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EXISTENCE THRESHOLDS AND RAMSEY PROPERTIES OF RANDOM POSETS

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Abstract. Let $\mathcal{P}(n)$ denote the power set of $[n]$, ordered by inclusion, and let $\mathcal{P}(n,p)$ denote the random poset obtained from $\mathcal{P}(n)$ by retaining each element from $\mathcal{P}(n)$ independently at random with probability p and discarding it otherwise.

Given *any* fixed poset F we determine the threshold for the property that $\mathcal{P}(n,p)$ contains F as an induced subposet. We also asymptotically determine the number of copies of a fixed poset F in $\mathcal{P}(n)$. Finally, we obtain a number of results on the Ramsey properties of the random poset $\mathcal{P}(n,p)$.

1. Introduction

Let $(P,\leq_P),(Q,\leq_Q)$ be posets. A *poset homomorphism* from $(P,\leq_P)$ to $(Q,\leq_Q)$ is a function $\phi: P\to Q$ such that for every $x,y\in P$, if $x\leq_P y$ then $\phi(x)\leq_Q\phi(y)$. We say that $(P,\leq_P)$ is a *subposet* of $(Q,\leq_Q)$ if there is an injective poset homomorphism from $(P,\leq_P)$ to $(Q,\leq_Q)$; otherwise, $(Q,\leq_Q)$ is said to be $(P,\leq_P)$*-free*. Further we say $(P,\leq_P)$ is an *induced subposet* of $(Q,\leq_Q)$ if there is an injective poset homomorphism ϕ from $(P,\leq_P)$ to $(Q,\leq_Q)$ such that for every x,y in P, $\phi(x)\leq_Q\phi(y)$ if and only if $x\leq_P y$. We shall sometimes use P as a shorthand for the poset $(P,\leq_P)$ when the partial order $\leq_P$ is clear from context, and write e.g. P-free for $(P,\leq_P)$-free.

Set $[n]:=\{1,2,\dots,n\}$, and denote by $\mathcal{P}(n)$ the power set of $[n]$. When viewed as a poset equipped with the inclusion relation, we refer to $\mathcal{P}(n)$ as the *Boolean lattice of dimension* n. Recall that given a poset P, a subset $\mathcal{A}\subseteq P$ is an *antichain* if all distinct $A,B\in\mathcal{A}$ are incomparable. A *chain of length* ℓ in P is an ℓ-subset of P in which all elements are comparable.

Many classical questions in graph theory have analogues in the setting of the Boolean lattice. For example, in graph theory, Turán-type questions ask what is the maximum number of edges a graph on n vertices may have if it does not contain any copy of a fixed graph H as a subgraph. The oldest result of this flavour in the study of the Boolean lattice is Sperner's theorem [30] which asserts that the size of the largest antichain in $\mathcal{P}(n)$ (i.e. the maximum size of a subposet of $\mathcal{P}(n)$ not containing a chain of length 2) is $\binom{n}{\lfloor n/2\rfloor}$. More generally, there has been much interest in determining the largest P-free subset of $\mathcal{P}(n)$ for a range of posets P (for a sample of such results see e.g. [1, 7, 10, 13, 21, 24]; furthermore, [12, 16] are surveys on the topic).

Recall that the Erdős–Rényi random graph $G_{n,p}$ is the n-vertex graph where each edge is present with probability p, independently of all other choices. In this paper we consider an analogue of the Erdős–Rényi random graph in the setting of posets: let $\mathcal{P}(n,p)$ be the induced subposet of $\mathcal{P}(n)$ obtained by independently including each element from $\mathcal{P}(n)$ in $\mathcal{P}(n,p)$ independently at random with probability p (and discarding it otherwise). The random poset model $\mathcal{P}(n,p)$ was first investigated by Rényi [26] who determined the probability threshold for the property that

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$\mathcal{P}(n,p)$ is not itself an antichain, thereby answering a question of Erdős. This model has also been studied with respect to a range of other properties. Answering a question of Osthus [23], a version of Sperner's theorem for $\mathcal{P}(n,p)$ was obtained independently by Balogh, Mycroft and Treglown [3] and by Collares Neto and Morris [9].[1] There have also been a number of results concerning the length of (the longest) chains in $\mathcal{P}(n,p)$ and related models of random posets (see for example, [6, 19, 20]).

Note too that natural questions concerning $\mathcal{P}(n,p)$ also arise when viewing it as a set system rather than a poset (see e.g. the random version of Katona's intersection theorem in [4]).

1.1. **The existence threshold for a subposet.** One of the fundamental questions in the study of the random graph $G_{n,p}$ concerns the values of p for which $G_{n,p}$ with high probability (w.h.p.) contains a given fixed graph H as a subgraph. Indeed, this was the first problem studied in a seminal paper of Erdős and Rényi [11], who determined the threshold for this problem in the case when H belongs to the class of *balanced* graphs. It took another twenty-one years before Bollobás [5] determined the threshold for general graphs.

It is natural to ask the analogous question in the setting of posets. That is: given a fixed poset P, for which values of p do we have that $\mathcal{P}(n,p)$ w.h.p. contains a copy of P as a subposet? In this paper we answer this question for every poset P. More precisely, we determine the critical value $c_\star(P)$ such that for $p = e^{-cn}$ and $c > c_\star(P)$ fixed, w.h.p. $\mathcal{P}(n,p)$ does not contain a copy of P, while for $p = e^{-cn}$ and $c < c_\star(P)$ fixed, w.h.p. $\mathcal{P}(n,p)$ contains an *induced* copy of P.

Whilst the analogous result in the setting of the random graph $G_{n,p}$ is not too difficult to state, in the Boolean lattice we must introduce several non-trivial concepts before we can give a formal statement of our main result. Thus, we defer its precise statement (Theorem 6.4) to Section 6. In Section 2 we give the intuition behind this result. We additionally prove that for almost all posets P with N elements, $c_\star(P) = (\log 2)/3 + O(1/\log N)$ (see Theorem 8.7).

We remark that Kreuter [20] considered a closely related question. Indeed, given a *distributive lattice* L he determined the threshold for the property that w.h.p. L can be *embedded* in $\mathcal{P}(n,p)$. That is, given $a, b \in L$, write $a \vee b$ for the *join* of a and b and $a \wedge b$ for the *meet* of a and b. Then an *embedding* of L into $\mathcal{P}(n,p)$ is an injective poset homomorphism ϕ from L to $\mathcal{P}(n,p)$ such that for all $a, b \in L$, $\phi(a \vee b) = \phi(a) \vee \phi(b)$ and $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$. Whilst the existence threshold we obtain for our problem has some features similar to that of Kreuter's, the two problems differ quite significantly.

1.2. **Counting subposets in the Boolean lattice.** In order to prove our existence threshold result (Theorem 6.4), in Section 4 we provide a correspondence between copies of a fixed poset P in $\mathcal{P}(n)$ and partitions of $[n]$. As a consequence of this we asymptotically determine the number of copies of P in $\mathcal{P}(n)$.

Theorem 1.1. *Let P be a fixed poset with m antichains (including the empty antichain). Then $\mathcal{P}(n)$ contains*

$$(1 + o(1))m^n$$

copies of P.

Thus, the number m of antichains in P is the parameter governing how many copies of P there are in the Boolean lattice. Theorem 1.1 also implies that the number of non-induced copies of P in $\mathcal{P}(n)$ is $o(m^n)$, since such a copy of P is simply another poset with strictly fewer antichains. Note that Axenovich and Walzer [2, Theorem 5] gave (asymptotically weaker) bounds on the number of copies of $\mathcal{P}(t)$ in $\mathcal{P}(n)$.

[1] This question had first been studied by Kohayakawa and Kreuter [18].

1.3. **Ramsey properties of random posets.** Ramsey-type problems have been extensively studied for graphs, and there has been interest in investigating similar phenomena in the Boolean lattice. As a consequence of the Hales–Jewett theorem, we have the following analogue of Ramsey's theorem for complete graphs in the Boolean setting: given any fixed $r, m \in \mathbb{N}$, if n is sufficiently large then in every r-colouring of the elements of $\mathcal{P}(n)$ there is a monochromatic copy of $\mathcal{P}(m)$ (see e.g. [25, p49]). In a recent paper, Axenovich and Walzer [2] gave bounds on the so-called *poset Ramsey number* for various posets: given posets F and F', the *poset Ramsey number* $R(F, F')$ is the smallest N such that any 2-colouring of the elements of $\mathcal{P}(N)$ contains a red induced copy of F or a blue induced copy of F'. They also considered multicolour variants of this Ramsey number. See also the very recent papers [22, 8].

In the case of the random graph $G_{n,p}$, we have a clear understanding of (symmetric) Ramsey properties. Indeed, seminal work of Rödl and Ruciński [27, 28, 29] determines the threshold for the property of $G_{n,p}$ being (H, r)-Ramsey for any fixed graph H and $r \in \mathbb{N}$. (We say that a graph G is (H, r)-*Ramsey* if every r-colouring of G yields a monochromatic copy of H in G.) However, far less is known about Ramsey properties of $\mathcal{P}(n, p)$.

In 1998, Kreuter [20] initiated the study of such questions for $\mathcal{P}(n, p)$ – however, in the setting of monochromatic embedded copies of a distributive lattice L. Given a lattice L and $r \in \mathbb{N}$ we say that a poset P is (L, r)-*embed-Ramsey* if whenever the elements of P are r-coloured, there exists an embedded monochromatic copy of L in P. Kreuter [20] determined the threshold for the property that $\mathcal{P}(n, p)$ is (C_ℓ, r)-embed-Ramsey, where here C_ℓ denotes a chain of length ℓ. He also raised the question of generalising this result to other sublattices. In particular, he asked for the probability threshold for the property that whenever $\mathcal{P}(n, p)$ is 2-coloured it contains an embedded monochromatic copy of $\mathcal{P}(2)$.

Given a poset F and $r \in \mathbb{N}$ we say that a poset P is (F, r)-*Ramsey* if whenever the elements of P are r-coloured, there exists a monochromatic copy of F in P. Similarly, given posets $F_1, \dots, F_r$ we say that a poset P is $(F_1, \dots, F_r)$-*Ramsey* if whenever the elements of P are r-coloured, for at least one $1 \le i \le r$, in P there exists a copy of F_i in colour i. In this paper we initiate the study of the following general question:

Question 1.2. *Given posets $F_1, \dots, F_r$, what values of p ensure that w.h.p. $\mathcal{P}(n, p)$ is $(F_1, \dots, F_r)$-Ramsey?*

Note that a copy of C_ℓ in $\mathcal{P}(n, p)$ is always an embedded copy; so in fact the aforementioned result of Kreuter answers Question 1.2 in the case when $F_1 = \cdots = F_r = C_\ell$. As an application of our existence threshold theorem (Theorem 6.4), we obtain a number of somewhat modest results concerning the Ramsey properties of random posets. For each poset P on at most 3 elements, (combined with Kreuter's result) we determine the critical value $c_{\mathrm{Ram}}(P)$ such that for $p = e^{-cn}$ and $c > c_{\mathrm{Ram}}(P)$ fixed, w.h.p. $\mathcal{P}(n, p)$ is not $(P, 2)$-Ramsey, while for $p = e^{-cn}$ and $c < c_{\mathrm{Ram}}(P)$ fixed, w.h.p. $\mathcal{P}(n, p)$ is $(P, 2)$-Ramsey. We give a number of results for other posets too, for example, the following result for $\mathcal{P}(2)$.

Theorem 1.3. *The following holds:*

(i) *If $c < 0.3250121326$ and $p = e^{-cn}$ then w.h.p. $\mathcal{P}(n, p)$ is $(\mathcal{P}(2), 2)$-Ramsey.*
(ii) *If $c > 0.3289037391$ and $p = e^{-cn}$ then w.h.p. $\mathcal{P}(n, p)$ is not $(\mathcal{P}(2), 2)$-Ramsey.*

Further Ramsey-type results are presented in Section 9. We suspect that Question 1.2 is likely to be extremely challenging in general. It would be very interesting to resolve the question fully in the case when $r = 2$ and $F_1 = F_2 = \mathcal{P}(2)$.

1.4. **Organisation of the paper.** The paper is organised as follows. In Section 2 we provide an intuitive outline of the existence threshold result (Theorem 6.4). Sections 3–5 introduce a number of concepts and auxiliary results that allow us to formally state and prove Theorem 6.4 in Section 6. Specifically, Section 3 introduces the crucial notions of extension families and shadows. In Section 4 we formally introduce a correspondence between partitions of $[n]$ and copies of a fixed poset P in $\mathcal{P}(n)$. This correspondence not only allows us to prove Theorem 1.1, but is also vital for the proof of Theorem 6.4. In Section 5 we introduce the notion of the weight of a partition of $[n]$ and describe its connection to shadows.

As discussed earlier, Theorem 6.4 determines the critical value $c_\star(P)$ such that for $p = e^{-cn}$ and $c > c_\star(P)$ fixed, w.h.p. $\mathcal{P}(n,p)$ does not contain a copy of P, while for $p = e^{-cn}$ and $c < c_\star(P)$ fixed, w.h.p. $\mathcal{P}(n,p)$ contains an induced copy of P. Whilst we provide an explicit formula for $c_\star(P)$ (see Definition 6.2 and Remark 6.3), computing $c_\star(P)$ by hand is in general awkward. In Section 7 we give a number of results (and heuristics) that provide bounds for $c_\star(P)$; this allows us to then compute $c_\star(P)$ for a number of posets P in Section 8. In Section 9 we turn our attention to Ramsey questions; we provide a range of results including general bounds, and bounds for the (P,Q)-Ramsey problem in $\mathcal{P}(n,p)$ for several specific pairs of posets P,Q. We conclude the paper with a number of open problems (see Section 10).

2. Intuition behind the existence threshold

In order to state the threshold for the property that $\mathcal{P}(n,p)$ contains a fixed P as a subposet, one requires several concepts. In particular, at first sight, it may seem rather hard to understand the intuition behind the formal statement of the threshold. In this section we describe the threshold in more informal terms in order to build up understanding for when we do finally state the precise result.

2.1. **Notation.** All posets considered in this paper are finite. Given a poset P, we denote by $\mathcal{A}(P)$ the family of all *antichains* of P. Note that we include the empty antichain in $\mathcal{A}(P)$. So for example, both $\emptyset$ and $\{\emptyset\}$ belong to $\mathcal{A}(\mathcal{P}(n))$! We write $a(P)$ for the cardinality of $\mathcal{A}(P)$. We will often enumerate the set of antichains as $\mathcal{A}(P) =: \{S_1, \dots, S_{a(P)}\}$; we always implicitly assume that $S_{a(P)} = \emptyset$.

A poset on a set of elements X is said to be *connected* if it is not possible to partition X into two non-empty sets X_1 and X_2 such that for every $x_1 \in X_1$ and $x_2 \in X_2$, the elements x_1 and x_2 are mutually incomparable. Equivalently, a poset is connected if its Hasse diagram is a connected graph.

Given integers m, n, we write $[m]^{[n]}$ for the collection of ordered m-partitions of $[n]$, i.e. the collection of all m-tuples $(A_1, A_2, \dots, A_m)$, where the A_is are pairwise disjoint subsets of $[n]$ whose union is $[n]$. Further, we write $[m]^{[n]}_\star$ for the collection of all $(A_1, A_2, \dots, A_m)$ in $[m]^{[n]}$ for which $A_j \neq \emptyset$ for all $j \in [m]$.

We use standard Landau notation throughout this paper. In a probabilistic setting, we say that an n-dependent event $E = E(n)$ occurs *with high probability (w.h.p.)* if $\mathbb{P}(E) \to 1$ as $n \to \infty$.

2.2. **A correspondence between partitions of $[n]$ and subposets of $\mathcal{P}(n)$.** A crucial feature of the existence threshold concerns a correspondence between partitions of $[n]$ and subposets of $\mathcal{P}(n)$. More precisely, fix a poset P, and consider an arbitrary fixed ordering $S_1, \dots, S_m$ of the elements of $\mathcal{A}(P)$ (so here $m := a(P)$). Now suppose we are given a partition $\mathbf{A} = (A_1, A_2, \dots, A_m) \in [m]^{[n]}_\star$. We use $\mathbf{A}$ to build an injective poset homomorphism $\phi :\ P \to \mathcal{P}_n$ as follows. Given $i \in P$, let $D(i) := \{i' \in P :\ i' \leq_P i\}$ denote the collection of elements of P which are less than or equal to i with respect to the partial order $\leq_P$. Then let ϕ be the map sending $i \in P$ to the subset $X_i \subseteq [n]$, where

$$X_i := \bigcup_{j:\ S_j \cap D(i) \neq \emptyset} A_j. \tag{2.1}$$

Set $f(\mathbf{A}) = \phi$. Roughly speaking, in Section 4 we show that f is a bijection from the set of partitions $[m]_\star^{[n]}$ to the set of all induced copies of P in $\mathcal{P}(n)$ (this is a slight simplification, see Lemmas 4.2 and 4.5).

Indeed, notice that for any i, $X_i = \bigcup_{i' \in D(i)} X_{i'}$ and thus ϕ ensures that comparable elements of P are mapped to comparable elements of $\mathcal{P}(n)$. Meanwhile, if i, i' are incomparable elements of P, since the definition of X_i involves the antichain $S_j := \{i\}$, and the definition of $X_{i'}$ involves the antichain $S_{j'} := \{i'\}$, we see that i and i' are mapped to incomparable elements of $\mathcal{P}(n)$.

Given an induced copy P' of P in $\mathcal{P}(n)$ we say that $\mathbf{A} := f^{-1}(P') \in [m]_\star^{[n]}$ is the partition of $[n]$ *associated with* P'. Write $\mathbf{A} = (A_1, \dots, A_m)$ and define $a_i := |A_i|/n$. We say that P' is a copy of P of $(a_1, \dots, a_m)$*-type*.

The partition associated with a copy P' of P in $\mathcal{P}(n)$ encodes structural information on how P' is positioned within $\mathcal{P}(n)$. To illustrate this, consider the case when $P := \mathcal{P}(2)$. Let $x_1, \dots, x_4$ denote the elements of $\mathcal{P}(2)$, where x_1 is the minimal element and x_4 is the maximal element. Note we have that $|\mathcal{A}(\mathcal{P}(2))| = 6$. By definition, every copy P' of $\mathcal{P}(2)$ in $\mathcal{P}(n)$ of $(1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$-type has its minimal element x_1 on the $n/6$-layer of $\mathcal{P}(n)$ (since x_1 lies in one antichain in $\mathcal{P}(2)$). The two middle elements x_2, x_3 of $\mathcal{P}(2)$ lie in the $n/2$-layer of $\mathcal{P}(n)$ (since they each lie in two antichains, in addition to the antichain that x_1 lies in). Finally, x_4 is positioned on the $5n/6$-layer (since it is only the empty antichain that does not contain one of $x_1, \dots, x_4$). The type of P' not only determines the layers of $\mathcal{P}(n)$ in which the elements of P' are located; it also gives us information about the way in which such elements 'overlap', when viewed as subsets of $[n]$. Indeed, continuing our running example above, suppose $x_2 \in \mathcal{P}(2)$ is mapped to the set $X_2 \in \mathcal{P}(n)$ and $x_3 \in \mathcal{P}(2)$ is mapped to the set $X_3 \in \mathcal{P}(n)$. Since $\{x_1\}$ and $\{x_2, x_3\}$ are the only antichains that intersect both $D(x_2)$ and $D(x_3)$, we know that $|X_2 \cap X_3| = n/3$.

Note that in such a copy P' of $\mathcal{P}(2)$ in $\mathcal{P}(n)$ we have a copy V' of the poset $V = \{x_1, x_2, x_3\}$ whose minimal element lies in the $n/6$-layer, and whose two other elements lie in the $n/2$-layer of $\mathcal{P}(n)$. We have that $\mathcal{A}(V) = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \emptyset\}$. It is easy to check that V' is in fact a copy of V in $\mathcal{P}(n)$ of $(1/6, 1/6, 1/6, 1/6, 1/3)$-type.

2.3. **Copies of P in $\mathcal{P}(n,p)$.** The correspondence mentioned in the previous section plays a crucial role in the existence threshold problem. To see this, suppose we first ask for the threshold for the property that $\mathcal{P}(n,p)$ contains a copy of $\mathcal{P}(2)$ of $(1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$-type. To ensure w.h.p. that $\mathcal{P}(n,p)$ contains such a copy of $\mathcal{P}(2)$, certainly we need p to be chosen so that the expected number of copies of $\mathcal{P}(2)$ of $(1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$-type in $\mathcal{P}(n,p)$ is at least 1. (Otherwise, Markov's inequality easily implies one cannot have such a copy of $\mathcal{P}(2)$ w.h.p.) Moreover, one also needs p to be such that the expected number of copies of V of the 'correct type' (i.e. $(1/6, 1/6, 1/6, 1/6, 1/3)$-type) in $\mathcal{P}(n,p)$ is at least 1. In fact, for any subposet F of $\mathcal{P}(2)$ one needs that the expected number of copies of F of the 'correct' type in $\mathcal{P}(n,p)$ is at least 1. It turns out these are the only barriers; if p is such that each of the above expectations is large, then indeed w.h.p. $\mathcal{P}(n,p)$ contains a copy of $\mathcal{P}(2)$ of $(1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$-type.

Now suppose we wish to find the threshold for the property that $\mathcal{P}(n,p)$ contains a copy of $\mathcal{P}(2)$ (i.e. the type of $\mathcal{P}(2)$ does not matter). Then, roughly speaking, we prove that this threshold p^* is the smallest $0 < p < 1$ such that there is some 6-tuple $(\alpha_1, \dots, \alpha_6)$ with each $\alpha_i > 0$ such that

- $\sum \alpha_i = 1$;
- the expected number of copies of $\mathcal{P}(2)$ in $\mathcal{P}(n,p)$ of $(\alpha_1, \dots, \alpha_6)$-type is at least one;
- given any subposet F of $\mathcal{P}(2)$, the expected number of copies of F of the 'correct' type is at least one.

More generally, the threshold p^* for the existence of *any* fixed poset P in $\mathcal{P}(n,p)$ is analogous. Indeed, it is the smallest $0 < p < 1$ such that there is some m-tuple $(\alpha_1, \dots, \alpha_m)$ with each $\alpha_i > 0$ such that

- $\sum \alpha_i = 1$;
- the expected number of copies of P in $\mathcal{P}(n,p)$ of $(\alpha_1, \dots, \alpha_m)$-type is at least one;
- given any subposet F of P, the expected number of copies of F of the 'correct' type is at least one.

(Recall here we define $m := a(P) = |\mathcal{A}(P)|$.)

3. Extension families and shadows

To rigorously build up the correspondence between partitions of $[n]$ and copies of a poset P in $\mathcal{P}(n)$ described in the last section, we require several definitions. In this section we shall define notions of *extension families* and *shadows* for antichains that will play a crucial role in both the proofs and the statements of our main results. Here (and elsewhere in the paper) we write $x <_P y$ as a shorthand for the statement "$x \leq_P y$ and $x \neq y$".

Definition 3.1 (Extension family)**.** *Let S be an antichain in $\mathcal{A}(P)$. The extension family of S in P is*

$$\mathrm{Ext}_P(S) := \{S' \in \mathcal{A}(P) : \ S \subsetneq S'\}.$$

Thus $\mathrm{Ext}_P(S)$ is the collection of all antichains in $\mathcal{A}(P)$ strictly containing S as a sub-antichain. So, for example, $\mathrm{Ext}_P(\emptyset) = \mathcal{A}(P) \setminus \{\emptyset\}$, and $\mathrm{Ext}_P(S) = \emptyset$ if and only if S is a maximal antichain. When the poset P is clear from context, we usually omit the subscript P and write $\mathrm{Ext}(S)$ for $\mathrm{Ext}_P(S)$.

Definition 3.2 (Shadow)**.** *Let $(P, \leq_P)$ be a poset, and $Q \subseteq P$. Given an antichain $S \in \mathcal{A}(P)$, its* (upper) shadow *$\partial_Q S$ is the set of all $y \in Q$ such that*

(i) there exists $x \in S$ with $x \leq_P y$,
(ii) y is $\leq_P$-minimal in Q with respect to property (i): that is, for every $z \in Q$ with $z <_P y$ and every $x \in S$, we have $x \not\leq_P z$.

Clearly we have $\partial_P S = S$ for every antichain $S \in \mathcal{A}(P)$. Going back to our running example $\mathcal{P}(2)$ from Section 2.2, the shadow of the antichain $\{x_1\}$ inside the subposet Λ of $\mathcal{P}(2)$ induced by $\{x_2, x_3, x_4\}$ is the antichain $\{x_2, x_3\}$; also, the shadow of the antichain $\{x_2, x_3\}$ inside the subposet induced by $\{x_3, x_4\}$ is the singleton antichain $\{x_3\}$. More generally we have:

Proposition 3.3. *For every $S \in \mathcal{A}(P)$ and $Q \subseteq P$, the shadow $\partial_Q S$ is an antichain in $(Q, \leq_P)$ (and thus in $(P, \leq_P)$).*

Proof. Property (ii) in Definition 3.2 implies elements of $\partial_Q S$ are pairwise $\leq_P$-incomparable. □

4. Poset embeddings and partitions

Let $(P, \leq_P), (Q, \leq_Q)$ be posets. Recall that a *poset homomorphism* from $(P, \leq_P)$ to $(Q, \leq_Q)$ is a function $\phi : \ P \to Q$ such that for every $x, y \in P$, if $x \leq_P y$ then $\phi(x) \leq_Q \phi(y)$. We denote by $\hom_P(Q)$ the set of poset homomorphism from $(P, \leq_P)$ to $(Q, \leq_Q)$, and by $\mathrm{inj} - \hom_P(Q)$ the set of injective homomorphisms.

We begin by showing that the number of copies of a subposet P inside $\mathcal{P}(n)$ may be estimated by counting certain partitions of $[n] = \{1, 2, \dots, n\}$. Given integers m, n, recall that we write $[m]^{[n]}$ for the collection of ordered m-partitions of $[n]$, i.e. the collection of all m-tuples $(A_1, A_2, \dots, A_m)$, where the A_is are pairwise disjoint subsets of $[n]$ whose union is $[n]$. Further, recall that we write $[m]_\star^{[n]}$ for the collection of all $(A_1, A_2, \dots, A_m)$ in $[m]^{[n]}$ for which $A_j \neq \emptyset$ for all $j \in [m]$. Note that

(4.1)
$$|[m]^{[n]}| = m^n \qquad \text{and, for } m \text{ fixed} \qquad |[m]_\star^{[n]}| = \sum_{i=0}^{m-1} (m-i)^n \binom{m}{i} (-1)^i = m^n + O\left((m-1)^n\right).$$

Our goal in this section is to prove the following:

Theorem 4.1. *Let P be a fixed poset with $a(P) = m$. Then*
$$\left|\mathrm{inj} - \hom_P(\mathcal{P}(n))\right| = m^n + O\left((m-1)^n\right).$$

We shall prove this result by constructing two maps: an injection from $\hom_P(\mathcal{P}(n))$ into $[m]^{[n]}$ (Lemma 4.2), and an injection from $[m]_\star^{[n]}$ into $\mathrm{inj} - \hom_P(\mathcal{P}(n))$ (Lemma 4.5). In addition to proving Theorem 4.1, these maps will play an important role when determining the existence threshold for copies of P in $\mathcal{P}(n)$, by establishing a correspondence between certain "weighted" copies of P and certain "weighted" m-partitions of $[n]$. (In the language of Section 2 we mean the correspondence between copies of P in $\mathcal{P}(n)$ of a given type and the associated partition of $[n]$.)

Let $(P, \leq_P)$ be a poset with $a(P) = m$. Assume without loss of generality that $P = [N]$, and that the antichains of P are enumerated as
$$\mathcal{A}(P) = \{S_1, S_2, \dots, S_m\},$$
where $S_m = \emptyset$ is the empty antichain. Suppose we are given $\phi \in \hom_P(\mathcal{P}(n))$. We use ϕ to build an m-partition of $[n]$ as follows.

(1) For $i \in [N]$, set $X_i := \phi(i)$.
(2) For every $i \in [N]$, set $Y_i := X_i \setminus \bigcup_{j <_P i} X_j$.
(3) For every $j \in [m-1]$, set $Z_j := \bigcap_{i \in S_j} Y_i$.
(4) For every $j \in [m-1]$, set $A_j := Z_j \setminus \left(\bigcup_{S_k \in \mathrm{Ext}(S_j)} Z_k\right)$.
(5) Finally, set $A_m := [n] \setminus \left(\bigcup_{j \in [m-1]} A_j\right)$.

So X_i is the subset of $[n]$ that ϕ maps i to; Y_i is the subset of $[n]$ that contains all elements of X_i that do not lie in any other X_j where $j <_P i$; given a non-empty antichain S_j, Z_j is set of elements of $[n]$ that lie in every Y_i, for each i in the antichain S_j. Now define $f_1(\phi) := (A_1, A_2, \dots, A_m)$.

Lemma 4.2. *The map f_1 is an injection* $\hom_P(\mathcal{P}(n)) \to [m]^{[n]}$.

Proof. We split the proof into two claims, from which the lemma is immediate.

Claim 4.3. *For every $\phi \in \hom_P(\mathcal{P}(n))$, $f_1(\phi) \in [m]^{[n]}$.*

Proof. The definition of A_m ensures that $\bigcup_{j \in [m]} A_j = [n]$, so all we need to show is that the A_j are pairwise disjoint. Consider A_{j_1} and A_{j_2} with j_1, j_2 distinct elements of $[m]$. Clearly A_m is disjoint from every other A_i so we may assume that $j_1, j_2 \leq m-1$.

If $S_{j_1} \in \mathrm{Ext}(S_{j_2})$, then by definition (step 4) A_{j_2} is disjoint from $Z_{j_1} \supseteq A_{j_1}$. We are similarly done if $S_{j_2} \in \mathrm{Ext}(S_{j_1})$. Thus we may assume that $S_{j_1} \not\subseteq S_{j_2}$ and $S_{j_2} \not\subseteq S_{j_1}$. In particular, there exist $i_1 \in S_{j_1} \setminus S_{j_2}$ and $i_2 \in S_{j_2} \setminus S_{j_1}$.

Suppose for a contradiction that there exists some $x \in [n]$ with $x \in A_{j_1} \cap A_{j_2}$. Then by definition (steps 3 and 4) $x \in A_{j_1} \subseteq Z_{j_1} = \bigcap_{i \in S_{j_1}} Y_i$. In particular we must have $x \in Y_{i_1}$. Similarly we have $x \in Y_{i_2}$. By definition (step 2), this implies the elements i_1 and i_2 are incomparable in $(P, \leq_P)$.

Now consider $S_{j_3} := S_{j_1} \cup S_{j_2}$. The paragraph above established that elements in $S_{j_1} \setminus S_{j_2}$ and $S_{j_2} \setminus S_{j_1}$ are mutually incomparable. Together with the fact that S_{j_1} and S_{j_2} are antichains, this implies that S_{j_3} is an antichain.

Thus S_{j_3} is an antichain which extends both of S_{j_1} and S_{j_2}. Further S_{j_3} is distinct from both S_{j_1} (since $i_2 \in S_{j_2} \setminus S_{j_1} \subseteq S_{j_3} \setminus S_{j_1}$) and S_{j_2} (since $i_1 \in S_{j_3} \setminus S_{j_2}$). What is more, since $x \in A_{j_1} \subseteq Z_{j_1} = \bigcap_{i \in S_{j_1}} Y_i$, we have that $x \in Y_i$ for every $i \in S_{j_1}$, and similarly $x \in Y_i$ for every $i \in S_{j_2}$. This implies that $x \in Z_{j_3} = \bigcap_{i \in S_{j_3}} Y_i$. Since $S_{j_3} \in \mathrm{Ext}(S_{j_1})$ and by definition (step 4) we have $A_{j_1} \subseteq Z_{j_1} \setminus Z_{j_3} \subseteq [n] \setminus \{x\}$, and $x \notin A_{j_1}$, which gives the desired contradiction. $\square$

Claim 4.4. *f_1 is injective.*

Proof. Let ϕ, ϕ' be distinct elements of $\hom_P(\mathcal{P}(n))$. Let X_i, Y_i, Z_j, A_j and X_i', Y_i', Z_j', A_j' be the families of subsets of $[n]$ in steps 1–4 of the definition of f_1 applied to ϕ and ϕ' respectively.

Since $\phi \neq \phi'$, there must be a $\leq_P$-minimal element $i \in P$ such that $\phi(i) = X_i \neq X_i' = \phi'(i)$ and for all $k <_P i$, $X_k = X_k'$. The symmetric difference of $X_i \triangle X_i'$ is nonempty and we may assume without loss of generality that there exists $x \in [n]$ with $x \in X_i \setminus X_i'$. By our $\leq_P$-minimality assumption, we have $x \notin X_k$ for all $k <_P i$, and thus $x \in Y_i$.

Let S_{j_1} be the antichain consisting of the singleton $\{i\}$. By definition (step 3) , we have $x \in Z_{j_1} = Y_i$. Now let S_{j_2} be a $\subseteq$-maximal antichain from $\mathrm{Ext}(S_{j_1}) \cup \{S_{j_1}\}$ with $x \in Z_{j_2}$ (i.e. $x \in Z_{j_2}$ and for every $S_{j_3} \in \mathrm{Ext}(S_{j_2})$, we have $x \notin Z_{j_3}$). By definition (step 4), we have $x \in A_{j_2}$. On the other hand, since $i \in S_{j_1} \subseteq S_{j_2}$, we have by definition (steps 4, 3) that $A_{j_2}' \subseteq Z_{j_2}' \subseteq Y_i' \subseteq X_i'$. Since $x \notin X_i'$, $x \notin A_{j_2}'$ and hence $A_{j_2} \neq A_{j_2}'$. The partitions $f_1(\phi)$ and $f_1(\phi')$ are thus different, as claimed. $\square$

$\square$

Now suppose we are given a partition $\mathbf{A} = (A_1, A_2, \ldots, A_m) \in [m]_\star^{[n]}$. We use $\mathbf{A}$ to build an injective poset homomorphism $\phi : P \to \mathcal{P}(n)$ by letting $\phi(i) = X_i$, where X_i defined in (2.1). Set $f_2(\mathbf{A}) := \phi$.

Lemma 4.5. *The map f_2 is an injection $[m]_\star^{[n]} \to \mathrm{inj} - \hom_P(\mathcal{P}(n))$. Moreover, for each $\mathbf{A} \in [m]_\star^{[n]}$, $f_2(A)$ is an induced copy of P in $\mathcal{P}(n)$.*

Proof. Again we split the proof of the lemma into two claims.

Claim 4.6. *For every $\mathbf{A} \in [m]_\star^{[n]}$, $\phi = f_2(\mathbf{A}) \in \mathrm{inj} - \hom_P(\mathcal{P}(n))$. Moreover $\phi(P)$ is an induced copy of P in $\mathcal{P}(n)$.*

Proof. Let $\mathbf{A} \in [m]_\star^{[n]}$, and let $\phi = f_2(\mathbf{A})$. If $i' \leq_P i$, then $D(i') \subseteq D(i)$, which by construction of ϕ implies $X_{i'} \subseteq X_i$. Thus $\phi : i \mapsto X_i$ is a poset homomorphism from $(P, \leq_P)$ to $(\mathcal{P}(n), \subseteq)$ as claimed.

It remains to show that ϕ is an injection and that $\phi(P)$ is an induced copy of P in $\mathcal{P}(n)$. It suffices to show that for $i_1, i_2 \in P$, if $i_1 \not\leq_P i_2$, then $X_{i_1} \not\subseteq X_{i_2}$. Indeed, assuming this fact, if $i_1 \neq i_2$, then either $i_1 \not\leq_P i_2$ or $i_2 \not\leq_P i_1$, and in either case, we have $X_{i_1} \neq X_{i_2}$; in particular, this ensures ϕ is injective and that strictly comparable pairs are mapped to strictly comparable pairs. The fact also guarantees incomparable pairs are mapped to incomparable pairs, as desired.

To prove this fact assume that $i_1 \not\leq_P i_2$. Then $i_1 \notin D(i_2)$. Let S_j be the antichain $\{i_1\}$. Then $S_j \cap D(i_1) \neq \emptyset$ and $S_j \cap D(i_2) = \emptyset$. It follows that $A_j \subseteq X_{i_1}$ and $A_j \cap X_{i_2} = \emptyset$. Since $\mathbf{A} \in [m]_\star^{[n]}$, we have $A_j \neq \emptyset$. This implies that $X_{i_1} \not\subseteq X_{i_2}$, as claimed. $\square$

Claim 4.7. *f_2 is injective.*

Proof. Let $\mathbf{A} = (A_1, \dots, A_m)$ and $\mathbf{A}' = (A'_1, \dots, A'_m)$ be distinct partitions from $[m]^{[n]}_\star$, and let $\phi = f_2(\mathbf{A})$ and $\phi' = f_2(\mathbf{A}')$. We claim $\phi \neq \phi'$.

If $A_m \neq A'_m$, then $\bigcup_{i\in P} \phi(i) = [n] \setminus A_m \neq [n] \setminus A'_m = \bigcup_{i\in P} \phi'(i)$, and hence $\phi \neq \phi'$ as required. Assume therefore that $A_m = A'_m$. Since $\mathbf{A}, \mathbf{A}'$ are distinct partitions of $[n]$ in which every part is non-empty, there exists $x \in [n]$ and distinct elements $j_1, j_2 \in [m-1]$ such that $x \in A_{j_1} \setminus A'_{j_1}$ and $x \in A'_{j_2} \setminus A_{j_2}$. Now S_{j_1}, S_{j_2} are distinct non-empty antichains in P. Assume without loss of generality that there exists some element $i_1 \in P$ with $i_1 \in S_{j_1} \setminus S_{j_2}$.

First assume that $S_{j_2} \cap D(i_1) = \emptyset$. By the definition of ϕ, $X'_{i_1} \subseteq [n] \setminus A'_{j_2} \subseteq [n] \setminus \{x\}$. On the other hand, since $S_{j_1} \cap D(i_1) \neq \emptyset$, we have $x \in A_{j_1} \subseteq X_{i_1}$. Thus $\phi(i_1) = X_{i_1} \neq X'_{i_1} = \phi'(i_1)$.

Second assume that there exists $i_2 \in S_{j_2}$ with $i_2 <_P i_1$, then by definition of f_2 we have $x \in A'_{j_2} \subseteq X'_{i_2}$. On the other hand for every $i' \leq_P i_2$ we have $i' <_P i_1$ and thus $i' \notin S_{j_1}$, implying $X_{i_2} \subseteq [n] \setminus A_{j_1}$. In particular $x \notin X_{i_2}$, and thus $\phi'(i_2) = X'_{i_2} \neq X_{i_2} = \phi(i_2)$.

It follows that ϕ and ϕ' are distinct members of $\hom_P(\mathcal{P}(n))$, as claimed. $\square$

$\square$

Remark 4.8. *The proof of Lemma 4.5 shows a little more, namely, f_2 remains an injection when viewed as a function $[m]^{[n]} \to \hom_P(\mathcal{P}(n))$ (because we only used $\mathbf{A} \in [m]^{[n]}_\star$ when showing that $f_2(\mathbf{A})$ is an injective poset homomorphism). It is not hard to show f_2 is in fact the inverse of f_1: for every $\mathbf{A} \in [m]^{[n]}$, we have $f_1(f_2(\mathbf{A})) = \mathbf{A}$.*

Proof of Theorem 4.1. By Lemmas 4.5 and 4.2

$$|[m]^{[n]}_\star| \leq |\mathrm{inj}-\hom_P(\mathcal{P}(n))| \leq |\hom_P(\mathcal{P}(n))| \leq |[m]^{[n]}|.$$

The theorem then follows from the estimates (4.1). $\square$

5. Shadows and weighted partitions

Building on the work in the previous section, we investigate weighted partitions and their interaction with shadows. Write $\triangle_m$ for the m-simplex

$$\triangle_m := \left\{ (\alpha_1, \alpha_2, \dots, \alpha_m) : \ \forall i,\ \alpha_i \geq 0 \text{ and } \sum_{i=1}^{m} \alpha_i = 1 \right\}.$$

We write $\triangle^*_m$ for the set of all $\boldsymbol{\alpha} \in \triangle_m$ whose coordinates are all non-zero. We follow the convention of using $\boldsymbol{\alpha}$ to denote the vector $(\alpha_1, \alpha_2, \dots, \alpha_m) \in \triangle_m$ (and vice versa). Similarly we use $\mathbf{A}$ to denote the partition $(A_1, A_2, \dots, A_m) \in [m]^{[n]}$ (and vice versa).

Definition 5.1 (Weighted partitions)**.** *The* weighting *of $\mathbf{A} \in [m]^{[n]}$ is*

$$w(\mathbf{A}) := \left(\frac{|A_1|}{n}, \frac{|A_2|}{n}, \dots, \frac{|A_m|}{n} \right).$$

The weighting $w(\mathbf{A})$ is an element of $\triangle_m$. Given $\boldsymbol{\alpha} \in \triangle_m$, we say that $\mathbf{A}$ is $\boldsymbol{\alpha}$-weighted *if $w(\mathbf{A}) = \boldsymbol{\alpha}$. We denote by $[m]^{[n]}_{\boldsymbol{\alpha}}$ the collection of all $\boldsymbol{\alpha}$-weighted $\mathbf{A} \in [m]^{[n]}$, and we say $\boldsymbol{\alpha}$ is* feasible *for n if this collection is non-empty (that is, if $n\boldsymbol{\alpha} \in (\mathbb{Z}_{\geq 0})^m$).*

Definition 5.2 (ε-close weightings)**.** *Given $\varepsilon > 0$, we say that two elements $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \triangle_m$ are* ε-close *if*

$$\|\boldsymbol{\alpha} - \boldsymbol{\beta}\|_\infty := \max\Big\{ |\alpha_i - \beta_i| : \ 1 \leq i \leq m \Big\} \leq \varepsilon.$$

We also say that a partition $\mathbf{A} \in [m]^{[n]}$ *is* ε-close *to being* $\boldsymbol{\alpha}$*-weighted if its weighting* $w(\mathbf{A})$ *is* ε*-close to* $\boldsymbol{\alpha}$*. We denote by* $[m]^{[n]}_{\boldsymbol{\alpha}\pm\boldsymbol{\varepsilon}}$ *the collection of all such* $\mathbf{A}$*. We say* $\boldsymbol{\alpha} \pm \boldsymbol{\varepsilon}$ *is* feasible *for* n *if this collection is non-empty.*

Definition 5.3. *Let* $(P, \leq_P)$ *be a poset together with a labelling of its antichains as* $\mathcal{A}(P) = \{S_1, S_2, \ldots, S_m\}$ *(where* S_m *is the empty antichain). Suppose we are given an ordered* m*-partition of* $[n]$*,* $\mathbf{A} = (A_1, A_2, \ldots, A_m)$*, with each set* A_i *associated to the antichain* S_i*. Given* $Q \subseteq P$ *and a labelling of the antichains in* $(Q, \leq_P)$ *as* $\mathcal{A}(Q) = \{T_1, T_2, \ldots, T_M\}$ *(where* T_M *is the empty antichain), the* M-partition of $[n]$ inherited from $\mathbf{A}$ *is* $\mathbf{B} = (B_1, B_2, \ldots, B_M)$*, where*

$$B_i := \bigcup_{j:\ \partial_Q S_j = T_i} A_j.$$

(Note that $\mathbf{B}$ *is a partition of* $[n]$ *by Proposition 3.3.) We call* $\mathbf{B}$ *the* Q-shadow of $\mathbf{A}$*, and denote it by* $\partial_Q(\mathbf{A})$*. Note that* $A_m \subseteq B_M$*.*

Observe that if $\mathbf{A}$ is $\boldsymbol{\alpha}$-weighted, then its Q-shadow $\mathbf{B} = \partial_Q(\mathbf{A})$ is $\boldsymbol{\beta}$-weighted, where $\boldsymbol{\beta}$ is given by

$$\beta_i := \sum_{j:\ \partial_Q S_j = T_i} \alpha_j \qquad \forall i \in [M]. \tag{5.1}$$

We call $\boldsymbol{\beta}$ the *weighting induced by* $\boldsymbol{\alpha}$ *in* Q, or *shadow* of $\boldsymbol{\alpha}$ in Q, and denote it by $\boldsymbol{\beta} =: \partial_Q(\boldsymbol{\alpha})$.

We can relate Q-shadows to our injection $f_1 :\ \mathrm{inj}-\mathrm{hom}_P(\mathcal{P}(n)) \to [m]^{[n]}$ as follows. Given $\phi \in \mathrm{inj}-\mathrm{hom}_P(\mathcal{P}(n))$, let $\phi_{|Q}$ denote the restriction of ϕ to Q (which is an element of $\mathrm{inj}-\mathrm{hom}_Q(\mathcal{P}(n))$). In a slight abuse of notation, we let $f_1(\phi_{|Q})$ denote the image of $\phi_{|Q}$ under f_1 defined with respect to Q.

Proposition 5.4. *Let* $(P, \leq_P)$ *be a poset, and let* $Q \subseteq P$*. Then for every* $\phi \in \mathrm{inj}-\mathrm{hom}_P(\mathcal{P}(n))$*, we have*

$$\partial_Q(f_1(\phi)) = f_1(\phi_{|Q}).$$

Proof. Assume without loss of generality that $P = [N]$ and $Q = [q] \subseteq [N]$. Further let $\mathcal{A}(P) = \{S_1, S_2, \ldots, S_m\}$ and $\mathcal{A}(Q) = \{T_1, T_2, \ldots, T_M\}$ be enumerations of the antichains in $(P, \leq_P)$ and $(Q, \leq_P)$ respectively with $S_m = T_M = \emptyset$ being the empty antichain.

To prove the proposition, we need to revisit the construction of f_1 and introduce some notation. Let X_i, Y_i, Z_j and A_j be as in the construction of f_1. Now

- for every $i \in [q]$, set $\tilde{X}_i := \phi_{|Q}(i)$ (note $\tilde{X}_i = X_i$),
- for every $i \in [q]$ set $\tilde{Y}_i := \tilde{X}_i \setminus \{\tilde{X}_j :\ j \in Q,\ j <_P i\}$,
- for every $j \in [M-1]$ set $\tilde{Z}_j := \bigcap_{i \in T_j} \tilde{Y}_i$,
- for every $j \in [M-1]$ set $\tilde{A}_j := \tilde{Z}_j \setminus \left(\bigcup_{T_k \in \mathrm{Ext}_Q(T_j)} \tilde{Z}_k\right)$,

and finally set $\tilde{A}_M := [n] \setminus \left(\bigcup_{j \in [M-1]} \tilde{A}_j\right)$. To prove the proposition we must show that for every $k \in [M]$, $\tilde{A}_k = \bigcup_{j:\ \partial_Q S_j = T_k} A_j$. To do this, we must first establish an important property of the partition $\mathbf{A} = (A_1, A_2, \ldots, A_m)$.

Claim 5.5. *For every* $j \in [m-1]$ *and every* $x \in A_j$*, we have that* $x \in \phi(i) = X_i$ *if and only if* $i' \leq_P i$ *for some* $i' \in S_j$*.*

Proof. Suppose $x \in A_j$. By construction of f_1, for every $i' \in S_j$ we have $x \in X_{i'} = \phi(i')$. As ϕ is a poset homomorphism, this implies $x \in X_i = \phi(i)$ for every $i \in P$ with $i' \leq_P i$.

For the reverse implication: let $i \in P \setminus S_j$ be such that $x \in X_i$ (we are done if $i \in S_j$). Without loss of generality, we may assume i is $\leq_P$-minimal with that property; that is, for every $i' \in P \setminus S_j$ with $i' <_P i$, we have $x \notin X_{i'}$.

Since $x \in A_j \subseteq \bigcap_{i' \in S_j} Y_{i'}$, we must have $x \in Y_{i'}$ for every $i' \in S_j$. In particular $x \notin X_{i''}$ for any $i'' \in P$ with $i'' <_P i'$ for some $i' \in S_j$. Thus i cannot be below any element of S_j in the partial order $\leq_P$.

Suppose for contradiction that i was incomparable with every element of S_j. Then $S_{j'} = S_j \cup \{i\}$ is an antichain in P extending S_j. Further, since $i' \not\leq_P i$ for any $i' \in S_j$ and since $x \notin X_{i'}$ for any $i' \in P \setminus S_j$ with $i' <_P i$ (by our minimality assumption on i), we have that $x \in Y_i$. We already know $x \in Z_j$, so we deduce that $x \in Z_{j'} = Z_j \cap Y_i$. But this implies $x \notin A_j \subseteq Z_j \setminus Z_{j'}$, a contradiction.

It follows that $i' \leq_P i$ for some $i' \in S_j$, as claimed. □

Claim 5.6. *Let $j \in [m]$. If $\partial_Q S_j = T_k$, then $A_j \subseteq \tilde{A}_k$.*

Proof. First suppose that $j = m$. In Remark 4.8 we observed that f_2 is the inverse of f_1. In particular, by definition of f_2, A_m is precisely the set of $x \in [n]$ such that $x \notin \phi(i)$ for all $i \in P = [N]$. But for each $i \in Q$ we have that $\phi(i) = X_i = \tilde{X}_i$. Thus by definition of the $\tilde{A}_{j'}$, this means $x \in A_m$ cannot be an element in $\tilde{A}_{j'}$ for any $j' \in [M-1]$. That is, $A_m \subseteq \tilde{A}_M$. Now by definition $\partial_Q S_M = \emptyset = T_M$. So this proves the claim in this case.

We may therefore assume that $j < m$. Now suppose that $\partial_Q S_j = T_k = \emptyset$, i.e. that $k = M$. Suppose for a contradiction that there exists $x \in [n]$ such that $x \in A_j$ and $x \notin \tilde{A}_M$. Then $x \in \tilde{A}_{k'}$ for some $k' < M$; this further implies $x \in \tilde{X}_i = \phi(i)$ for some $i \in [q] = Q$. We may assume i is $\leq_Q$-minimal in Q with respect to this property. By Claim 5.5, there is some $i' \in S_j$ such that $i' \leq_P i$. This property together with the definition of $\partial_Q S_j$ (and our assumption of $\leq_Q$-minimality for i) ensures $\partial_Q S_j \neq \emptyset$, a contradiction. Therefore $x \in \tilde{A}_M$. Thus $A_j \subseteq \tilde{A}_M$, as desired.

Finally, suppose $T_k = \partial_Q S_j$ with $T_k, S_j \neq \emptyset$. By definition of $T_k = \partial_Q S_j$, for every $i' \in T_k$ there exists $i \in S_j$ with $i \leq_P i'$. By Claim 5.5, this implies $x \in \tilde{X}_{i'} = X_{i'}$ for every $i' \in T_k$.

Now consider $i'' \in T_k$ and $i' \in Q \setminus T_k$ with $i' <_P i''$. If $x \in \tilde{X}_{i'} = X_{i'}$, then we must have $i \leq_P i'$ for some $i \in S_j$, contradicting $i'' \in T_k = \partial_Q S_j$. Thus $x \notin \tilde{X}_{i'}$ for any $i' \in Q$ such that $i' <_P i''$, and hence $x \in \tilde{Y}_{i''}$. This in turn implies $x \in \tilde{Z}_k = \bigcap_{i'' \in T_k} \tilde{Y}_{i''}$.

Now for any extension $T_{k'}$ of T_k in Q there exists $i'' \in T_{k'} \setminus T_k$. Suppose $x \in \tilde{X}_{i''} = X_{i''}$. By Claim 5.5, this implies there exists $i \in S_j$ with $i \leq_P i''$. Since $i'' \in Q$ and $i'' \notin T_k = \partial_Q(S_j)$, this implies there exists $i' \in T_k$ with $i' <_P i''$ by the definition of $\partial_Q(S_j)$, contradicting the fact that $T_{k'}$ is an antichain. Thus for any extension $T_{k'}$ of T_k in Q and every $i'' \in T_{k'} \setminus T_k$, $x \notin \tilde{X}_{i''}$, implying in turn that $x \notin \tilde{Z}_{k'}$.

In particular we have shown that $x \in \tilde{A}_k = \tilde{Z}_k \setminus \left(\bigcup_{T_{k'} \in \mathrm{Ext}_Q(T_k)} \tilde{Z}_{k'} \right)$. Since $x \in A_j$ was arbitrary, we have $A_j \subseteq \tilde{A}_k$ as claimed. □

As $f_1(\phi) = (A_1, A_2, \ldots, A_m)$ and $f_1(\phi_{|Q}) = (\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_M)$ both form partitions of $[n]$ (by Lemma 4.2), Claim 5.6 immediately implies that $\tilde{A}_k = \bigcup_{j : \partial_Q S_j = T_k} A_j$ for all $k \in [M]$. It follows that $\partial_Q(f_1(\phi)) = f_1(\phi_{|Q})$ as desired, proving the proposition. □

Having made these definitions and related shadows to partitions for subposets, our final goal in this section is to estimate the number of $\boldsymbol{\alpha}$-weighted partitions in $[m]^{[n]}$. For this purpose, we introduce the *entropy* of a weighting $\boldsymbol{\alpha} \in \triangle_m$ to be

$$H_m(\boldsymbol{\alpha}) := \sum_{j=1}^{m} -\alpha_j \log \alpha_j. \tag{5.2}$$

(Note that here log denotes the natural logarithm and $0\log 0 := 0$.) The entropy function H_m is a well-studied object in combinatorics and discrete probability. It has a maximum value of $\log m$ in $\triangle_m$, uniquely attained at $\boldsymbol{\alpha} = \left(\frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right)$.

Proposition 5.7. *Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \triangle_m$ be fixed.*

Then for any sequence $\varepsilon = \varepsilon(n) \to 0$ such that $\boldsymbol{\alpha} \pm \boldsymbol{\varepsilon}$ is feasible for every n, we have

$$\left|[m]^{[n]}_{\boldsymbol{\alpha}\pm\boldsymbol{\varepsilon}}\right| = \exp\Big(H_m(\boldsymbol{\alpha})n + O\left(\log n\right)\Big).$$

Proof. For every $\boldsymbol{\beta} \in \triangle_m$ which is feasible for n, we have by Stirling's estimate for the factorial that

$$\left|[m]^{[n]}_{\boldsymbol{\beta}}\right| = \binom{n}{\beta_1 n, \beta_2 n, \dots, \beta_{m-1} n} = \exp\Big(H_m(\boldsymbol{\beta})n + O(\log n)\Big). \tag{5.3}$$

Now there are at most

$$(2\varepsilon n + 1)^{m-1} = e^{O(\log \varepsilon n)} \tag{5.4}$$

weightings $\boldsymbol{b}$ which are both $\boldsymbol{\varepsilon}$-close to $\boldsymbol{\alpha}$ and feasible for m. Let $\mathcal{B}$ denote the family of all such $\boldsymbol{\beta}$. By continuity of the function H_m and the fact that $\varepsilon = o(1)$, for all $\boldsymbol{\beta} \in \mathcal{B}$ we have $H_m(\boldsymbol{\beta}) = H_m(\boldsymbol{\alpha}) + o(1)$. Putting it all together, we have

$$\left|[m]^{[n]}_{\boldsymbol{\alpha}\pm\boldsymbol{\varepsilon}}\right| = \sum_{\boldsymbol{\beta}\in\mathcal{B}} \left|[m]^{[n]}_{\boldsymbol{\beta}}\right| = \left|\mathcal{B}\right| \exp\Big(H_m(\boldsymbol{\alpha})n + O(\log n)\Big) = \exp\Big(H_m(\boldsymbol{\alpha})n + O(\log n)\Big),$$

as desired. □

6. The existence threshold

6.1. **The existence threshold.** In this section, we shall determine the existence threshold for copies of a fixed poset $(P, \leq_P)$ in a random subposet of $(\mathcal{P}(n), \subseteq)$. Explicitly, let $p = e^{-cn}$. We let $\mathcal{P}(n,p)$ be a p-random subset of $\mathcal{P}(n)$, obtained by retaining each element of $\mathcal{P}(n)$ independently at random with probability p. This gives rise to a random poset $(\mathcal{P}(n,p), \subseteq)$. For which c does this poset contain w.h.p. a copy of P — i.e. for which c is $\mathrm{inj}-\mathrm{hom}_P(\mathcal{P}(n,p))$ w.h.p. non-empty?

To answer this question, we need to introduce two definitions.

Definition 6.1. *Let $(P, \leq_P)$ be a poset with $a(P) = m$. Let $Q \subseteq P$ be a subposet of P with $Q \neq \emptyset$ and $a(Q) = M$. Given a weighting $\boldsymbol{\alpha} \in \triangle_m$, let $\boldsymbol{\beta} = \partial_Q(\boldsymbol{\alpha})$ be the weighting from $\triangle_M$ induced by $\boldsymbol{\alpha}$ in Q (that is, $\boldsymbol{\beta}$ is the weighting obtained when applying (5.1)). We define the* critical exponent *of Q in P with respect to $\boldsymbol{\alpha}$ to be*

$$c_{\boldsymbol{\alpha},P}(Q) := \frac{H_M(\boldsymbol{\beta})}{|Q|}.$$

Definition 6.2. *Let $(P, \leq_P)$ be a poset with $a(P) = m$. The* critical exponent *of P is defined to be*

$$c_\star(P) := \sup\Big\{c \in \mathbb{R}_{\geq 0} : \ \exists \boldsymbol{\alpha} \in \triangle_m \ s.t.\ \forall Q : \emptyset \neq Q \subseteq P,\ c \leq c_{\boldsymbol{\alpha},P}(Q)\Big\}.$$

Remark 6.3. *Equivalently, since $\triangle_m$ is compact and $\min\{c_{\boldsymbol{\alpha},P}(Q) :\ \emptyset \neq Q \subseteq P\}$ is a continuous function of $\boldsymbol{\alpha}$, we can express the critical exponent as:*

$$c_\star(P) = \max_{\boldsymbol{\alpha}\in\triangle_m} \min_{\emptyset\neq Q\subseteq P} c_{\boldsymbol{\alpha},P}(Q).$$

Theorem 6.4. *Let $(P, \leq_P)$ be a finite poset. For $c > 0$ fixed and $p = p(n) = e^{-cn}$, the following hold:*

(i) if $c > c_\star(P)$, then w.h.p. $(P, \leq_P)$ is not a subposet of $(\mathcal{P}(n,p), \subseteq)$;
(ii) if $c < c_\star(P)$, then w.h.p. $(P, \leq_P)$ is an induced subposet of $(\mathcal{P}(n,p), \subseteq)$.

The proof of parts (i) and (ii) of Theorem 6.4 occupy the next two subsections. Before we dive into these proofs, however, we should like to outline the main idea behind Theorem 6.4, which is to look at certain "weighted" copies of P in $\mathcal{P}(n)$.

Definition 6.5 (Weighted copies of posets)**.** *Let $(P, \leq_P)$ be a poset with $a(P) = m$. We define the* weight *of $\phi \in \mathrm{inj} - \mathrm{hom}_P(\mathcal{P}(n))$ to be*

$$w_\phi := w(f_1(\phi)),$$

*where f_1 is the injection $\mathrm{inj} - \mathrm{hom}_P(\mathcal{P}(n)) \to [m]^{[n]}$ given in Lemma 4.2 and w is the weighting from Definition 5.1. Given $\boldsymbol{\alpha} \in \triangle_m$, we say ϕ is an $\boldsymbol{\alpha}$-*weighted *copy of P in $\mathcal{P}(n)$ if $w_\phi = \boldsymbol{\alpha}$.*

By (5.3) the expected number of $\boldsymbol{\alpha}$-weighted copies of $(P, \leq_P)$ in $(\mathcal{P}(n,p), \subseteq)$ is

$$e^{H_m(\alpha)n+O(\log n)}p^{|P|} = e^{|P|\left(c_{\boldsymbol{\alpha},P}(P)-c\right)n+O(\log n)}.$$

Thus certainly for a fixed feasible weighting $\boldsymbol{\alpha}$, if $c > c_{\boldsymbol{\alpha},P}(P)$ then w.h.p. no such $\boldsymbol{\alpha}$-weighted copies exist. Further by Proposition 5.4, an $\boldsymbol{\alpha}$-weighted copy of P can only exist in $\mathcal{P}(n,p)$ if for every $Q \subseteq P$ a $\boldsymbol{\beta}$-weighted copy of Q exists, where $\boldsymbol{\beta} = \partial_Q(\boldsymbol{\alpha})$. This leads us to require $c \leq \min_{Q \subseteq P} c_{\boldsymbol{\alpha},P}(Q)$, and to the statement of the theorem.

6.2. **Proof of Theorem 6.4, part (i).** Let $(P, \leq_P)$ be a poset with $a(P) = m$. Suppose $c = c_\star(P) + \eta$, for some $\eta > 0$. For every $Q \subseteq P$, both $c_{\boldsymbol{\alpha},P}(Q)$ and $\partial_Q(\boldsymbol{\alpha})$ are continuous functions of $\boldsymbol{\alpha}$ in the compact set $\triangle_m$. Since there are finitely many subposets Q: $\emptyset \neq Q \subseteq P$, there exist constants $\varepsilon_1, \varepsilon_2 > 0$ such that if $\|\boldsymbol{\alpha} - \boldsymbol{\alpha}'\|_\infty < \varepsilon_1$, then for every $Q \subseteq P$

$$\|\partial_Q(\boldsymbol{\alpha}) - \partial_Q(\boldsymbol{\alpha}')\|_\infty < \varepsilon_2 \tag{6.1}$$

and

$$|c_{\boldsymbol{\alpha},P}(Q) - c_{\boldsymbol{\alpha}',P}(Q)| < \frac{\eta}{2} \tag{6.2}$$

both hold.

For $\boldsymbol{\alpha} \in \triangle_m$, denote by $B_{\varepsilon_1}(\boldsymbol{\alpha})$ the open ℓ_∞-ball in $\triangle_m$ of radius ε_1 centered at $\boldsymbol{\alpha}$. As $\triangle_m$ is compact, there exists some finite set $\mathcal{C} \subseteq \triangle_m$ such that the collection $\left\{B_{\varepsilon_1}(\boldsymbol{\alpha}) : \boldsymbol{\alpha} \in \mathcal{C}\right\}$ constitutes an open cover for $\triangle_m$. (In fact, since $\triangle_m$ has measure $1/m!$ and each $B_{\varepsilon_1}(\boldsymbol{\alpha})$ has measure at least $(\varepsilon_1/m)^{m-1}$, it is not hard to show that one can take $|\mathcal{C}| \leq \left(\frac{C}{\varepsilon_1}\right)^m$ for some absolute constant $C > 0$.)

Now fix $n \in \mathbb{N}$. For each $\boldsymbol{\alpha} \in \mathcal{C}$, let $\tilde{B}_{\boldsymbol{\alpha}}$ denote the collection of $\boldsymbol{\alpha}'$ in $B_{\varepsilon_1}(\boldsymbol{\alpha})$ which are feasible for n. Pick $\boldsymbol{\alpha} \in \mathcal{C}$. By definition of $c_\star(P)$, we have that $c_{\boldsymbol{\alpha},P}(Q) \leq c_\star(P)$ for some $Q \in P$. Set $M := a(Q)$. For every $\boldsymbol{\alpha}' \in \tilde{B}_{\boldsymbol{\alpha}}$, we have $\|\boldsymbol{\alpha}' - \boldsymbol{\alpha}\|_\infty < \varepsilon_1$. By (6.1) this implies that $\partial_Q(\boldsymbol{\alpha}')$ is ε_2-close to $\partial_Q(\boldsymbol{\alpha})$. Further by (6.2) we have

$$c_{\boldsymbol{\alpha}',P}(Q) \leq c_{\boldsymbol{\alpha},P}(Q) + \frac{\eta}{2} \leq c_\star(P) + \frac{\eta}{2} = c - \frac{\eta}{2}. \tag{6.3}$$

Combining (6.1), (5.4), (5.3), and (6.3), the expected number of $\boldsymbol{\beta}$-weighted injective poset homomorphisms $\phi \in \mathrm{inj} - \mathrm{hom}_Q(\mathcal{P}(n,p))$ with $\boldsymbol{\beta} \in \{\partial_Q(\boldsymbol{\alpha}') : \boldsymbol{\alpha}' \in \tilde{B}_{\boldsymbol{\alpha}}\}$ is at most

$$(2\varepsilon_2 n + 1)^{M-1} e^{|Q|\left(c-\frac{\eta}{2}\right)n+O(\log n)} p^{|Q|} = e^{-\frac{|Q|\eta}{2}n+O(\log n)} = o(1).$$

By Markov's inequality, we deduce that w.h.p. no such copy exists, which in turns implies by Proposition 5.4 that w.h.p. $\mathcal{P}(n,p)$ contains no $\boldsymbol{\alpha}'$-weighted copy of P for $\boldsymbol{\alpha}' \in \tilde{B}_{\boldsymbol{\alpha}}$. Since $\boldsymbol{\alpha} \in \mathcal{C}$ was arbitrary and $\mathcal{C}$ is finite, we deduce that w.h.p. $\mathcal{P}(n,p)$ contains no $\boldsymbol{\alpha}'$-weighted copy of P for $\boldsymbol{\alpha}' \in \bigcup_{\boldsymbol{\alpha} \in \mathcal{C}} \tilde{B}_{\boldsymbol{\alpha}}$. By construction of $\mathcal{C}$, this latter union covers all weightings $\boldsymbol{\alpha}' \in \triangle_m$ which are feasible for n. Thus we deduce that for $c = c_\star(P) + \eta$, $\eta > 0$, and $p = e^{-cn}$ the random poset $\mathcal{P}(n,p)$ is w.h.p. P-free. This concludes the proof of Theorem 6.4, part (i). □

6.3. **Proof of Theorem 6.4, part (ii).** Let $(P, \leq_P)$ be a poset with $a(P) = m$. Suppose $c = c_\star(P) - \eta$ for some $\eta > 0$. By definition of $c_\star(P)$, there exists $\boldsymbol{\alpha}_\star \in \triangle_m$ such that for all Q: $\emptyset \neq Q \subseteq P$, we have $c_{\boldsymbol{\alpha}_\star,P}(Q) \geq c_\star(P)$. When n is sufficiently large there exists a weighting $\boldsymbol{\alpha}_f = \boldsymbol{\alpha}_f(n) \in \triangle_m^*$ (i.e. $\boldsymbol{\alpha}_f$ has non-zero coordinates) such that $\boldsymbol{\alpha}_f$ is feasible for n and $\|\boldsymbol{\alpha}_f - \boldsymbol{\alpha}_\star\|_\infty < \varepsilon_1$, where $\varepsilon_1 > 0$ is the constant given in Section 6.2. By (6.2), for all non-empty $Q \subseteq P$, we have

$$c_{\boldsymbol{\alpha}_f,P}(Q) \geq c_\star(P) - \frac{\eta}{2} = c + \frac{\eta}{2}. \tag{6.4}$$

Write $\Phi_{\boldsymbol{\alpha}_f,Q}(\mathcal{P}(n))$ and $\Phi_{\boldsymbol{\alpha}_f,Q}(\mathcal{P}(n,p))$ for the collections of all $\partial_Q(\boldsymbol{\alpha}_f)$-weighted $\phi \in \text{inj}-\text{hom}_Q(\mathcal{P}(n))$ and $\phi \in \text{inj}-\text{hom}_Q(\mathcal{P}(n,p))$ respectively. By (5.3), for all non-empty $Q \subseteq P$,

$$\mathbb{E}\big|\Phi_{\boldsymbol{\alpha_f},Q}(\mathcal{P}(n,p))\big| = \big|\Phi_{\boldsymbol{\alpha_f},Q}(\mathcal{P}(n))\big|p^{|Q|} = e^{|Q|c_{\boldsymbol{\alpha}_f,P}(Q)n+O(\log n)}e^{-c|Q|n} \overset{(6.4)}{\geq} e^{\frac{|Q|\eta}{2}n+O(\log n)}, \tag{6.5}$$

which tends to infinity as $n \to \infty$. We shall use the celebrated Janson inequalities to show $\mathbb{P}\Big(\big|\Phi_{\boldsymbol{\alpha_f},P}(\mathcal{P}(n,p))\big| = 0\Big)$ is small. To do this, we must first introduce some notation.

Given $\phi_1, \phi_2 \in \Phi_{\boldsymbol{\alpha_f},P}(\mathcal{P}(n))$ and non-empty $Q_1, Q_2 \subseteq P$, we say that the pair (ϕ_1, ϕ_2) is (Q_1, Q_2)-*intersecting* if $\phi_1(Q_1) = \phi_2(Q_2) = \phi_1(P) \cap \phi_2(P)$. Let $I(Q_1, Q_2)$ denote the collection of all (Q_1, Q_2)-intersecting pairs (ϕ_1, ϕ_2), and define

$$D := \sum_{Q_1,Q_2 \neq \emptyset} \left(\sum_{(\phi_1,\phi_2) \in I(Q_1,Q_2)} \mathbb{P}\Big(\phi_1(P) \cup \phi_2(P) \subseteq \mathcal{P}(n,p)\Big) \right).$$

Set $\mu := \mathbb{E}\big|\Phi_{\boldsymbol{\alpha_f},P}(\mathcal{P}(n,p))\big|$. We now apply the following inequalities due to Janson (see e.g. [14]).

Proposition 6.6 (Janson inequalities)**.**

$$\mathbb{P}\Big(\big|\Phi_{\boldsymbol{\alpha_f},P}(\mathcal{P}(n,p))\big| = 0\Big) \leq \begin{cases} e^{-\frac{\mu^2}{2D}} & \textit{if } D \geq \mu, \\ e^{-\mu+\frac{D}{2}} & \textit{otherwise.} \end{cases}$$

It thus remains to bound D. Observe first of all that

$$I(Q_1, Q_2) = \big|\Phi_{\boldsymbol{\alpha_f},P}(\mathcal{P}(n))\big|^2 \Big/ \big|\Phi_{\boldsymbol{\alpha_f},Q_1}(\mathcal{P}(n))\big|, \tag{6.6}$$

since each copy of P in $\mathcal{P}(n)$ specifies a unique copy of Q_1 and Q_2, and since all copies 'look the same', being given by partitions with exactly the same weight. Next, note that for any (Q_1, Q_2)-intersecting pair (ϕ_1, ϕ_2) we have

$$\mathbb{P}\Big(\phi_1(P) \cup \phi_2(P) \subseteq \mathcal{P}(n,p)\Big) = p^{2|P|-|Q_1|}. \tag{6.7}$$

Putting (6.6), (6.7) and (6.5) together, we have

$$\sum_{(\phi_1,\phi_2) \in I(Q_1,Q_2)} \mathbb{P}\Big(\phi_1(P) \cup \phi_2(P) \subseteq \mathcal{P}(n,p)\Big) = \frac{\mu^2}{\mathbb{E}\big|\Phi_{\boldsymbol{\alpha_f},Q_1}(\mathcal{P}(n,p))\big|} \leq \mu^2 e^{-\frac{|Q_1|\eta}{2}n+O(\log n)}.$$

Plugging this back into the definition of D we have

$$D \leq \sum_{Q_1 \subseteq P:\ Q_1 \neq \emptyset} \ \sum_{Q_2 \subseteq P:\ |Q_2|=|Q_1|} \mu^2 e^{-\frac{|Q_1|\eta}{2}n+O(\log n)} < 2^{2|P|}\mu^2 e^{-\frac{\eta}{2}n+O(\log n)}.$$

In particular, we have $\exp\big(-\mu^2/2D\big) = o(1)$. When $D < \mu$, since we showed in (6.5) that $\mu \to \infty$ (as $n \to \infty$), we also have $\exp(-\mu + D/2) \leq \exp(-\mu/2) = o(1)$. It then follows from the Janson inequalities (Proposition 6.6) that

$$\mathbb{P}\Big(\big|\Phi_{\boldsymbol{\alpha_f},P}(\mathcal{P}(n,p))\big| = 0\Big) = o(1).$$

Thus for $c = c_\star(P) - \eta$, $\eta > 0$, and $p = e^{-cn}$, the random poset $\mathcal{P}(n,p)$ contains w.h.p. the image of an $\boldsymbol{\alpha}_f$-weighted injective poset homorphism $\phi :\ P \to \mathcal{P}(n)$. In particular, P is w.h.p. a subposet of $\mathcal{P}(n,p)$.

Moreover, recall that $\boldsymbol{\alpha}_f \in \triangle_m^*$. That is, the weight of the copy P' of P we obtain in $\mathcal{P}(n,p)$ lies in $[m]^*$. Since f_2 is the inverse of f_1, the moreover part of Lemma 4.5 implies that P' is in fact an induced copy of P. This concludes the proof of Theorem 6.4, part (ii). □

Remark 6.7. *The proof of Theorem 6.4 parts (i) and (ii) shows something slightly stronger than we claimed. Namely, instead of having $\eta > 0$ fixed, we can run through the same arguments with $\eta = K \log n/n$ for some sufficiently large constant $K > 0$. A little analysis shows we can take values $\varepsilon_1, \varepsilon_2 = O(\eta)$ and still satisfy* (6.1) *and* (6.2)*. Using the bound $|\mathcal{C}| = O\left((\varepsilon_1)^{-m}\right)$ and choosing K sufficiently large to beat the $O(\log n)$ error terms, one then gets that*

- *if $p \leq e^{-c_\star(P)n - K\log n}$, then w.h.p. $\mathcal{P}(n,p)$ contains no copy of P;*
- *if $p \geq e^{-c_\star(P)n + K\log n}$, then w.h.p. $\mathcal{P}(n,p)$ contains an induced copy of P.*

Remark 6.8. *The proof of Theorem 6.4 can also be used to derive a more general result about the existence of induced copies of P with a specific embedding in $\mathcal{P}(n)$. Given a weighting $\boldsymbol{\alpha}$, we may define $c_{\boldsymbol{\alpha}}(P) := \min_{\emptyset \neq Q \subseteq P} c_{\boldsymbol{\alpha},P}(Q)$. Our proof demonstrates that $c_{\boldsymbol{\alpha}}(P)$ is the threshold for the existence of an $\boldsymbol{\alpha}$-weighted copy of P in $(\mathcal{P}(n,p), \subseteq)$.*

Here we also note that since $c_{\boldsymbol{\alpha}}(P)$ is continuous in $\boldsymbol{\alpha}$, there will be many different embeddings of $\mathcal{P}$ in $(\mathcal{P}(n,p), \subseteq)$ as soon as p is strictly larger than the value given by $c_\star(P)$.

7. Computing $c_\star(P)$ in practice: general bounds and heuristics

Theorem 6.4 gives us the location of the threshold for the appearance of copies of a given poset $(P, \leq_P)$ inside the random poset $(\mathcal{P}(n,p), \subseteq)$ in terms of the parameter

$$c_\star(P) = \max_{\boldsymbol{\alpha} \in \triangle_{a(P)}} \left(\min_{\emptyset \neq Q \subseteq P} \frac{H_{a(Q)}(\partial_Q(\boldsymbol{\alpha}))}{|Q|} \right). \tag{7.1}$$

In practice, this parameter is somewhat awkward to compute by hand, even for very small examples. Of course, as it is the maximum of a continuous function over a compact set, we can obtain good computational approximations for its value — though one should note that the complexity will certainly grow exponentially in P (since we are optimising the value of $\boldsymbol{\alpha}$ over an $a(P)$-dimensional space, and minimising over all $2^{|P|} - 1$ nonempty subsets $Q \subseteq P$).

In this section we prove some general bounds on $c_\star(P)$ and discuss heuristics for computing its value exactly — heuristics that in particular were used to determine $c_\star(P)$ for the examples in the next section. We begin by establishing some useful properties of the set of optimal weightings. Given a poset P, let $Opt(P)$ denote the collection of weightings $\boldsymbol{\alpha} \in \triangle_{a(P)}$ for which equality is attained in (7.1), i.e. such that $c_\star(P) = \min\Big\{c_{\boldsymbol{\alpha},P}(Q) :\ \emptyset \neq Q \subseteq P\Big\}$.

Proposition 7.1. *$Opt(P)$ is a convex subset of $\triangle_{a(P)}$.*

Proof. Suppose $\boldsymbol{\alpha}, \boldsymbol{\beta} \in Opt(P)$. Given $\theta \in [0,1]$, consider the weighting $\boldsymbol{\gamma} = \theta\boldsymbol{\alpha} + (1-\theta)\boldsymbol{\beta}$. Clearly $\boldsymbol{\gamma} \in \triangle_{a(P)}$. Fix any subposet Q with $\emptyset \neq Q \subseteq P$. Since $\partial_Q :\ \boldsymbol{x} \mapsto \delta_Q(\boldsymbol{x})$ is a linear operator from $\triangle_{a(P)}$ to $\triangle_{a(Q)}$ and since the map $x \mapsto -x\log x$ is concave, we have

$$\begin{aligned} c_{\boldsymbol{\gamma},P}(Q) &= \frac{1}{|Q|} H_{a(Q)}(\partial_Q(\boldsymbol{\gamma})) = \frac{1}{|Q|} H_{a(Q)}\left(\theta\partial_Q\left(\boldsymbol{\alpha}\right) + (1-\theta)\partial_Q\left(\boldsymbol{\beta}\right)\right) \\ &\geq \frac{1}{|Q|}\left(\theta H_{a(Q)}\left(\partial_Q(\boldsymbol{\alpha})\right) + (1-\theta) H_{a(Q)}\left(\boldsymbol{\partial_Q(\beta)}\right)\right) = \theta c_{\boldsymbol{\alpha},P}(Q) + (1-\theta) c_{\boldsymbol{\beta},P}(Q) \geq c_\star(P). \end{aligned}$$

In particular it follows from the definition of $c_\star(P)$ in (7.1) that $\boldsymbol{\gamma} \in Opt(P)$. Thus $Opt(P)$ is a convex set as claimed. □

A *poset-automorphism* of $(P, \leq_P)$ is a bijection $\phi: P \to P$ such that both ϕ and its inverse are poset homorphisms. Write $\mathrm{Aut}(P)$ for the set of all poset-automorphisms of $(P, \leq_P)$. Then each $\phi \in \mathrm{Aut}(P)$ induces a permutation on the elements of $\mathcal{A}(P) = \{S_1, S_2, \ldots, S_{a(P)}\}$, with S_j sent to the antichain $\phi(S_j) = \{\phi(i) : i \in S_j\}$. Similarly, each $\phi \in \mathrm{Aut}(P)$ gives rise to a permutation of the space of weightings $\triangle_{a(P)}$ via permutation of the coordinates, with $\boldsymbol{\alpha}$ sent to the weighting $\phi(\boldsymbol{\alpha})$ defined by $\phi(\alpha)_i = \alpha_j$ if $\phi(S_i) = S_j$.

Given a poset $(P, \leq_P)$, its *reverse* is the poset $(P, \leq_{R(P)})$, where $x \leq_{R(P)} y$ if and only if $y \leq_P x$. Thus the reverse of a poset is simply the poset obtained by reversing all inequalities. We say a poset $(P, \leq_P)$ is *reverse-symmetric* if there exists a bijection ϕ from $(P, \leq_P)$ to its reverse $(P, \leq_{R(P)})$ such that both ϕ and its inverse are poset homomorphisms. We refer to such a function ϕ, if it exists, as *reverse automorphism* of P, and let $R-\mathrm{Aut}(P)$ denote the set of all reverse automorphisms. Analogously to ordinary automorphisms, reverse automorphisms induce permutations on $\mathcal{A}(P)$ and $\triangle_{a(P)}$.

Proposition 7.2. *(i)* $\phi\left(\mathit{Opt}(P)\right) = \mathit{Opt}(P)$ *for all* $\phi \in \mathrm{Aut}(P)$.
(ii) If P *is reverse-symmetric, then* $\psi\left(\mathit{Opt}(P)\right) = \mathit{Opt}(P)$ *for all* $\psi \in R - \mathrm{Aut}(P)$.

Proof. Clearly for any non-empty $Q \subseteq P$ and $\phi \in \mathrm{Aut}(P)$ we have

$$c_{\phi(\boldsymbol{\alpha}),P}(Q) = c_{\boldsymbol{\alpha},P}(\phi(Q)).$$

In particular, $\boldsymbol{\alpha} \in \mathit{Opt}(P)$ if and only if $\phi(\boldsymbol{\alpha}) \in \mathit{Opt}(P)$. Similarly, for any $\psi \in R - \mathrm{Aut}(P)$ we have

$$c_{\psi(\boldsymbol{\alpha}),P}(Q) = c_{\boldsymbol{\alpha},R(P)}(\psi(Q)).$$

Now observe that the random poset $R(\mathcal{P}(n,p))$ has the same distribution as $\mathcal{P}(n,p)$. In particular, this immediately implies that for every reverse-symmetric poset P,

$$c_\star(P) = c_\star(R(P)),$$

and further, since P and $R(P)$ are isomorphic, that $\mathit{Opt}(P) = \mathit{Opt}(R(P))$. Together with the preceding equality, this shows that $\boldsymbol{\alpha} \in \mathit{Opt}(P)$ if and only if $\psi(\boldsymbol{\alpha}) \in \mathit{Opt}(P)$. ☐

Corollary 7.3. *For every poset* P*, there exists an optimal weighting* $\boldsymbol{\alpha} \in \mathit{Opt}(P)$ *such that* $\boldsymbol{\alpha}$ *is invariant under the action of automorphisms of* $(P, \leq_P)$*. Further if* P *is reverse-symmetric then this optimal weighting can in addition be taken to be invariant under the actions of reverse-automorphisms of* $(P, \leq_P)$*.*

Proof. Let $\mathrm{AutInv}(P)$ denote the collection of weightings in $\triangle_{a(P)}$ that are invariant under the action of automorphisms $\phi \in \mathrm{Aut}(P)$. Similarly, let $R-\mathrm{AutInv}(P)$ denote the collection of weightings invariant under the action of automorphisms $\phi \in R-\mathrm{Aut}(P)$ (if any such automorphism exists — otherwise let $R-\mathrm{AutInv}(P)$ denote the whole of $\triangle_{a(P)}$). By considering the uniform weighting $\boldsymbol{u}$, we have that $\mathrm{AutInv}(P)$ and $R-\mathrm{AutInv}(P)$ are non-empty sets, and it is easy to see that both of them form closed convex subsets of $\triangle_{a(P)}$.

Given $\boldsymbol{\alpha} \in \mathit{Opt}(P)$, consider

$$\tilde{\boldsymbol{\alpha}} := \mathbb{E}_{\phi \in \mathrm{Aut}(P)} \phi(\boldsymbol{\alpha}).$$

By Proposition 7.2(i) and the convexity of $\mathit{Opt}(P)$, $\tilde{\boldsymbol{\alpha}}$ is an element of $\mathit{Opt}(P)$, and is easily seen to be invariant under the action of automorphisms from $\mathrm{Aut}(P)$. Thus $\mathrm{AutInv}(P) \cap \mathit{Opt}(P)$ is a non-empty closed and convex subset of $\triangle_{a(P)}$, being the non-empty intersection of closed, convex sets. Repeating the same argument mutatis mutandis starting with $\boldsymbol{\alpha} \in \mathrm{AutInv}(P) \cap \mathit{Opt}(P)$, we obtain in a similar way that $R - \mathrm{AutInv}(P) \cap \mathrm{AutInv}(P) \cap \mathit{Opt}(P)$ is a non-empty (closed and convex) subset of $\triangle_{a(P)}$. The corollary follows. ☐

For a non-connected poset P, we can reduce the computation of $c_\star(P)$ to its components.

Lemma 7.4. *If P is a poset which can be partitioned into non-empty sets P_1 and P_2 of mutually incomparable elements, then*

$$c_\star(P) = \min\{c_\star(P_1), c_\star(P_2)\}.$$

Proof. Since the P_i are subposets of P we have $c_\star(P) \le c' := \min\{c_\star(P_1), c_\star(P_2)\}$.

It is easy to see that $c_\star(P) \ge c'$ also. Indeed, given any $c < c'$, let $p = e^{-cn}$ and note that $\mathcal{P}(n,p) = \mathcal{P}(n,p_1) \cup \mathcal{P}(n,p_2)$ for some p_1, p_2 where $p_1, p_2 \ge e^{-c''n}$ and $c \le c'' < c'$. As $c'' < c_\star(P_1)$, w.h.p. there is a copy P_1' of P_1 in $\mathcal{P}(n,p_1)$. Notice that there is a constant $d = d(P_1)$ so that $\mathcal{P}(n)$ contains a copy of $\mathcal{P}(n-d)$ that avoids P_1'. Further, as $c'' < c_\star(P_2)$, w.h.p. $\mathcal{P}(n-d,p_2)$ contains a copy of P_2. The last two sentences together imply that w.h.p. there is a copy P_2' of P_2 in $\mathcal{P}(n,p_2)$ that does not intersect P_1'. Together P_1' and P_2' yield a copy of P in $\mathcal{P}(n,p)$, as desired. □

Next we consider some special classes of posets P where $Opt(P)$ consists of a single, very simple weighting. By the remark immediately after the definition of entropy (equation (5.2)), we have that for all posets P, $c_{\boldsymbol{\alpha},P}(P) \le \frac{\log a(P)}{|P|}$ with equality attained if and only if $\boldsymbol{\alpha}$ is the *uniform weighting* $\boldsymbol{u} = \boldsymbol{u}(P) = (\frac{1}{a(P)}, \frac{1}{a(P)}, \dots, \frac{1}{a(P)})$. In particular

$$c_\star(P) \le \frac{\log a(P)}{|P|} \tag{7.2}$$

holds for all P.

Remark 7.5. *Set $c := \log(a(P))/|P|$. Then (by Theorem 1.1) $p = e^{-cn}$ is the threshold at which the expected number of copies of P in $\mathcal{P}(n,p)$ becomes large. Further, for any P, the 'typical' copies of P in $\mathcal{P}(n)$ are precisely those with weightings close to $\boldsymbol{u}(P)$.*

Definition 7.6. *A poset $(P, \le_P)$ is* uniformly balanced *if $c_\star(P) = \frac{\log a(P)}{|P|}$.*

Remark 7.7. *By the remark before* (7.2)*, if P is uniformly balanced then necessarily $Opt(P) = \{\boldsymbol{u}(P)\}$. A more general form of this result is proved in Proposition 7.10 below.*

Checking whether a given poset $(P, \le_P)$ is uniformly balanced is easy (though possibly tedious if the poset is large): one runs over all possible non-empty subsets Q of P, computing the value $c_{\boldsymbol{u},P}(Q)$ and checking whether or not it is greater or equal to $\frac{\log a(P)}{|P|}$.

A finite poset P is *bounded* if it contains a unique $\le_P$-minimal and a unique $\le_P$-maximal element. In such posets, we denote the unique minimum and maximum elements by $\min(P)$ and $\max(P)$ respectively. For bounded posets P, the upper bound $c_\star(P) \le \log(a(P))/|P|$ in (7.2) may be improved as follows.

Proposition 7.8. *Let $(P, \le_P)$ be a bounded poset on $|P| \ge 2$ elements. Let $\mathcal{Q}$ be the collection of subposets Q of P containing both* $\min(P)$ *and* $\max(P)$*. Then*

$$c_\star(P) \le \max_{x \in [\frac{1}{a(P)}, \frac{1}{2}]} \min\left\{H_2(x), \min_{Q \in \mathcal{Q}} -\frac{1}{|Q|}\left(2x\log x + (1-2x)\log\left(\frac{1-2x}{a(Q)-2}\right)\right)\right\}. \tag{7.3}$$

Proof. Denote by Q_- and Q_+ the one-element subposets of $(P, \le_P)$ induced by $\{\min(P)\}$ and $\{\max(P)\}$ respectively. By definition of $c_\star(P)$, we have

$$c_\star(P) \le \max_{\boldsymbol{\alpha} \in \Delta_m} \min\left\{c_{\boldsymbol{\alpha},P}(Q_-), c_{\boldsymbol{\alpha},P}(Q_+), \min_{Q \in \mathcal{Q}} c_{\boldsymbol{\alpha},P}(Q)\right\}. \tag{7.4}$$

Consider a weighting $\boldsymbol{\alpha} \in \Delta_m$ of $\mathcal{A}(P)$. Then by definition

$$c_{\boldsymbol{\alpha},P}(Q_-) = H_2(\alpha_{\{\min(P)\}}) \qquad \text{and} \qquad c_{\boldsymbol{\alpha},P}(Q_+) = H_2(\alpha_\emptyset). \tag{7.5}$$

Observe that $\partial_Q(\boldsymbol{\alpha})_\emptyset = \alpha_\emptyset$ and $\partial_Q(\boldsymbol{\alpha})_{\{\min(P)\}} = \alpha_{\{\min(P)\}}$.[2] Since $x \mapsto -x\log x$ is strictly concave in $[0,1]$, we have that for any $Q \in \mathcal{Q}$,

$$c_{\boldsymbol{\alpha},P}(Q) \leq \frac{1}{|Q|}\Big(-\alpha_{\{\min(P)\}}\log(\alpha_{\{\min(P)\}}) - \alpha_\emptyset \log(\alpha_\emptyset) \\ - (a(Q)-2)\left(\frac{1-\alpha_{\{\min(P)\}}-\alpha_\emptyset}{a(Q)-2}\right)\log\left(\frac{1-\alpha_{\{\min(P)\}}-\alpha_\emptyset}{a(Q)-2}\right)\Big), \tag{7.6}$$

with equality holding if and only if $\partial_Q(\boldsymbol{\alpha})_S = (1-\alpha_\emptyset - \alpha_{\{\min(P)\}})/(a(Q)-2)$ for all antichains $S \in \mathcal{A}(Q) \setminus \{\emptyset, \{\min(P)\}\}$. Setting $x := \alpha_\emptyset$, $y := \alpha_{\{\min(P)\}}$, $R := \{(x,y) \in [0,1]^2 : \ x+y \leq 1\}$ and combining (7.4), (7.5) and (7.6) we get

$$c_\star(P) \leq \max_{(x,y)\in R} \min\left\{H_2(x), H_2(y), \min_{Q\in\mathcal{Q}} \frac{1}{|Q|}\left(-y\log y - x\log x - (1-x-y)\log\left(\frac{1-x-y}{a(Q)-2}\right)\right)\right\}.$$

It is then an easy exercise in optimisation to show that the maximum on the right hand side is attained on the line $x = y$, and that for $x = y$ the maximum is obtained for some x satisfying $\frac{1}{a(P)} \leq x \leq \frac{1}{2}$, yielding the claimed upper bound on $c_\star(P)$. □

Definition 7.9. *A bounded poset $(P, \leq_P)$ with $|P| \geq 2$ elements is* balanced *if $c_\star(P) = H_2(x_\star)$, where $x_\star = x_\star(P)$ is the unique solution in $[1/a(P), 1/2]$ to the equation*

$$H_2(x) = -\frac{1}{|P|}\left(2x\log x + (1-2x)\log\left(\frac{1-2x}{a(P)-2}\right)\right).$$

Proposition 7.10. *If P is balanced then $Opt(P) = \{\boldsymbol{b}(P)\}$, where $\boldsymbol{b} = \boldsymbol{b}(P)$ is defined so that $b_S := x_\star(P)$ if $S = \emptyset, \{min(P)\}$, and $b_S := (1-2x_\star(P))/(a(P)-2)$ otherwise.*

Proof. Let $f_1(x)$ and $f_2(x,y)$ be the functions defined by $f_1(x) := -\frac{1}{|P|}\left(2x\log x + (1-2x)\log\left(\frac{1-2x}{a(P)-2}\right)\right)$ for $x \in [0,1]$ and $f_2(x,y) := \frac{1}{|P|}\left(-y\log y - x\log x - (1-x-y)\log\left(\frac{1-x-y}{a(P)-2}\right)\right)$ for $(x,y) \in R$, where $R := \{(x,y) \in [0,1]^2 : \ x+y \leq 1\}$. As observed in the proof of Proposition 7.8,

$$\max_{(x,y)\in R} \min\{H_2(x), H_2(y), f_2(x,y)\} = \max_{x\in[0,1]} \{H_2(x), f_1(x)\} = H_2(x_\star) = f_2(x_\star, x_\star), \tag{7.7}$$

and this common maximum is uniquely attained at $x = y = x_\star$.

Consider any $\boldsymbol{\alpha} \in Opt(P)$. Let $x' := \alpha_\emptyset$, $y' := \alpha_{\{\min(P)\}}$. Thus $c_{\boldsymbol{\alpha},P}(\{\max(P)\}) = H_2(x')$ and $c_{\boldsymbol{\alpha},P}(\{\min(P)\}) = H_2(y')$. Further, by strict concavity of the function $x \mapsto -x\log x$, $c_{\boldsymbol{\alpha},P}(P) \leq f_2(x',y')$ and this inequality is strict unless $\alpha_S = (1-x'-y')/(a(P)-2)$ for all antichains $S \in \mathcal{A}(P)$ with $S \neq \emptyset, \{\min(P)\}$. Since $\boldsymbol{\alpha} \in Opt(P)$, we thus have

$$c_\star(P) = \min_{\emptyset \neq Q \subseteq P} c_{\boldsymbol{\alpha},P}(Q) \leq \min\left\{H_2(x'), H_2(y'), f_2(x',y')\right\} \overset{(7.7)}{\leq} H_2(x_\star) = c_\star(P).$$

By the uniqueness of the maximum in (7.7) the last inequality immediately implies $x' = y' = x_\star$. For these values of x', y', the first inequality is strict unless $\alpha_S = (1-2x_\star)/(a(P)-2)$ for all antichains $S \in \mathcal{A}(P)$, as observed above. Thus $\boldsymbol{\alpha} \in Opt(P)$ implies $\boldsymbol{\alpha} = \boldsymbol{b}$, as desired. □

The next simple proposition shows that a uniformly balanced poset is balanced; note the reverse is not true (see the next section for an example).

[2]Note that here and elsewhere, we abuse notation slightly in order to ease the exposition. Explicitly, if the antichains of P are enumerated as $\mathcal{A}(P) = \{S_1, S_2, \ldots, S_{a(P)}\}$, we use the antichain S_i in place of i as an index for weightings in $\triangle_{a(P)}$. So for example $\alpha_{\{\min(P)\}} := \alpha_i$, where S_i is the antichain $\{\min(P)\}$.

Proposition 7.11. *Suppose P is a bounded poset on $|P| \geq 2$ elements. If P is uniformly balanced then it is balanced.*

Proof. Let $f_1(x)$ be as in the proof of Proposition 7.10. Then by (7.3) we have

$$c_\star(P) \leq \max_{x \in [\frac{1}{a(P)}, \frac{1}{2}]} \min\{H_2(x), f_1(x)\} \leq \frac{\log a(P)}{|P|}, \tag{7.8}$$

where the rightmost inequality follows from the concavity of the function $x \mapsto -x \log x$. So if P is uniformly balanced there is equality in (7.8). Thus,

$$c_\star(P) = \max_{x \in [\frac{1}{a(P)}, \frac{1}{2}]} \min\Big\{H_2(x), f_1(x)\Big\} = H_2(x_\star) = f_1(x_\star),$$

and so P is balanced. □

Informally, a poset P is uniformly balanced if the first copy of P to appear in $\mathcal{P}(n,p)$ is a 'typical' copy of P in $\mathcal{P}(n)$. On the other hand a poset P is balanced if the first copy P' of P to appear in $\mathcal{P}(n,p)$ is a 'squashed' version of a typical copy, in the following sense: let $x_\star = x_\star(P)$ be as above. Then the sets $X_{\min(P)}$ and $X_{\max(P)}$ in P' corresponding to the $\leq_P$-extremal elements $\min(P)$ and $\max(P)$ are sitting in layers $x_\star n$ and $(1-x_\star)n$ rather than in layers $\frac{1}{a(P)}n$ and $(1-\frac{1}{a(P)})n$ as they would in a typical copy. (Recall that $\frac{1}{a(P)} \leq x_\star(P) \leq 1/2$, so this means P' has been pushed towards the middle layer relative to a typical copy of P in $\mathcal{P}(n)$.)

Giving an explicit value for $c_\star(P)$ when P is balanced is not straightforward — indeed we have $c_\star(P) = H_2(x_\star)$, where $x_\star = x_\star(P)$ is the unique solution to the equation

$$-x \log x - (1-x)\log(1-x) = \frac{1}{|P|}\left(-2x \log x - (1-2x)\log\left(\frac{1-2x}{a(P)-2}\right)\right)$$

in the interval $[\frac{1}{a(P)}, \frac{1}{2}]$. To show P is balanced likewise entails some non-trivial computations: one must consider the weighting $\boldsymbol{b} = \boldsymbol{b}(P)$ introduced in Proposition 7.10. Running over all non-empty subposets $Q \subseteq P$, one must then check that $c_{\boldsymbol{b},P}(Q) \geq c_{\boldsymbol{b},P}(P)$, which involves delicate algebraic manipulations of entropic expressions involving $x_\star$. This can be done by hand for some small or nicely structured examples, but requires computer assistance for even moderately-sized posets P (unless they are very nicely structured indeed). However if one is content with numerical approximations for $c_\star(P)$, then balanced posets are certainly quite easy to handle with the aid of a computer.

What, however, does one do if P is *not* balanced? To prove a lower bound of the form $c \leq c_\star(P)$, we must find a 'good' $\boldsymbol{\alpha} \in \triangle_{a(P)}$ such that $c_{\boldsymbol{\alpha},P}(Q) \geq c$ for all non-empty $Q \subseteq P$. This is in principle an $(a(P)-1)$-dimensional problem, but using our earlier observations about the shape $Opt(P)$, we can significantly reduce the dimension of the search-space.

Explicitly, let $\mathcal{A}(P) = \{S_1, S_2, \ldots S_m\}$. Then Corollary 7.3 says there exists $\boldsymbol{\alpha} \in Opt(P)$ such that for every $\phi \in \mathrm{Aut}(P)$ and $S_i \in \mathcal{A}(P)$,

$$\alpha_{S_i} = \alpha_{\phi(S_i)}, \tag{7.9}$$

and if P is reverse-symmetric we in addition have that for every $\psi \in R-\mathrm{Aut}(P)$ and $S_i \in \mathcal{A}(P)$,

$$\alpha_{S_i} = \alpha_{\psi(S_i)}. \tag{7.10}$$

These inequalities can be used to reduce the number of unknown variables when solving the optimisation problem (7.1) to determine $c_\star(P)$.

Finally, suppose we believe that we can identify a balanced 'core' in P, that is some unique non-empty $Q_\star \subseteq P$ such that $(Q_\star, \leq_P)$ is balanced and such that we believe it is the non-existence of a copy of $Q_\star$ which is the last obstruction to the appearance of a copy of P — i.e. as soon as copies of $Q_\star$ exist in $\mathcal{P}(n,p)$, then so will copies of P. This entails

$$c_\star(P) = c_\star(Q) = c_{\boldsymbol{b},Q}(Q),$$

where $\boldsymbol{b} = \boldsymbol{b}(Q_\star) \in \triangle_{a(Q_\star)}$ is the (unique) optimal balanced weighting of $Q_\star$ as in Definition 7.9.

Corollary 7.12. *If $Q_\star$ is a balanced subposet of P with $c_\star(Q_\star) = c_\star(P)$, then for every $\boldsymbol{\alpha} \in Opt(P)$,*

$$\partial_{Q_\star}(\boldsymbol{\alpha}) = \boldsymbol{b}, \tag{7.11}$$

where $\boldsymbol{b} \in \triangle_{a(Q_\star)}$ is the optimal balanced weighting of $Q_\star$. □

The heuristic from (7.11) in conjunction with (7.9) and (7.10) can aid our computations of lower bounds for $c_\star(P)$ by giving us extra constraints on the coordinates of an optimal weighting $\boldsymbol{\alpha} \in \triangle_{a(P)}$. It is worth noting that the existence of a (presumed) balanced 'core' is of course also helpful for obtaining upper bounds on $c_\star(P)$, as noted in Proposition 7.8.

Given our work in this section, and considering the behaviour for small examples, it is natural to wonder (a) whether or not $Opt(P)$ is always invariant under the action of automorphisms of P, and (b) whether or not $Opt(P)$ always consists of a single point. The answer to both of these questions turns out to be no. Let H_1 be the poset obtained from C_7 by adding two new element x_1 and x_2, such that $x_1 <_{H_1} y_4$ and $y_4 <_{H_1} x_2$, with no other added relations, where y_4 is the middle element of the C_7. We next assign all antichains from C_7 the same weight w_1 and all which contain either x_1 or x_2 the weight w_2. It is now straightforward, but a bit tedious, to check that $c_*(H_1) = c_*(C_7)$ and that this value is achieved for an interval of values for w_2, thereby giving a negative answer to the first question.

We can answer the second question by using a similar construction. Let H_2 be obtained from C_t, for odd $t \geq 11$, by adding a copy of $C(1,2)$ and $C(2,1)$, and specifying that the middle element y of the chain is below the copy of $C(1,2)$ and above the $C(2,1)$. For this poset we can assign a uniform weight to the antichains in the C_t and a reversal invariant weight to the antichain which contain elements not in the C_t. Here the weight of the two maximal elements of the $C(1,2)$ can be made slightly different without changing the threshold. Apart from working out the explicit entropies this can be seen by considering the up and down-sets from y in $\mathcal{P}(n)$; these are both copies of $\mathcal{P}(n/2)$, and since our c is strictly less than half of $c_*(C(1,2))$ both will contain copies of $C(1,2)$. By Remark 6.8 we will also have copies where the maximal elements receive different weights.

8. Computing $c_\star(P)$ in practice: some concrete examples

8.1. **Chains, stars and wide diamonds.** Let $C(n_1, n_2, \ldots, n_t)$ denote the poset whose elements come from t pairwise disjoint sets $V_1, V_2, \ldots, V_t$ with $|V_i| = n_i$ and $x <_P y$ precisely when $x \in V_i$ and $y \in V_j$ for some i, j with $1 \leq i < j \leq t$. When $n_1 = n_2 = \ldots n_t = 1$, we write C_t as a shorthand for $C(1, 1, \ldots, 1)$. Thus C_t is the chain of length t, and is one of the simplest posets there is as far as computing the parameter $c_\star(P)$ is concerned.

Theorem 8.1. *For all integers $t \geq 2$, C_t is uniformly balanced and satisfies*

$$c_\star(C_t) = \frac{\log(t+1)}{t}.$$

This result in fact already follows from the work of Kreuter [20], since every copy of the poset C_t in $\mathcal{P}(n,p)$ is also an embedded copy of C_t viewed as a distributive lattice. Nevertheless, it is worth giving a proof here as an illustration of the general technique for determining $c_\star(P)$ when P is uniformly balanced.

Proof. We may identify C_t with the integer set $[t] = \{1, 2, \ldots, t\}$ equipped with the usual order relation $<$. The antichains of C_t are then $\mathcal{A}(C_t) = \{\{1\}, \{2\}, \ldots, \{t\}, \emptyset\}$. Consider the uniform weighting $\boldsymbol{u} \in \triangle_{a(C_t)}$, and a non-empty subposet $Q \subseteq P$ with elements $i_1 < i_2 < \ldots < i_q$. This subposet is also a chain, of length q, and its antichains are the singletons from Q together with the empty antichain. Set $i_0 := 0$ and $i_{q+1} := t+1$, and for each $j \in [q+1]$ let $x_j := (i_j - i_{j-1})/(t+1)$. Then $\boldsymbol{\beta} = \partial_Q(\boldsymbol{u})$ satisfies $\beta_{\{i_j\}} = x_j$ for each $j \in [q]$ and $\beta_\emptyset = x_{q+1}$. By elementary properties of entropy, subject to the constraints above, for a fixed q we have that $H_{a(Q)}(\boldsymbol{\beta})$ is minimised by $Q = [q]$, $x_1 = x_2 = \cdots = x_q = 1/(t+1)$ and $x_{t+1} = 1 - q/(t+1)$. Thus, as $q \leq t$, we have

$$
\begin{aligned}
c_{\boldsymbol{u},C_t}(Q) = -\frac{1}{q}\sum_{j=1}^{q+1} x_j \log x_j &\geq \frac{1}{q}\left(\frac{q}{t+1}\log(t+1) - \left(\frac{t+1-q}{t+1}\right)\log\left(\frac{t+1-q}{t+1}\right)\right) \\
&= \frac{1}{t+1}\log(t+1) - \left(\frac{1}{q} - \frac{1}{t+1}\right)\log\left(\frac{t+1-q}{t+1}\right) \geq \frac{\log(t+1)}{t} = c_{\boldsymbol{u},C_t}(C_t),
\end{aligned}
$$

where the last inequality follows from the fact the expression on the left hand side is a non-increasing function of q and $q \leq t$. It follows from (7.2) that $c_\star(C_t) = \log(t+1)/t$ as claimed. ☐

Our second general family are the star posets $C(1,t)$, which have a common threshold for $t \geq 2$.

Theorem 8.2. *Let $x_\star$ denote the unique solution in $[\frac{1}{5}, \frac{1}{2}]$ to the equation*

$$(1-x)\log 2 - H_2(x) = 0. \tag{8.1}$$

Then for all $t \geq 2$, we have

$$c_\star(C(1,t)) = H_2(x_\star) \approx 0.5357390.$$

Proof. For the upper bound, consider the 2-leaved star $C(1,2)$. Let $\boldsymbol{\alpha} \in \mathit{Opt}(C(1,2))$, and set $x := \alpha_{\{\min(C(1,2))\}}$. Then by standard properties of the entropy function, we have

$$
\begin{aligned}
c_\star(C(1,2)) &\leq \min\left\{c_{\boldsymbol{\alpha},C(1,2)}(\{\min(C(1,2))\}), c_{\boldsymbol{\alpha},C(1,2)}(C(1,2))\right\} \\
&\leq \min\left\{H_2(x), \frac{1}{3}\left(-x\log x - (1-x)\log\left(\frac{1-x}{4}\right)\right)\right\}.
\end{aligned} \tag{8.2}
$$

Now the function $F_1 :\ x \mapsto H_2(x)$ is increasing in $[0, \frac{1}{2}]$ and decreasing in $[\frac{1}{2}, 1]$, while the function $F_2 :\ x \mapsto \frac{1}{3}\left(-x\log x - (1-x)\log\left(\frac{1-x}{4}\right)\right)$ is increasing in $[0, \frac{1}{5}]$ and decreasing in $[\frac{1}{5}, 1]$. Further, we have $F_1(\frac{1}{5}) < F_2(\frac{1}{5})$ and $F_1(\frac{1}{2}) > F_2(\frac{1}{2})$. Thus there is a unique solution in the interval $[\frac{1}{5}, \frac{1}{2}]$ to the equation $F_1(x) = F_2(x)$. Rearranging terms, we see that this unique solution is precisely the value $x_\star$ from the statement of the theorem. Together with (8.2) this implies

$$c_\star(C(1,2)) \leq F_1(x_\star) = H_2(x_\star).$$

Since $c_\star(C(1,t))$ is non-increasing in t this establishes the upper bound in the theorem.

For the lower bound, we use a direct probabilistic argument. Let c be a fixed constant with $c < H_2(x_\star)$, and let $p = e^{-cn}$. Then by a standard Chernoff bound, w.h.p. $\mathcal{P}(n,p)$ contains an element A_0 with $|A_0| = \lfloor x_\star n \rfloor$. Condition on this event, consider then $Y = \mathcal{P}(n,p) \cap \{X \in \mathcal{P}(n) : A_0 \subsetneq X\}$. With our conditioning $|Y|$ is a binomially distributed random variable with expected value

$$\left(2^{(1-x_\star)n} - 1\right)p = e^{n\left((1-x_\star)\log 2 - c\right) + o(1)} = e^{n\left(H_2(x_\star) - c\right) + o(1)},$$

where in the last inequality we use the defining equation (8.1) for $x_\star$. Thus for $c < H_2(x_\star)$ fixed, the expectation above is (exponentially) large, and a standard Chernoff bound shows that for any fixed t, w.h.p. $|Y| \geq t$. In particular this shows that for any fixed t, w.h.p. $\mathcal{P}(n,p)$ contains a copy of $C(1,t)$ (by considering A_0 and any t elements from Y), and $c_\star(C(1,t)) \geq H_2(x_\star)$ as claimed. ☐

The poset $C(1,2,1)$ is in fact the Boolean lattice $\mathcal{P}(2)$, and better known in an extremal setting as the *diamond*. Extending the methods of the previous proof we next determine the threshold for the appearance for the broader class of *t-wide diamonds* $C(1,t,1)$, $t \geq 2$, which again share a common threshold for $t \geq 2$.

Theorem 8.3. *Let $x_\star$ denote the unique solution in $[\frac{1}{6}, \frac{1}{4}]$ to the equation*

$$2(1-3x)\log 2 - H_2(2x) = 0. \tag{8.3}$$

Then for all $t \geq 2$, we have

$$c_\star(C(1,t,1)) = (1-2x_\star)\log 2 \approx 0.389429.$$

Proof. For the upper bound, let us consider the diamond $C(1,2,1) = \mathcal{P}(2)$. This is a bounded poset. Label the elements of $C(1,2,1)$ as $1,2,3,4$ with $1 = \max(\mathcal{P}(2))$ and $4 = \min(\mathcal{P}(2))$. Applying Proposition 7.8, we obtain

$$\begin{aligned}
c_\star(\mathcal{P}(2)) &\leq \max_{x\in[\frac{1}{6},\frac{1}{2}]} \min \left\{ H_2(x), \min_{Q\in\{\{1,4\},\mathcal{P}(2)\}} -\frac{1}{|Q|}\left(2x\log x + (1-2x)\log\left(\frac{1-2x}{a(Q)-2}\right)\right)\right\} \\
&\leq \max_{x\in[\frac{1}{6},\frac{1}{2}]} \min \left\{ -\frac{1}{2}\left(2x\log x + (1-2x)\log(1-2x)\right), -\frac{1}{4}\left(2x\log x + (1-2x)\log\left(\frac{1-2x}{4}\right)\right\}\right) \\
&=: \max_{x\in[\frac{1}{6},\frac{1}{2}]} \min\left\{F_1(x), F_2(x)\right\}.
\end{aligned}$$

Now the function $F_1(x)$ is increasing in $[\frac{1}{6}, \frac{1}{3}]$ and decreasing in $[\frac{1}{3}, \frac{1}{2}]$, while the function $F_2(x)$ is decreasing in $[\frac{1}{6}, \frac{1}{2}]$. Further, we have $F_1(\frac{1}{6}) < F_2(\frac{1}{6})$ and $F_1(\frac{1}{4}) = \frac{3}{4}\log 2 > \frac{5}{8}\log 2 = F_2(\frac{1}{4})$. Thus there is a unique solution in the interval $[\frac{1}{6}, \frac{1}{4}]$ to the equation $F_1(x) = F_2(x)$. Rearranging terms, we see that this unique solution is precisely the value $x_\star$ from the statement of the theorem. We then have

$$c_\star(\mathcal{P}(2)) \leq \max_{x\in[\frac{1}{6},\frac{1}{2}]} \min\left\{F_1(x), F_2(x)\right\} = F_1(x_\star) = (1-2x_\star)\log 2,$$

where for the equality we used $F_1(x_\star) = \frac{1}{2}H_2(2x_\star) + x_\star\log 2$. Since $c_\star(C(1,t,1))$ is non-increasing in t, this establishes the upper bound in the theorem.

For the lower bound, we use a direct probabilistic argument. Let c be a fixed constant with $c < (1-2x_\star)\log 2$ and set $p = e^{-cn}$. Let N denote the number of pairs of sets (A,B) with $A, B \in \mathcal{P}(n,p)$, $A \subset B$ and $n - |B| = |A| = \lfloor x_\star n \rfloor$. We have

$$\begin{aligned}
\mathbb{E}N &= \binom{n}{\lfloor x_\star n\rfloor, \lfloor x_\star n\rfloor} p^2 = \exp\left(\left(-2x_\star\log x_\star - (1-2x_\star)\log(1-2x_\star) - 2c\right)n + o(n)\right) \\
&= \exp\left(n\left(2(1-2x_\star)\log 2 - 2c\right) + o(n)\right),
\end{aligned} \tag{8.4}$$

which by our assumption on c tends to infinity as $n \to \infty$. Now

$$\begin{aligned}
\mathbb{E}\left(N^2\right) &\leq (\mathbb{E}N)^2 + \left(2\binom{n-\lfloor x_\star n\rfloor}{\lfloor x_\star n\rfloor}p + 1\right)\mathbb{E}N \\
&= (\mathbb{E}N)^2 + \mathbb{E}N\left(\exp\left(\left(H_2(2x_\star) - H_2(x_\star) + 2x_\star\log 2 - c\right)n + o(n)\right) + 1\right).
\end{aligned} \tag{8.5}$$

Using $H_2(2x_\star) = 2(1-3x_\star)\log 2$, we have

$$\begin{aligned}
(2(1-2x_\star)\log 2 - 2c) - (H_2(2x_\star) - H_2(x_\star) + 2x_\star\log 2 - c) &= H_2(x_\star) - c \\
&> H_2(x_\star) - (1-2x_\star)\log 2 > 0.
\end{aligned} \tag{8.6}$$

Combining (8.4), (8.5) and (8.6) we have that $\mathbb{E}\left(N^2\right) = (1+o(1))\left(\mathbb{E}N\right)^2$, and hence that $\mathrm{Var}N = o(\mathbb{E}N^2)$. A simple application of Chebyshev's inequality then tells us that N is concentrated around its mean and in particular that w.h.p. there exists some pair (A_0, B_0) satisfying $A_0, B_0 \in \mathcal{P}(n,p)$, $|A_0| = n - |B_0| \lfloor x_\star n \rfloor$ and $A_0 \subseteq B_0$. Conditioning on this event, consider the binomially distributed random variable $Y = \mathcal{P}(n,p) \cap \{X \in \mathcal{P}(n) : A_0 \subsetneq X \subsetneq B\}$. Then with our conditioning $|Y|$ is a binomially distributed random variable with expected value at least

$$\left(2^{(1-2x_\star)n} - 2\right) p = e^{n\left((1-2x_\star)\log 2 - c\right) + o(1)}.$$

Since $c < (1-2x_\star)\log 2$, the expectation above is (exponentially) large, and a standard Chernoff bound shows that for any fixed t, w.h.p. $|Y| \geq t$. In particular this shows that for any fixed t, w.h.p. $\mathcal{P}(n,p)$ contains a copy of $C(1,t,1)$ (by considering A_0, B_0 and any t elements from Y), and $c_\star(C(1,t,1)) \geq (1-2x_\star)\log 2$ as claimed. □

The trick we used in the proofs of the lower bounds for $c_\star(P)$ in Theorem 8.2 and Theorem 8.3 of first finding suitable images for the top and/or bottom elements of P in $\mathcal{P}(n,p)$ is applicable more generally. Given a poset Q, let $C(1,Q)$ denote the poset obtained by adding a new element b to Q together with the relations $b < q$ for all $q \in Q$. Similarly, let $C(1,Q,1)$ denote the poset obtained from Q by adding two new elements b, t together with the relations $b < q < t$ for all $q \in Q$.

Proposition 8.4. *Let Q be a finite poset.*

(i) Let $x_\star$ be the unique solution in $[0, 1/2]$ to

$$(1-x_\star)c_\star(Q) - H_2(x_\star) = 0.$$

Then $c_\star(C(1,Q)) \geq H_2(x_\star)$.

(ii) Let $x_\star$ be the unique solution in $[0, 1/2]$ to

$$2(1-2x_\star)c_\star(Q) - 2x\log 2 - H_2(x_\star) = 0.$$

Then $c_\star(C(1,Q,1)) \geq H_2(x_\star)$.

Note that in both cases it is easy to see that the solution $x_\star$ is unique: we have a decreasing linear function fighting against a concave entropy function that attains its maximum at $x = 1/2$, and $c_\star(Q) \leq \log 2$, so considering the values of the functions at 0 and 1/2 shows the solution will occur in this interval.

Proof. Identical to the lower bound proofs in Theorems 8.2, 8.3 with the single change that instead of counting the binomially distributed number of points Y in the subcube above A_0/between A_0 and B_0 that we are investigating, we use instead the fact that p is above the threshold for the existence of a copy of Q in a subcube of that dimension. □

Finally, we record bounds on $c_\star(P)$ for $P = C_\ell(t) = C(t,t,\ldots,t)$, which is the poset obtained for the chain of length ℓ by replacing each element by an antichain of size t.

Proposition 8.5. *For all $\ell, t \in \mathbb{N}$, we have*

$$\frac{\log 2}{\ell} \leq c_\star(C_\ell(t)) \leq \frac{\log 2}{\ell} + \frac{\log\left(\ell - (\ell-1)2^{-t}\right)}{\ell t}.$$

Note the lower bound is asymptotically tight as $t \to \infty$.

Proof. For the lower bound, let $\sqcup_{i=1}^{\ell} A_i$ be an equipartition of $[n]$. Suppose $c < (\log 2)/\ell$ is fixed, and set $p = e^{-cn}$. Let $Y_i = \mathcal{P}(n,p) \cap \{X \in \mathcal{P}(n) : \bigcup_{j<i} A_j \subseteq X \subsetneq \bigcup_{j\leq i} A_j\}$. Clearly the $|Y_i|$ are independent binomially distributed random variables, each with $\mathbb{E}Y_i = \left(2^{|A_i|} - 1\right)p = \exp\left(\left(\frac{\log 2}{\ell} - c\right)n + o(n)\right)$. In particular our choice of c ensures that w.h.p. $|Y_i| > t$ for all i. Taking any t elements from each of the Y_i then yields a copy of $C_\ell(t)$ in $\mathcal{P}(n,p)$. It follows that $c_\star(C_\ell(t)) \geq (\log 2)/\ell$ as claimed.

For the upper bound, we simply appeal to (7.2), noting that $a\left(C_\ell(t)\right) = \ell 2^t - (\ell - 1)$. □

8.2. **Universality.** Our next aim is to establish the existence of a universality threshold c_u, such that for c smaller than c_u almost all fixed posets appear in $\mathcal{P}(n,p)$.

In 1975 Kleitman and Rotschild [17] gave a structural description of a typical poset on N elements. Given a ground set V_N of N elements, define a class of posets A_N on V_N as follows. For every member P of A_N, we have a partition of V_N into three antichains L_1, L_2, L_3 such that $||L_i| - N/4| < \sqrt{N}\log(N)$ for $i = 1, 3$ and $||L_i| - N/2| < \log(N)$ for $i = 2$, together with the following poset relations: for every $x \in L_1$ and $y \in L_3$, $x <_P y$; for every $x \in L_1$ and $y \in L_2$, either $x <_P y$ or x and y are incomparable in P; likewise for every $x \in L_2$ and $y \in L_3$, either $x <_P y$ or x and y are incomparable in P.

Theorem 8.6 (Kleitman and Rotschild, 1975)**.** *Asymptotically almost every poset on a set V_N of N (labelled) elements belongs to A_N, i.e. $\lim_{N\to\infty} \frac{|\mathcal{P}_N|}{|A_N|} = 1$, where $\mathcal{P}_N$ denotes the collection of all posets on V_N.*

One consequence of this theorem is that if we consider the uniform probability measure on A_N, by making each relation between elements from L_1, L_2 and L_2, L_3 exist with probability $1/2$, then the corresponding random poset will be contiguous with the uniform distribution on $\mathcal{P}_N$, in the sense that the asymptotic 0/1 events of the two distributions agree. See [15] for a more detailed discussion of contiguous random models.

Using the Kleitman–Rotschild theorem we can establish a universality result for the appearance of posets on N elements as subposets of $\mathcal{P}(n,p)$.

Theorem 8.7. *Almost all posets P on N elements satisfy*

$$c_\star(P) = \frac{\log 2}{3} + O\left(\frac{1}{\log N}\right),$$

where the lower order term is positive.

Proof. By Theorem 8.6, it is enough to show that for a uniformly chosen random poset P from A_N we have that w.h.p. $c_\star(P) = (\log 2)/3 + O(1/\log N)$.

For the lower bound, note that every poset P in A_N is clearly a subposet of $C(N,N,N) = C_3(N)$. Thus $c_\star(P) \geq c_\star(C_3(N))$, which by Proposition 8.5 is at least $(\log 2)/3$.

For the upper bound, it is an easy exercise in discrete probability to show that there is a constant $b > 0$, such that w.h.p. a uniformly chosen random poset P from A_N contains a copy of $C(t,t,t) = C_3(t)$, where $t = \lceil b\log N\rceil$. For such a P, we thus have $c_\star(P) \leq c_\star(C(t,t,t))$, which by Proposition 8.5 is at most $(\log 2)/3 + O(1/\log N)$. ☐

8.3. **Small examples.** We now turn to examples of computations of the exact or approximate value of $c_\star(P)$ for various small posets P, including some of the posets used for our Ramsey results in the next section.

Write V for the 3-element poset on $\{A,B,C\}$ with $A <_V B$, $A <_V C$. Thus $V = C(1,2)$. Set also Λ to be the reverse of V, i.e. the 3-element poset on $\{A,B,C\}$ with $A >_\Lambda B, C$. Theorem 8.2 determined $c_\star(V)$. Now for any poset P, $c_\star(P) = c_\star(R(P))$ (since $R(\mathcal{P}(n,p))$ and $\mathcal{P}(n,p)$ have the same distribution). Thus Theorem 8.2 also determined $c_\star(\Lambda)$. Combining this with Theorem 8.1 we thus have the existence thresholds for all posets P on at most 3 elements.

Next, let Λ' be the poset obtained from Λ by adding two elements D, E and the relations $B >_{\Lambda'} D$ and $C >_{\Lambda'} E$. Let Y denote the poset on $\{A,B,C,D\}$ defined by the relations $A <_Y B$ and $B <_Y C, D$ (so $Y = C(1,1,2)$. Let Y' denote the poset obtained from Y by adding two new elements E, F and the relations $A <_{Y'} E <_{Y'} F$. Let Y'' be the poset obtained from Y' by adding four new elements G, H, I, J and the relations $G <_{Y''} A$ and $G <_{Y''} H <_{Y''} I <_{Y''} J$.

Further let T_2 denote the binary tree of height 3, which is the poset obtained from Y' by adding a new element G and the relation $E <_{T_2} G$. Let F denote the fish-like poset obtained from T_2 by identifying the elements D and F.

We will also need results about the existence thresholds of the 'long Y' $C(1,1,1,2)$, of $C(2,3,2)$, of the kite-like poset $C(1,1,2,1)$, of the double diamond DD, which is obtained from $C(1,2,2,1)$ by removing one of the inequalities between the elements in the second and third layer. Finally we will need to know the existence thresholds for $C(1,2,1,2,1)$ (a diamond on top of a diamond), and for the poset H defined by the Hasse diagram in Figure 2.

In Figure 1 we display the Hasse diagrams for some of these posets. Using the bounds from the previous sections we have computed the threshold for all connected posets on at most four elements, and some of the additional examples which we will use in our results on Ramsey thresholds. In some cases we have to settle for upper and lower bounds on the threshold. These results are compiled in Table 1. For uniform and balanced posets we state a numerical version of the exact threshold in the lower bound column of the table. For general posets we state the best lower bound we have found by a numerical procedure, and the best upper bound found by either using the simple upper bound $\frac{\log a(P)}{|P|}$ or the best upper bound for a subposet for P. In the final column of the table we state which class the posets belongs to, and we include the label Exact for posets which do not belong to our general classes but for which we can nonetheless determine the exact value of the threshold (in terms of the solution to an equation involving entropy functions).

We have already seen several examples of families of posets with identical thresholds, in particular $C(1,t)$ and $C(1,t,1)$ for $t \geq 2$, and for these we only include the smallest member in the table. There are a few posets P for which the bounds on $c_\star(P)$ that we obtain are very close, and where the thresholds should in fact be the same, for instance Y and Y'.

Conjecture 8.8. $c_\star(Y) = c_\star(Y')$.

9. Ramsey thresholds for posets

9.1. **Ramsey exponents.** Given non-empty posets P, Q, R, we say that R is (P,Q)-*Ramsey* if in every 2-colouring of the elements of R, there is either a copy of P in colour 1 or a copy of Q in colour 2. We write $R \to (P,Q)$ if R is (P,Q)-Ramsey, and $R \not\to (P,Q)$ otherwise. The *poset Ramsey number* $R(P,Q)$ of the pair (P,Q) is defined to be the least $N \in \mathbb{N}$ such that $\mathcal{P}(N) \to (P,Q)$. Recall from the introduction that this number exists and is finite for every (P,Q). In this section, we consider the problem of determining the range of $p = e^{-cn}$ for which w.h.p. $\mathcal{P}(n,p) \to (P,Q)$.

Define the *lower* and *upper Ramsey exponents* of (P,Q) $c_{\mathrm{Ram}^-}(P,Q)$ and $c_{\mathrm{Ram}^+}(P,Q)$ to be

$$c_{\mathrm{Ram}^-}(P,Q) := \sup\Big\{c > 0 : \ \mathcal{P}(n, e^{-cn}) \to (P,Q) \text{ holds w.h.p.}\Big\}$$

and

$$c_{\mathrm{Ram}^+}(P,Q) := \inf\Big\{c > 0 : \ \mathcal{P}(n, e^{-cn}) \not\to (P,Q) \text{ holds w.h.p.}\Big\}.$$

Clearly

$$0 \leq c_\star\left(\mathcal{P}(R(P,Q))\right) \leq c_{\mathrm{Ram}^-}(P,Q) \leq c_{\mathrm{Ram}^+}(P,Q) \leq \max\left\{c_\star(P), c_\star(Q)\right\},$$

so these exponents are well-defined. If $c_{\mathrm{Ram}^-}(P,Q) = c_{\mathrm{Ram}^+}(P,Q)$, then with say that their common value is the *critical Ramsey exponent* for (P,Q), and denote it by $c_{\mathrm{Ram}}(P,Q)$.

Conjecture 9.1. *For every pair of fixed posets* (P,Q)*,* $c_{\mathrm{Ram}}(P,Q)$ *exists.*

More generally, rather than a pair of posets (P,Q) we may consider Ramsey problems for pairs of families of posets $(\mathcal{P}, \mathcal{Q})$. We write $R \to (\mathcal{P}, \mathcal{Q})$ if R is $(\mathcal{P}, \mathcal{Q})$-Ramsey, i.e. if every 2-colouring of R contains a copy of a member of $\mathcal{P}$ in colour 1 or a member of $\mathcal{Q}$ in colour 2. We extend

Name	L.b.	U.b.	Class
$C(2)$	0.549306...		Uniform
$V = C(1,2)$	0.53573885...		Exact
A2	0.51986038...		Uniform
$C(2,2)$	0.48647753...		Uniform
$C(3)$	0.462098...		Uniform
A1	0.4620981202	0.4620981203	General
Λ'	0.455914351	0.46051702	General
$C(1,2,1)$	0.447699551...		Balanced
Y	0.44769950088	0.44793987	General
Y'	0.44769951418	0.44793987	General
T_2	0.4474689916	0.44793987	General
F	0.43238626	0.43984289	General
$C(2,1,2)$	0.415888308...		Uniform
$C(1,2,2)$	0.415507009	0.4158883	General
$C(4)$	0.402359...		Uniform
$C(1,1,2,1)$	0.3891411	0.38918203	General
$C(1,1,1,2)$	0.3891411	0.38918203	General
Y''	0.38890390	0.38918203	General
DD	0.3816641132...		Balanced
$C(2,3,2)$	0.376783	0.3770081	General
$\mathcal{P}(3)$	0.36356411...		Uniform
$C(1,2,1,2,1)$	0.3289037390...		Uniform
H	0.3250121326	0.328903	General

TABLE 1. Thresholds for small posets

our definitions of poset Ramsey numbers and Ramsey exponents from poset pairs (P, Q) to poset family pairs $(\mathcal{P}, \mathcal{Q})$ in the natural way.

9.2. **General bounds.** Let P, Q be posets such that P has a unique $\leq_P$-maximal element and Q has a unique $\leq_Q$-minimal element. Define the *Q–on–P* poset $T = T(P, Q)$ by taking disjoint copies of P and Q, identifying $\max(P)$ with $\min(Q)$ and adding the relation $p \leq_T q$ for every $p \in P$, $q \in Q$.

Theorem 9.2. *For every pair of fixed posets P, Q such that P has a unique $\leq_P$-maximal element and Q has a unique $\leq_Q$-minimal element, we have*

$$c_{\mathrm{Ram}^+}(P, Q) \leq c_\star(T(P, Q)).$$

Proof. Let $c > c_\star(T(P, Q))$. Then for $p = e^{-cn}$, w.h.p. $\mathcal{P}(n, p)$ contains no copy of $T(P, Q)$. Condition on this event and colour the elements of $\mathcal{P}(n, p)$ as follows. Given an element $x \in P(n, p)$, assign it colour 2 if it is the unique maximal element in a (not necessarily induced) copy of P in $\mathcal{P}(n, p)$, and otherwise assign it colour 1.

Clearly in this colouring there is no monochromatic copy of P in colour 1, since by construction the maximal element is in colour 2. Further, there is no monochromatic copy Q' of Q in colour 2, otherwise there must be a copy of P' of P in $\mathcal{P}(n, p)$ such that $\max(P') = \min(Q')$. But then $P' \cup Q'$ contains a copy of $T(P, Q)$, a contradiction. □

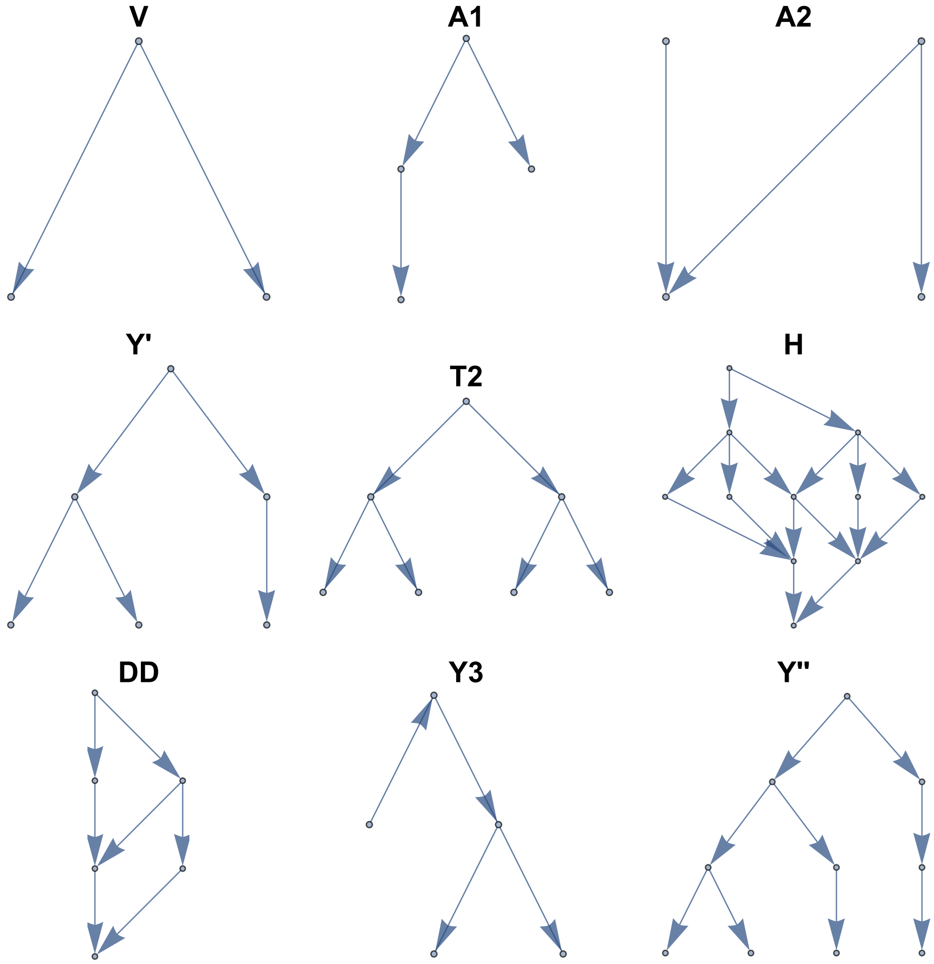

FIGURE 1. Small posets

Given posets P, Q we may define their *lexicographic product* $P \times_{\text{lex}} Q$ to be the poset on $P \times Q := \{(p,q) : \ p \in P, q \in Q\}$ with partial order $\leq$ defined by $(p,q) \leq (p',q')$ if and only if either $p <_P p'$ or $p = p'$ and $q \leq_Q q'$.

Theorem 9.3. *For every pair of fixed posets P, Q, we have*

$$c_\star(P \times_{lex} Q) \leq c_{\text{Ram}^-}(P, Q).$$

Proof. We claim that $P \times_{\text{lex}} Q$ is (P,Q)-Ramsey. Indeed consider any 2-colouring of $P \times Q$. If for any $p \in P$ the set $\{p\} \times Q$ is monochromatic in colour 2, then this gives us a copy of Q in colour 2 inside $P \times_{\text{lex}} Q$. Otherwise for every $p \in P$ there exists $q_p \in Q$ such that (p, q_p) received colour 1. Then the set $\{(p, q_p) : \ p \in P\}$ gives us a copy of P in colour 1 inside $P \times_{\text{lex}} Q$.

Thus $P \times_{\text{lex}} Q \to (P, Q)$ as claimed, and the theorem follows immediately from that fact. ☐

9.3. **Specific posets.** In this subsection, we give bounds on the Ramsey exponents for (P,Q) for various pairs of small posets P and Q.

Theorem 9.4 (Kreuter [20])**.** *For all $s, t \geq 2$, $c_{\text{Ram}}(C_s, C_t) = c_\star(C_{s+t-1})$.*

Proof. For the upper bound, we have by Theorem 9.2 that $c_{\text{Ram}^+}(C_s, C_t) \leq c_\star(T(C_s, C_t)) = c_\star(C_{s+t-1})$. For the lower bound, observe that by the pigeonhole principle $C_{s+t-1} \to (C_s, C_t)$. ☐

Theorem 9.5. $c_{\mathrm{Ram}}(V,V) = c_\star(T_2)$.

Proof. For the upper bound, we use a slight variant of the colouring given in the proof of Theorem 9.2. Assign an element $X \in \mathcal{P}(n,p)$ the colour 1 if there exists $Y, Z \in \mathcal{P}(n,p)$ with $X \subsetneq Y, Z$ (i.e. if X is the minimal element of a copy of V in $\mathcal{P}(n,p)$), and otherwise assign X the colour 2. By construction, there is no monochromatic copy of V in colour 2. Suppose now there exists a monochromatic copy of V in colour 1. By construction of our colouring, this implies the existence of one of the following: a copy of the binary tree T_2 of height 3, a copy of the poset F obtained from T_2 by identifying the elements D and E, or a copy of $C(1,2,2)$. In particular, this shows

$$c_{\mathrm{Ram}^+}(V,V) \leq \max\{c_\star(T_2), c_\star(F), c_\star(C(1,2,2)\} = c_\star(T_2),$$

where the last equality follows from the bounds given in Section 8.3. For the lower bound, it is easily checked that $T_2 \to (V,V)$, whence $c_{\mathrm{Ram}^-}(V,V) \geq c_\star(T_2)$. □

Clearly a poset H is (P,Q)-Ramsey if and only if its reverse $R(H)$ is $(R(P),R(Q))$-Ramsey for the pair of reverse posets $R(P), R(Q)$. Since $R(\mathcal{P}(n,p))$ has exactly the same distribution as $\mathcal{P}(n,p)$, the Ramsey exponents for (P,Q) and $(R(P),R(Q))$ are equal for all pairs (P,Q). In particular, Theorem 9.5 also determines $c_{\mathrm{Ram}}(\Lambda,\Lambda)$. Thus Theorems 9.4–9.5 together determine the critical Ramsey exponents for all pairs (P,P) with $|P| \leq 3$. For mixed pairs (P,Q), we can give the following bounds on the Ramsey exponents.

Theorem 9.6. *The following hold:*

(i) $c_\star(Y') \leq c_{\mathrm{Ram}^-}(C_2,V) \leq c_{\mathrm{Ram}^+}(C_2,V) \leq c_\star(Y)$;
(ii) $c_\star(C(2,3,2)) \leq c_{\mathrm{Ram}^-}(\Lambda,V) \leq c_{\mathrm{Ram}^+}(\Lambda,V) \leq c_\star(C(2,1,2))$;
(iii) $c_\star(Y'') \leq c_{\mathrm{Ram}^-}(C_3,V) \leq c_{\mathrm{Ram}^+}(C_3,V) \leq c_\star(T(C_3,V)) = c_\star(C(1,1,1,2))$;
(iv) $c_\star(C(2,1,2)) \leq c_{\mathrm{Ram}^-}(\{V,\Lambda\},\{V,\Lambda\}) \leq c_{\mathrm{Ram}^+}(\{V,\Lambda\},C_2) \leq c_\star(\Lambda')$;
(v) $c_\star(DD) \leq c_{\mathrm{Ram}^-}(\mathcal{P}(2),C_2) \leq c_{\mathrm{Ram}^+}(\mathcal{P}(2),C_2) \leq c_\star(T(C_2,\mathcal{P}(2))) = c_\star(C(1,1,2,1))$.

Proof. (i) For the upper bound, Theorem 9.2 implies $c_{\mathrm{Ram}^+}(C_2,V) \leq c_\star(T(C_2,V)) = c_\star(Y)$. For the lower bound, it is easily checked that $Y' \to (C_2,V)$.

(ii) For the upper bound, Theorem 9.2 implies $c_{\mathrm{Ram}^+}(\Lambda,V) \leq c_\star(T(\Lambda,V)) = c_\star(C(2,1,2))$.

For the lower bound, we claim that $C(2,3,2) \to (\Lambda,V)$. Indeed, suppose the bottom two elements of $C(2,3,2)$ both received colour 1. Since $C(3,2) \to (C_1,V)$, this would give us either a Λ in colour 1 or a V in colour 2. On the other hand, suppose that the bottom two elements of $C(2,3,2)$ both received colour 2. Since $C(3,2) \to (\Lambda, C_1 \sqcup C_1)$, this would give us either a Λ in colour 1 or a V in colour 2.

We may thus assume that one of the bottom elements of $C(2,3,2)$ receives colour 1 and the other receives colour 2. By symmetry, the same is true of the top elements of $C(2,3,2)$. By the pigeonhole principle, at least two elements in the middle layer of $C(2,3,2)$ are in the same colour, say 1. Thus we have a C_3 (and hence a Λ) in colour 1. Thus $C(2,3,2) \to (\Lambda,V)$ as claimed.

(iii) For the upper bound, Theorem 9.2 implies $c_{\mathrm{Ram}^+}(C_3,V) \leq c_\star(T(C_3,V)) = c_\star(C(1,1,1,2))$.

For the lower bound, we claim that $Y'' \to (C_3,V)$. Indeed, suppose the bottom element of Y'' is in colour 2. If both of the branches of Y'' above this minimum element contain elements in colour 2, then we have a copy of V in colour 2. Otherwise one of the branches receives only the colour 1, and hence gives us a copy of C_3 in colour 1.

Assume therefore that the bottom element of Y'' is in colour 1. One of the branches of Y'' above this bottom element is a copy of Y', which as we observed in part (i) is (C_2,V)-Ramsey. Thus in that branch we either get a copy of V in colour 2 or a copy of C_2 in colour 1, which together with the bottom element of Y'' gives us a copy of C_3 in colour 2.

(iv) For the upper bound, assign each vertex in $\mathcal{P}(n,p)$ the colour 1 if it is the top element of a copy of C_2 in $\mathcal{P}(n,p)$, and assign it the colour 2 otherwise. Clearly in such a colouring there can be no copy of C_2 in colour 2. Further a copy of V in colour 1 would require the existence of a copy of Y, while a copy of Λ in colour 1 would require the existence of Λ' or $\mathcal{P}(2)$. Thus we have

$$c_{\mathrm{Ram}^+}(\{\Lambda, V\}, C_2) \leq \max\left\{c_\star(Y), c_\star(\mathcal{P}(2)), c_\star(\Lambda')\right\} = c_\star(\Lambda').$$

For the lower, bound, by considering the colour of the middle element, it is easy to see that $C(2,1,2)$ is $(\{V,\Lambda\},\{V,\Lambda\})$-Ramsey.

(v) For the upper bound, Theorem 9.2 implies $c_{\mathrm{Ram}^+}(C_2, \mathcal{P}(2)) \leq c_\star(T(C_2, \mathcal{P}(2))) = c_\star(C(1,1,2,1))$.

For the lower bound, we claim the double diamond DD is $(\mathcal{P}(2), C_2)$-Ramsey. Indeed, suppose the bottom element of DD receives colour 2. If any element above it is in colour 2 we have a C_2 in colour 2. Otherwise DD contains a copy of $\mathcal{P}(2)$ in colour 1. By reverse-symmetry, we are similarly done if the top element of DD receives colour 2.

On the other hand, suppose both the bottom and the top elements of DD are in colour 1. Then if any two of the other elements of DD are in colour 1 we have a copy of $\mathcal{P}(2)$ in colour 1. Otherwise, at least three of the 'middle' elements of $\mathcal{P}(2)$ are in colour 2, and two of these will give us a copy of C_2 in colour 2.

□

Next we turn our attention to the Ramsey problem for the diamond $\mathcal{P}(2)$. Let H be the poset defined by the Hasse diagram in Figure 2.

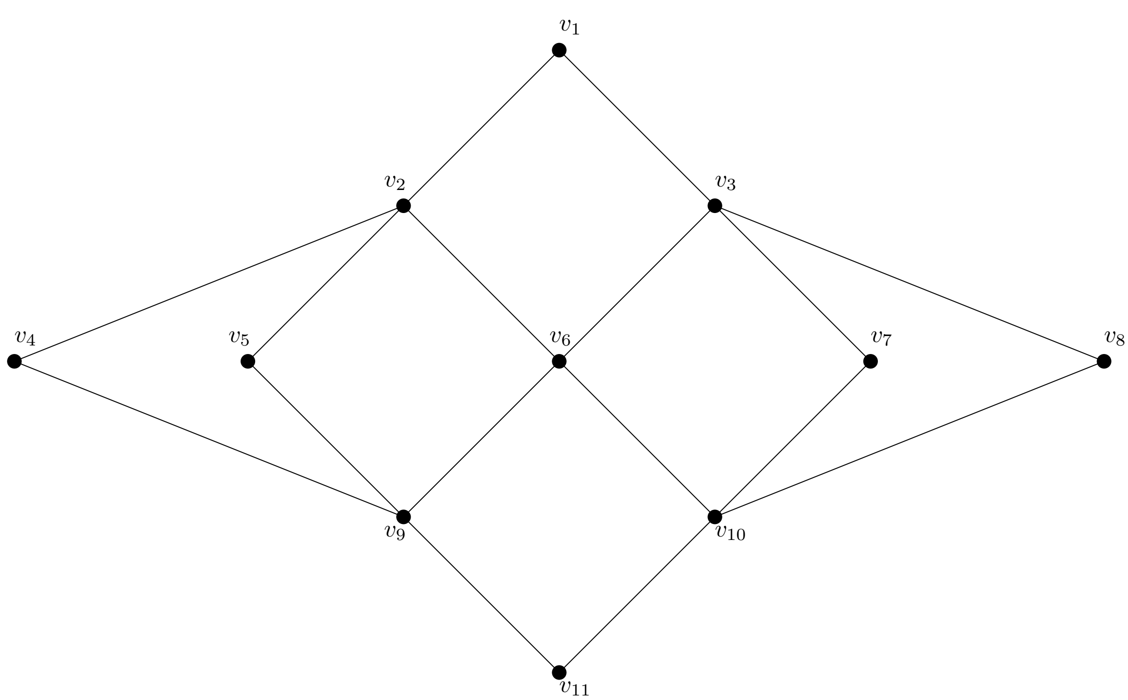

Figure 2. The Hasse diagram of the poset H

Theorem 9.7. $c_\star(H) \leq c_{\mathrm{Ram}^-}(\mathcal{P}(2), \mathcal{P}(2)) \leq c_{\mathrm{Ram}^+}(\mathcal{P}(2), \mathcal{P}(2)) \leq c_\star(C(1,2,1,2,1))$

Proof. The upper bound on $c_{\mathrm{Ram}^+}(\mathcal{P}(2),\mathcal{P}(2))$ is an immediate consequence of Theorem 9.2 and the fact that $T(\mathcal{P}(2),\mathcal{P}(2)) = C(1,2,1,2,1)$. For the lower bound, it suffices to show that H is $(\mathcal{P}(2),\mathcal{P}(2))$-Ramsey.

Consider any red-blue colouring of the elements of H. Suppose for a contradiction that there does not exist a monochromatic copy of $\mathcal{P}(2)$. Without loss of generality we may assume that **the maximal element v_1 in H is coloured red**.

(a) If the minimal element v_{11} is coloured red, then at most one element in $H \setminus \{v_1, v_{11}\}$ is coloured red. Indeed, otherwise we obtain a red copy of $\mathcal{P}(2)$ in H. However, then $H \setminus \{v_1, v_{11}\}$ contains two disjoint copies of $\mathcal{P}(2)$, at least one of which is blue, a contradiction. Thus, **the minimal element v_{11} is coloured blue**.
(b) If both v_9 and v_{10} are coloured blue, then v_2, v_3, v_6 are coloured red (else we get a blue $\mathcal{P}(2)$). However, then v_1, v_2, v_3, v_6 induce a red $\mathcal{P}(2)$ in H. So at least one of v_9 and v_{10} is coloured red. Without loss of generality assume v_9 **is coloured red**.
(c) Suppose v_2 is blue. Then at most one of v_4, v_5, v_6 is blue (else we obtain a blue copy of $\mathcal{P}(2)$ with maximal element v_2 and minimal element v_{11}). However, at most one of v_4, v_5, v_6 is red (else we obtain a red copy of $\mathcal{P}(2)$ with maximal element v_1 and minimal element v_9). This is a contradiction, so v_2 **is coloured red**.
(d) This implies v_3 **is coloured blue** (else v_1, v_2, v_3, v_9 induce a red $\mathcal{P}(2)$).
(e) By symmetry with step (c), this implies v_{10} **is blue**.
(f) Note that if v_6 is red, together with v_1, v_2 and v_9 it induces a red C_4, (which contains $\mathcal{P}(2)$ as a subposet). If v_6 is blue, together with v_3, v_{10} and v_{11} it induces a blue C_4. In either case we obtain a monochromatic copy of $\mathcal{P}(2)$, a contradiction. Thus, H is indeed $(\mathcal{P}(2),\mathcal{P}(2))$-Ramsey, as claimed.

□

Finally we note that Theorem 8.6 shows that most posets have height 3 and this makes it possible to find an interval which contains the Ramsey threshold for almost all posets.

Theorem 9.8. *There exists constants $\frac{\log 2}{5} \le c^-_{ru} \le c^+_{ru} \le \frac{\log 2}{3}$ such that almost all posets P on N elements satisfy $c^-_{ru} \le c_{\mathrm{Ram}^-}(P,P) \le c_{\mathrm{Ram}^+}(P,P) \le c^+_{ru} + O(1/\log N)$.*

Proof. For every poset P we have that $c_{\mathrm{Ram}^+}(P,P) \le c_\star(P)$. Thus Theorem 8.7 implies the upper bound.

Recall from Theorem 8.6 that asymptotically almost every poset on a fixed set of N elements belongs to A_N, and that every poset P in A_N is a subposet of $C(N,N,N) = C_3(N)$. So

$$c_{\mathrm{Ram}^-}(C_3(N), C_3(N)) \le c_{\mathrm{Ram}^-}(P,P) \le c_{\mathrm{Ram}^+}(P,P).$$

For $c < \frac{\log(2)}{5}$ and $p = e^{-cn}$, by Proposition 8.5, $\mathcal{P}(n,p)$ w.h.p. contains a copy of $C_5(2N-1)$; any two-colouring of $C_5(2N-1)$ will contain a monochromatic copy of $C_3(N)$ and hence also a monochromatic copy of P. Thus the two exponents lie in the stated interval. □

We believe that this result can be sharpened to give a single universality exponent. As we noted in the proof of Theorem 8.7, there is a constant b such that w.h.p. a poset P from A_N contains a copy of $C_3(t)$ with $t = \lceil b \log N \rceil$. Hence, in order to be (P,P)-Ramsey the random poset must also be $(C_3(t), C_3(t))$-Ramsey and hence $c^+_{ru} \le c_{\mathrm{Ram}^+}(C_3(t))$. Since the sequences $c_{\mathrm{Ram}^-}(C_3(N), C_3(N))$ and $c_{\mathrm{Ram}^+}(C_3(N)), C_3(N))$ are both bounded and non-increasing in N, they both converge to limits that give a lower bound on c^-_{ru} and an upper bound on c^+_{ru} respectively. However by Conjecture 9.1 these limits should be the same, which would imply the following.

Conjecture 9.9. *There exists a constant $c_{ru} = \lim_{N\to\infty} c_{\mathrm{Ram}^-}(C_3(N))$ such that almost all posets P on N elements have $c_{ru} \le c_{\mathrm{Ram}^-}(P,P) \le c_{\mathrm{Ram}^+}(P,P) \le c_{ru} + O(1/\log N)$*

Question 9.10. *What is the value of* $\lim_{N\to\infty} c_{\mathrm{Ram}^-}(C_3(N))$*?*

10. Open problems

In addition to Conjecture 9.1 about the existence of Ramsey exponents and the obvious problem of tightening our Ramsey results, many other open problems remain.

Question 10.1.

- *Is* $C(n, n)$ *uniformly balanced for all* n*?*
- *Is* $C(n_1, n_2, n_1)$ *uniformly balanced if* $n_1 \geq n_2$*?*

Let Z_t be the poset whose Hasse graph is obtained from a path on t vertices by giving the edges alternating directions. The number of antichains in Z_t is given by the Fibonacci numbers, which implies that $c_\star(Z_t) \leq \log \frac{1+\sqrt{5}}{2} + O(t^{-1})$. On the other hand, $(\log 2)/2$ is a lower bound for $c_\star(Z_t)$, since Z_t is a subposet of $C(t/2, t/2)$. This leaves a small gap which it would be nice to close.

Question 10.2. *What is* $c_\star(Z_t)$*?*

Something which we have touched upon in the paper, albeit indirectly, is the size of the connected components of $\mathcal{P}(n, p)$.

Question 10.3. *What is the size of the largest connected component of* $\mathcal{P}(n, p)$*?*

At the common threshold for the stars $C(1, t)$ the components size becomes unbounded. For larger values of $p = \exp(-cn)$ it would be interesting to compare the size of largest component to $2^n e^{-cn}$, the expected number of elements in $\mathcal{P}(n, p)$.

As we have seen, once we pass the threshold for the existence of P, the collection of 'profiles' of copies of P that occur with positive probability in $\mathcal{P}(n, p)$ begins to expand. We have also given examples where this set of embeddings does not consist of a single point, even at $c_\star(P)$. It would be interesting to identify conditions which ensure that there is a unique embedding at $c_\star(P)$, and to give some quantitative large deviation bounds for the occurring copies of P.

Finally, let us note that determining the Ramsey threshold for $\mathcal{P}(d)$ exactly seems hard, much like the deterministic question of finding $R(\mathcal{P}(d_1), \mathcal{P}(d_2))$. In [2] various bounds were given and it was shown that $R(\mathcal{P}(3), \mathcal{P}(3))$ is either 7 or 8. As part of our own investigation into Ramsey problems for posets we proved the following:

Theorem 10.4. $R(\mathcal{P}(3), \mathcal{P}(3)) = 7$

In order to prove this we created a Boolean satisfiability version of the problem. Here we have one Boolean variable for each element of $\mathcal{P}(d)$. For each $\mathcal{P}(3)$ in $\mathcal{P}(d)$ we create two clauses, expressing that at least one variable in a $\mathcal{P}(3)$ must be set to True and at least one to False, thereby avoiding a monochromatic copy of $\mathcal{P}(3)$. For $d \leq 6$ satisfying assignments for these Boolean formulae are easily found by a standard SAT-solver like MiniSat, while for $d = 7$ the formula is found to be unsatisfiable.

Added in proof. After submitting this paper, we learned that Theorem 1.1 could be derived from an old result of Stanley. Stanley [31] showed that for any poset P, there is a bijection between the family of antichains in P and $\hom_P(\mathcal{P}(1))$, the family of homomorphisms from P to $\mathcal{P}(1)$. As there is a natural bijection between $\hom_P(\mathcal{P}(n))$ and $(\hom_P(\mathcal{P}(1)))^n$, this gives an alternative proof of Theorem 1.1.

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