An infinite-dimensional 2-generated primitive axial algebra of Monster type

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AN INFINITE-DIMENSIONAL 2-GENERATED PRIMITIVE AXIAL ALGEBRA OF MONSTER TYPE

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Abstract. Rehren proved in [13, 14] that a primitive 2-generated axial algebra of Monster type (α, β), over a field of characteristic other than 2, has dimension at most eight if $\alpha \notin \{2\beta, 4\beta\}$. In this note we construct an infinite-dimensional 2-generated primitive axial algebra of Monster type $(2, \frac{1}{2})$ over an arbitrary field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2, 3$. This shows that the second special case, $\alpha = 4\beta$, is a true exception to Rehren's bound.

1. Introduction

Let $\mathbb{F}$ be a field and let $\mathcal{S}$ be a finite subset of $\mathbb{F}$ with $1 \in \mathcal{S}$. A *fusion law* on $\mathcal{S}$ is a map

$$\star\colon \mathcal{S} \times \mathcal{S} \to 2^{\mathcal{S}}.$$

An *axial algebra* over $\mathbb{F}$ with *spectrum* $\mathcal{S}$ and fusion law $\star$ is a commutative non-associative $\mathbb{F}$-algebra V generated by a set $\mathcal{A}$ of nonzero idempotents (called *axes*) such that, for each $a \in \mathcal{A}$,

(Ax1) $ad(a) : v \mapsto av$ is a semisimple endomorphism of V with spectrum contained in $\mathcal{S}$;

(Ax2) for every $\lambda, \mu \in \mathcal{S}$, the product of a λ-eigenvector and a μ-eigenvector of ad_a is the sum of δ-eigenvectors, for $\delta \in \lambda \star \mu$.

Furthermore, V is called *primitive* if

(Ax3) $V_1 = \langle a \rangle$.

An axial algebra over $\mathbb{F}$ is said to be *of Monster type* (α, β) if it satisfies the fusion law $\mathcal{M}(\alpha, \beta)$ given in Table 1, with $\alpha, \beta \in \mathbb{F} \setminus \{0, 1\}$, with $\alpha \neq \beta$.

Axial algebras were introduced by Hall, Rehren and Shpectorov [7, 8] in order to axiomatise some key features of certain classes of algebras, such as the weight-2 components of OZ-type vertex operator algebras, Jordan algebras, Matsuo algebras and Majorana algebras (see [9] and the introductions of [7], [14] and [4]). In this paper we shall assume that the underlying field $\mathbb{F}$ has characteristic different from

$\star$	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1, 0$	β
β	β	β	β	$1, 0, \alpha$

Table 1. Fusion law $\mathcal{M}(\alpha, \beta)$

2. This case is of particular interest for finite group theorists, since most of the finite simple groups can be faithfully represented as groups generated by certain involutory automorphisms, called Miyamoto involutions [11], of such algebras. In particular, the Griess algebra (see [5]) is a real axial algebra of Monster type $(\frac{1}{4}, \frac{1}{32})$ and the Myiamoto involutions of this algebra (also called Majorana involutions) are precisely the involutions of type $2A$ in the Monster, i.e. those whose centraliser in the Monster is the double cover of the Baby Monster.

The classification of 2-generated axial algebras has a fundamental rôle in the development of the theory of axial algebras. In a pioneering work [12], Norton classified subalgebras of the Griess algebra generated by two axes. Norton showed that there are exactly nine isomorphism classes of such subalgebras, corresponding to the nine conjugacy classes of dihedral subgroups of the Monster generated by two involutions of type $2A$. These algebras have been proven to be, up to isomorphisms, the only 2-generated primitive real axial algebras of Monster type $(\frac{1}{4}, \frac{1}{32})$ and are now known as Norton-Sakuma algebras ([15], [10], [7], [2]). In the minimal non-associative case of axial algebras of Jordan type η, the classification has been obtained by Hall, Rehren and Shpectorov in [8]. Note that axial algebras of Jordan type η are also axial algebras of Monster type (α, β) when $\eta \in \{\alpha, \beta\}$. In [13] and [14], Rehren started a systematic study of axial algebras of Monster type (α, β). In particular, he showed that, when $\alpha \notin \{2\beta, 4\beta\}$ every 2-generated primitive axial algebra of Monster type (α, β) has dimension at most 8.

This note is part of a project of the authors aimed at classifying all 2-generated primitive axial algebras of Monster type. In particular, in [2] and [3], the authors extend Rehren's result showing that, if $(\alpha, \beta) \neq (2, \frac{1}{2})$, then any symmetric 2-generated primitive axial algebra of Monster type (α, β) has dimension at most 8 (*symmetric* means that the map that swaps the generating axes extends to an algebra automorphism). Here we prove that the case $(\alpha, \beta) = (2, \frac{1}{2})$ is indeed an exception.

Theorem 1.1. *For every field* $\mathbb{F}$ *of characteristic different from* 2 *and* 3*, there exists an infinite-dimensional* 2*-generated primitive axial algebra of Monster type* $(2, \frac{1}{2})$ *over* $\mathbb{F}$*.*

2. The algebra HW

Let $\mathbb{F}$ be a field of characteristic other than 2. Let HW be the infinite-dimensional $\mathbb{F}$-vector space with basis $\{a_i, \sigma_j \mid i \in \mathbb{Z}, j \in \mathbb{Z}_+\}$,

$$HW := \bigoplus_{i\in\mathbb{Z}} \mathbb{F}a_i \oplus \bigoplus_{j\in\mathbb{Z}_+} \mathbb{F}\sigma_j.$$

Set $\sigma_0 = 0$. Define a commutative non-associative product on HW by linearly extending the following values on the basis elements:

(HW1) $a_i a_j := \frac{1}{2}(a_i + a_j) + \sigma_{|i-j|}$;

(HW2) $a_i \sigma_j := -\frac{3}{4}a_i + \frac{3}{8}(a_{i-j} + a_{i+j}) + \frac{3}{2}\sigma_j$;

(HW3) $\sigma_i \sigma_j := \frac{3}{4}(\sigma_i + \sigma_j) - \frac{3}{8}(\sigma_{|i-j|} + \sigma_{i+j})$.

In particular, $a_i^2 = \frac{1}{2}(a_i + a_i) + \sigma_0 = a_i$, so each a_i is an idempotent.

We call HW the *highwater algebra* because it was discovered in Venice during the disastrous floods in November 2019. In what follows, double angular brackets denote algebra generation while single brackets denote linear span.

Theorem 2.1. *If* $\mathrm{char}(\mathbb{F}) \neq 3$ *then* $HW = \langle\langle a_0, a_1 \rangle\rangle$ *is a primitive axial algebra of Monster type* $(2, \frac{1}{2})$.

Manifestly, this result implies Theorem 1.1.

If $\mathrm{char}(\mathbb{F}) = 3$ then $2 = \frac{1}{2}$ and so the concept of an algebra of Monster type $(2, \frac{1}{2})$ is not defined. However, the four-term decomposition typical for algebras of Monster type still exists, and so HW in characteristic 3 is an example of an axial decomposition algebra as defined in [1]. We also prove that HW in this case is a Jordan algebra. Note, however, that because a lot of structure constants in HW become zero in characteristic 3, HW is no longer generated by a_0 and a_1. In fact, every pair of distinct axes a_i, a_j generates a 3-dimensional subalgebra, linearly spanned by a_i, a_j, and $\sigma_{|i-j|}$, and isomorphic to the algebra $Cl^{00}(\mathbb{F}^2, b_2)$ (see [8, Theorem (1.1)]).

We start with a number of observations concerning the properties of HW. First of all, we show that it is not a simple algebra. Consider the linear map $\lambda : HW \to \mathbb{F}$ defined on the basis of HW as follows: $\lambda(a_i) = 1$ for all $i \in \mathbb{Z}$ and $\lambda(\sigma_j) = 0$ for all $j \in \mathbb{Z}_+$.

Lemma 2.2. *The map* λ *is a homomorphism of algebras.*

Proof. It suffices to show that λ is multiplicative, i.e., $\lambda(uv) = \lambda(u)\lambda(v)$ for all $u, v \in HW$. Since this equality is linear in both u and v, it suffices to check it for the elements of the basis. If $u = a_i$ and $v = a_j$ then $\lambda(a_i a_j) = \lambda(\frac{1}{2}(a_i + a_j) + \sigma_{|i-j|}) = \frac{1}{2} + \frac{1}{2} = 1 = 1 \cdot 1 = \lambda(a_i)\lambda(a_j)$. If $u = a_i$ and $v = \sigma_j$ then $\lambda(a_i \sigma_j)) = \lambda(-\frac{3}{4} a_i + \frac{3}{8}(a_{i-j} + a_{i+j}) + \frac{3}{2}\sigma_j) = -\frac{3}{4} + \frac{3}{8} + \frac{3}{8} = 0 = 1 \cdot 0 = \lambda(a_i)\lambda(\sigma_j)$. Finally, if $u = \sigma_i$ and $v = \sigma_j$ then $\lambda(\sigma_i \sigma_j)) = \lambda(\frac{3}{4}(\sigma_i + \sigma_j) - \frac{3}{8}(\sigma_{|i-j|} + \sigma_{i+j})) = 0 = \lambda(\sigma_i)\lambda(\sigma_j)$. So the equality holds in all cases and so λ is indeed an algebra homomorphism. □

Such a homomorphism is usually called a weight function and its existence shows that HW is a baric (or weighted) algebra. Since λ is a homomorphism, its kernel J is an ideal of codimension 1.

Using λ, we can also define a bilinear form on HW. Namely, for $u, v \in HW$, we set $(u, v) := \lambda(u)\lambda(v)$. It is immediate that this is a bilinear form; furthermore, it associates with the algebra product. Indeed, for $u, v, w \in HW$, we have $(uv, w) = \lambda(uv)\lambda(w) = \lambda(u)\lambda(v)\lambda(w) = \lambda(u)\lambda(vw) = (u, vw)$. In the theory of axial algebras such forms are called Frobenius forms. The form $(\cdot, \cdot)$ further satisfies the property that $(a_i, a_i) = \lambda(a_i)\lambda(a_i) = 1 \cdot 1 = 1$, which is often required in the definition of a Frobenius form.

The next observation to make is that HW is quite symmetric. Let D be the infinite dihedral group acting naturally on $\mathbb{Z}$. For $\rho \in D$, let ϕ_ρ be the linear map that fixes all σ_j and sends a_i to a_{i^ρ}. Then ϕ_ρ is an automorphism of HW and the map $\rho \mapsto \phi_\rho$ defines a faithful representation of D as an automorphism group of HW. When $\mathrm{char}(\mathbb{F}) \neq 3$, we claim that $Aut(HW) \cong D$. For this, we need the following observation.

Lemma 2.3. *If* $\mathrm{char}(\mathbb{F}) \neq 3$*, the elements* $\{a_i \mid i \in \mathbb{Z}\}$ *are the only nontrivial idempotents in* HW.

Proof. Let
$$e=\sum_{i\in\mathbb{Z}} r_i a_i + \sum_{j\in\mathbb{Z}_+} s_j\sigma_j$$
be a nontrivial idempotent of HW. Suppose by contradiction that e is not one of the a_i's. First, suppose that e involves both nonzero terms a_i and nonzero terms σ_j. Select the maximum i with $r_i\neq 0$ and, similarly, the maximum j with $s_j\neq 0$. Then, by (HW2), $e=e^2$ involves a_{i+j}, which is a contradiction. Thus, e cannot contain both a_i's and σ_j's. If e contains no a_i's, select the maximum j with $s_j\neq 0$. Then it follows from (HW3) that $e=e^2$ involves σ_{2j}, which is again a contradiction. It remains to consider the case where e only involves a_i's. Clearly, it must involve two different a_i. Say, let i be maximum such that $r_i\neq 0$ and i' be minimum such that $r_{i'}\neq 0$. Then $i\neq i'$ and (HW1) yields that $e=e^2$ involves $\sigma_{i-i'}$; a contradiction. □

Proposition 2.4. *If* $\mathrm{char}(\mathbb{F})\neq 3$*, then* $Aut(HW)\cong D$.

Proof. Let $\varphi\in \mathrm{Aut}(HW)$. By Lemma 2.3, φ induces a permutation on the set $\{a_i \mid i\in\mathbb{Z}\}$ and consequently, by $(HW1)$, φ induces a permutation on the set $\{\sigma_j \mid j\in\mathbb{Z}_+\}$. Now observe that the action on the latter set has to be trivial. Indeed, for $\sigma\in\{\sigma_j\mid j\in\mathbb{Z}_+\}$, define the graph Γ_σ with vertices $\{a_i,\mid i\in\mathbb{Z}\}$, where a_i is adjacent to a_k if and only if $\sigma=a_ia_k-\frac{1}{2}(a_i+a_k)$. It is easy to see that if $\sigma=\sigma_j$ then Γ_σ has exactly j connected components. On the other hand, if $\sigma^\varphi=\sigma'$ then $\Gamma_\sigma^\varphi=\Gamma_{\sigma'}$. Since no two graphs Γ_σ are isomorphic, we conclude that indeed φ fixes all σ_j.

In particular, the entire group $\mathrm{Aut}(HW)$ fixes σ_1, and so it acts on the infinite string graph Γ_{σ_1}. Since this action is faithful, we conclude that $\mathrm{Aut}(HW)\cong D$. □

We are aiming to show that the a_i are axes satisfying the fusion law $\mathcal{M}(2,\frac{1}{2})$. Since D is transitive on the a_i's, it suffices to check this for just one of them, say $a=a_0$. We start with the eigenvalues and eigenspaces of ad_a.

Select $j\in\mathbb{Z}_+$ and set $U=\langle a,a_{-j},a_j,\sigma_j\rangle$. It is immediate to see that U is invariant under ad_a, the latter being represented by the following matrix:
$$\begin{pmatrix} 1 & 0 & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & 0 & 1\\ \frac{1}{2} & 0 & \frac{1}{2} & 1\\ -\frac{3}{4} & \frac{3}{8} & \frac{3}{8} & \frac{3}{2}\end{pmatrix}$$
This has characteristic polynomial $x^4-\frac{7}{2}x^3+\frac{7}{2}x^2-x=(x-1)x(x-2)(x-\frac{1}{2})$ and eigenspaces $U_1=\langle a\rangle$, $U_0=\langle u_j\rangle$, $U_2=\langle v_j\rangle$, $U_{\frac{1}{2}}=\langle w_j\rangle$, where
$$\begin{aligned} u_j &:= 6a-3(a_{-j}+a_j)+4\sigma_j\\ v_j &:= 2a-(a_{-j}+a_j)-4\sigma_j \qquad (1)\\ w_j &:= a_{-j}-a_j.\end{aligned}$$
The only exception to this statement arises when $\mathrm{char}(\mathbb{F})=3$. In this case, $2=\frac{1}{2}$ and so U_2 and $U_{\frac{1}{2}}$ merge into a single 2-dimensional eigenspace $U_2=U_{\frac{1}{2}}=\langle v_j,w_j\rangle$.

In all cases, we can write that $U=\langle a,u_j,v_j,w_j\rangle$, that is, a_{-j}, a_j and σ_j can be expressed via these vectors. From this we deduce the following.

Lemma 2.5. *The adjoint map* ad_a *is semisimple on* HW *and*

(a) *if* $\mathrm{char}(\mathbb{F}) \neq 3$ *then the spectrum of* ad_a *is* $\{1, 0, 2, \frac{1}{2}\}$ *and the eigenspaces are* $HW_1 = \langle a \rangle$, $HW_0 = \langle u_j \mid j \in \mathbb{Z}_+ \rangle$, $HW_2 = \langle v_j \mid j \in \mathbb{Z}_+ \rangle$, *and* $HW_{\frac{1}{2}} = \langle w_j \mid j \in \mathbb{Z}_+ \rangle$*;*

(b) *if* $\mathrm{char}(\mathbb{F}) = 3$ *then the spectrum is* $\{1, 0, \frac{1}{2}\}$ *and the eigenspaces are* $HW_1 = \langle a \rangle$, $HW_0 = \langle u_j \mid j \in \mathbb{Z}_+ \rangle$, *and* $HW_{\frac{1}{2}} = \langle v_j, w_j \mid j \in \mathbb{Z}_+ \rangle$.

In order to avoid the complication arising in characteristic 3, we will use the notation $HW_u := \langle u_j \mid j \in \mathbb{Z}_+ \rangle$, $HW_v := \langle v_j \mid j \in \mathbb{Z}_+ \rangle$, and $HW_w = \langle w_j \mid j \in \mathbb{Z}_+ \rangle$ calling these subspaces the u-, v-, and w-parts of HW, respectively. A similar terminology will be used for sums of these subspaces. Thus, in all characteristics we have the decomposition

$$HW = \langle a \rangle \oplus HW_u \oplus HW_v \oplus HW_w.$$

Let us relate this decomposition to the ideal J.

Lemma 2.6. $J = HW_u \oplus HW_v \oplus HW_w$.

Proof. By inspection, the weight function is zero on $HW_u \oplus HW_v \oplus HW_w$, so this sum is contained in J. Now the equality is forced because both J and $HW_u \oplus HW_v \oplus HW_w$ have codimension 1 in HW. □

So J is the uvw-part of HW.

The stabiliser D_a of a in D has order 2 and it is generated by the involution τ sending every a_i to a_{-i} (and fixing every σ_j). From this, it is immediate to see that the following holds.

Lemma 2.7. *The involution* τ *acts as identity on* $\langle a \rangle \oplus HW_u \oplus HW_v$ *and as minus identity on* HW_w*. In particular,* $HW_\tau = \langle a \rangle \oplus HW_u \oplus HW_v$ *is the fixed subalgebra of* τ*.*

We find a further subalgebra by intersecting HW_τ with the ideal J. Namely, we set $V := HW_\tau \cap J = HW_u \oplus HW_v$, the uv-part of HW. Clearly, V is an ideal of HW_τ.

The vectors u_j and v_j form a basis of V. We will also use a second basis. For $j \in \mathbb{Z}_+$, let $c_j := 2a - (a_{-j} + a_j)$. These we combine with the elements σ_j.

Let us, first of all, record the following, see the definition of u_j and v_j in (1).

Lemma 2.8. *For all* $j \in \mathbb{Z}_+$*, we have* $u_j = 3c_j + 4\sigma_j$ *and* $v_j = c_j - 4\sigma_j$*.*

It is also easy to express c_j and σ_j via u_j and v_j. This means that the set of all vectors c_j and σ_j is a basis of V. In the following lemma we compute the products for this basis. Note that we only need to compute the products $c_i c_j$ and $c_i \sigma_j$, because $\sigma_i \sigma_j$ are given in (HW3). It will be convenient to use the following notation: for $i, j \in \mathbb{Z}_+$, we let $c_{i,j} := -2c_i - 2c_j + c_{|i-j|} + c_{i+j}$ and $\sigma_{i,j} := -2\sigma_i - 2\sigma_j + \sigma_{|i-j|} + \sigma_{i+j}$. For example, (HW3) can now be restated as: $\sigma_i \sigma_j = -\frac{3}{8}\sigma_{i,j}$.

Lemma 2.9. *For* $i, j \in \mathbb{Z}_+$*, we have that* $c_i c_j = 2\sigma_{i,j}$ *and* $c_i \sigma_j = \frac{3}{8} c_{i,j}$*.*

Proof. Using (HW1), $c_i c_j = (2a - (a_{-i} + a_i))(2a - (a_{-j} + a_j)) = 4a - 2(\frac{1}{2}a_{-i} + \frac{1}{2}a + \sigma_i + \frac{1}{2}a_i + \frac{1}{2}a + \sigma_i) - 2(\frac{1}{2}a + \frac{1}{2}a_{-j} + \sigma_j + \frac{1}{2}a + \frac{1}{2}a_j + \sigma_j) + (\frac{1}{2}a_{-i} + \frac{1}{2}a_{-j} + \sigma_{|i-j|} + \frac{1}{2}a_i + \frac{1}{2}a_{-j} + \sigma_{i+j} + \frac{1}{2}a_{-i} + \frac{1}{2}a_j + \sigma_{i+j} + \frac{1}{2}a_i + \frac{1}{2}a_j + \sigma_{|i-j|})$. Here all a, $a_{\pm i}$, $a_{\pm j}$ magically cancel yielding the first claim.

Similarly, using (HW2), $c_i\sigma_j = (2a - (a_{-i} + a_i))\sigma_j = 2(-\frac{3}{4}a + \frac{3}{8}(a_{-j} + a_j) + \frac{3}{2}\sigma_j) - (-\frac{3}{4}a_{-i} + \frac{3}{8}(a_{-i-j} + a_{-i+j}) + \frac{3}{2}\sigma_j - \frac{3}{4}a_i + \frac{3}{8}(a_{i-j} + a_{i+j}) + \frac{3}{2}\sigma_j)$. Here the σ_j cancel and the second claim follows after the terms are rearranged. □

Manifestly, all these products have a very uniform structure. Note that both $c_{i,j}$ and $\sigma_{i,j}$ are symmetric in i and j. In particular, it follows that $c_i\sigma_j = \frac{3}{8}c_{i,j} = \frac{3}{8}c_{j,i} = c_j\sigma_i$. Another interesting consequence of this lemma is that the involution ρ negating all c_j and fixing all σ_j is an automorphism of V. We will however need another involution, and for this we need to compute products for the other basis, the vectors u_j and v_j.

Similarly to the above, we introduce $u_{i,j} := -2u_i - 2u_j + u_{|i-j|} + u_{i+j}$ and $v_{i,j} := -2v_i - 2v_j + v_{|i-j|} + v_{i+j}$. Lemma 2.8 gives us the following.

Lemma 2.10. *For all $i, j \in \mathbb{Z}_+$, we have $u_{i,j} = 3c_{i,j} + 4\sigma_{i,j}$ and $v_{i,j} = c_{i,j} - 4\sigma_{i,j}$.*

Now we compute the products of the vectors u_j and v_j.

Lemma 2.11. *For all $i, j \in \mathbb{Z}_+$, we have that $u_iu_j = 3u_{i,j}$, $u_iv_j = -3v_{i,j}$, and $v_iv_j = -u_{i,j}$.*

Proof. By Lemma 2.8, $u_iu_j = (3c_i + 4\sigma_i)(3c_j + 4\sigma_j) = 9c_ic_i + 12c_i\sigma_j + 12\sigma_ic_j + 16\sigma_i\sigma_j$. Using Lemma 2.9, this is $9(2\sigma_{i,j}) + 12(\frac{3}{8}c_{i,j}) + 12(\frac{3}{8}c_{i,j}) + 16(-\frac{3}{8}\sigma_{i,j}) = 9c_{i,j} + 12\sigma_{i,j} = 3u_{i,j}$. For the last step, we used Lemma 2.10.

Similarly, $u_iv_j = (3c_i + 4\sigma_i)(c_j - 4\sigma_j) = 3c_ic_j + 4\sigma_ic_j - 12c_j\sigma_j - 16\sigma_i\sigma_j = 3(2\sigma_{i,j}) + 4(\frac{3}{8}c_{i,j}) - 12(\frac{3}{8}c_{i,j}) - 16(-\frac{3}{8}\sigma_{i,j}) = -3c_{i,j} + 12\sigma_{i,j} = -3v_{i,j}$ and $v_iv_j = (c_i - 4\sigma_i)(c_j - 4\sigma_j) = c_ic_j - 4\sigma_ic_j - 4c_i\sigma_j + 16\sigma_i\sigma_j = 2\sigma_{i,j} - 4(\frac{3}{8}c_{i,j}) - 4(\frac{3}{8}c_{i,j}) + 16(-\frac{3}{8}\sigma_{i,j}) = -3c_{i,j} - 4\sigma_{i,j} = -u_{i,j}$. □

Just like Lemma 2.9 implied that V admits an involution ρ, this lemma gives us that the involution θ fixing every u_j and negating every v_j is an automorphism of V.

We are now ready to establish the fusion law for the axis a.

Proof of Theorem 2.1. Recall that, when $\mathrm{char}(\mathbb{F}) \neq 3$, we have that $\langle a\rangle = HW_1$, $HW_u = HW_0$, $HW_v = HW_2$, and $HW_w = HW_{\frac{1}{2}}$ are the eigenspaces of ad_a. So we need to see how these subspaces behave under the product in HW.

First of all, clearly, $\langle a\rangle HW_u = 0$, $\langle a\rangle HW_v = HW_v$ and $\langle a\rangle HW_w = HW_w$. Now we focus on the products of the u-, v-, and w-parts, which are all contained in J, and so their products are also contained in J, avoiding $\langle a\rangle$. Furthermore, the involution τ induces a grading on J, under which $V = HW_u \oplus HW_v$ is the even part and H_w is the odd part. Hence HW_uHW_w and HW_vHW_w are contained in HW_w, while $HW_wHW_w \subseteq V = HW_u \oplus HW_v$. It remains to check the products within V. Similarly, the involution θ induces a grading on V, under which HW_u is even and HW_v is odd. Hence $HW_uHW_u \subseteq HW_u$, $HW_uHW_v \subseteq HW_v$, and $HW_vHW_v \subseteq HW_u$. (Of course, these three inclusions can be seen directly from Lemma 2.11.) Hence, when $\mathrm{char}(\mathbb{F}) \neq 3$, a satisfies the fusion law of Table 2, which is slightly stricter than the fusion law $\mathcal{M}(2, \frac{1}{2})$.

Since D is transitive on the a_i, all of them are axes satisfying the law in Table 2. To complete the proof of Theorem 2.1, it remains to show that $HW = \langle\langle a_0, a_1\rangle\rangle$. Let $H := \langle\langle a_0, a_1\rangle\rangle$. Note that $\sigma_1 = a_0a_1 - \frac{1}{2}(a_0 + a_1) \in H$. Also, $\frac{3}{8}a_{-1} = a_0\sigma_1 + \frac{3}{4}a_0 - \frac{3}{8}a_1 - \frac{3}{2}\sigma_1 \in H$. Assuming that $\mathrm{char}(\mathbb{F}) \neq 3$, this gives us that

$\star$	1	0	2	$\frac{1}{2}$
1	1		2	$\frac{1}{2}$
0		0	2	$\frac{1}{2}$
2	2	2	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$0, 2$

TABLE 2. Fusion law for HW

$\star$	1	0	$\frac{1}{2}$
1	1		$\frac{1}{2}$
0			$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0

TABLE 3. Fusion law for HW in characteristic 3

$a_{-1} \in H$. Clearly, $= \langle\langle a_0, a_1 \rangle\rangle$ is invariant under the involution $\pi \in D$ switching a_0 and a_1. Now we also see that $H = \langle\langle a_{-1}, a_0, a_1 \rangle\rangle$ is invariant under the involution $\tau \in D_a$. Since $D = \langle \tau, \pi \rangle$, this makes H invariant under all of D, and so H contains all axes a_i. Clearly, this means that $H = HW$. This completes the proof of Theorem 2.1. □

Let us discuss what happens when $\mathrm{char}(\mathbb{F}) = 3$. The fusion that we established for the decomposition $HW = \langle a \rangle \oplus HW_u \oplus HW_v \oplus HW_w$ remains true, but it becomes even stricter because so many coefficients are multiples of 3. For example, $HW_u HW_u = 0 = HW_u HW_v$. On the other hand, $2 = \frac{1}{2}$ in characteristic 3, so we need to merge HW_v and HW_w into a single eigenspace $HW_{\frac{1}{2}} = HW_v \oplus HW_w$. Let us see what fusion we get inside this eigenspace. We already know that $HW_v HW_v \subseteq HW_u$.

Lemma 2.12. *If* $\mathrm{char}(\mathbb{F}) = 3$ *then* $HW_v HW_w = 0$ *and* $HW_w HW_w \subseteq HW_u$.

Proof. Indeed, for $i, j \in \mathbb{Z}_+$, $v_i w_j = (2a - (a_{-i} + a_i) - 4\sigma_i)(a_{-j} - a_j)$. Taking into account that $a w_j = \frac{1}{2} w_j$ and that $\sigma_i w_h = 0$ when $\mathrm{char}(\mathbb{F}) = 3$ (see (HW2)), we obtain that $v_i w_j = (a_{-j} - a_j) - (a_{-i} + a_i)(a_{-j} - a_j) = (a_{-j} - a_j) - (\frac{1}{2}(a_{-i} + a_{-j}) + \sigma_{|i-j|}) + (\frac{1}{2}(a_i + a_{-j}) + \sigma_{i+j}) - (\frac{1}{2}(a_{-i} + a_j) + \sigma_{i+j}) + (\frac{1}{2}(a_i + a_j) + \sigma_{|i-j|}) = 0$, because everything cancels here.

Also, $w_i w_j = (a_{-i} - a_i)(a_{-j} - a_j) = (\frac{1}{2}(a_{-i} + a_{-j}) + \sigma_{|i-j|}) - (\frac{1}{2}(a_i + a_{-j}) + \sigma_{i+j}) - (\frac{1}{2}(a_{-i} + a_j) + \sigma_{i+j}) + (\frac{1}{2}(a_i + a_j) + \sigma_{|i-j|}) = 2\sigma_{|i-j|} - 2\sigma_{i+j} = \frac{1}{2} u_{|i-j|} - \frac{1}{2} u_{i+j}$. The last equality holds because $u_k = 4\sigma_k$ when $\mathrm{char}(\mathbb{F}) = 3$ (see (1)). □

So we can see now that in characteristic 3 the axes a_i in HW satisfy the fusion law in Table 3. In particular, HW is an algebra of Jordan type $\frac{1}{2}$. Let us prove that, in fact, HW is a Jordan algebra when $\mathrm{char}(\mathbb{F}) = 3$. For this we need to verify the Jordan identity $x(yx^2) = (xy)x^2$ for all $x, y \in HW$. For an element

$$x = \sum_{i \in \mathbb{Z}} r_i a_i + \sum_{j \in \mathbb{Z}_+} s_j \sigma_j \in HW,$$

we call $a(x) := \sum_{i\in\mathbb{Z}} r_i a_i$ the a-part of x and we call $\sigma(x) := \sum_{j\in\mathbb{Z}_+} s_j\sigma_j$ the σ-part of x. According to (HW2) and (HW3), when $\mathrm{char}(\mathbb{F}) = 3$, we have $\sigma_j HW = 0$ for all $j \in \mathbb{Z}_+$. In particular, we have that $xy = a(x)a(y)$.

Lemma 2.13. *If $x, y \in HW$, with $x = a(x)$ and $y = a(y)$, then $a(xy) = \frac{\lambda(y)}{2}x + \frac{\lambda(x)}{2}y$.*

Proof. Indeed, let $x = \sum_{i\in\mathbb{Z}} r_i a_i$ and $y = \sum_{i\in\mathbb{Z}} t_i a_i$. Note that $\lambda(x) = \sum_{i\in Z} r_i$ and $\lambda(y) = \sum_{i\in\mathbb{Z}} t_i$. Now we have from (HW1) that

$$\begin{aligned} a(xy) &= \sum_{i,j\in\mathbb{Z}} r_i t_j \frac{1}{2}(a_i + a_j) \\ &= \sum_{i,j\in\mathbb{Z}} r_i t_j \frac{1}{2} a_i + \sum_{i,j\in\mathbb{Z}} r_i t_j \frac{1}{2} a_j \\ &= \frac{1}{2}(\sum_{j\in\mathbb{Z}} t_j)(\sum_{i\in\mathbb{Z}} r_i a_i) + \frac{1}{2}(\sum_{i\in\mathbb{Z}} r_i)(\sum_{j\in\mathbb{Z}} t_j a_j) \\ &= \frac{\lambda(y)}{2}x + \frac{\lambda(x)}{2}y. \end{aligned}$$

□

In particular, this lemma implies that $a(x^2) = \lambda(x)a(x)$ for all $x \in HW$.

Theorem 2.14. *If $\mathrm{char}(\mathbb{F}) = 3$ then HW is a Jordan algebra.*

Proof. For $x, y \in HW$, we have

$$x(yx^2) = a(x)(a(y)a(x^2)) = a(x)(a(y)\lambda(x)a(x)) = \lambda(x)a(x)(a(y)a(x)) =$$
$$(a(x)a(y))\lambda(x)a(x) = (a(x)a(y))a(x^2) = (xy)x^2.$$

Note that we used that HW is commutative. □

Note also that with the same argument used for Theorem 2.14 one can prove the following more general result.

Proposition 2.15. *Every commutative baric algebra B, with weight function ω, such that*

(a) *$B = A \oplus I$, where A is a subspace and I is an ideal,*
(b) *$IB = 0$, and*
(c) *for all $a, b \in A$, $ab - \frac{1}{2}(\omega(b)a + \omega(a)b) \in I$,*

is a Jordan algebra.

Conversely, take arbitrary vector spaces A and I over $\mathbb{F}$ (of characteristic not 2), an arbitrary linear map $\omega : A \to \mathbb{F}$, and an arbitrary symmetric bilinear map $\sigma : A \times A \to I$. Define the algebra product on $B = A \oplus I$ by $BI = 0$ and $ab := \frac{1}{2}(\omega(b)a + \omega(a)b) + \sigma(a, b)$ for $a, b \in A$. Then this algebra B satisfies the assumptions of Proposition 2.15 and hence B is a baric Jordan algebra. Note that ω will be the weight function of B if we extend it to B via $\omega(I) = 0$. (Note that this may not be the only choice for the weight function.)

Manifestly, HW in characteristic 3 can be obtained by this construction with $\omega = \lambda$.

Now that we have proved our main results, let us tie the loose ends.

Proposition 2.16. (a) *The form $(\cdot,\cdot)$ is, up to a scalar factor, the only Frobenius form on HW.*
(b) *Every proper ideal of HW is contained in J, which is the radical of HW.*

Proof. We refer to [6] for a detailed discussion of the radical of an axial algebra, projection form and projection graph.

The form $(\cdot,\cdot)$ that we introduced on HW is the projection form, because it is a Frobenius form satisfying $(a_i, a_i) = 1$ for all $i \in \mathbb{Z}$. The projection graph has all a_i as vertices with an edge between a_i and a_j, $i \neq j$, whenever $(a_i, a_j) \neq 0$. In fact, one can easily check that $(a_i, a_j) = 1$ for all $i, j \in \mathbb{Z}$, so the projection graph of HW is the complete graph. In particular, it is connected. It now follows from [6, Proposition 4.19] that HW has only one Frobenius form up to a scalar factor. The same connectivity property implies, by [6, Corollary 4.15], that every proper ideal of HW is contained in the radical of HW (the largest ideal not containing axes). Finally, by [6, Corollary 4.11], the radical of HW coincides with the radical of the projection form $(\cdot,\cdot)$, and that is J. □

Recall that we earlier introduced automorphisms ρ and θ of the subalgebra $V = HW_u \oplus HW_v$. In the final result of the section we indicate the connection between these two involutions.

Lemma 2.17. *The linear transformation $\psi : V \to V$ defined by:*

$$c_j \mapsto \frac{1}{2}v_j = \frac{1}{2}c_j - 2\sigma_j \text{ and } \sigma_j \mapsto -\frac{1}{8}u_j = -\frac{3}{8}c_i - \frac{1}{2}\sigma_j,$$

for $j \in \mathbb{Z}_+$, is an automorphism of V. Furthermore, $\rho^\psi = \theta$.

This follows from the formulae in Lemmas 2.9 and 2.11. Note that ψ is also an involution because the matrix

$$\begin{pmatrix} \frac{1}{2} & -2 \\ -\frac{3}{8} & -\frac{1}{2} \end{pmatrix}$$

squares to identity. This means that $\langle \rho, \psi \rangle \cong D_8$. Interestingly, the subalgebra V appears to have symmetries independent of the entire algebra HW, because the automorphisms from $\langle \rho, \psi \rangle$ do not seem to extend to HW.

3. Discussion

In this final section we pose some questions related to HW. Recall that it is known that every symmetric 2-generated primitive algebra of Monster type (α, β) over a field of characteristic other that 2 is at most 8-dimensional unless $(\alpha, \beta) = (2, \frac{1}{2})$.

Question 3.1. *Is HW the only infinite-dimensional primitive algebra of Monster type? Is it the only such algebra having dimension greater than 8?*

The answer to this question could depend on knowing all ideals of HW.

Question 3.2. *Does HW contain any nonzero ideals of finite dimension? Is it possible to classify all ideals in HW?*

Note that J is not the only proper nonzero ideal of HW. It can be shown that the ideal generated by all elements σ_j has codimension 1 in J (and hence codimension 2 in HW).

In addition to finding all ideals of HW, it is also interesting to find all of its subalgebras and their automorphisms.

Question 3.3. *Which is the full automorphism group of HW in characteristic 3? Which is the automorphism group of V? Which automorphisms of V extend to larger subalgebras of HW?*

Finally, the fact that HW is a baric algebra looks exciting. One can see easily that a baric algebra of Monster type satisfies the fusion law as in Table 2, but with 2 and $\frac{1}{2}$ substituted with arbitrary α and β.

Question 3.4. *Are there examples of baric algebras of Monster type (α, β) for other values of α and β? Is it possible to classify all such algebras?*

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