\eta + s_0)u'_{R1} - u_{R1} = -\frac{3}{2}s_1 u'_{R0}, \quad \eta > 0, \tag{25b}$$

subject to the coupling conditions

$$u_{L1}(0) = u_{R1}(0) = 0, \tag{25c}$$

$$u'_{L1}(0) = u'_{R1}(0), \tag{25d}$$

and the matching conditions to region $\mathbf{II_L}$ and to region $\mathbf{II_R}$, respectively, which are readily obtained as,

$$u_{L1}(\eta) \to 0 \quad \text{as} \quad \eta \to -\infty, \tag{25e}$$

$$u_{R1}(\eta) \to 0 \quad \text{as} \quad \eta \to \infty. \tag{25f}$$

In considering the coupled problem (25), we first observe that $1 + \frac{1}{2}(\eta + s_0)^2$ is a solution to the homogeneous Part of both (25a) and (25b). With this observation, together with the method of variation of parameters, we can write the general solutions to (25a) and (25b) as,

$$u_{L1}(\eta) = d_1\hat{u}(\eta) + d_2\bar{u}(\eta) - \frac{s_1}{2\sqrt{\pi}}\exp\left(-\left(\frac{\eta + s_0}{2}\right)^2\right) + u_{p2}(\eta), \quad \eta \le 0, \tag{26a}$$

$$u_{R1}(\eta) = \bar{d}_1\hat{u}(\eta) + \bar{d}_2\bar{u}(\eta) - \frac{s_1}{2\sqrt{\pi}}\exp\left(-\left(\frac{\eta + s_0}{2}\right)^2\right), \quad \eta \ge 0, \tag{26b}$$

where $d_1, d_2, \bar{d}_1$ and $\bar{d}_2$ are arbitrary constants to be determined and the function $u_{p2}(\eta)$ is given by

$$u_{p2}(\eta) = \frac{\hat{u}(\eta)}{2}\int_\eta^0 I_1(\lambda)d\lambda - \frac{\bar{u}(\eta)}{2}\int_\eta^0 I_2(\lambda)d\lambda, \quad \eta \le 0, \tag{27}$$

with functions

$$\hat{u}(\eta) = \sqrt{\pi}\left(1 + \frac{(\eta + s_0)^2}{2}\right)\operatorname{erf}\left(\frac{\eta + s_0}{2}\right) + (\eta + s_0)\exp\left(-\left(\frac{\eta + s_0}{2}\right)^2\right), \tag{28a}$$

$$\bar{u}(\eta) = 1 + \frac{(\eta + s_0)^2}{2}, \tag{28b}$$

$$I_1(\eta) = \exp\left(\left(\frac{\eta + s_0}{2}\right)^2\right)\bar{u}(\eta)f(u_{L0}(\eta)), \tag{28c}$$

$$I_2(\eta) = \exp\left(\left(\frac{\eta + s_0}{2}\right)^2\right)\hat{u}(\eta)f(u_{L0}(\eta)). \tag{28d}$$

Next, an application of condition (25c) requires

$$d_2 = \left(\frac{s_1}{\sqrt{\pi}} - 2d_1 s_0\right)\frac{e^{-\frac{{s_0}^2}{4}}}{({s_0}^2 + 2)} - d_1\sqrt{\pi}\operatorname{erf}\left(\frac{s_0}{2}\right), \tag{29}$$

$$\bar{d}_2 = \left(\frac{s_1}{\sqrt{\pi}} - 2\bar{d}_1 s_0\right)\frac{e^{-\frac{{s_0}^2}{4}}}{({s_0}^2 + 2)} - \bar{d}_1\sqrt{\pi}\operatorname{erf}\left(\frac{s_0}{2}\right), \tag{30}$$

whilst, applying the matching conditions (25e) and (25f) requires

$$d_2 = \sqrt{\pi}\left(d_1 + \frac{1}{2}\hat{d}_1\right), \tag{31}$$

$$\bar{d}_2 = -\sqrt{\pi}\bar{d}_1, \tag{32}$$

with the constant $\hat{d}_1$ given by

$$\hat{d}_1 = \int_{-\infty}^{0} \left(\sqrt{\pi} I_1(\lambda) + I_2(\lambda)\right) d\lambda. \tag{33}$$

As $u'_{p2}(0) = 0$, an application of the coupling condition (25d) determines $d_1 = \bar{d}_1$ (and thus $d_2 = \bar{d}_2$) which finally requires that

$$s_1 = \frac{1}{4}\left(\sqrt{\pi}({s_0}^2 + 2)\left(1 - \mathrm{erf}\left(\frac{s_0}{2}\right)\right) e^{\frac{{s_0}^2}{4}} - 2s_0\right)\hat{d}_1, \tag{34}$$

after which (using(23)), d_1, $\bar{d}_1$, d_2, $\bar{d}_2$ follow from (29), (30), (31) and (32).

Thus, we have determined that the two-term expansions for $u(\eta, t)$ in region $\mathbf{I_L}$ and region $\mathbf{I_R}$ are given by

$$u(\eta,t) = \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{\eta + s_0}{2}\right)\right] + t\left(d_1\hat{u}(\eta) + d_2\bar{u}(\eta) - \frac{s_1}{2\sqrt{\pi}}\exp\left[-\left(\frac{\eta + s_0}{2}\right)^2\right] + u_{p2}(\eta)\right) + o(t), \tag{35}$$

as $t \to 0^+$ with $\eta = O(1)^-$, and

$$u(\eta,t) = \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{\eta + s_0}{2}\right)\right] + t\left(d_1\hat{u}(\eta) + d_2\bar{u}(\eta) - \frac{s_1}{2\sqrt{\pi}}\exp\left[-\left(\frac{\eta + s_0}{2}\right)^2\right]\right) + o(t), \tag{36}$$

as $t \to 0^+$, with $\eta = O(1)^+$, whilst the two-term expansion for $s(t)$ is given by

$$s(t) = s_0 t^{\frac{1}{2}} + s_1 t^{\frac{3}{2}} + o(t^{\frac{3}{2}}), \tag{37}$$

as $t \to 0^+$. Here the constants d_1, d_2, s_0 and s_1 are given by (31), (29), (23) and (34), respectively, and the functions $\hat{u}(\eta)$, $\bar{u}(\eta)$, $I_1(\lambda)$, $I_2(\lambda)$ and $u_{p2}(\eta)$ are given by (28) and (27), respectively. It is worth noting that we have obtained the two term small-time expansions for $s(t)$ without needing to know the precise asymptotic structure of the solution in regions $\mathbf{II_L}$ and $\mathbf{II_R}$. The matching conditions with regions $\mathbf{I_L}$ and $\mathbf{I_R}$, respectively, were sufficient. The asymptotic expansion in regions $\mathbf{II_L}$ and $\mathbf{II_R}$ are now obtained to complete the small-time asymptotic structure.

3.2 Region $\mathbf{II_L}$

First, from (35) and (36), we observe that for $(-\eta) \gg 1$,

$$u(\eta,t) \sim 1 - \frac{1}{\sqrt{\pi}}\frac{1}{|\eta + s_0|}\exp\left(-\left(\frac{\eta + s_0}{2}\right)^2\right)(1 - O((\eta + s_0)^{-2})), \tag{38}$$

as $t \to 0^+$, and for $\eta \gg 1$,

$$u(\eta, t) \sim \frac{1}{\sqrt{\pi}} \frac{1}{(\eta + s_0)} \exp\left(- \left(\frac{\eta + s_0}{2} \right)^2 \right) (1 - O((\eta + s_0)^{-2})), \tag{39}$$

as $t \to 0^+$. Now, as $\eta \to -\infty$ we move out of region $\mathbf{I_L}$ and into region $\mathbf{II_L}$, in which, via (12c), $y = O(1)^-$ and $u(y,t) = 1 + o(1)$ as $t \to 0^+$. The structure of the expansion in region $\mathbf{I_L}$, for $(-\eta) \gg 1$, (given by (38)) suggests that in region $\mathbf{II_L}$ we write

$$u(y,t) = 1 - e^{-\frac{H(y,t)}{t}}, \tag{40}$$

and expand in the form,

$$H(y,t) = H_0(y) + t^{\frac{1}{2}} H_1(y) + t \ln t H_2(y) + t H_3(y) + o(t), \tag{41}$$

as $t \to 0^+$ with $y = O(1)^-$ and $H_0(y) > 0$ (the $t \ln t$ term arises from the algebraic prefactor of the exponential term in (38)). We substitute expansions (40) and (41) into equation (8a) to obtain (on solving at each order of t in turn)

$$\begin{aligned} u(y,t) = 1 - \exp\Bigg(& - \frac{y^2}{4t} - \frac{1}{t^{\frac{1}{2}}} \left(\frac{s_0}{2} y + D_1 (-y)^{\frac{1}{2}} \right) - D_2 \ln t \\ & - \left(\frac{(1 - 2D_2)}{2} \ln(-y) + \frac{s_0 D_1}{2} \frac{1}{(-y)^{\frac{1}{2}}} + \frac{{D_1}^2}{4} \frac{1}{y} + D_3 \right) + o(1) \Bigg), \end{aligned} \tag{42}$$

as $t \to 0^+$, with $y = O(1)^-$, and where D_1, D_2 and D_3 are arbitrary constants to be determined. It remains to match expansion (42) in region $\mathbf{II_L}$ (as $y \to 0^-$) with expansion (38) in region $\mathbf{I_L}$ (as $\eta \to -\infty$). On applying Van Dyke's matching principle [19], we readily obtain that

$$D_1 = 0, \qquad D_2 = -\frac{1}{2}, \qquad D_3 = \frac{1}{2} \ln \pi + \frac{{s_0}^2}{4}. \tag{43}$$

Thus, the expansion in region $\mathbf{II_L}$ is given by

$$u(y,t) = 1 - \exp\left(- \frac{y^2}{4t} - \frac{y s_0}{2 t^{\frac{1}{2}}} + \frac{1}{2} \ln t - \left(\ln(-y) + \frac{1}{2} \ln \pi + \frac{{s_0}^2}{4} \right) + o(1) \right), \tag{44}$$

as $t \to 0^+$, with $y = O(1)^-$. Furthermore, we conclude from (44) that this expansion remains uniform for $(-y) \gg 1$ as $t \to 0^+$.

3.3 Region $\mathbf{II_R}$

Next, as $\eta \to \infty$, we move out of region $\mathbf{I_R}$ and into region $\mathbf{II_R}$, in which, via (12d), $y = O(1)^+$ and $u(y,t) = o(1)$ as $t \to 0^+$. The structure of the expansion in region $\mathbf{I_R}$, for $\eta \gg 1$, (given by (39)) suggests that in region $\mathbf{II_R}$ we write

$$u(y,t) = e^{-\frac{\bar{H}(y,t)}{t}}, \tag{45}$$

and expand in the form,

$$\bar{H}(y,t) = \bar{H}_0(y) + t^{\frac{1}{2}} \bar{H}_1(y) + t \ln t \bar{H}_2(y) + t \bar{H}_3(y) + o(t), \tag{46}$$

as $t \to 0^+$ with $y = O(1)^+$ and $\bar{H}_0(y) > 0$ (the $t \ln t$ term arises from the algebraic prefactor of the exponential term in (39)). Substitution of (45) and (46) into equation (8a) gives (on solving at each order of t in turn)

$$u(y,t) = \exp\left(-\left(\frac{y^2}{4t}\right) - \frac{1}{t^{\frac{1}{2}}}\left(\frac{s_0}{2}y + \bar{D}_1 y^{\frac{1}{2}}\right) - \bar{D}_2 \ln t \right.$$
$$\left. - \left(\frac{(1-2\bar{D}_2)}{2}\ln y + \frac{s_0\bar{D}_1}{2}\frac{1}{y^{\frac{1}{2}}} + \frac{\bar{D}_1^2}{4}\frac{1}{y} + \bar{D}_3\right) + o(1)\right), \tag{47}$$

as $t \to 0^+$, with $y = O(1)^+$, and where $\bar{D}_1$, $\bar{D}_2$ and $\bar{D}_3$ are arbitrary constants to be determined. It remains to match expansion (47) in region $\mathbf{II_R}$ (as $y \to 0^+$) with expansion (39) in region $\mathbf{I_R}$ (as $\eta \to \infty$). On applying Van Dyke's matching principle [19], we readily obtain that

$$\bar{D}_1 = 0, \qquad \bar{D}_2 = -\frac{1}{2}, \qquad \bar{D}_3 = \frac{1}{2}\ln\pi + \frac{{s_0}^2}{4}. \tag{48}$$

Thus, the expansion in region $\mathbf{II_R}$ is given by

$$u(y,t) = \exp\left(-\frac{y^2}{4t} - \frac{ys_0}{2t^{\frac{1}{2}}} + \frac{1}{2}\ln t - \left(\ln y + \frac{1}{2}\ln\pi + \frac{{s_0}^2}{4}\right) + o(1)\right), \tag{49}$$

as $t \to 0^+$ and $y = O(1)^+$. Furthermore, we conclude from (44) that this expansion remains uniform for $y \gg 1$ as $t \to 0^+$.

The asymptotic structure of the solution to QIVP as $t \to 0^+$ is now complete with the expansions (44), (35), (36) and (49) in regions $\mathbf{II_L}$, $\mathbf{I_L}$, $\mathbf{I_R}$ and $\mathbf{II_R}$. We next use this information to enable us to develop the asymptotic structure of the solution to QIVP as $|y| \to \infty$ with $t = O(1)$. However, before proceeding to this, it is of interest to examine the form of $\dot{s}(t)$ in the small-time limit for all $u_c \in (0,1)$. It follows from expression (37) that

$$\dot{s}(t) \sim \frac{1}{2}s_0 t^{-\frac{1}{2}} + \frac{3}{2}s_1 t^{\frac{1}{2}} \quad \text{as} \quad t \to 0^+, \tag{50}$$

with s_0 and s_1 given by equations (23) and (34) respectively. In particular, we observe from (23) that s_0 is monotonic decreasing in u_c with

$$s_0 \to \infty \ \text{ as } \ u_c \to 0^+, \quad s_0 = 0 \ \text{ when } \ u_c = \frac{1}{2} \quad \text{and} \quad s_0 \to -\infty \ \text{ as } \ u_c \to 1^-. \tag{51}$$

Thus, the leading term in (50) reveals that $\dot{s}(t)$ has an integrable singularity as $t \to 0^+$, with

$$\dot{s}(t) \to +\infty \quad \text{as} \quad t \to 0^+, \tag{52}$$

when $0 < u_c < 1/2$, whilst,

$$\dot{s}(t) \to -\infty \quad \text{as} \quad t \to 0^+, \tag{53}$$

when $1/2 < u_c < 1$. When $u_c = 1/2$, a transition occurs with $\dot{s}(t)$ not singular and

$$\dot{s}(t) \to 0 \quad \text{as} \quad t \to 0^+. \tag{54}$$

3.4 The case of a cut-off Fisher reaction

We observe that (52), (53) and (54) agree with the numerical solutions for QIVP obtained for the cut-off Fisher reaction function in section 2, as illustrated in Figures 3 and 4. Moreover, it is straightforward to establish (via (33) and (34)) that for $u_c = 1/2$, $s_1 = s_1^* > 0$. Therefore $\dot{s}(t) \to 0^+$ as $t \to 0^+$. In addition, it is interesting to note from expression (50) that when u_c is close to $1/2$ a local minimum point in the graph of $\dot{s}(t)$ against t bifurcates singularly from $t = 0$ as u_c decreases through $u_c = 1/2$. In particular, the local minimum point when $u_c < 1/2$ is located when $t = t_m \sim \frac{1}{3}s_0/s_1 > 0$. As $u_c \to \frac{1}{2}^-$, $\frac{1}{3}s_0/s_1 \sim \frac{2}{3}\sqrt{\pi}(1-2u_c)/s_1^* + O((1-2u_c)^2)$ where $s_1^* \simeq 0.28$ is approximated numerically using (33) and (34). The location of the minimum point increases as u_c decreases, until $u_c \approx 0.2$ when t_m is no longer small and in fact the local minimum point ceases to exist at this sufficiently low value of u_c. This is also in agreement with the numerical solution of section 2 and in particular Figures 3 and 4. A comparison of $\dot{s}(t)$ and $u(y,t)$ as computed from (35), (36), (44) and (49) with the full numerical solution to QIVP obtained for the cut-off Fisher reaction function is readily made (but for brevity is not presented here). This demonstrates the full agreement with the small-time asymptotic structure of the solution obtained in this section and the numerical solution obtained in section 2 for t small.

4 Asymptotic solution to QIVP as $|y| \to \infty$ with $t = O(1)$

We now develop the structure of the solution to QIVP as $|y| \to \infty$ with $t = O(1)$.

4.1 Region $\mathbf{III_L}$

We begin in region $\mathbf{III_L}$, where $y \to -\infty$ with $t = O(1)$. The structure of the expansion in region $\mathbf{II_L}$, for $(-y) \gg 1$, (given by (44)) suggests that in region $\mathbf{III_L}$ we write

$$u(y,t) = 1 - e^{-y^2\Phi(y,t)}, \tag{55}$$

and expand in the form,

$$\Phi(y,t) = \Phi_0(t) + \frac{1}{y}\Phi_1(t) + \frac{\ln(-y)}{y^2}\Phi_2(t) + \frac{1}{y^2}\Phi_3(t) + o\left(y^{-2}\right), \tag{56}$$

as $y \to -\infty$ with $t = O(1)$ and $\Phi_0(t) > 0$. On substitution from expansions (55) and (56) into equation (8a) we obtain a system of equations at successive orders which we solve in turn to give

$$\Phi_0(t) = \frac{1}{(4t + C_0)}, \qquad \Phi_1(t) = \frac{(2s(t) + C_1)}{(4t + C_0)}, \qquad \Phi_2(t) = C_2, \tag{57a}$$

$$\dot{\Phi}_3(t) = \dot{s}(t)\left(\frac{2s(t) + C_1}{4t + C_0}\right) + \frac{(2 + 4C_2)}{(4t + C_0)} - \left(\frac{2s(t) + C_1}{4t + C_0}\right)^2 - f'(1), \tag{57b}$$

where C_0, C_1, C_2 and the constant associated with integrating equation (57b), C_3, are constants to be determined. Note that $\Phi_1(t)$ and $\Phi_3(t)$ both depend on the function $s(t)$ which remains undetermined when $t = O(1)$. We now match the expansion in region $\mathbf{III_L}$, given by substituting expressions (56) and (57) into (55) (as $t \to 0^+$), with expansion (44) in region $\mathbf{II_L}$ (as $y \to -\infty$). On applying Van Dyke's matching principle [19] we find

$$C_0 = 0, \qquad C_1 = 0, \qquad C_2 = -1, \qquad C_3 = \frac{1}{2}\ln\pi. \tag{58}$$

Thus, the expansion in region $\mathbf{III_L}$ is given by

$$u(y,t) = 1 - \exp\left(-\frac{y^2}{4t} - y\frac{s(t)}{2t} - \ln(-y) - \left(\frac{s(t)^2}{4t} - \frac{1}{2}\ln t - f'(1)t + \frac{1}{2}\ln\pi\right) + o(1)\right), \quad (59)$$

as $y \to -\infty$ with $t = O(1)$. Furthermore, we note that the uniformity of expansion (59) as $y \to -\infty$ when $t \gg 1$ is dependent on the order of $s(t)$ as $t \gg 1$. This will be discussed further in section 5 when we investigate the asymptotic solution to QIVP as $t \to \infty$.

4.2 Region $\mathbf{III_R}$

We next consider the corresponding region $\mathbf{III_R}$ where we determine the structure of the solution to QIVP as $y \to \infty$ with $t = O(1)$. The structure of the expansion in region $\mathbf{II_R}$, for $y \gg 1$, (given by (44)) suggests that in region $\mathbf{III_R}$ we write

$$u(y,t) = e^{-y^2\bar{\Phi}(y,t)}, \quad (60)$$

and expand in the form,

$$\bar{\Phi}(y,t) = \bar{\Phi}_0(t) + \frac{1}{y}\bar{\Phi}_1(t) + \frac{\ln y}{y^2}\bar{\Phi}_2(t) + \frac{1}{y^2}\bar{\Phi}_3(t) + o\left(y^{-2}\right), \quad (61)$$

as $y \to \infty$ with $t = O(1)$ and $\bar{\Phi}_0(t) > 0$. On substitution from expansions (60) and (61) into equation (8a) we obtain a system of equations at successive orders of y which we solve in turn to give

$$\bar{\Phi}_0(t) = \frac{1}{(4t + \bar{C}_0)}, \qquad \bar{\Phi}_1(t) = \frac{(2s(t) + \bar{C}_1)}{(4t + \bar{C}_0)}, \qquad \bar{\Phi}_2(t) = \bar{C}_2, \quad (62a)$$

$$\dot{\bar{\Phi}}_3(t) = \dot{s}(t)\left(\frac{2s(t) + \bar{C}_1}{4t + \bar{C}_0}\right) + \frac{(2 + 4\bar{C}_2)}{(4t + \bar{C}_0)} - \left(\frac{2s(t) + \bar{C}_1}{4t + \bar{C}_0}\right)^2, \quad (62b)$$

where $\bar{C}_0$, $\bar{C}_1$, $\bar{C}_2$ and the constant associated with integrating equation (62b), $\bar{C}_3$, are constants to be determined. We now match the expansion in region $\mathbf{III_R}$, given by substituting expressions (62) and (61) into (60) (as $t \to 0^+$), with expansion (49) in region $\mathbf{II_R}$ (as $y \to \infty$). On applying Van Dyke's matching principle [19] we find

$$\bar{C}_0 = 0, \qquad \bar{C}_1 = 0, \qquad \bar{C}_2 = -1, \qquad \bar{C}_3 = \frac{1}{2}\ln\pi. \quad (63)$$

Thus, the expansion in region $\mathbf{III_R}$ is given by

$$u(y,t) = \exp\left(-\frac{y^2}{4t} - y\frac{s(t)}{2t} - \ln y - \left(\frac{s(t)^2}{4t} - \frac{1}{2}\ln t + \frac{1}{2}\ln\pi\right) + o(1)\right), \quad (64)$$

as $y \to \infty$ with $t = O(1)$. As before, the uniformity of expansion (64) as $y \to \infty$ when $t \gg 1$ is dependent on the order of $s(t)$ as $t \gg 1$. Finally, we are now in a position to consider the structure of the solution to QIVP as $t \to \infty$.

5 Asymptotic solution to QIVP as $t \to \infty$

We now develop the structure of the solution to QIVP as $t \to \infty$. Guided by the numerical results in section 2, we anticipate that

$$s(t) = \sum_{i=0}^{3} c_i \phi_i(t) + o(\phi_3(t)) \quad \text{as} \quad t \to \infty, \tag{65}$$

where $\phi_0(t) = t$, $\phi_1(t)$, $\phi_2(t) = 1$ and $\phi_3(t)$ are a gauge sequence as $t \to \infty$, and the constants c_0, c_1, c_2, c_3 are to be determined, with $c_0 > 0$. We begin by developing the structure of the solution to QIVP as $t \to \infty$ at leading order, uniform for $y \in \mathbb{R}$. We anticipate that the structure of the solution to QIVP as $t \to \infty$ will have two principal asymptotic regions in $y < 0$, and two principal asymptotic regions in $y > 0$. An examination of the leading order balances in the exponent of expansions (59) and (64) when $t \gg 1$ (using (65)), together with the connection conditions (8e) and (8f) determine the principal asymptotic structure as:

$$\text{region } \mathbf{IV_L}: \quad y = O(t)^- \text{ with } u = 1 + o(1) \text{ as } t \to \infty, \tag{66a}$$
$$\text{region } \mathbf{IV_R}: \quad y = O(t)^+ \text{ with } u = o(1) \text{ as } t \to \infty, \tag{66b}$$
$$\text{region } \mathbf{V_L}: \quad y = O(1)^- \text{ with } u = O(1) \text{ as } t \to \infty, \tag{66c}$$
$$\text{region } \mathbf{V_R}: \quad y = O(1)^+ \text{ with } u = O(1) \text{ as } t \to \infty. \tag{66d}$$

5.1 Regions $\mathbf{IV_L}$, $\mathbf{V_L}$, $\mathbf{IV_R}$ and $\mathbf{V_R}$

The expansion (59) in region $\mathbf{III_L}$ will remain uniform for $t \gg 1$ provided that $(-y) \gg t$, but fails when $y = O(t)^-$ as $t \to \infty$. Hence, we begin in region $\mathbf{IV_L}$, in which, via (66a), we introduce the scaled coordinate $w = \frac{y}{t} = O(1)^-$ as $t \to \infty$. The structure of the expansion in region $\mathbf{III_L}$, for $t \gg 1$, (given by (59)) suggests that in region $\mathbf{IV_L}$, we write

$$u(w,t) = 1 - \exp\Big(- t\,(G_0(w) + o(1))\Big), \tag{67}$$

as $t \to \infty$ with $w = O(1)^-$ and $G_0(w) > 0$. On substitution of expansions (65) and (67) into equation (8a) we obtain the following boundary value problem, namely,

$$\big(G_0'\big)^2 - (w + c_0)G_0' + G_0 = -f'(1), \quad w < 0, \tag{68a}$$
$$G_0(w) > 0, \quad w < 0, \tag{68b}$$
$$G_0(w) \sim \left(\frac{w + c_0}{2}\right)^2 - f'(1) \quad \text{as} \quad w \to -\infty, \tag{68c}$$
$$G_0(w) = O(w) \quad \text{as} \quad w \to 0^-. \tag{68d}$$

Here condition (68c) represents the matching condition between expansion (67) in region $\mathbf{IV_L}$ when $(-w) \gg 1$, and expansion (59) in region $\mathbf{III_L}$ as $t \to \infty$ with $(-y) \gg t$ whilst condition (68d) represents the matching condition between expansion (67) in region $\mathbf{IV_L}$ when $w = O(t^{-1})^-$, and region $\mathbf{V_L}$ when $y = O(t)^-$ via (66c). Equation (68a) has a family of linear solutions

$$G_0(w) = a_1(w + c_0 - a_1) - f'(1) \quad \forall w < 0, \tag{69}$$

for any $a_1 \in \mathbb{R}$, and an envelope solution

$$G_0(w) = \left(\frac{w + c_0}{2}\right)^2 - f'(1) \quad \forall w < 0. \tag{70}$$

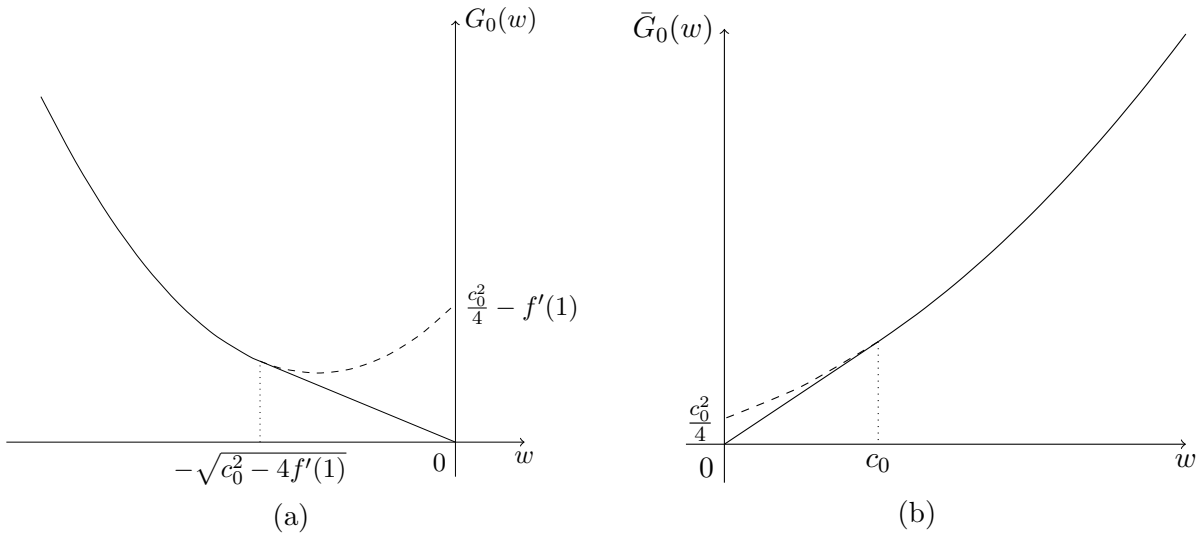


Figure 7: (a) A sketch of the leading order term $G_0(w)$ in the exponential expansion of the large-time solution to QIVP when $y = O(t)^-$. (b) A sketch of the leading order term $\bar{G}_0(w)$ in the exponential expansion of the large-time solution to QIVP when $y = O(t)^+$.

It is also possible for a combination of (69) and (70) to represent 'envelope-linear' solutions to equation (68a), which also remain continuous and differentiable. Applying the matching conditions (68c) and (68d) determines that for each $c_0 > 0$, the solution to the boundary value problem (68) is given by the 'envelope-linear' solution

$$G_0(w) = \begin{cases} \left(\frac{w+c_0}{2}\right)^2 - f'(1), & w < -\sqrt{c_0^2 - 4f'(1)}, \\ \left(\frac{c_0 - \sqrt{c_0^2 - 4f'(1)}}{2}\right) w, & -\sqrt{c_0^2 - 4f'(1)} \leq w < 0. \end{cases} \tag{71}$$

A sketch of $G_0(w)$, for a fixed $c_0 > 0$, is given in Figure 7(a). For completeness we note that although $G_0(w)$ and $G_0'(w)$ are continuous, $G_0''(w)$ is discontinuous at the point $w = -\sqrt{c_0^2 - 4f'(1)}$. Therefore, a thin transition region must exist about the point $w = -\sqrt{c_0^2 - 4f'(1)}$ where the second derivative in equation (8a) is retained at leading order to smooth out this discontinuity. Moreover, region $\mathbf{IV_L}$ will then be replaced by three regions, namely, region $\mathbf{IV_L^a}$, with $-\infty < w < -\sqrt{c_0^2 - 4f'(1)} - o(1)^+$, region $\mathbf{T_L}$, a thin transition region about the point $w = -\sqrt{c_0^2 - 4f'(1)}$ and region $\mathbf{IV_L^b}$, with $-\sqrt{c_0^2 - 4f'(1)} + o(1)^+ < w < 0$. As we are only interested in the leading order structure in each expansion for now, we will return to consider these regions in more detail in §5.3.

Now, as $w \to 0^-$ we move out of region $\mathbf{IV_L}$ and into region $\mathbf{V_L}$, in which, via (66c), $u = O(1)$ with $y = O(1)^-$ as $t \to \infty$. In this region we therefore expand as

$$u(y,t) = \hat{u}_{L0}(y) + O(\psi_L(t)) \quad \text{as} \quad t \to \infty, \tag{72}$$

with $y = O(1)^-$, $\hat{u}_{L0}(y) > 0$ ([18], equation (22b)) and where $\psi_L(t) = o(1)$ as $t \to \infty$. On substitution from expansions (65) and (72) into equation (8a), we obtain at leading order as $t \to \infty$,

$$\hat{u}_{L0}'' + c_0 \hat{u}_{L0}' + f(\hat{u}_{L0}) = 0, \tag{73a}$$

which must be solved subject to the boundary condition (8e) at $y = 0$, together with the matching condition with region $\mathbf{IV_L}$ as $y \to -\infty$. Using (72) and (71), these conditions require,

$$\hat{u}_{L0}(0^-) = u_c, \tag{73b}$$

$$\hat{u}_{L0}(y) \to 1 \quad \text{as} \quad y \to -\infty. \tag{73c}$$

Due to the coupling condition (8f) across $y = 0$, it is necessary now to formulate the leading order problem in the corresponding regions when $y > 0$ as $t \to \infty$.

The expansion (64) in region $\mathbf{III_R}$ will remain uniform for $t \gg 1$ provided that $y \gg t$, but fails when $y = O(t)^+$ as $t \to \infty$. Hence, we now consider region $\mathbf{IV_R}$, in which, via (66b), we introduce the scaled coordinate $w = \frac{y}{t} = O(1)^+$ as $t \to \infty$. The structure of the expansion in region $\mathbf{III_R}$, for $t \gg 1$, (given by (64)) suggests that in region $\mathbf{IV_R}$, we write

$$u(w,t) = \exp\Big(- t \left(\bar{G}_0(w) + o(1)\right)\Big), \tag{74}$$

as $t \to \infty$ with $w = O(1)^+$ and $\bar{G}_0(w) > 0$. On substitution of expansion (74) into equation (8a) we obtain the following boundary value problem, namely,

$$\left(\bar{G}_0'\right)^2 - (w + c_0)\bar{G}_0' + \bar{G}_0 = 0, \quad w > 0, \tag{75a}$$

$$\bar{G}_0(w) > 0, \quad w > 0, \tag{75b}$$

$$\bar{G}_0(w) \sim \left(\frac{w + c_0}{2}\right)^2 \quad \text{as} \quad w \to \infty, \tag{75c}$$

$$\bar{G}_0(w) = O(w) \quad \text{as} \quad w \to 0^+. \tag{75d}$$

Here condition (75c) represents the matching condition between expansion (74) in region $\mathbf{IV_R}$ when $w \gg 1$, and expansion (64) in region $\mathbf{III_R}$ as $t \to \infty$ when $y \gg t$ whilst condition (75d) represents the matching condition between expansion (74) in region $\mathbf{IV_R}$ when $w = O(t^{-1})^+$, and region $\mathbf{V_R}$ when $y = O(t)^+$ via (66d). For each $c_0 > 0$, the boundary value problem (75) has the unique solution

$$\bar{G}_0(w) = \begin{cases} \left(\frac{w+c_0}{2}\right)^2, & w > c_0, \\ c_0 w, & 0 < w \le c_0. \end{cases} \tag{76}$$

A sketch of $\bar{G}_0(w)$ for a fixed $c_0 > 0$ is given in Figure 7(b). For completeness we note that although $\bar{G}_0(w)$ and $\bar{G}_0'(w)$ are continuous, $\bar{G}_0''(w)$ is discontinuous at the point $w = c_0$. Hence, a thin transition region about the point $w = c_0$ is required in which the second derivative in equation (8a) is retained at leading order to smooth out the discontinuity. This requires that region $\mathbf{IV_R}$ is replaced by three regions, namely, region $\mathbf{IV_R^a}$, with $c_0 + o(1) < w < \infty$, region $\mathbf{T_R}$, a thin transition region about the point $w = c_0$ and region $\mathbf{IV_R^b}$, with $0 < w < c_0 - o(1)$. As before, we will consider these regions in more detail in §5.2.

Now, as $w \to 0^+$ we move out of region $\mathbf{IV_R}$ and into region $\mathbf{V_R}$, in which, via (66d), $u = O(1)$ and $y = O(1)^+$ as $t \to \infty$. In this region we must therefore expand as

$$u(y,t) = \hat{u}_{R0}(y) + O(\psi_R(t)) \quad \text{as} \quad t \to \infty, \tag{77}$$

with $y = O(1)^+$, $\hat{u}_{R0}(y) > 0$ ([18], equation (22b)) and $\psi_R(t) = o(1)$ as $t \to \infty$. On substitution from expansions (65) and (77) into equation (8a), we obtain at leading order as $t \to \infty$,

$$\hat{u}_{R0}'' + c_0 \hat{u}_{R0}' = 0, \tag{78a}$$

which must be solved subject to the boundary condition (8e) at $y = 0$, together with the matching condition with region $\mathbf{IV_R}$ as $y \to \infty$. Using (72) and (71), these conditions require,

$$\hat{u}_{R0}(0^+) = u_c, \tag{78b}$$

$$\hat{u}_{R0}(y) \to 0 \quad \text{as} \quad y \to \infty. \tag{78c}$$

Finally, the boundary value problems (73) and (78) must be solved subject to the coupling condition (8f) across $y = 0$, which requires

$$\hat{u}'_{L0}(0^-) = \hat{u}'_{R0}(0^+). \tag{79}$$

The coupled nonlinear boundary value problem, given by (73), (78) and (79), across regions $\mathbf{V_L}$ and $\mathbf{V_R}$ is precisely the nonlinear boundary value problem satisfied by the PTW structure considered in Part I with v replaced by c_0. Thus, we immediately conclude that

$$\hat{u}_{R0}(y) = U_T(y), \quad y \geq 0, \tag{80a}$$
$$\hat{u}_{L0}(y) = U_T(y), \quad y < 0, \tag{80b}$$

and that c_0 is now determined as,

$$c_0 = v^*(u_c), \tag{80c}$$

where $U_T : \mathbb{R} \to \mathbb{R}$ is the PTW solution to QIVP at cut-off $u_c \in (0, 1)$, which has propagation speed $v^*(u_c)$. For convenience, we recall from Theorem 1.1 of Part I that

$$U_T(y) = u_c e^{-v^*(u_c)y} \quad \forall y \in [0, \infty), \tag{81a}$$

and

$$U_T(y) \sim 1 - A_{-\infty} e^{\lambda_+(v^*(u_c))y} \quad \text{as} \quad y \to -\infty, \tag{81b}$$

where $\lambda_+(v^*(u_c)) = \frac{1}{2}\left(-v^*(u_c) + \sqrt{v^*(u_c)^2 - 4f'(1)}\right)$, and $A_{-\infty}$ is a global constant depending upon u_c. This completes the asymptotic structure of the solution to QIVP as $t \to \infty$ at leading order.

5.2 Regions $\mathbf{IV^a_R}$, $\mathbf{T_R}$, $\mathbf{IV^b_R}$ and $\mathbf{V_R}$

To develop the solution to QIVP to higher order we must first return to region $\mathbf{T_R}$, the localised transition region in which $w = v^*(u_c) + o(1)$ as $t \to \infty$. It follows from the leading order term in the expansion in region $\mathbf{IV_R}$ (given by (76), (78) and (80c)) that to examine region $\mathbf{T_R}$ we must introduce the scaled coordinate $\zeta = (w - v^*(u_c))t^{\frac{1}{2}}$ and expand $u(\zeta, t)$ in the form

$$u(\zeta, t) = \left(\bar{F}_0(\zeta) + o(1)\right) \exp\left(- tv^*(u_c)^2 - t^{\frac{1}{2}}\zeta v^*(u_c)\right), \tag{82}$$

as $t \to \infty$ with $\zeta = O(1)$ and $\bar{F}_0(\zeta) > 0$. On substitution of expansions (82) and (65) into equation (8a) we obtain

$$t\dot{\phi}_1(t)\left(v^*(u_c)c_1\bar{F}_0\right) + \left(-\frac{1}{2}\zeta\bar{F}'_0 - \bar{F}''_0\right) + o(1) = 0, \qquad -\infty < \zeta < \infty. \tag{83}$$

The only non-trivial dominant balance requires that we set, without loss of generality

$$\phi_1(t) = \ln t. \tag{84}$$

Thus, the leading order equation in region $\mathbf{T_R}$ is given by

$$\bar{F}''_0 + \frac{1}{2}\zeta\bar{F}'_0 - \gamma\bar{F}_0 = 0, \qquad -\infty < \zeta < \infty, \tag{85}$$

with $\gamma = v^*(u_c)c_1$. To obtain the full boundary value problem for $\bar{F}_0(\zeta)$ we require matching conditions as $\zeta \to -\infty$ with region $\mathbf{IV_R^b}$ and as $\zeta \to \infty$ with region $\mathbf{IV_R^a}$. Therefore, we next return to region $\mathbf{IV_R^b}$. The structure of the expansion in region $\mathbf{V_R}$, for $y \gg 1$, (given by (77), (80a) and (81a)) dictates that in region $\mathbf{IV_R^b}$ we expand in the form

$$u(w,t) = \exp\left(-t\left(v^*(u_c)w - \frac{1}{t}\hat{G}(w) + o\left(\frac{1}{t}\right)\right)\right), \tag{86}$$

as $t \to \infty$ with $O(t^{-1}) < w < v^*(u_c) - O(t^{-\frac{1}{2}})$. We substitute expansion (86) into equation (8a) to obtain, on solving at each order in turn,

$$u(w,t) = \exp\Big(- tv^*(u_c)w + v^*(u_c)c_1 \ln\left(v^*(u_c) - w\right) + \bar{d} + o\left(1\right)\Big), \tag{87}$$

as $t \to \infty$ with $O(t^{-1}) < w < v^*(u_c) - O(t^{-\frac{1}{2}})$ and where the constants c_1 and $\bar{d}$ are to be determined. On matching expansion (87) in region $\mathbf{IV_R^b}$ (as $w \to v^*(u_c)^-$) with expansion (82) in region $\mathbf{T_R}$ (as $\zeta \to -\infty$), via Van Dyke's matching principle [19], we readily obtain that

$$c_1 = 0, \tag{88}$$

after which we must have

$$\bar{F}_0(\zeta) = e^{\bar{d}} + o(1) \quad \text{as} \quad \zeta \to -\infty. \tag{89}$$

To determine $\bar{d}$ we next match expansion (87) (with (88)) in region $\mathbf{IV_R^b}$ (as $w \to 0^+$) with expansion (81a) in region $\mathbf{V_R}$ (as $y \to \infty$). On applying Van Dyke's matching principle [19], we require that

$$\bar{d} = \ln u_c. \tag{90}$$

Thus, via (87), (88) and (90), the expansion in region $\mathbf{IV_R^b}$ is given by

$$u(w,t) = \exp\Big(- tv^*(u_c)w + \ln u_c + o(1)\Big), \tag{91}$$

as $t \to \infty$ with $O(t^{-1}) < w < v^*(u_c) - O(t^{-\frac{1}{2}})$. In addition (89) becomes

$$\bar{F}_0(\zeta) = u_c + o(1) \quad \text{as} \quad \zeta \to -\infty. \tag{92}$$

We next consider region $\mathbf{IV_R^a}$. The structure of the expansion in region $\mathbf{III_R}$, as $t \to \infty$ with $y \gg t$, (given by (64)) and the form of $s(t)$ as $t \to \infty$ (given by (65) with c_1 now determined by (88)), suggests that in region $\mathbf{IV_R^a}$ we write

$$u(w,t) = e^{-t\bar{G}(w,t)}, \tag{93}$$

and expand in the form,

$$\bar{G}(w,t) = \left(\frac{w + v^*(u_c)}{2}\right)^2 + \frac{\ln t}{t}\bar{G}_1(w) + \frac{1}{t}\bar{G}_2(w) + o(t^{-1}), \tag{94}$$

as $t \to \infty$ with $w > v^*(u_c) + O(t^{-\frac{1}{2}})$. On substitution from (93) and (94) into equation (8a) we obtain a series of boundary value problems which we solve at each order of t in turn to obtain

$$u(w,t) = \exp\Big(- t\left(\frac{w + v^*(u_c)}{2}\right)^2 - \frac{1}{2}\ln t - \bar{G}_2(w) + o(1)\Big), \tag{95}$$

as $t \to \infty$ with $w > v^*(u_c) + O(t^{-\frac{1}{2}})$ and where the function $\bar{G}_2(w)$ is indeterminate, being globally dependent on the evolution at earlier stages when $t = O(1)$ and $y = O(1)$. However, to match with expansion $\mathbf{III_R}$ (as $t \to \infty$ with $y \gg t$), we require

$$\bar{G}_2(w) \sim c_2\left(\frac{w + v^*(u_c)}{2}\right) + \ln w + \frac{1}{2}\ln \pi \quad \text{as} \quad w \to \infty. \tag{96}$$

In addition the structure of the expansion in region $\mathbf{T_R}$, as given by (82), requires, for matching to be possible, that,

$$\bar{G}_2(w) \sim \bar{\alpha}_1 \ln\big(w - v^*(u_c)\big) + \bar{\alpha}_2 \quad \text{as} \quad w \to v^*(u_c)^+, \tag{97}$$

for some constants $\bar{\alpha}_1, \bar{\alpha}_2$ to be determined. We now match in detail the expansion in region $\mathbf{IV^a_R}$, given by (95) and (97) (as $w \to v^*(u_c)^+$), with expansion (82) in region $\mathbf{T_R}$ (as $\zeta \to \infty$). On applying Van Dyke's matching principle [19] we find that

$$\bar{\alpha}_1 = 1, \tag{98}$$

after which,

$$\bar{F}_0(\zeta) = \bar{\sigma}\zeta^{-1}e^{-\frac{\zeta^2}{4}}\,(1 + o(1)) \quad \text{as} \quad \zeta \to \infty, \tag{99}$$

where $\bar{\sigma} = e^{-\bar{\alpha}_2}$. Hence, on collecting (85), (88), (92) and (99) we obtain the boundary value problem in region $\mathbf{T_R}$ for $\bar{F}_0(\zeta)$ as,

$$\bar{F}_0'' + \frac{1}{2}\zeta\bar{F}_0' = 0, \qquad -\infty < \zeta < \infty, \tag{100a}$$

$$\bar{F}_0(\zeta) > 0, \qquad -\infty < \zeta < \infty, \tag{100b}$$

$$\bar{F}_0(\zeta) = \bar{\sigma}\zeta^{-1}e^{-\frac{\zeta^2}{4}}\,(1 + o(1)) \quad \text{as} \quad \zeta \to \infty, \tag{100c}$$

$$\bar{F}_0(\zeta) = u_c + o(1) \quad \text{as} \quad \zeta \to -\infty. \tag{100d}$$

This boundary value problem has a solution only when

$$\bar{\sigma} = \frac{u_c}{\sqrt{\pi}}, \tag{101}$$

with the solution being unique, and given by,

$$\bar{F}_0(\zeta) = \frac{1}{2}u_c \operatorname{erfc}\left(\frac{\zeta}{2}\right) \quad \forall -\infty < \zeta < \infty. \tag{102}$$

It follows from (101) that

$$\bar{\alpha}_2 = -\ln\frac{u_c}{\sqrt{\pi}}. \tag{103}$$

It is now instructive to summarize the structure in regions $\mathbf{IV^a_R}$, $\mathbf{T_R}$ and $\mathbf{IV^b_R}$. The expansion in region $\mathbf{IV^a_R}$ is given by (95) together with the asymptotic conditions

$$\bar{G}_2(w) \sim \begin{cases} \ln\big(w - v^*(u_c)\big) - \ln\frac{u_c}{\sqrt{\pi}}, & \text{as} \quad w \to v^*(u_c)^+, \\ c_2\left(\frac{w+v^*(u_c)}{2}\right) + \ln w + \frac{1}{2}\ln\pi, & \text{as} \quad w \to \infty, \end{cases} \tag{104}$$

whilst in region $\mathbf{T_R}$

$$u(\zeta, t) = \left(\frac{1}{2}u_c \operatorname{erfc}\left(\frac{\zeta}{2}\right) + o(1)\right)\exp\Big(-tv^*(u_c)^2 - t^{\frac{1}{2}}\zeta v^*(u_c)\Big), \tag{105}$$

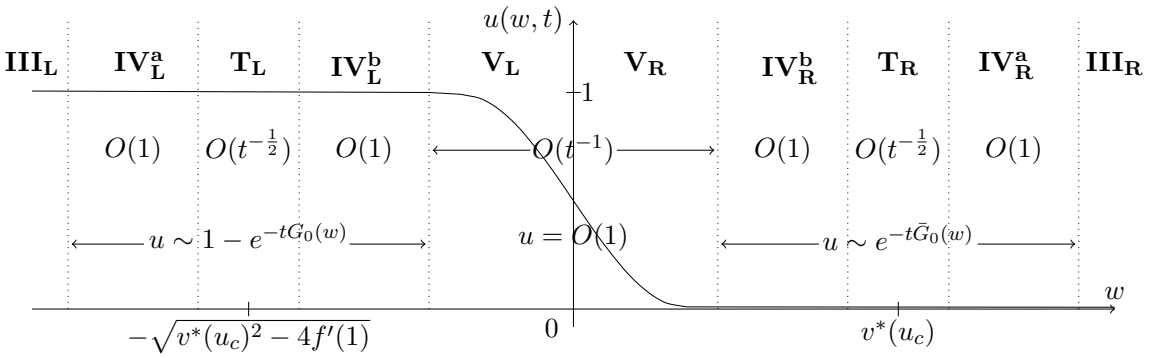


Figure 8: A schematic representation of the location and thickness of the asymptotic regions in the solution to QIVP as $t \to \infty$. Here the the leading order terms in the exponential form of the solution $G_0(w)$ and $\bar{G}_0(w)$ are given by (71) and (76), respectively. Additionally, there are thin transition regions at $w = -\sqrt{v^*(u_c)^2 - 4f'(1)}$ and at $w = v^*(u_c)$. Note that regions $\mathbf{III_L}$ and $\mathbf{III_R}$ are far field regions for $|w| \gg 1$ as $t \to \infty$.

as $t \to \infty$ with $\zeta = O(1)$, and in region $\mathbf{IV_R^b}$

$$u(w,t) = \exp\bigg(- tv^*(u_c)w + \ln u_c + o(1)\bigg), \tag{106}$$

as $t \to \infty$ with $O(t^{-1}) < w < v^*(u_c) - O(t^{-\frac{1}{2}})$. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Figure 8.

We next consider the structure of the expansion in region $\mathbf{T_R}$ in more detail. Via (105), we observe that for $(-\zeta) \gg 1$,

$$u(\zeta,t) \sim \exp\left(-tv^*(u_c)^2 - t^{\frac{1}{2}}v^*(u_c)\zeta + \ln\left(u_c\left(1 + \frac{1}{\sqrt{\pi}}\frac{1}{\zeta}e^{-\frac{\zeta^2}{4}}\right)\right)\right), \tag{107}$$

as $t \to \infty$, which demands that in region $\mathbf{IV_R^b}$, to continue the expansion in (106), we must write

$$u(w,t) = u_c e^{-twv^*(u_c)} + t^{-\frac{1}{2}}\bar{G}(w,t)\exp\left(-\frac{t(w+v^*(u_c))^2}{4}\right), \tag{108}$$

as $t \to \infty$ with $O(t^{-1}) < w < v^*(u_c) - O(t^{-\frac{1}{2}})$ and $\bar{G}(w,t) = O(1)$ as $t \to \infty$. On substituting from expansion (108) into equation (8a), and simplifying, we obtain

$$\bar{G}_t - \frac{1}{2}t^{-1}\bar{G} - t^{-2}\bar{G}_{ww} = O\left(t^{\frac{1}{2}}\dot{\phi}_3(t)\exp\left(-t\left(wv^*(u_c) - \frac{(w+v^*(u_c))^2}{4}\right)\right)\right), \tag{109}$$

as $t \to \infty$ with $O(t^{-1}) < w < v^*(u_c) - O(t^{-\frac{1}{2}})$. We will later verify that the right-hand side of equation (109) is exponentially small as $t \to \infty$ in this region. Hence, to obtain a structured balance in (109), we must expand $\bar{G}(w,t)$ in the form

$$\bar{G}(w,t) = \bar{G}_0(w) + t^{-1}\bar{G}_1(w) + o\left(t^{-1}\right), \tag{110}$$

as $t \to \infty$ with $O(t^{-1}) < w < v^*(u_c) - O(t^{-\frac{1}{2}})$ and on substitution into (109) we obtain at leading order

$$\bar{G}_0'' + \bar{G}_1 = 0, \tag{111}$$

with $O(t^{-1}) < w < v^*(u_c) - O(t^{-\frac{1}{2}})$. We conclude that $\bar{G}_0(w)$ is indeterminate and represents a further globally determined function. Therefore, the expansion in region $\mathbf{IV^b_R}$ is, from equations (108) and (110),

$$u(w,t) = u_c e^{-twv^*(u_c)} + t^{-\frac{1}{2}}\bar{G}_0(w)(1+O(t^{-1}))\exp\left(-\frac{t(w+v^*(u_c))^2}{4}\right), \tag{112}$$

as $t \to \infty$ with $O(t^{-1}) < w < v^*(u_c) - O(t^{-\frac{1}{2}})$. We now match the expansion (112) in region $\mathbf{IV^b_R}$ (as $w \to v^*(u_c)^-$), with expansion (107) in region $\mathbf{T_R}$ (as $\zeta \to -\infty$), in detail. On applying Van Dyke's matching principle [19] we require

$$\bar{G}_0(w) = -\frac{u_c}{\sqrt{\pi}}(w - v^*(u_c))^{-1} + o(w - v^*(u_c))^{-1} \quad \text{as} \quad w \to v^*(u_c)^-. \tag{113}$$

We next return to region $\mathbf{V_R}$. First, a balance between expansion (72) in region $\mathbf{V_L}$ and expansion (77) in region $\mathbf{V_R}$, across the connection at $y = 0$, requires

$$\psi_L(t) = \psi_R(t) = \psi(t), \tag{114}$$

where $\psi(t) = o(1)$ as $t \to \infty$. Now, the induced correction term in expansion (77) in region $\mathbf{V_R}$ from region $\mathbf{IV^b_R}$ when $0 < w \ll 1$, must have, via (112),

$$\psi(t) = O\left(t^\gamma e^{-\frac{v^*(u_c)^2 t}{4}}\right), \tag{115}$$

as $t \to \infty$, with constant γ to be determined. Thus, without loss of generality we set

$$\psi(t) = t^\gamma e^{-\frac{v^*(u_c)^2 t}{4}}. \tag{116}$$

Hence, in region $\mathbf{V_R}$ we develop expansion (77) in the form

$$u(y,t) = U_T(y) + t^\gamma e^{-\frac{v^*(u_c)^2 t}{4}} u_1(y)(1+o(1)), \tag{117}$$

as $t \to \infty$ with $y = O(1)^+$. On substitution of expansion (117) into equation (8a), and cancelling at leading order, we obtain

$$-\frac{1}{4}v^*(u_c)^2 u_1 - v^*(u_c)u_1' - u_1'' + o(1) = c_3 U_T'(y) t^{-\gamma}\dot{\phi}_3(t) e^{\frac{v^*(u_c)^2 t}{4}}, \tag{118}$$

as $t \to \infty$ with $y = O(1)^+$. The non-trivial balance in (118) requires that we set, without loss of generality

$$\dot{\phi}_3(t) = t^\gamma e^{-\frac{v^*(u_c)^2 t}{4}}, \tag{119}$$

and we note that this now confirms that the right-hand side of (109) is exponentially small as $t \to \infty$. The corresponding problem for $u_1(y)$ is then

$$u_1'' + v^*(u_c)u_1' + \frac{1}{4}v^*(u_c)^2 u_1 = -c_3 U_T'(y), \quad y > 0, \tag{120a}$$

$$u_1(0^+) = 0, \tag{120b}$$

where the condition (120b) is required for the boundary condition (8e) to be satisfied. The problem for $u_1(y)$, given by (120), must be solved subject to the matching condition with region $\mathbf{IV^b_R}$. Before formulating this matching condition, we consider the corresponding structure in regions $\mathbf{IV^a_L}, \mathbf{T_L}, \mathbf{IV^b_L}$ and $\mathbf{V_L}$. Thus, we now move to region $\mathbf{IV^a_L}$.

5.3 Regions $\mathbf{IV_L^a}$, $\mathbf{T_L}$, $\mathbf{IV_L^b}$ and $\mathbf{V_L}$

The structure of the expansion in region $\mathbf{III_L}$ as $t \to \infty$ with $(-y) \gg t$ (given by (59)), the structure of $s(t)$ as $t \to \infty$ (given by (65) with c_0 and c_1 given by (80c) and (88) respectively) and the leading order behaviour in regions $\mathbf{IV_L^a}$ and $\mathbf{IV_L^b}$ (given by (67) and (71)), suggests that in region $\mathbf{IV_L^a}$ we write

$$u(w,t) = 1 - e^{-tG(w,t)}, \tag{121}$$

and expand in the form,

$$G(w,t) = \left(\frac{w + v^*(u_c)}{2}\right)^2 - f'(1) + \frac{\ln t}{t} G_1(w) + \frac{1}{t} G_2(w) + o(t^{-1}), \tag{122}$$

as $t \to \infty$ with $w < -\sqrt{v^*(u_c)^2 - 4f'(1)} - O(t^{-\frac{1}{2}})$. On substitution of (121) and expansion (122) into equation (8a) we obtain a sequence of boundary value problems which we solve at each order to obtain

$$u(w,t) = 1 - \exp\left(- t\left(\left(\frac{w + v^*(u_c)}{2}\right)^2 - f'(1)\right) - \frac{1}{2}\ln t - G_2(w) + o(1)\right), \tag{123}$$

as $t \to \infty$ with $w < -\sqrt{v^*(u_c)^2 - 4f'(1)} - O(t^{-\frac{1}{2}})$, and where the function $G_2(w)$ is indeterminate, being globally dependent on the evolution at earlier stages when $t = O(1)$ and $y = O(1)$. However, to match with expansion $\mathbf{III_L}$ (as $t \to \infty$ with $(-y) \gg t$), we require

$$G_2(w) \sim c_2\left(\frac{w + v^*(u_c)}{2}\right) + \ln(-w) + \frac{1}{2}\ln \pi \quad \text{as} \quad w \to -\infty. \tag{124}$$

We next examine region $\mathbf{T_L}$. It follows from the structure of the expansion in region $\mathbf{IV_L^a}$, as $w \to (-\sqrt{v^*(u_c)^2 - 4f'(1)})^-$ (given by (123)), that in region $\mathbf{T_L}$ we must introduce the scaled coordinate $\zeta = \left(w + \sqrt{v^*(u_c)^2 - 4f'(1)}\right) t^{\frac{1}{2}}$ and expand $u(\zeta, t)$ in the form

$$\begin{aligned} u(\zeta,t) =& 1 - (F_0(\zeta) + o(1)) \exp\left(- t\left(\left(\frac{v^*(u_c) - \sqrt{v^*(u_c)^2 - 4f'(1)}}{2}\right)^2 \right.\right. \\ & \left.\left. - f'(1)\right) - t^{\frac{1}{2}}\zeta\left(\frac{v^*(u_c) - \sqrt{v^*(u_c)^2 - 4f'(1)}}{2}\right)\right), \end{aligned} \tag{125}$$

as $t \to \infty$ with $\zeta = O(1)$. On substitution of expansion (125) into equation (8a) we obtain at leading order

$$F_0'' + \frac{1}{2}\zeta F_0' = 0, \qquad -\infty < \zeta < \infty. \tag{126}$$

To obtain the full boundary value problem for $F_0(\zeta)$ we require matching conditions as $\zeta \to \pm\infty$. To that end, the structure of the expansion in region $\mathbf{T_L}$, as given by (125), requires, for matching to be possible, with expansions (123) and (124) in region $\mathbf{IV_L^a}$, that

$$G_2(w) \sim \alpha_1 \ln\left|w + \sqrt{v^*(u_c)^2 - 4f'(1)}\right| + \alpha_2, \tag{127}$$

as $w \to (-\sqrt{v^*(u_c)^2 - 4f'(1)})^-$ for some constants α_1, α_2 to be determined. We now match in detail the expansion in region $\mathbf{IV_L^a}$, given by (123) and (127), as $w \to (-\sqrt{v^*(u_c)^2 - 4f'(1)})^-$, with expansion (125) in region $\mathbf{T_L}$, as $\zeta \to -\infty$. On applying Van Dyke's matching principle [19] it immediately follows that

$$\alpha_1 = 1, \tag{128}$$

after which we must have

$$F_0(\zeta) = \sigma\zeta^{-1}e^{-\frac{\zeta^2}{4}}\,(1+o(1)) \quad \text{as} \quad \zeta \to -\infty, \tag{129}$$

where $\sigma = e^{-\alpha_2}$. We next consider the matching condition as $\zeta \to \infty$. The structure of the expansion in region $\mathbf{V_L}$, for $(-y) \gg 1$, (given by (72), (80b) and (81b)) dictates that in region $\mathbf{IV_L^b}$ we must expand in the form

$$u(w,t) = 1 - \exp\left(-t\left(\frac{v^*(u_c) - \sqrt{v^*(u_c)^2 - 4f'(1)}}{2}\right)w + \tilde{G}(w) + o\,(1)\right), \tag{130}$$

as $t \to \infty$ with $-\sqrt{v^*(u_c)^2 - 4f'(1)} + O(t^{-\frac{1}{2}}) < w < O(t^{-1})^-$. We substitute expansion (130) into equation (8a) to obtain, on solving at each order in turn,

$$u(w,t) = 1 - \exp\left(-t\left(\frac{v^*(u_c) - \sqrt{v^*(u_c)^2 - 4f'(1)}}{2}\right)w + d + o\,(1)\right), \tag{131}$$

as $t \to \infty$ with $-\sqrt{v^*(u_c)^2 - 4f'(1)} + O(t^{-\frac{1}{2}}) < w < O(t^{-1})^-$ and where the constant d is to be determined. On matching expansion (131) in region $\mathbf{IV_L^b}$ (as $w \to 0^-$) with expansion (81b) in region $\mathbf{V_L}$ (as $y \to -\infty$), via Van Dyke's matching principle [19], we readily obtain that

$$d = \ln A_{-\infty}. \tag{132}$$

Thus, via (131) and (132), the expansion in region $\mathbf{IV_L^b}$ is given by

$$u(w,t) = 1 - \exp\left(-t\left(\frac{v^*(u_c) - \sqrt{v^*(u_c)^2 - 4f'(1)}}{2}\right)w + \ln A_{-\infty} + o\,(1)\right), \tag{133}$$

as $t \to \infty$ with $-\sqrt{v^*(u_c)^2 - 4f'(1)} + O(t^{-\frac{1}{2}}) < w < O(t^{-1})^-$. On matching expansion (133) in region $\mathbf{IV_L^b}$ (as $w \to (-\sqrt{v^*(u_c)^2 - 4f'(1)})^-$) with expansion (125) in region $\mathbf{T_L}$ (as $\zeta \to \infty$), we obtain the condition

$$F_0(\zeta) = A_{-\infty} + o(1) \quad \text{as} \quad \zeta \to \infty. \tag{134}$$

Hence, on collecting (126), (129) and (134) we obtain the boundary value problem in region $\mathbf{T_L}$ for $F_0(\zeta)$ as,

$$F_0'' + \frac{1}{2}\zeta F_0' = 0, \qquad -\infty < \zeta < \infty, \tag{135a}$$

$$F_0(\zeta) > 0, \qquad -\infty < \zeta < \infty, \tag{135b}$$

$$F_0(\zeta) = \sigma\zeta^{-1}e^{-\frac{\zeta^2}{4}}\,(1+o(1)) \quad \text{as} \quad \zeta \to -\infty, \tag{135c}$$

$$F_0(\zeta) = A_{-\infty} + o(1) \quad \text{as} \quad \zeta \to \infty. \tag{135d}$$

This boundary value problem has a solution only when

$$\sigma = \frac{A_{-\infty}}{\sqrt{\pi}}, \tag{136}$$

with the solution being unique, and given by,

$$F_0(\zeta) = \frac{1}{2}A_{-\infty}\left(1 + \operatorname{erf}\left(\frac{\zeta}{2}\right)\right) \qquad \forall -\infty < \zeta < \infty. \tag{137}$$

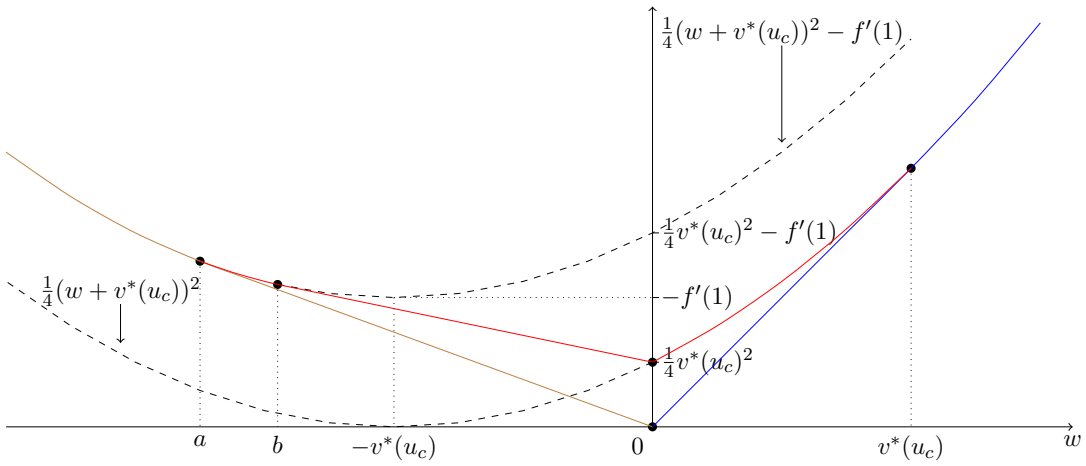


Figure 9: Sketches of the exponent in the large-time solution to QIVP. Sketches of the leading order term $G_0(w)$ when $w<0$ (brown), in expansions (123) and (141), in regions $\mathbf{IV_L^a}$ and $\mathbf{IV_L^b}$, respectively; sketches of the leading order term $\bar{G}_0(w)$ when $w>0$ (blue), in expansions (95) and (106) in regions $\mathbf{IV_R^a}$ and $\mathbf{IV_R^b}$, respectively; and sketches of the exponential corrections (red) in regions $\mathbf{IV_L^b}$ $(a<w<0)$ and $\mathbf{IV_R^b}$ $(0<w<v^*(u_c))$, respectively. Here we have used the notation $a=-\sqrt{v^*(u_c)^2-f'(1)}$ and $b=-2\sqrt{-f'(1)}$.

It follows from (136) that

$$\alpha_2 = -\ln\frac{A_{-\infty}}{\sqrt{\pi}}. \tag{138}$$

It is again instructive to summarize the structure in regions $\mathbf{IV_L^a}$, $\mathbf{T_L}$ and $\mathbf{IV_L^b}$. The expansion in region $\mathbf{IV_L^a}$ is given by (123) together with the asymptotic conditions

$$G_2(w) \sim \begin{cases} \ln|w+\sqrt{v^*(u_c)^2-4f'(1)}| - \ln\frac{A_{-\infty}}{\sqrt{\pi}}, & \text{as } w \to \left(-\sqrt{v^*(u_c)^2-4f'(1)}\right)^-, \\ \left(\frac{w+v^*(u_c)}{2}\right) + \ln|w| + \frac{1}{2}\ln\pi, & \text{as } w\to-\infty, \end{cases} \tag{139}$$

whilst in region $\mathbf{T_L}$,

$$\begin{aligned} u(\zeta,t) =& 1 - \left(\frac{1}{2}A_{-\infty}\left(1+\operatorname{erf}\left(\frac{\zeta}{2}\right)\right)+o(1)\right) \\ &\times \exp\left(-t\left(\left(\frac{v^*(u_c)-\sqrt{v^*(u_c)^2-4f'(1)}}{2}\right)^2 - f'(1)\right)\right. \\ &\left. -t^{\frac{1}{2}}\zeta\left(\frac{v^*(u_c)-\sqrt{v^*(u_c)^2-4f'(1)}}{2}\right)\right), \end{aligned} \tag{140}$$

as $t\to\infty$ with $\zeta=O(1)$, and in region $\mathbf{IV_L^b}$

$$u(w,t) = 1 - \exp\left(-t\left(\frac{v^*(u_c)-\sqrt{v^*(u_c)^2-4f'(1)}}{2}\right) + \ln A_{-\infty} + o(1)\right), \tag{141}$$

as $t\to\infty$ with $-\sqrt{v^*(u_c)^2-4f'(1)}+O(t^{-\frac{1}{2}}) < w < O(t^{-1})^-$. A schematic representation of the location and thickness of the asymptotic regions as $t\to\infty$ is given in Figure 8.

We next consider the structure of the expansion in region $\mathbf{T_L}$ in closer detail. Via (140), we observe that for $\zeta \gg 1$,

$$u(\zeta,t) \sim 1 - \exp\left(-t\left(\left(\frac{v^*(u_c) - \sqrt{v^*(u_c)^2 - 4f'(1)}}{2}\right)^2 - f'(1)\right)\right.$$
$$\left. -t^{\frac{1}{2}}\zeta\left(\frac{v^*(u_c) - \sqrt{v^*(u_c)^2 - 4f'(1)}}{2}\right) + \ln\left(A_{-\infty}\left(1 - \frac{1}{\sqrt{\pi}}\frac{1}{\zeta}e^{-\frac{\zeta^2}{4}}\right)\right)\right), \tag{142}$$

as $t \to \infty$, which demands that in region $\mathbf{IV_L^b}$, to continue the expansion in (141), we must write

$$u(w,t) = 1 - A_{-\infty}\exp\left[-t\left(\frac{v^*(u_c) - \sqrt{v^*(u_c)^2 - 4f'(1)}}{2}\right)w\right] + t^{-\hat{\beta}}G(w,t)e^{-tH(w)}, \tag{143}$$

as $t \to \infty$ with $-\sqrt{v^*(u_c)^2 - 4f'(1)} + O(t^{-\frac{1}{2}}) < w < O(t^{-1})^-$ and $G(w,t) = O(1)$ as $t \to \infty$. Here $\hat{\beta}$ is a constant to be determined and

$$H(w) > \frac{1}{2}\left(v^*(u_c) - \sqrt{v^*(u_c)^2 - 4f'(1)}\right)w, \tag{144}$$

for all $-\sqrt{v^*(u_c)^2 - 4f'(1)} < w < 0$. On substituting from expansion (143) with (144) into equation (8a) we obtain

$$G\left(H_w^2 - (w + v^*(u_c)H_w + H + f'(1)\right) + O(t^{-1})$$
$$= O\left(t^{\gamma+\hat{\beta}}\exp\left(-t\left(\left(\frac{v^*(u_c) - \sqrt{v^*(u_c)^2 - 4f'(1)}}{2}\right)w + \frac{1}{4}v^*(u_c)^2 - H(w)\right)\right)\right), \tag{145}$$

as $t \to \infty$ with $-\sqrt{v^*(u_c)^2 - 4f'(1)} + O(t^{-\frac{1}{2}}) < w < O(t^{-1})^-$. To obtain a non-trivial balance at leading order as $t \to \infty$ we suppose that the function $H(w)$ is such that the right-hand side of equation (145) is exponentially small as $t \to \infty$, and we will later verify this as consistent. Thus, at leading order, we obtain the following boundary value problem in region $\mathbf{IV_L^b}$ for $H(w)$,

$$H_w^2 - (w + v^*(u_c))H_w + H = -f'(1), \tag{146a}$$
$$0 < H(w) - \frac{1}{2}\left(v^*(u_c) - \sqrt{v^*(u_c)^2 - 4f'(1)}\right)w < \frac{1}{4}v^*(u_c)^2, \tag{146b}$$

with $-\sqrt{v^*(u_c)^2 - 4f'(1)} < w < 0$ and which must be solved subject to

$$H(w) \to \frac{1}{4}v^*(u_c)^2 \quad \text{as} \quad w \to 0^-, \tag{146c}$$
$$H(w) \sim \frac{1}{4}(w + v^*(u_c))^2 - f'(1), \quad \text{as} \quad w \to (-\sqrt{v^*(u_c)^2 - 4f'(1)})^+. \tag{146d}$$

Here the lower bound of inequality (146b) follows from (144) whilst the upper bound ensures the right-hand side of equation (145) is exponentially small as $t \to \infty$. Condition (146c) is required so that the correction term in expansion (143) is of the appropriate order to enable matching of (143) in region $\mathbf{IV_L^b}$ (as $w \to 0^-$) with expansion (72), (80b), (81b), (114) and (116), in region $\mathbf{V_L}$ (as $y \to -\infty$). Condition (146d) represents the matching condition between the expansion in region $\mathbf{IV_L^b}$ as $w \to (-\sqrt{v^*(u_c)^2 - 4f'(1)})^+$ (given by (143)) and the expansion in region

$\mathbf{T_L}$ as $\zeta \to \infty$ (given by (142)). Recalling that for each $u_c \in (0,1)$ then $v^*(u_c) \in (0,2)$, the boundary value problem (146) has the unique solution

$$H(w) = \begin{cases} H_{L1}(w), & -\sqrt{v^*(u_c)^2 - 4f'(1)} < w < -2\sqrt{-f'(1)}, \\ H_{L2}(w), & -2\sqrt{-f'(1)} \le w < 0, \end{cases} \tag{147a}$$

with

$$H_{L1}(w) = \frac{1}{4}(w + v^*(u_c))^2 - f'(1) \quad \text{and} \quad H_{L2}(w) = \frac{1}{4}v^*(u_c)^2 + \left(\frac{1}{2}v^*(u_c) - \sqrt{-f'(1)}\right) w, \tag{147b}$$

and where we also determine, via asymptotic matching, that $\hat{\beta} = \frac{1}{2}$ for $-\sqrt{v^*(u_c)^2 - 4f'(1)} + O(t^{-\frac{1}{2}}) < w < -2\sqrt{-f'(1)} - O(t^{-\frac{1}{2}})$. A sketch of the exponents in expansions (95) and (106), (123) and (141) in regions $\mathbf{IV^a_R}, \mathbf{IV^b_R}$, $\mathbf{IV^a_L}$ and $\mathbf{IV^b_L}$, respectively, is given in Figure 9. We note that although $H(w)$ and $H'(w)$ are continuous for all $-\sqrt{v^*(u_c)^2 - 4f'(1)} < w < 0$, the second derivative $H''(w)$ is discontinuous at the point $w = -2\sqrt{-f'(1)}$. Hence, a thin transition region about the point $w = -2\sqrt{-f'(1)}$ is required in which the second derivative in equation (8a) is retained at leading order to smooth out the discontinuity. However, this region is passive, and for brevity will not be considered here. It remains to determine $G(w,t)$ in region $\mathbf{IV^b_L}$. To that end, since $G(w,t) = O(1)$ as $t \to \infty$ with $w = O(1)^-$, we must expand $G(w,t)$ in the form

$$G(w,t) = G_0(w) + t^{-\lambda} G_1(w) + o\left(t^{-\lambda}\right), \tag{148}$$

as $t \to \infty$ with $-\sqrt{v^*(u_c)^2 - 4f'(1)} + O(t^{-\frac{1}{2}}) < w < O(t^{-1})$ and substitute from expansion (143) (with (147) and (148)) into equation (8a). When $-\sqrt{v^*(u_c)^2 - 4f'(1)} < w < -2\sqrt{-f'(1)}$ we find $\lambda = 1$ and at leading order $G_0(w)$ remains indeterminate when $-\sqrt{v^*(u_c)^2 - 4f'(1)} < w < -2\sqrt{-f'(1)}$ and represents a further globally determined function. However, when $-2\sqrt{-f'(1)} < w < 0$, we require that $\lambda = 1$ and at leading order we obtain

$$\left(w + 2\sqrt{-f'(1)}\right) G_0' = -\hat{\beta} G_0, \tag{149}$$

which gives, on integration,

$$G_0(w) = \frac{\left(2\sqrt{-f'(1)}\right)^{\hat{\beta}} A_L}{\left(w + 2\sqrt{-f'(1)}\right)^{\hat{\beta}}}, \tag{150}$$

with $-2\sqrt{-f'(1)} < w < 0$, where $A_L \neq 0$ is a globally determined constant. Therefore, the expansion in region $\mathbf{IV^b_L}$ is developed to,

$$u(w,t) = 1 - A_{-\infty} \exp\left(- t \left(\frac{v^*(u_c) - \sqrt{v^*(u_c)^2 - 4f'(1)}}{2} \right) w \right) + \hat{u}(w,t), \tag{151}$$

as $t \to \infty$. Here

$$\hat{u}(w,t) = t^{-\beta_1} \big(G_0(w) + o(1)\big) \exp\left(-t \left(\frac{1}{4}\left(w + v^*(u_c)\right)^2 - f'(1) \right) \right), \tag{152}$$

when $-\sqrt{v^*(u_c)^2 - 4f'(1)} + O(t^{-\frac{1}{2}}) < w < -2\sqrt{-f'(1)} - O(t^{-\frac{1}{2}})$, with

$$G_0(w) \sim \frac{A_{-\infty}}{\sqrt{\pi}} \left(w + \sqrt{v^*(u_c)^2 - 4f'(1)}\right)^{-1}, \tag{153}$$

as $w \to (-\sqrt{v^*(u_c)^2 - 4f'(1)})^+$ and

$$\beta_1 = \frac{1}{2}, \tag{154}$$

on matching with region $\mathbf{T_L}$. However,

$$\begin{aligned}\hat{u}(w,t) =& \frac{\left(2\sqrt{-f'(1)}\right)^{\beta_2} A_L}{\left(w + 2\sqrt{-f'(1)}\right)^{\beta_2}} t^{-\beta_2}\big(1 + o(1)\big) \\ & \times \exp\left(-t\left(\frac{1}{4}v^*(u_c)^2 + \left(\frac{1}{2}v^*(u_c) - \sqrt{-f'(1)}\right)w\right)\right),\end{aligned} \tag{155}$$

when $-2\sqrt{-f'(1)} + O(t^{-\frac{1}{2}}) < w < O(t^{-1})^-$, and with β_2 undetermined at this stage. It is important to recall that the change in structure of $\hat{u}(w,t)$ across $w = -2\sqrt{-f'(1)}$ is accommodated in a transition region when $w = -2\sqrt{-f'(1)} \pm O(t^{-\frac{1}{2}})$. This region is passive and its details may be omitted here.

We can now return to region $\mathbf{V_L}$. It follows from (72) with (80b), (81b), (114) and (116), that in region $\mathbf{V_L}$ we must develop expansion (72) in the form

$$u(y,t) = U_T(y) + t^{\gamma}\exp\left(-\frac{1}{4}v^*(u_c)^2 t\right)u_1(y)(1 + o(1)), \tag{156}$$

as $t \to \infty$ with $y = O(1)^-$. On substituting from expansions (65) and (156) into equation (8a), and cancelling at leading order, we obtain

$$u_1'' + v^*(u_c)u_1' + \left(\frac{1}{4}v^*(u_c)^2 + f'(U_T(y))\right)u_1 = -c_3 U_T'(y), \quad y < 0, \tag{157a}$$

$$u_1(0^-) = 0, \tag{157b}$$

where the condition (157b) is required for the boundary condition (8e) to be satisfied. It remains to match expansion (156) in region $\mathbf{V_L}$ (as $y \to -\infty$) with expansion (151) in region $\mathbf{IV_L^b}$ (as $w \to 0^-$). On applying Van Dyke's matching principle [19], we readily obtain this matching condition as

$$u_1(y) \sim A_L \exp\left(\left(\sqrt{-f'(1)} - \frac{1}{2}v^*(u_c)\right)y\right) \quad \text{as} \quad y \to -\infty, \tag{157c}$$

with β_2 now determined as

$$\beta_2 = -\gamma. \tag{158}$$

On collecting (120) and (157), in addition to the derivative continuity condition (8f) at $y = 0$, we obtain the following boundary value problem for $u_1(y)$,

$$u_1'' + v^*(u_c)u_1' + \frac{1}{4}v^*(u_c)^2 u_1 = -c_3 U_T'(y), \quad y > 0, \tag{159a}$$

$$u_1'' + v^*(u_c)u_1' + \left(\frac{1}{4}v^*(u_c)^2 + f'(U_T(y))\right)u_1 = -c_3 U_T'(y), \quad y < 0, \tag{159b}$$

$$u_1(y) \sim A_L \exp\left(\left(\sqrt{-f'(1)} - \frac{1}{2}v^*(u_c)\right)y\right) \quad \text{as} \quad y \to -\infty, \tag{159c}$$

$$u_1(0^-) = u_1(0^+) = 0, \tag{159d}$$

$$u_1'(0^-) = u_1'(0^+), \tag{159e}$$

which must be solved subject, in addition, to the matching condition on $u_1(y)$ as $y \to \infty$ with expansion (112) in region $\mathbf{IV_R^b}$. We begin in $y < 0$, with the inhomogeneous linear equation

(159b). Since $U_T(y)$ satisfies the equation $U_T''(y) + v^*(u_c)U_T'(y) + f_c(U_T(y)) = 0$, a particular integral for (159b) is readily deduced to be proportional to $U_T'(y)$, and so the general solution to (159b) may be written as

$$u_1(y) = E_0\phi_+(y) + E_1\phi_-(y) - 4\frac{c_3}{v^*(u_c)^2}U_T'(y), \quad y \leq 0, \tag{160}$$

with $\phi_+(y), \phi_-(y) : (-\infty, 0] \to \mathbb{R}$ basis functions for the homogeneous part of equation (159b) chosen so that

$$\phi_+(y) \sim \exp\left(\left(\sqrt{-f'(1)} - \frac{1}{2}v^*(u_c)\right)y\right), \tag{161a}$$

$$\phi_-(y) \sim \exp\left(-\left(\sqrt{-f'(1)} + \frac{1}{2}v^*(u_c)\right)y\right), \tag{161b}$$

as $y \to -\infty$, whilst E_0 and E_1 are arbitrary constants to be determined. It follows from (81b), (161) and an application of condition (159c) that we must have

$$E_0 = A_L, \qquad E_1 = 0. \tag{162}$$

Moreover, on applying condition (159d) (where we have evaluated $U_T'(0)$ via (81a)) we obtain

$$c_3 = -\frac{A_L v^*(u_c)\phi_+(0)}{4u_c}. \tag{163}$$

Thus, on collecting expressions (160), (162) and (163) we have

$$u_1(y) = A_L\phi_+(y) + \frac{A_L\phi_+(0)}{v^*(u_c)u_c}U_T'(y), \quad y < 0. \tag{164}$$

We next consider $u_1(y)$ with $y > 0$. The general solution to the inhomogeneous linear equation (159a) (using equations (81a) and (163)) is readily found to be

$$u_1(y) = (E_3 + E_4 y)\, e^{-\frac{1}{2}v^*(u_c)y} - A_L\phi_+(0)e^{-v^*(u_c)y}, \quad y \geq 0, \tag{165}$$

with arbitrary constants E_3 and E_4 determined, via application of the coupling conditions (159d) and (159e), as

$$E_3 = A_L\phi_+(0), \tag{166}$$

$$E_4 = A_L\left(\phi_+'(0) + \phi_+(0)\left(\frac{1}{2}v^*(u_c) - \frac{f_c^+}{v^*(u_c)u_c}\right)\right), \tag{167}$$

with $A_L \neq 0$. Finally, we match the expansion in region $\mathbf{V_R}$ (as $y \to \infty$) with the expansion in region $\mathbf{IV_R^b}$ (as $w \to 0^+$). Now, when $E_4 = 0$, we obtain the matching condition

$$\bar{G}_0(w) \sim A_L\phi_+(0) \quad \text{as} \quad w \to 0^+, \tag{168}$$

and

$$\gamma = -\frac{1}{2}(= -\beta_2). \tag{169}$$

However, when $E_4 \neq 0$, we obtain the matching condition

$$\bar{G}_0(w) \sim E_4 w \quad \text{as} \quad w \to 0^+, \tag{170}$$

and

$$\gamma = -\frac{3}{2}(= -\beta_2). \tag{171}$$

Also, it follows from expression (163) (since $A_L \neq 0$) that $c_3 = 0$ if and only if $\phi_+(0) = 0$. Therefore, we have the following cases, namely;

Case (I) $\phi_+(0) \neq 0$. In this case

$$c_3 \neq 0,$$

and

$$E_4 = 0 \text{ with } \gamma = -\frac{1}{2}(= -\beta_2) \quad \text{or} \quad E_4 \neq 0 \text{ with } \gamma = -\frac{3}{2}(= -\beta_2).$$

Case (II) $\phi_+(0) = 0$. In this case $\phi'_+(0) \neq 0$ and

$$c_3 = 0,$$

whilst $E_4 \neq 0$, and so

$$\gamma = -\frac{3}{2}(= -\beta_2).$$

We next consider the basis function $\phi_+ : (-\infty, 0] \to \mathbb{R}$. For fixed $u_c \in (0,1)$ the initial value problem for $\phi_+ : (-\infty, 0] \to \mathbb{R}$ is given by

$$\phi''_+ + v^*(u_c)\phi'_+ + \left(\frac{1}{4}v^*(u_c)^2 + f'(U_T(y))\right)\phi_+ = 0, \quad y < 0, \tag{172a}$$

$$\phi_+(y) \sim \exp\left(\left(\sqrt{-f'(1)} - \frac{1}{2}v^*(u_c)\right)y\right) \quad \text{as} \quad y \to -\infty. \tag{172b}$$

We reduce the problem (172) to normal form by setting $\phi_+(y) = \psi_+(y)\exp\left(-\frac{1}{2}v^*(u_c)y\right)$ with $\psi_+ : (-\infty, 0] \to \mathbb{R}$ now satisfying the initial value problem

$$\psi''_+ + f'(U_T(y))\psi_+ = 0, \quad y < 0, \tag{173a}$$

$$\psi_+(y) \sim \exp\left(\sqrt{-f'(1)}y\right) \quad \text{as} \quad y \to -\infty. \tag{173b}$$

This can now be solved numerically to find $\psi_+(0)$ and $\psi'_+(0)$ which we then use to obtain $\phi_+(0)$ and $\phi'_+(0)$, after which the occurrence of case (I) or case (II) is determined.

The asymptotic structure of the solution to QIVP as $t \to \infty$ is now complete with the expansions in regions $\mathbf{IV^a_L}$, $\mathbf{T_L}$, $\mathbf{IV^b_L}$, $\mathbf{V_L}$, $\mathbf{V_R}$, $\mathbf{IV^b_R}$, $\mathbf{T_R}$ and $\mathbf{IV^a_R}$ providing a uniform approximation to the solution of QIVP as $t \to \infty$. On collecting expressions (65), (80c), (84), (88) and (119) we have obtained, in particular, that

$$\dot{s}(t) \;=\; v^*(u_c) + c_3 t^\gamma \exp\left(-\frac{1}{4}v^*(u_c)^2 t\right) + o\left(t^\gamma \exp\left(-\frac{1}{4}v^*(u_c)^2 t\right)\right) \quad \text{as} \quad t \to \infty, \tag{174}$$

where the constants c_3 and γ depend upon whether case (I) or case (II) is pertaining for the given KPP reaction function and the cut-off value $u_c \in (0,1)$. Hence, via the method of matched asymptotic coordinate expansions, we have been able to obtain the correction term to the asymptotic propagation speed $v^*(u_c)$ of the developing PTW structure in the solution to QIVP as $t \to \infty$. In addition, with $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ being the solution to QIVP, it follows from expansions (95), (104), (105), (112), (117), (123), (139), (140), (151), (156) in regions $\mathbf{IV^a_L}$, $\mathbf{IV^b_L}$, $\mathbf{IV^a_R}$, $\mathbf{IV^b_R}$, $\mathbf{T_L}$, $\mathbf{T_R}$, $\mathbf{V_L}$ and $\mathbf{V_R}$ that,

$$u(y,t) = U_T(y) + E(y,t), \tag{175}$$

as $t \to \infty$ for $y \in \mathbb{R}$, with $E(y,t)$ linearly exponentially small in t as $t \to \infty$, uniformly for $y \in \mathbb{R}$. In particular, on any closed bounded interval I,

$$E(y,t) = O\left(t^\gamma e^{-\frac{1}{4}v^{*2}(u_c)t}\right), \tag{176}$$

as $t \to \infty$ uniformly for $y \in \mathrm{I}$. A significant point to note here, is that, for KPP reaction functions satisfying (2), in the absence of cut-off, the corresponding correction terms in (174), (175) and (176) are only algebraically small in t as $t \to \infty$, being of $O(t^{-1})$ (see, for example, Leach and Needham [14]).

To illustrate these results we consider a simple example of KPP reaction function $f : \mathbb{R} \to \mathbb{R}$ which satisfies (2), and has

$$f(u) = \lambda(1-u), \quad u \geq \frac{1}{2}\left(1 + \frac{\lambda}{(1+\lambda)}\right), \tag{177}$$

with $\lambda > 0$ fixed. With the cut-off value

$$u_c \in \left[\frac{1}{2}\left(1 + \frac{\lambda}{(1+\lambda)}\right), 1\right), \tag{178}$$

then, in this example, $f_c : \mathbb{R} \to \mathbb{R}$ is given by

$$f_c(u) = \begin{cases} 0, & u \in (-\infty, u_c], \\ \lambda(1-u), & u \in (u_c, \infty), \end{cases} \tag{179}$$

and

$$f'(1) = -\lambda, \quad f_c^+ = \lambda(1-u_c). \tag{180}$$

For this example, we can readily obtain the PTW explicitly as $U_T : \mathbb{R} \to \mathbb{R}$ given by

$$U_T(y) = \begin{cases} 1 - (1-u_c)\exp\left(\left(\frac{\sqrt{v^*(u_c)^2+4\lambda}-v^*(u_c)}{2}\right)y\right), & y \leq 0, \\ u_c e^{-v^*(u_c)y}, & y > 0, \end{cases} \tag{181}$$

with propagation speed

$$v^*(u_c) = \sqrt{\lambda}\frac{(1-u_c)}{\sqrt{u_c}}. \tag{182}$$

Now, via (172), the basis function $\phi_+ : (-\infty, 0] \to \mathbb{R}$ satisfies

$$\phi_+'' + v^*(u_c)\phi_+' + \left(\frac{1}{4}v^*(u_c)^2 - \lambda\right)\phi_+ = 0, \quad y < 0, \tag{183a}$$

$$\phi_+(y) \sim \exp\left(\left(\sqrt{\lambda} - \frac{1}{2}v^*(u_c)\right)y\right) \quad \text{as} \quad y \to -\infty, \tag{183b}$$

which has solution

$$\phi_+(y) = \exp\left(\left(\sqrt{\lambda} - \frac{1}{2}v^*(u_c)\right)y\right), \quad y \leq 0. \tag{184}$$

Thus we obtain via (184)

$$\phi_+(0) = 1, \quad \phi_+'(0) = \sqrt{\lambda} - \frac{1}{2}v^*(u_c), \tag{185}$$

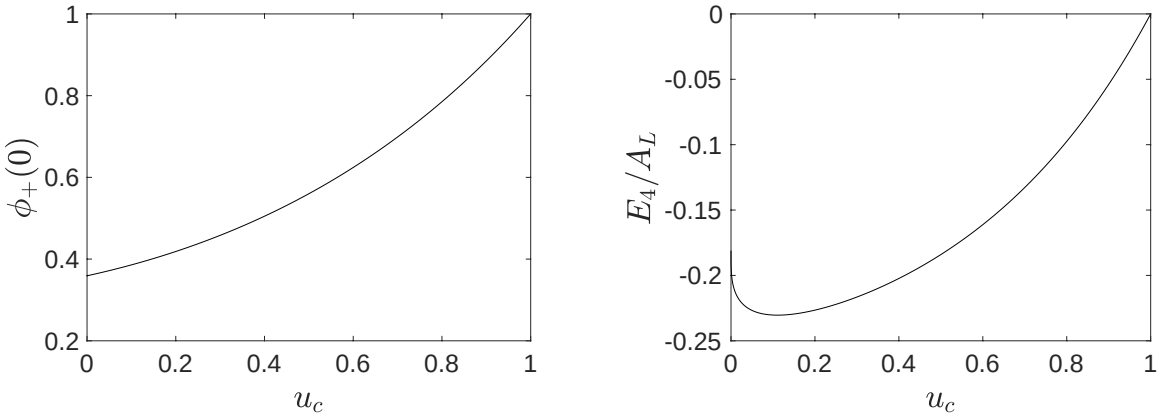
and so,

$$E_4 = A_L\sqrt{\lambda}\,(1 - \sqrt{u_c}) \neq 0. \tag{186}$$

Thus, the particular reaction function (179) falls into case (I) which has

$$\dot{s}(t) \;=\; v^*(u_c) + c_3 t^{-\frac{3}{2}}\exp\left(-\frac{1}{4}v^*(u_c)^2 t\right) + o\left(t^{-\frac{3}{2}}\exp\left(-\frac{1}{4}v^*(u_c)^2 t\right)\right) \quad \text{as } t \to \infty, \tag{187}$$

with $c_3 \neq 0$, and $v^*(u_c)$ given by (182). Similarly, in this example, both (175) and (176) have $\gamma = -3/2$.



(a) (b)

Figure 10: A graph of (a) $\phi_+(0)$ and (b) $E_4/A_L = \phi'_+(0) + \phi_+(0)(v^*(u_c)/2 - (1-u_c)/v^*(u_c))$ corresponding to the cut-off Fisher reaction function (10). These are obtained by solving (188) numerically for a range of values of $u_c \in (0,1)$ and are used to determine the precise form of the correction to $\dot{s}(t)$ as $t \to \infty$, given by equation (174).

5.4 The case of a cut-off Fisher reaction

To conclude this section we focus on the particular case of the cut-off Fisher reaction function (10) for fixed cut-off $u_c \in (0,1)$. For this example, via (173), $\psi_+ : (-\infty, 0] \to \mathbb{R}$ satisfies

$$\psi''_+ + (1 - 2U_T(y))\,\psi_+ = 0, \quad y < 0, \tag{188a}$$

$$\psi_+(y) \sim e^y \quad \text{as} \quad y \to -\infty. \tag{188b}$$

We obtain numerical approximations of $\psi_+(0)$ and $\psi'_+(0)$ from were we deduce $\phi_+(0)$ and $\phi'_+(0)$. This is readily achieved by solving (188) together with the nonlinear boundary value problem determining $U_T(y)$ (see equation (11) in Part I of this series) numerically over an interval $y \in [-M, 0]$ for $M \in \mathbb{R}^+$ using the Matlab initial value solver `ode45`, taking $v = v^*(u_c)$. The values of $v^*(u_c)$ and M are determined numerically as detailed in Part I of this series of papers. As 'initial condition' we employ $(U_T, U'_T, \psi_+, \psi'_+) = (1-\epsilon, -\lambda_+(v^*(u_c))\epsilon, e^{-M}, e^{-M})$, where $\epsilon = 10^{-10}$ and prescribe an absolute and relative ODE tolerance of 10^{-13}.

Figure 10 examines the behaviour of $\phi_+(0)$ and $E_4/A_L = \phi'_+(0) + \phi_+(0)(1/2v^*(u_c) - (1-u_c)/v^*(u_c))$ for a range of values of u_c. It suggests that $\phi_+(0)$ and E_4 are both non-zero and therefore the particular reaction function (10) falls into case (I) with $c_3 \neq 0$, $\gamma = -3/2$ and where $\dot{s}(t)$ has the asymptotic expression

$$\dot{s}(t) \sim v^*(u_c) - \frac{A_L v^*(u_c)\phi_+(0)}{4u_c} t^{-\frac{3}{2}} \exp\left(-\frac{1}{4}v^*(u_c)^2 t\right) \quad \text{as} \quad t \to \infty. \tag{189}$$

We observe that the asymptotic expression (189) qualitatively agrees with the numerical solutions for QIVP obtained for the cut-off Fisher reaction function in section 2: Figures 3 and 4 suggest that the correction to $\dot{s}(t)$ is exponentially small in t as $t \to \infty$ while Figure 1 makes clear that the exponential decay rate decreases with the increasing value of u_c. However, a quantitative test of the validity of (189) is challenging because we do not have sufficient precision to allow the numerical solver to resolve exponentially small terms in the numerical solution; as such we are unable to accurately compare (189) directly with numerical solutions to estimate the global constant A_L.

6 Conclusions

In this series of papers we have considered an evolution problem for a reaction-diffusion process when the reaction function is of standard KPP-type, but experiences a cut-off in the reaction rate below the normalised cut-off concentration $u_c \in (0,1)$. We have formulated this evolution problem in terms of the moving boundary initial-boundary value problem QIVP. In the companion paper we considered PTW solutions $U_T(y) = u(y,t)$ to QIVP. In this paper we concentrated on examining whether a PTW evolves in the large-time solution to QIVP and when this is found to be the case, determining the rate of convergence of the solution to the PTW. Key to this study is $y = x - s(t) = 0$ which represents the location of the moving boundary where $u = u_c$. We used the method of matched asymptotic coordinate expansions to develop the detailed asymptotic structure of the solution to QIVP in the small-time ($t = o(1)$), intermediate-time ($t = O(1)$) and large-time ($t \to \infty$) regimes for arbitrary cut-off $u_c \in (0,1)$. We first determined that the asymptotic structure of $u(y,t)$ in the small-time regime has two regions in $y < 0$, and two regions in $y > 0$ and is given by expansions (44), (35), (36) and (49). The two-term asymptotic expression (37) for the function $s(t)$ can be derived from the inner left and inner right regions, where $y = o(1)^-$ and $y = o(1)^+$, in addition to the leading order boundary conditions. This reveals that as $t \to 0^+$, $\dot{s}(t)$ has an integrable singularity which depends on the cut-off u_c. Here $\dot{s}(t) \to +\infty$ when $u_c \in (0, \frac{1}{2})$, whilst, $\dot{s}(t) \to -\infty$ when $u_c \in (\frac{1}{2}, 1)$ with a transition case where $\dot{s}(t) \to 0$ when $u_c = \frac{1}{2}$. We then employed the asymptotic structure of $u(y,t)$ in the outer left and right regions, where $y = O(1)^-$ and $y = O(1)^+$, for $t = o(1)$ to determine the asymptotic structures of $u(y,t)$ when $|y| \to \infty$ for $t = O(1)$. The latter is key to deriving the asymptotic structure of $u(y,t)$ as $t \to \infty$ which consists of two principal regions in $y < 0$ and two principal regions in $y > 0$ and given by the asymptotic expressions (95), (104), (105), (112), (117), (123), (139), (140), (151), (156), with the asymptotic structure of $s(t)$ as $t \to \infty$ being determined simultaneously and given by the asymptotic expression (174). This systematic approach allows to establish that the solution to QIVP converges to the PTW solution as $t \to \infty$ at a rate that is linearly exponentially small in t with the exact form dependent on the particular underlying KPP-type reaction function $f(u)$ and the cut-off value $u_c \in (0,1)$. Thus, introducing an arbitrary cut-off into the reaction significantly modifies the rate of convergence of the large-time solution onto the PTW (from an algebraic to an exponential rate). Consequently, the presence of a cut-off significantly shortens the time for the solution to QIVP to converge to the PTW. We anticipate that the approach developed in this paper will be readily adaptable to corresponding problems, when the KPP-type cut-off reaction function is replaced by a broader class of cut-off reaction functions.

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A Numerical scheme

We approximate $u(y,t)$ and $s(t)$ by piecewise linear functions $u_d(y_i, t_j)$ and $s_d(t_j)$, defined on evenly spaced space and time grids given by $\{y_i = -M + i\Delta y\}_{i=0}^{I+\mathcal{I}}$ and $\{t_j = j\Delta t\}_{j=0}^{J}$ with $y_I = 0$ and $t_J = T$. We use explicit finite differences to approximate (8a) by

$$U_i^{j+1} - U_i^j = \mu\left(U_{i+1}^j - 2U_i^j + U_{i-1}^j\right) + \nu\left(S^{j+1} - S^j\right)\left(U_{i+1}^j - U_{i-1}^j\right) + \Delta t f_c(U_i^j), \qquad (190)$$

for $i = 2, \ldots, I-1, I+1, \ldots, I+\mathcal{I}-1$, $j = 1, \ldots J$, $\mu = \Delta t/\Delta y^2$ and $\nu = 1/(2\Delta y)$, where $U_i^j = u_d(y_i, t_j)$ and $S_i^j = s_d(t_j)$ respectively approximate $u(y_i, t_j)$ and $s(t_j)$. We then use (8d), (8e) and (8f) to set

$$U_0^j = 1, \quad U_{2I}^j = 0, \quad U_I^j = u_c, \quad U_{I+1}^j + U_{I-1}^j = 2u_c, \quad \text{for } j = 1, \ldots J. \tag{191}$$

We solve the resulting sparse linear algebraic system of equations for the unknowns U_i^j and S^j with $i = 2, \ldots, I-1, I+1, \ldots, I+\mathcal{I}-1$ and $j = 1, \ldots, J$ in an evolutionary manner starting from

$$\{U_i^0\}_{i=1}^{I-1} = 1, \quad \{U_i^0\}_{i=I}^{I+\mathcal{I}} = 0, \quad S^0 = 0, \tag{192}$$

corresponding to the initial conditions (8c) and (8g). We choose $\Delta y = 5 \times 10^{-3}$ and $\Delta t = 0.4\Delta y^2$ to ensure the stability of the explicit method. We take I and $\mathcal{I}$ sufficiently large to ensure that any error arising from truncating the right-hand and left-hand boundary does not affect the solution in the interior. In practice, we have found that choosing I and $\mathcal{I}$ so that $e^{\lambda_+(v^*(u_c))y_0}, e^{-v^*(u_c)y_{I+\mathcal{I}}} \lesssim 5 \times 10^{-5}$ (corresponding to the asymptotic behaviour of the PTW as described by equation (81)) provides reasonable accuracy. Comparison with results obtained for a spatial resolution of $\Delta y = 10^{-3}$ resulted in a less than 0.5% difference in $u_d(y_i, t_j)$ and $s_d(t_j)$.

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