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# $L^p$-BOUNDEDNESS FOR THE HILBERT TRANSFORM AND MAXIMAL OPERATOR ALONG A CLASS OF NONCONVEX CURVES

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ABSTRACT. Some sufficient conditions on a real polynomial $P$ and a convex function $\gamma$ are given in order for the Hilbert transform and maximal operator along $(t, P(\gamma(t)))$ to be bounded on $L^p$, for all $p$ in $(1, \infty)$, with bounds independent of the coefficients of $P$. The same conclusion is shown to hold for the corresponding hypersurface in $\mathbb{R}^{d+1}$ ($d \geq 2$) under weaker hypotheses on $\gamma$.

## INTRODUCTION

Given an integer $d \geq 2$ and a map $\Gamma : \mathbb{R} \to \mathbb{R}^d$ we define operators $H$ and $\mathcal{M}$ by

$$Hf(x) := p.v. \int_{-\infty}^{\infty} f(x - \Gamma(t)) \, \frac{dt}{t},$$

$$\mathcal{M}f(x) := \sup_{h>0} h^{-1} \left| \int_0^h f(x - \Gamma(t)) \, dt \right|,$$

for appropriate functions $f$ on $\mathbb{R}^d$. We shall refer to $H$ as the (global) Hilbert transform along $\Gamma$ and $\mathcal{M}$ as the (global) maximal operator along $\Gamma$. The question of whether these operators are bounded on $L^p$ has received much attention in the last thirty years. In particular, for which $\Gamma$ and what range of $p$ can we achieve this? (Of course $\mathcal{M}$ is bounded on $L^\infty$, and so we choose to omit this triviality from subsequent theorems on $\mathcal{M}$). We begin with the case that $\Gamma$ is a 'polynomial curve' in $\mathbb{R}^d$. The following theorem is well known.

**Theorem 0.1.** ([S]) *Let* $\Gamma(t) = (P_1(t), \dots, P_d(t))$, *where* $P_1, \dots, P_d$ *are real polynomials on* $\mathbb{R}$. *Then* $H$ *and* $\mathcal{M}$ *are bounded on* $L^p$ *for all* $p \in (1, \infty)$, *with bounds independent of the coefficients of* $P_1, \dots, P_d$.

A somewhat related problem is the case when $\Gamma$ is of finite type, that is to say $\{\Gamma^{(k)}(0) : k \geq 1\}$ spans $\mathbb{R}^d$. Here we must consider the local versions $H_{loc}$ and $\mathcal{M}_{loc}$ of the operators $H$ and $\mathcal{M}$, where the integral defining $H$ is restricted to $(-1, 1)$ and the supremum in the definition of $\mathcal{M}$ is restricted to $h$ in $(0, 1)$.

**Theorem 0.2.** ([SW]) *If* $\Gamma$ *is of finite type then* $H_{loc}$ *and* $\mathcal{M}_{loc}$ *are bounded on* $L^p$ *for all* $p \in (1, \infty)$.

We may then ask what happens in the case that $\Gamma$ is not of finite type. This brings us to the simplest case of this kind, where we have $d = 2$, $\Gamma(t) = (t, \gamma(t))$ and $\gamma$ is allowed to be infinitely flat at the origin. One such (nonconvex) $\gamma$ was

constructed in [SW] for which $\mathcal{M}_{loc}$ is unbounded on $L^p$ for any $p \in (1, \infty)$. Despite this, positive results are possible for such flat curves when, in particular, we consider convex $\gamma$. If we restrict our attention to curves $\gamma$ satisfying:

$$(0.1) \qquad \gamma \in C^2(0, \infty), \text{ convex on } [0, \infty) \quad \text{and} \quad \gamma(0) = \gamma'(0) = 0,$$

and extend $\gamma$ to a function on $\mathbb{R}$ by stipulating that it must be either even or odd, then the following notions naturally arise.

**Definition 0.3.**    (1) A function $f : \mathbb{R} \to \mathbb{R}$ belongs to $\mathcal{C}_1$ if there exists $\lambda \in (1, \infty)$ such that for each $t > 0$ we have $f(\lambda t) \geq 2f(t)$. Such an $f$ is said to be doubling.

   (2) A differentiable function $f : \mathbb{R} \to \mathbb{R}$ belongs to $\mathcal{C}_2$ if there exists $\varepsilon_0 > 0$ such that for $t > 0$, $f'(t) \geq \varepsilon_0 f(t)/t$. Such an $f$ is said to be infinitesimally doubling, and if $f$ is nondecreasing on $(0, \infty)$ then $f \in \mathcal{C}_2$ implies $f \in \mathcal{C}_1$.

We shall also need the function $h$ defined for $t > 0$ by $h(t) := t\gamma'(t) - \gamma(t)$. Notice that because $\gamma$ is convex and $\gamma(0) = 0$ we get the important fact that

$$(0.2) \qquad t\gamma'(t) \geq \gamma(t) \text{ for all } t > 0$$

(and hence $h$ is nonnegative). We now state a series of known results in this setting.

**Theorem 0.4.** ([Ca et al]) *Suppose $\gamma$ is even and satisfies (0.1), and $p \in (1, \infty)$. Then $H$ is $L^p$ bounded if and only if $\gamma' \in \mathcal{C}_1$.*

The $L^2$ result in Theorem 0.4 was proved earlier in [NVWWe]. This is of course the end of the matter for $H$ when $\gamma$ is convex and even. In the odd case, the current situation is less satisfactory. We have:

**Theorem 0.5.** ([NVWWe]) *Suppose $\gamma$ is odd and satisfies (0.1). Then $H$ is $L^2$ bounded if and only if $h \in \mathcal{C}_1$.*

This theorem of course means that, for each $p \in (1, \infty)$, $h \in \mathcal{C}_1$ is a necessary condition for $H$ to be $L^p$ bounded. However, it was demonstrated in [CChVWWa] that this condition is far from sufficient. There they construct a $\gamma$ such that $h \in \mathcal{C}_1$ yet $H$ is unbounded on $L^p$ for any $p \in (1, \infty)$ not equal to 2. Some known sufficient conditions in the odd case are given in the following:

**Theorem 0.6.** *Suppose $\gamma$ is odd and satisfies (0.1), and $p \in (1, \infty)$.*
   (1) ([Ca et al]) *If $\gamma' \in \mathcal{C}_1$ then $H$ is $L^p$ bounded.*
   (2) ([CChVWWa]) *If $h \in \mathcal{C}_2$ then $H$ is $L^p$ bounded.*

For $\mathcal{M}$, a necessary and sufficient condition for $L^p$ boundedness in geometric terms is not known. It was demonstrated in [St] (see also [SeWWrZ]) that a convex $\gamma$ exists for which $\mathcal{M}$ is unbounded on $L^p$ for all $p \in (1, \infty)$. There is however an analogue of Theorem 0.6:

**Theorem 0.7.** *Suppose $\gamma$ satisfies (0.1) and $p \in (1, \infty)$.*
   (1) ([Ca et al]) *If $\gamma' \in \mathcal{C}_1$ then $\mathcal{M}$ is $L^p$ bounded.*
   (2) ([CChVWWa]) *If $h \in \mathcal{C}_2$ then $\mathcal{M}$ is $L^p$ bounded.*

*Remark.* The case where a convex curve on $[0, \infty)$ is extended to be either even or odd is encompassed by the notion of a biconvex balanced curve given in [CoRdF]. There it is shown that if the derivative of such a curve satisfies a doubling condition then, for all $p \in (1, \infty)$, we get $L^p$ boundedness of both $H$ and $\mathcal{M}$ (and also the associated maximal Hilbert transform).

We shall now present the main result of this short note.

**Theorem 0.8.** *Suppose $P$ is a real polynomial and $\gamma$ is convex on $[0, \infty)$, twice differentiable, either even or odd, $\gamma(0) = 0$, and $\gamma'(0) \geq 0$. If $\Gamma(t) = (t, P(\gamma(t)))$, $p \in (1, \infty)$, and either (1) $P'(0)$ is zero, or (2) $P'(0)$ is nonzero and $\gamma' \in \mathcal{C}_1$, then*

$$\|Hf\|_p \leq C\|f\|_p \quad and \quad \|\mathcal{M}f\|_p \leq C\|f\|_p.$$

*Moreover the constant $C$ depends only on $p$, $\gamma$, and the degree of $P$.*

*Remarks.* (1) By taking $\gamma(t) = t$ we recover a form of Theorem 0.1 since we can then suppose $P'(0) = 0$. Our proof does not require the 'lifting' technique used in [S] to prove Theorem 0.1. Also, taking $P(s) = s$ we recover Theorem 0.6(1), Theorem 0.7(1), and the sufficiency part of Theorem 0.4.

(2) Some examples of nonconvex curves were studied in [Wr], and later these were generalised somewhat through a technical theorem in [VWWr]. Although the class of curves in Theorem 0.8 falls within the scope of [VWWr], the bounds obtained from the technical theorem in [VWWr] depend on the coefficients of $P$. Furthermore, our proof is more direct in this setting.

We shall see that ideas in our proof of Theorem 0.8 can be used for certain hypersurfaces instead of curves. Specifically, if $d \geq 2$ and $\Gamma : \mathbb{R}^d \to \mathbb{R}^{d+1}$ parameterises a hypersurface, then we associate to this the corresponding Hilbert transform and maximal operator by

$$Hf(x) := p.v. \int_{\mathbb{R}^d} f(x - \Gamma(y))K(y)\, dy,$$

$$\mathcal{M}f(x) := \sup_{h>0} h^{-d} \left| \int_{|y| \in [0,h]} f(x - \Gamma(y))\, dy \right|,$$

where $K : \mathbb{R}^d \to \mathbb{R}$ is a Calderón-Zygmund kernel; that is $K$ is of class $C^\infty$ on $\mathbb{R}^d \setminus \{0\}$, $K(ty) = t^{-d}K(y)$ for each $t > 0$ and $y \in \mathbb{R}^d$, and $\int_{|y| \in [a,b]} K(y)\, dy = 0$ for each $0 < a < b$. Then we have the following theorem.

**Theorem 0.9.** *Suppose $P$ is a real polynomial and $\gamma$ is convex on $[0, \infty)$, twice differentiable, either even or odd, $\gamma(0) = 0$, and $\gamma'(0) \geq 0$. If $\Gamma(y) = (y, P(\gamma(|y|)))$ and $p \in (1, \infty)$ then*

$$\|Hf\|_p \leq C\|f\|_p \quad and \quad \|\mathcal{M}f\|_p \leq C\|f\|_p.$$

*Moreover the constant $C$ depends only on $p$, $d$, $\gamma$, and the degree of $P$.*

*Remark.* The case $P(s) = s$ was proved in [KWWrZ]. Notice how in this case the convexity of $\gamma$ suffices for $L^p$ boundedness, which is in stark contrast to the case $d = 1$ that we alluded to earlier.

*Notation.* For positive $A$ and $B$, $A \lesssim B$ and $B \gtrsim A$ mean $A \leq CB$, where $C$ is an absolute constant which may depend on $p$, $\gamma$, $d$, and the degree of $P$ but is independent of the coefficients of $P$. Also, $A \sim B$ denotes $A \lesssim B \lesssim A$.

*Overview.* In the next section we make a suitable decomposition of our operators based on key results concerning polynomials of one variable. The next section contains the fundamental results for the proof of Theorem 0.8. In the last section we prove Theorem 0.9.

## 1. Preliminaries and reductions

Let $P(s) = \sum_{k=1}^{n} p_k s^k$ be a real polynomial of degree $n$, where $n \geq 2$ (it is without loss of generality that we suppose $P(0) = 0$).

If $\lambda$ is the doubling constant for $(\gamma^j)'$ then define $\rho := \max\{3, \lambda\}$ and the 'dyadic' version $M$ of $\mathcal{M}$ by

$$Mf(x) = \sup_{k \in \mathbb{Z}} \rho^{-k} \left| \int_{[\rho^k, \rho^{k+1}]} f(x_1 - t, x_2 - P(\gamma(t))) \, dt \right|,$$

Since $\mathcal{M}f \lesssim M|f|$, we shall from now on deal with $M$.

We now discuss the decomposition of $(0, \infty)$ crucial to the proof of Theorem 0.8. The ideas here originated from work in [CRWr] (see also [FGWr]). First let $z_1, \ldots, z_n$ be the roots of $P$ ordered as $0 = |z_1| \leq |z_2| \leq \ldots \leq |z_n|$. Our decomposition will depend on $A \sim 1$, whose value we fix later. Firstly, we include $G_1 = (0, A^{-1}|z_2|]$. Then, for $j \in \{2, \ldots, n-1\}$, if the interval $(A|z_j|, A^{-1}|z_{j+1}|]$ is nonempty this is also included and called $G_j$. Finally, we include $G_n = [A|z_n|, \infty)$. Now let $\mathfrak{J} := \{1\} \cup \{n\} \cup \bigcup_{G_j \neq \emptyset}\{j\}$. Observe that $(0, \infty) \setminus \bigcup_{j \in \mathfrak{J}} G_j$ can be written as $\bigcup_{k \in \mathfrak{K}} D_k$ where the $D_k$'s are disjoint and, moreover, each $D_k = (\alpha_k, \beta_k)$ enjoys the property that $\alpha_k \sim \beta_k$. The notation is suggestive since the $D_k$'s resemble dyadic intervals and, as we are thinking of $A$ as 'large', the $G_j$'s are 'long' intervals, or gaps of $P$. Our decomposition is then:

$$(1.1) \qquad (0, \infty) = \bigcup_{j \in \mathfrak{J}} \gamma|_{(0,\infty)}^{-1}(G_j) \cup \bigcup_{k \in \mathfrak{K}} \gamma|_{(0,\infty)}^{-1}(D_k).$$

We of course then get the corresponding decomposition of $\mathbb{R}$ by taking symmetric versions of the intervals in the above decomposition. If $I$ is a subset of $(0, \infty)$ then define $H_I$ and $M_I$ by

$$H_I f(x) := \int_{|t| \in \gamma|_{(0,\infty)}^{-1}(I)} f(x - \Gamma(t)) \, \frac{dt}{t},$$

$$M_I f(x) := \sup_{k \in \mathbb{Z}} \rho^{-k} \left| \int_{t \in [\rho^k, \rho^{k+1}] \cap \gamma|_{(0,\infty)}^{-1}(I)} f(x - \Gamma(t)) \, dt \right|.$$

It is easy to see that each $H_{D_k}$ and $M_{D_k}$ are $L^p$ bounded. After an application of Minkowski's inequality, this is true if $\gamma^{-1}(\beta_k) \lesssim \gamma^{-1}(\alpha_k)$. This follows because (0.2) implies

$$\log \frac{\gamma^{-1}(\beta_k)}{\gamma^{-1}(\alpha_k)} = \int_{\gamma^{-1}(\alpha_k)}^{\gamma^{-1}(\beta_k)} \frac{dt}{t} = \int_{\alpha_k}^{\beta_k} \frac{1}{\gamma^{-1}(s)\gamma'(\gamma^{-1}(s))} \, ds \leq \int_{\alpha_k}^{\beta_k} \frac{ds}{s} = \log \frac{\beta_k}{\alpha_k} \lesssim 1.$$

Along with the fact that the cardinalities of $\mathfrak{J}$ and $\mathfrak{K}$ are $\lesssim 1$, Theorem 0.8 will follow once we verify that $H_{G_j}$ and $M_{G_j}$ are $L^p$ bounded (with bounds independent of the coefficients of $P$), for each $j \in \mathfrak{J}$. So for the rest of this paper, we fix $j \in \mathfrak{J}$, and for $k \in \mathbb{Z}$ we define $I_k := [1, \rho] \cap \rho^{-k}\gamma|_{(0,\infty)}^{-1}(G_j)$, and measures $H_k$ and $\mu_k$ by:

$$\langle H_k, \psi \rangle := \int_{|t| \in I_k} \psi(\rho^k t, P(\gamma(\rho^k t))) \, \frac{dt}{t}, \quad \langle \mu_k, \psi \rangle := \int_{I_k} \psi(\rho^k t, P(\gamma(\rho^k t))) \, dt,$$

for $\psi \in \mathcal{S}(\mathbb{R}^2)$. In order to analyse $H_{G_j}$ and $M_{G_j}$, we need to understand the behaviour of $P$ on $G_j$. The following lemma is essentially contained in [CRWr] and [FGWr].

**Lemma 1.1.** *There exists a number $C_n > 1$ such that for any $A \geq C_n$ and any $j \in \mathfrak{J}$,*

(1) $|P(s)| \sim |p_j||s|^j$ *for* $|s| \in G_j$;

(2) $\frac{P'(s)}{P(s)} > 0$ *for* $s \in G_j$, $\frac{P'(s)}{P(s)} < 0$ *for* $-s \in G_j$;

(3) $\frac{|P'(s)|}{|P(s)|} \sim \frac{1}{|s|}$ *for* $|s| \in G_j$;

*and for any $j \in \mathfrak{J} \setminus \{1\}$,*

(4) $\frac{P''(s)}{P(s)} > 0$ *and* $\frac{P''(s)}{P(s)} \sim \frac{1}{s^2}$ *for* $|s| \in G_j$.

*Proof.* For (1)-(3) see Lemma 2.1 in [FGWr] and Lemma 2.5 of [CRWr]. For (4), let $\mathbb{N}_n := \{1, \ldots, n\}$ and define $S_1 := \{(l_1, l_2) \in \mathbb{N}_n \times \mathbb{N}_n : l_1 < l_2 \text{ and } l_2 \leq j\}$ and $S_2 := \{(l_1, l_2) \in \mathbb{N}_n \times \mathbb{N}_n : l_1 < l_2 \text{ and } l_2 \geq j + 1\}$. Then write

$$\frac{P''(s)}{P(s)} = 2 \sum_{l_1 < l_2} \frac{1}{(s - z_{l_1})(s - z_{l_2})}$$

$$= 2 \sum_{(l_1, l_2) \in S_1} \frac{1}{(s - z_{l_1})(s - z_{l_2})} + 2 \sum_{(l_1, l_2) \in S_2} \frac{1}{(s - z_{l_1})(s - z_{l_2})}$$

$$=: I + II.$$

Let $\Re[z]$ denote the real part of $z$ and suppose $A > 10$. Then, for $(l_1, l_2) \in S_1$,

$$\Re\left[\frac{1}{(s - z_{l_1})(s - z_{l_2})}\right] = \frac{\Re[(s - z_{l_1})(s - z_{l_2})]}{|s - z_{l_1}|^2|s - z_{l_2}|^2}$$

$$= \frac{s^2 - \Re[(z_{l_1} + z_{l_2})]s + \Re[z_{l_1} z_{l_2}]}{|s - z_{l_1}|^2|s - z_{l_2}|^2}$$

$$\geq \frac{(1 - 2A^{-1} - A^{-2})}{(1 + A^{-1})^4} \frac{1}{s^2},$$

where the last inequality follows because $|z_{l_k}| \leq A^{-1}|s|$ for $k = 1, 2$.

If $l \leq j$ then $|s - z_l| \geq (1 - A^{-1})|s|$ and if $l \geq j + 1$ then $|s - z_l| \geq (A - 1)|s| \geq (1 - A^{-1})|s|$. Therefore, if $(l_1, l_2) \in S_2$ then

$$\frac{1}{|s - z_{l_1}||s - z_{l_2}|} \leq \frac{1}{A(1 - A^{-1})^2} \frac{1}{s^2}.$$

If $C_n'$ is twice the cardinality of $S_1$ and $C_n''$ is twice the cardinality of $S_2$ then

$$\frac{P''(s)}{P(s)} = \Re\left[\frac{P''(s)}{P(s)}\right] = \Re[I] + \Re[II]$$

$$\geq \left(\frac{C_n'(1 - 2A^{-1} - A^{-2})}{(1 + A^{-1})^4} - \frac{C_n''}{A(1 - A^{-1})^2}\right)\frac{1}{s^2}.$$

It is now clear that there is some $C_n > 1$ for which the first assertion of (4) and the lower bound in the remaining assertion follow for $A \geq C_n$. The upper bound is much easier and we leave the details to the reader. $\square$

By (the proof of) Lemma 1.1, we can choose $A$ so that for all $|s| \in G_j$,

(1.2) $\qquad |P(s)| \leq 2|p_j|s^j \quad$ and $\quad \frac{1}{2}j|p_j|s^{j-1} \leq |P'(s)| \leq 2j|p_j|s^{j-1}$.

In the light of Lemma 1.1 it is an appropriate moment to discuss our method of proof of the $L^p$ boundedness of $H_{G_j}$ and $M_{G_j}$, and hence Theorem 0.8. Firstly,

$P'(0)$ being zero is equivalent to $G_1$ being empty. Heuristically Lemma 1.1 tells us that on $G_j$ the curve $(t, P(\gamma(t)))$ is essentially $(t, |p_j|\gamma(t)^j)$. Of course, when $j = 1$ some stronger condition than convexity is necessary. When $G_1$ is nonempty, under the assumption $\gamma' \in \mathcal{C}_1$, we will be able to follow the proof in [Ca et al] or [CoRdF] to get $L^p$ bounds for our operators on $G_1$. We stress here that, under the assumption $h \in \mathcal{C}_2$ (or the stronger condition $\gamma' \in \mathcal{C}_2$), the method of proof in [CChVWWa] fails to work for our operators on $G_1$. Fundamental to the argument in [CChVWWa] are dilation matrices and estimates on the Fourier transform of certain measures. However the fact that Lemma 1.1(4) does not hold for $j = 1$ means we are unable to achieve such estimates. For $j \geq 2$ either the approach in [Ca et al] (and also [CoRdF]) or [CChVWWa] is available to us because $(\gamma^j)' \in \mathcal{C}_2$. Therefore $(\gamma^j)' \in \mathcal{C}_1$ and the $h$-function associated to $\gamma^j$ belongs to $\mathcal{C}_2$.

The following proposition, which can be found on page 384 of [CZ], lays down the bare essentials of a combination of ideas from [CChVWWa], [Ca et al] and [CoRdF]. We use this to prove $L^p$ bounds for $H_{G_j}$ and $M_{G_j}$, and state it as follows:

**Proposition 1.2.** ([CZ]) *Suppose* $\{A_k\}_{k\in\mathbb{Z}} \subseteq GL(2, \mathbb{R})$ *satisfies*

$$(1.3) \qquad\qquad \|A_{k+1}^{-1} A_k\| \leq \alpha < 1.$$

*Suppose* $\{\nu_k\}_{k\in\mathbb{Z}}$ *is a family of measures satisfying*

$$(1.4) \qquad\qquad A_{k+1}^{-1} \mathrm{supp}\nu_k \subseteq B,$$

*for some fixed ball* $B$,

$$(1.5) \qquad\qquad \widehat{\nu_k}(0) = 0,$$

*and*

$$(1.6) \qquad |\widehat{\nu_k}(\xi)| \leq C|A_k^*\xi|^{-1} \quad \text{for } \xi \text{ outside some cone } \triangle_k.$$

*If* $T_k$ *is defined by* $\widehat{T_k f}(\xi) = \chi_{\triangle_k}(\xi)\hat{f}(\xi)$ *and satisfies*

$$(1.7) \qquad \left\| \left( \sum_{k\in\mathbb{Z}} |T_k f|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p \quad \text{for} \quad p \in (1, \infty),$$

*then* $f \mapsto \sum_{k\in\mathbb{Z}} \nu_k * f$ *is bounded on* $L^p$ *for* $p \in (1, \infty)$ *with bound depending only on* $\alpha, B, C$ *and* $C_p$.

## 2. $L^p$ BOUNDS FOR $M_{G_j}$ AND $H_{G_j}$

For $t > 0$ let

$$A(t) := \begin{pmatrix} t & 0 \\ 0 & |p_j|\gamma(t)^j \end{pmatrix}.$$

Define the family of dilations $\{A_k\}_{k\in\mathbb{Z}}$ by $A_k := A(\rho^k)$, where we recall that $\rho = \max\{3, \lambda\}$ and $\lambda$ is the doubling constant for $(\gamma^j)'$.

We begin with $M_{G_j}$ and create cancellation by introducing measures $\sigma_k$ defined by:

$$\langle \sigma_k, \psi \rangle := \frac{\widehat{\mu_k}(0)}{|A_{k+1}B|} \int_{A_{k+1}B} \psi(x)\, dx,$$

where $B := \{x \in \mathbb{R}^2 : |x| < 10\}$. To complete the setup of Proposition 1.2, we define $\nu_k := \varepsilon_k(\mu_k - \sigma_k)$, where $\{\varepsilon_k\} \subseteq \{-1, 1\}$. Now (0.2) implies that $\gamma(t)^j/\gamma(s)^j \leq t/s$

whenever $s \geq t > 0$, and therefore (1.3) holds with $\alpha = 2/\rho < 1$. By (1.2), if $t \in I_k$ then $|P(\gamma(\rho^k t))| \leq 2|p_j||\gamma(\rho^k t)^j \leq 2|p_j||\gamma(\rho^{k+1})^j$. Thus,

$$\text{supp}\mu_k = \{(\rho^k t, P(\gamma(\rho^k t))) : t \in I_k\} \subseteq A_{k+1}B.$$

Of course $\sigma_k$ is supported in $A_{k+1}B$, therefore so is $\nu_k$ and we have (1.4). It is trivial to verify (1.5). To deal with (1.6) and (1.7) we define $\triangle_k$ to be the set of $\xi = (\xi_1, \xi_2)$ in $\mathbb{R}^2$ satisfying:

$$(2.1) \qquad 4|p_j|(\gamma^j)'(\rho^{k+1}) > \frac{|\xi_1|}{|\xi_2|} > \frac{1}{4}|p_j|(\gamma^j)'(\rho^k).$$

The following lemma is well known.

**Lemma 2.1.** ([NSW]) *Let $\{\tau_k\}_{k\in\mathbb{Z}}$ be a sequence of positive real numbers such that for some $R > 1$, $\tau_{k+1} \geq R\tau_k$ for all $k \in \mathbb{Z}$. Let $M > 1$ and define $\triangle_k$ to be the set of all $\xi \in \mathbb{R}^2$ satisfying $M^{-1}\tau_k \leq |\xi_1||\xi_2|^{-1} \leq M\tau_{k+1}$. If $\widehat{T_k f} = \chi_{\triangle_k}\hat{f}$ then*

$$\left\|\left(\sum_{k\in\mathbb{Z}} |T_k f|^2\right)^{1/2}\right\|_p \leq C_p\|f\|_p,$$

*for all $p \in (1, \infty)$.*

It is immediate from Lemma 2.1 that we now have (1.7) (note there is no issue of the constant $C_p$ depending on $p_j$ because $|p_j|(\gamma^j)'(\rho^{k+1})(|p_j|(\gamma^j)'(\rho^k))^{-1} \geq 2$). If we can prove (1.6) then we are done. Indeed, $M_{G_j}f \leq \left(\sum_k |\mu_k - \sigma_k|^2\right)^{1/2} + \sup_k |\sigma_k * f|$. In $L^p$ norm, the former term is $\lesssim \|f\|_p$ by using a standard Rademacher function argument and the fact that the conclusion of Proposition 1.2 holds with bounds independent of $\varepsilon$, and the latter term is $\lesssim \|f\|_p$ by Proposition 2.2 of [CChVWWa] and the fact that $|\widehat{\mu_k}(0)| \lesssim 1$.

Before we prove (1.6) in Lemma 2.3 we need the following:

**Lemma 2.2.** *For all $j \in \mathfrak{J} \setminus \{1\}$, the function*

$$t \longmapsto P''(\gamma(\rho^k t))\gamma'(\rho^k t)^2 + P'(\gamma(\rho^k t))\gamma''(\rho^k t)$$

*is singled-signed on $I_k$.*

*Proof.* By (2) and (4) of Lemma 1.1, it must be the case that $P'$ and $P''$ have the same sign on $G_j$. The convexity of $\gamma$ implies $\gamma''(\rho^k t)$ is nonnegative for $t \in I_k$ and so the result follows. $\qquad\square$

**Lemma 2.3.** *If $\xi \notin \triangle_k$ then $|\widehat{\nu_k}(\xi)| \lesssim |A_k\xi|^{-1}$.*

*Proof.* Since

$$(2.2) \qquad |\widehat{\sigma_k}(\xi)| \lesssim |\widehat{\chi_B}(A_{k+1}\xi)| \lesssim |A_{k+1}\xi|^{-1} \lesssim |A_k\xi|^{-1},$$

we are left to find a decay estimate for $\widehat{\mu_k}$. Let $\theta(t) = \rho^k t\xi_1 + P(\gamma(\rho^k t))\xi_2$ for $t \in I_k$. Suppose first that $|\xi_1| > 4|p_j|(\gamma^j)'(\rho^{k+1})|\xi_2|$. Then, by (1.2),

$$|\theta'(t)| \geq \rho^k|\xi_1| - |P'(\gamma(\rho^k t))|\gamma'(\rho^k t)\rho^k|\xi_2| \geq \rho^k|\xi_1| - 2|p_j|(\gamma^j)'(\rho^k t)\rho^k|\xi_2| \gtrsim \rho^k|\xi_1|.$$

Now $\theta''(t) = [P''(\gamma(\rho^k t))\gamma'(\rho^k t)^2 + P'(\gamma(\rho^k t))\gamma''(\rho^k t)]\rho^{2k}\xi_2$. For any $j \neq 1$, Lemma 2.2 implies that $\theta''$ is singled-signed on $I_k$ and therefore we have that $\theta'$ is monotone on $I_k$. We now invoke van der Corput's lemma (see [S]) for these $j$'s to get $|\widehat{\mu_k}(\xi)| \lesssim (\rho^k|\xi_1|)^{-1} \lesssim |A_k\xi|^{-1}$, where the last inequality follows from (0.2). The situation for $j = 1$ will be dealt with momentarily.

If now $|\xi_1| < \frac{1}{4}|p_j|(\gamma^j)'(\rho^k)|\xi_2|$ then we use (1.2) to get

$$\begin{aligned}
|\theta'(t)| &\geq \frac{1}{2}|p_j|(\gamma^j)'(\rho^k t)\rho^k|\xi_2| - \rho^k|\xi_1| \\
&\geq \frac{1}{4}|p_j|(\gamma^j)'(\rho^k t)\rho^k|\xi_2| \geq \frac{1}{4}|p_j|(\gamma^j)'(\rho^k)\rho^k|\xi_2|.
\end{aligned}$$

Another application of van der Corput's lemma and then (0.2) gives

$$|\widehat{\mu_k}(\xi)| \lesssim (|p_j|\gamma(\rho^k)^{j-1}\gamma'(\rho^k)\rho^k|\xi_2|)^{-1} \lesssim |A_k\xi|^{-1},$$

which completes the proof for $j \neq 1$.

For $j = 1$ we again begin with $|\xi_1| > 4|p_1|\gamma'(\rho^{k+1})|\xi_2|$. Of course we still get $|\theta'(t)| \gtrsim \rho^k|\xi_1|$ for $t \in I_k$. Using this and integration by parts (which is how the standard proof of van der Corput's lemma proceeds),

$$|\widehat{\mu_k}(\xi)| \lesssim (\rho^k|\xi|)^{-1} + \int_{I_k} \frac{|\theta''(t)|}{\theta'(t)^2}\, dt.$$

Note $\int_{I_k} |\theta''(t)|/\theta'(t)^2\, dt$ is less than

$$\int_{I_k} \frac{\rho^{2k}|\xi_2||P'(\gamma(\rho^k t))|\gamma''(\rho^k t)}{\theta'(t)^2}\, dt + \int_{I_k} \frac{\rho^{2k}|\xi_2||P''(\gamma(\rho^k t))|\gamma'(\rho^k t)^2}{\theta'(t)^2}\, dt =: \alpha_1 + \alpha_2.$$

For $\alpha_1$ we introduce $\phi(t) = \rho^k t|\xi_1| + |p_1|\gamma(\rho^k t)|\xi_2|$ for $t \in I_k$. Note, $\phi'(t) \sim \rho^k|\xi_1| \lesssim |\theta'(t)|$ and, again using (1.2), we see that

$$\alpha_1 \lesssim \int_{I_k} \frac{\phi''(t)}{\phi'(t)^2}\, dt \lesssim (\rho^k|\xi_1|)^{-1}.$$

For $\alpha_2$, first we write

$$\alpha_2 \leq \int_{I_k} \rho^k|P''(\gamma(\rho^k t))|\gamma'(\rho^k t)\frac{\gamma'(\rho^{k+1})\rho^k|\xi_2|}{\theta'(t)^2}\, dt \lesssim (|p_1|\rho^k|\xi_1|)^{-1}\int_{G_1} |P''(s)|\, ds.$$

Suppose $P'' \geq 0$ on $[s_1, s_2] \subseteq G_1$. Then $\int_{[s_1,s_2]} |P''(s)|\, ds = P'(s_2) - P'(s_1) \lesssim |p_1|$ by Lemma 1.1. Similarly if $P'' < 0$ on $[\tilde{s}_1, \tilde{s}_2] \subseteq G_1$. Since $G_1$ splits into $\lesssim 1$ disjoint such intervals, we get $\alpha_2 \lesssim (\rho^k|\xi_1|)^{-1}$. Now, (0.2) implies $(\rho^k|\xi_1|)^{-1} \lesssim |A_k\xi|^{-1}$, so we have $|\widehat{\mu_k}(\xi)| \lesssim |A_k\xi|^{-1}$ in the case $|\xi_1| > 4|p_1|\gamma'(\rho^{k+1})|\xi_2|$.

Finally, suppose $|\xi_1| < \frac{1}{4}|p_1|\gamma'(\rho^k)|\xi_2|$. Yet another application of (1.2) gives

$$|\theta'(t)| \geq \frac{1}{4}|p_1|\gamma'(\rho^k t)\rho^k|\xi_2| \geq \frac{1}{4}|p_1|\gamma'(\rho^k)\rho^k|\xi_2|,$$

for $t \in I_k$. With $\alpha_1, \alpha_2$, and $\phi$ as above we have $\phi'(t) \sim |p_1|\gamma'(\rho^k t)\rho^k|\xi_2| \lesssim |\theta'(t)|$. The same argument used previously for $\alpha_1$ gives $\alpha_1 \lesssim (|p_1|\gamma'(\rho^k)\rho^k|\xi_2|)^{-1}$. Also

$$\begin{aligned}
\alpha_2 &\lesssim \int_{I_k} \rho^k|P''(\gamma(\rho^k t))|\gamma'(\rho^k t)\frac{1}{|p_1|\gamma'(\rho^k t)\rho^k|\xi_2|}\, dt \\
&\lesssim (|p_1|\gamma'(\rho^k)\rho^k|\xi_2|)^{-1}\int_{G_1} |p_1|^{-1}|P''(s)|\, ds \lesssim (|p_1|\gamma'(\rho^k)\rho^k|\xi_2|)^{-1}.
\end{aligned}$$

By (0.2) it follows that $|\widehat{\mu_k}(\xi)| \lesssim (|p_1|\gamma'(\rho^k)\rho^k|\xi_2|)^{-1} \lesssim |A_k\xi|^{-1}$, and this completes the proof of Lemma 2.3. $\qquad\square$

Finally, for $H_{G_j}$ we apply Proposition 1.2 with $A_k$ and $\triangle_k$ unchanged, and $\nu_k$ equal to $H_k$. Since (1.5) is true, we only need check (1.6). Firstly, if $\gamma$ is even then this is almost immediate from the work done in the proof of Lemma 2.3. Indeed,

this and a standard integration by parts argument gives us the decay for the integral over $I_k$, while the integral over $-I_k$ is simply a reflection in the vertical axis of the integral over $I_k$. For odd $\gamma$, we claim that Lemma 2.2 holds on $-I_k$ as well. To see this, simply observe that $P'$ and $P''$ have opposing signs on $-G_j$, by (2) and (4) of Lemma 1.1, and couple this with the fact that $\gamma'' \leq 0$ on $(-\infty, 0)$. Now, (1.6) will follow if we carry out the argument used in the proof of Lemma 2.3 and the standard integration by parts argument just mentioned. This completes the proof of Theorem 0.8.

## 3. The hypersurface

We again decompose $(0, \infty)$ as in (1.1). If $H_{D_k}$ and $M_{D_k}$ are defined in the analogous way, then

$$\int_{|y| \in \gamma^{-1}(D_k)} |K(y)| \, dy \lesssim \int_{S^{d-1}} |K(\omega)| \int_{r \in \gamma^{-1}(D_k)} \frac{dr}{r} \, d\sigma(\omega) \lesssim 1,$$

and therefore these operators are bounded on $L^p$. So we fix $j \in \mathfrak{J}$ and turn our attention to showing $H_{G_j}$ and $M_{G_j}$ are $L^p$ bounded operators. Taking $\rho := d + 2$ and $I_k$ as before, define $H_k$ and $\mu_k$ by:

$$\langle H_k, \psi \rangle := \int_{|y| \in I_k} \psi(\rho^k y, P(\gamma(\rho^k |y|))) K(y) \, dy,$$

$$\langle \mu_k, \psi \rangle := \int_{|y| \in I_k} \psi(\rho^k y, P(\gamma(\rho^k |y|))) \, dy,$$

for $\psi \in \mathcal{S}(\mathbb{R}^{d+1})$. Also, put $A_k := A(\rho^k)$ where, for $t > 0$, $A(t)$ is the $d+1$ by $d+1$ diagonal matrix with $(r, r)$-entry equal to $|p_j| \gamma(t)^j$ when $r = d+1$, and $t$ otherwise.

**Lemma 3.1.** $|\widehat{H_k}(\xi)| + |\widehat{\mu_k}(\xi)| \lesssim |A_k \xi|^{(1-d)/2}$ for $\xi \neq 0$.

*Proof.* We just prove the decay estimate for $\widehat{H_k}$ because the corresponding result for $\widehat{\mu_k}$ can be proved in the same way. If $\xi = (\xi', \xi_{d+1})$ then

$$\widehat{H_k}(\xi) = \int_{|y| \in I_k} e^{i(\rho^k y \cdot \xi' + P(\gamma(\rho^k |y|))\xi_{d+1})} K(y) \, dy$$

$$= \int_{r \in I_k} e^{iP(\gamma(\rho^k r))\xi_{d+1}} \int_{S^{d-1}} e^{i\rho^k r \omega \cdot \xi'} K(\omega) \, d\sigma(\omega) \frac{dr}{r}.$$

It is well known (see, for example [S]) that because $K$ is smooth away from the origin, for $r \in I_k$,

$$\left| \int_{S^{d-1}} e^{i\rho^k r \xi' \cdot \omega} K(\omega) \, d\sigma(\omega) \right| \lesssim (\rho^k r |\xi'|)^{(1-d)/2} \lesssim (\rho^k |\xi'|)^{(1-d)/2}.$$

Therefore the claim follows for $|p_j| \gamma(\rho^k)^j |\xi_{d+1}| \leq 4\rho^k |\xi'|$. Suppose then that $|p_j| \gamma(\rho^k)^j |\xi_{d+1}| \geq 4\rho^k |\xi'|$. Fix $\omega \in S^{d-1}$ and let $\theta(r) = \rho^k r \omega \cdot \xi' + P(\gamma(\rho^k r))\xi_{d+1}$ for $r \in I_k$. Then (1.2) and (0.2) imply

$$|\theta'(r)| \geq \frac{1}{2} |p_j| (\gamma^j)'(\rho^k r) \rho^k |\xi_{d+1}| - \rho^k |\xi'| \gtrsim |p_j| \gamma(\rho^k)^j |\xi_{d+1}|.$$

It follows that

$$\left| \int_{r \in I_k} e^{i\theta(r)} \frac{dr}{r} \right| \lesssim (|p_j| \gamma(\rho^k)^j |\xi_{d+1}|)^{-1} \lesssim |A_k \xi|^{-1}$$

(as in the proof of Lemma 2.3 this follows by van der Corput's lemma for $j \in \mathfrak{J} \setminus \{1\}$, and the substitute argument for $j = 1$). This completes the proof of Lemma 3.1. $\square$

We can now use Proposition 1.2 (or a weaker form, given that we in fact have uniform decay estimates) to complete the proof of Theorem 0.9.

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