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Critical plural logic*

Salvatore Florio and Øystein Linnebo

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Abstract

What is the relation between some things and the set of these things? Mathematical practice does not provide a univocal answer. On the one hand, it relies on ordinary plural talk, which is implicitly committed to a traditional form of plural logic. On the other hand, mathematical practice favors a liberal view of definitions which entails that traditional plural logic must be restricted. We explore this predicament and develop a “critical” alternative to traditional plural logic.

1 Introduction

English and other natural languages contain plural expressions, which allow us to talk about many objects simultaneously, for example:

- (1) The students cooperate.
- (2) The natural numbers are infinite.

How should such sentences be analyzed? A common strategy does without plurals and relies on sets.¹ Plural talk is eliminated in favor of singular talk about sets. For example, sentence (2) is analyzed as:

- (3) The set $\{x : x \text{ is a natural number}\}$ is infinite.

In recent years, there has been a surge of interest in an alternative strategy that makes uses of plural logic. This is a logical system that takes plurals at face value. When analyzing language,

*This article draws on material from our forthcoming book *The Many and the One: A Philosophical Study*, which is a comprehensive study of the logic, meaning, and metaphysics of plurals. Here we follow a particular thread concerned with the relation between pluralities and sets, relying especially on Chapters 2, 4 and 12. For useful comments and discussion, we would like to thank two anonymous referees, José Ferreirós, Peter Fritz, Simon Hewitt, David Nicolas, Alex Oliver, Agustín Rayo, Sam Roberts, Timothy Smiley, Eric Snyder, Hans Robin Solberg, Stewart Shapiro, and Gabriel Uzquiano, as well as audiences at the Fifth International Meeting of the Association for the Philosophy of Mathematical Practice in Zurich and the International Conference for Philosophy of Science and Formal Methods in Philosophy in Gdańsk.

¹See, e.g., Quine 1982 and Resnik 1988.

there is no need to eliminate the plural resources of English in favor of talk about sets or any other singular resources. Rather, the plural resources can be retained as primitive, not understood in terms of anything else.²

Assuming the alternative strategy is right, we have (at least) two different ways to talk about many objects simultaneously: plurally and by means of singular talk about sets. What is the relation between these two ways? That is, what is the relation between some things and the set of these things? In this article, we are especially interested in how mathematical practice bears on these questions. At the center of our discussion are:

- (i) Cantor’s and Gödel’s appeals to plurals to explain the notion of a set;
- (ii) a liberal view of mathematical definitions, espoused by Cantor and others, which entails that every plurality defines a set.

As we explain, this liberal view requires us to replace the traditional logic of plurals with a more “critical” plural logic.

Two larger questions pervade our discussion. The first question concerns how, and on what basis, we should choose a “correct” logic—in this particular case, a logic of plurals. We argue that the choice of a plural logic is entangled with some hard questions in the philosophy and foundations of mathematics. The second larger question concerns what and how philosophers can learn from studying mathematical practice. Mathematical practice is not always internally consistent. On the one hand, it is implicitly committed to a traditional form of traditional plural logic, at least insofar as this practice relies on ordinary plural language. On the other hand, mathematical practice favors a liberal view of definitions which entails that traditional plural logic must be restricted. We must therefore be extremely careful when attempting to extract philosophical lessons from mathematical practice. A detailed analysis of a broadly philosophical character is needed to adjudicate between the conflicting implicit commitments.

2 Plural logic

Let us begin by describing a language that may be used to regiment a wide range of natural language uses of plurals and to represent many valid patterns of reasoning that essentially involve plural expressions. This language is associated with what is known in the philosophical literature as PFO+, which is short for *plural first-order logic plus plural predicates*. In one variant or another, it is the most common regimenting language for plurals in philosophical logic.³

²This surge draws much of its inspiration from seminal work by George Boolos (1984; 1985), but Russell 1903’s notion of a “class as many” is an important anticipation.

³For systems that employ the notation for variables adopted here, see Rayo 2002 and Linnebo 2003. An ancestor of this notation is found in Burgess and Rosen 1997. Variants of the system represent plural variables by means

We start with the the standard language of first-order logic and expand it by making the following additions.

- A. Plural terms, comprising plural variables (vv, xx, yy, \dots , and variously indexed variants thereof) and plural constants (aa, bb, \dots , and variants thereof), roughly corresponding to the natural language pronoun ‘they’ and to plural proper names, respectively.
- B. Quantifiers that bind plural variables ($\forall vv, \exists xx, \dots$).
- C. A binary predicate \prec for plural membership, corresponding to the natural language ‘is one of’ or ‘is among’. This predicate is treated as logical.
- D. Symbols for collective plural predicates with numerical superscripts representing the predicate’s arity ($P^1, P^2, \dots, Q^1, \dots$, and variously indexed variants thereof). Examples of collective plural predicates are ‘...cooperate’, ‘...gather’, ‘...meet’, ‘...outnumber ...’, ‘...are infinite’. For economy, we leave the arity unmarked.

To illustrate the use of PFO+, let us provide some examples of regimentation based on this language.

- (4) Some students cooperated.
- (5) $\exists xx (\forall y (y \prec xx \rightarrow S(y)) \wedge C(xx))$
- (6) Bunsen and Kirchhoff laid the foundations of spectral analysis.
- (7) $\exists xx (\forall y (y \prec xx \leftrightarrow (y = b \vee y = k)) \wedge L(xx))$
- (8) Some critics admire only one another.
- (9) $\exists xx (\forall x (x \prec xx \rightarrow C(x)) \wedge \forall x \forall y [(x \prec xx \wedge A(x, y)) \rightarrow (y \prec xx \wedge x \neq y)])$

The formal system PFO+ comes equipped with logical axioms and rules of inference aimed at capturing correct reasoning in the fragment of natural language that is being regimented. The axioms and rules associated with the logical vocabulary of ordinary first-order logic are the usual ones. For example, one could rely on introduction and elimination rules for each logical expression. The plural quantifiers are governed by axioms or rules analogous to those governing the first-order quantifiers.

Plural logic is often taken to include some further, very intuitive axioms. First, every plurality is non-empty:

of other typographical conventions: boldface letters (Oliver and Smiley), capitalized letters (McKay), or singular variables pluralized with an ‘s’ (Yi).

(Non-empty) $\forall xx \exists y y \prec xx$

Then, there is an axiom scheme of indiscernibility stating that coextensive pluralities satisfy the same formulas:

(Indiscernibility) $\forall xx \forall yy [\forall x (x \prec xx \leftrightarrow x \prec yy) \rightarrow (\varphi(xx) \leftrightarrow \varphi(yy))]$

(The formula φ may contain parameters. So, strictly speaking, we have the universal closure of each instance of the displayed axiom scheme. Henceforth, we assume this reading for similar axiom schemes.) This is a plural analogue of Leibniz’s law of the indiscernibility of identicals, and as such, the scheme needs to be restricted to formulas $\varphi(xx)$ that don’t set up intensional contexts.

Finally, there is the unrestricted axiom scheme of *plural comprehension*, an intuitive principle that provides information about what pluralities there are. Informally, for any formula $\varphi(x)$ containing x but not xx free, we have an axiom stating that if $\varphi(x)$ is satisfied by at least one thing, then there are the things each of which satisfies $\varphi(x)$:

(P-Comp) $\exists x \varphi(x) \rightarrow \exists xx \forall x (x \prec xx \leftrightarrow \varphi(x))$

We refer to an axiomatization of plural logic based on the principles just described as *traditional plural logic*. This is to emphasize its prominence in the literature. We believe traditional plural logic is implicit in our ordinary use of plural language—and thus also in mathematical practice, which sometimes relies on such language.

Philosophers often proceed to make strong claims about this deductive system: it is aptly and rightly called “plural logic”, because its principles have the same privileged status as is widely accorded to ordinary first-order logic.⁴ Thus, plural logic is *pure logic*, not set theory in disguise. Although this is not the place for a thorough discussion of what counts as pure logic, we wish to make some brief remarks, focusing on three features.

One aspect of logicity is *topic-neutrality*. This is based on the simple, intuitive idea that logical principles should be applicable to reasoning about any subject matter. By contrast, non-logical principles are only applicable to particular domains. The laws of physics, for instance, concern the physical world and do not apply in reasoning about natural numbers or other abstract entities. Plural logic seems to satisfy this intuitive notion of topic-neutrality: the validity of the principles of plural logic does not appear to be restricted to specific domains. As partial evidence for the topic-neutrality of plural logic, one may point out that, when available, pluralization as a morphological transformation does not depend in any systematic way on the kind of objects one

⁴See, e.g., Boolos 1985 and Hossack 2000.

speaks about; for example, both concrete and abstract count nouns exhibit plural forms. While we agree that plural logic *in some form* is topic neutral, we deny that *traditional* plural logic has this status. Specifically, we argue that the unrestricted plural comprehension scheme is valid only for special kinds of domains—loosely speaking, domains that are properly circumscribed.

A second aspect of logicity is *ontological innocence*: a logical truth should not carry any ontological commitments. (There is usually one exception: the existence of at least one object. But even this commitment is generally tolerated only so as to streamline logical theory, not endorsed for doctrinal reasons.) Plural logic is widely held to be ontologically innocent in this sense.⁵ The plural existential quantifier expresses that there is one or more objects *of the sort to which one is already committed*; it does not introduce any *new* ontological commitment. In other work, however, we dispute this widely held view, arguing that there is an important sense in which plural logic does carry non-trivial commitments of a broadly ontological sort.⁶ We choose not to enter into this debate here.

A third aspect of logicity is the idea that logical notions and principles permit a special kind of epistemic primacy. Logical notions can be grasped without relying on non-logical notions. Likewise, logical truths, if knowable, can be known independently of non-logical truths, including those of mathematics. The claim that the principles of plural logic enjoy this kind of epistemic primacy will be challenged in what follows. As advertised, we take plural logic to be entangled with certain broadly set-theoretic principles.

3 Plural logic vs. a simple set theory: a formal comparison

What is the relation between some things and their set? Let us begin with a formal comparison of plural logic and set theory, which will clarify some important technical aspects of the question. In later sections, we will address more philosophical issues concerning the relation between some things and their set.

Assume we start with an ordinary singular first-order language whose quantifiers range over certain objects. Let us refer to these objects as *individuals*. We are interested in ways to talk simultaneously about many individuals. The most familiar option, at least to anyone with some training in mathematics, is to use set theory. A set is a *single object* that has zero or more *elements*. Talking about a single set thus provides a way to talk about all of its elements simultaneously. We can, for example, convey information about Russell and Whitehead by talking about their set {Russell, Whitehead}. The information that they are philosophers can

⁵Defenses of the innocence of plural logic are put forth, among others, by Boolos (1984, 1985), Yi (1999, 2002, 2005, 2006), Hossack (2000), Oliver and Smiley (2001, 2016), Rayo (2002), and McKay (2006).

⁶See Florio and Linnebo 2016 and Florio and Linnebo forthcoming, Chapter 8.

be conveyed by saying that every element of the set is a philosopher. Suppose, more generally, that we want to talk about the φ 's, where φ is a formula of our language that is satisfied by at least one object. According to the present strategy, this can be achieved by talking about the associated set, namely $\{x : \varphi(x)\}$.

However, it is not obvious that this strategy always works. After all, the lesson of the set-theoretic paradoxes is that not every formula defines a set. (We assume classic logic.) The most famous example is Russell's paradox of the set of all sets that are not elements of themselves. Consider the formula that serves as a condition for membership in this would-be set: $x \notin x$. Suppose this formula defines a set R . Now ask: is R an element of itself? The answer is affirmative if and only if R satisfies the membership condition. In other words: $R \in R$ if and only if $R \notin R$. But this is a contradiction!

Thankfully, the problem posed by the set-theoretic paradoxes can be put off, at least for a little while. The paradoxes do not arise when we consider only sets of individuals drawn from a fixed first-order domain. And for present purposes, this is all we need. Let us therefore consider a very simple set theory, which satisfies our present needs but does not give rise to paradoxes.

We need to distinguish between individuals and sets of individuals. To do so, it is convenient to use a *two-sorted* language. Such languages are easily explained because they are implicit in various mathematical practices. For example, in geometry we often use one set of variables to range over points (say, p_1, p_2, \dots) and another set of variables to range over lines (say, l_1, l_2, \dots). We adopt a similar approach to our simple set theory, letting lower-case variables range over individuals (x, y, \dots) and upper-case variables (X, Y, \dots) range over sets of individuals. We refer to these as *individual variables* and *set variables*, respectively. If desired, we can add constants of either sort. There are sortal restrictions on the formation rules. For instance, the language has a membership predicate ' \in ' whose first argument can only be an individual term and whose second argument can only be a set term. Thus, ' $a \in X$ ' means that the individual a is an element of the set X . In addition to the ordinary identity predicate, which can be flanked by any two individual terms, our extended language contains a set identity predicate, which can be flanked only by set terms. For convenience, we write this predicate as the usual identity sign. Given the restrictions just mentioned, it is impermissible to make identity claims involving both an individual and a set term (such as ' $a = X$ '). We call this language \mathcal{L}_{SST} and we let $\mathcal{L}_{\text{SST}}^+$ be the language obtained from \mathcal{L}_{SST} by adding predicates that take set terms as arguments. This is an optional extra to which we will return.

We now formulate our simple set theory, SST, based on the axioms and rules of two-sorted

classical logic. First, we adopt the axiom of extensionality for sets:

$$(S\text{-Ext}) \quad \forall x(x \in X \leftrightarrow x \in Y) \rightarrow X = Y$$

Then, we adopt an axiom scheme of set comprehension:

$$(S\text{-Comp}) \quad \exists X \forall x(x \in X \leftrightarrow \varphi(x))$$

where X does not occur free in φ . The theory SST+ is obtained by adapting the axioms and rules of SST to the richer language $\mathcal{L}_{\text{SST}}^+$. Notice how Russell’s paradox is blocked by the use of separate sorts for individuals and their sets. In our two-sorted language, the membership condition for the offending set, namely $x \notin x$, cannot even be formulated. The same can be seen to hold for the other set-theoretic paradoxes.

Consider a domain of individuals to which both plural logic and the simple set theory are applicable. We thus have two different ways to talk about many objects simultaneously. As we will now show, these the two different ways of talking share a common structure.

To begin with, the two languages share a common stock of variables x_i that take as their values one individual at a time. Further, each language has an additional stock of variables that are used to convey information about (loosely speaking) collections of individuals: plural variables xx_i , which take as their values many individuals simultaneously, or set variables X_i , which take as their values a single set of individuals. Finally, each language has a predicate for membership in a collection: $x_i \prec xx_j$ for “ x_i is one of xx_j ” or $x_i \in X_j$ for “ x_i is an element of X_j ”. These observations suggest that it should be straightforward to translate back and forth between the two languages. One can simply replace \prec with \in and xx_j with X_j , and *vice versa*.

However, there are two wrinkles to be ironed out:

- \mathcal{L}_{SST} has an identity predicate that can be flanked by set terms, whereas the language of PFO has no identity predicate that can be flanked by plural terms.
- SST postulates an empty set, whereas PFO has an axiom stating that every plurality is non-empty.

Fortunately, both problems are easily overcome. It is possible to define a translation from each language to the other such that each sentence and its translation convey the same information, at least as far as the individuals are concerned. The only difference is that one sentence does so by utilizing plural resources, while the other uses set-theoretic resources.⁷

⁷This observation is due to Boolos (1984). For an exposition, see Florio and Linnebo forthcoming, Appendix 4.A.

The translations satisfy the following important conditions:

- (i) Each translation is recursive, that is, there is an effective algorithm for carrying out the translation.
- (ii) Each translation respects logical structure; for example, the translation of a negation $\neg\varphi$ is the negation of the translation of φ .
- (iii) Every axiom of each of the two theories is translated as a theorem of the other theory; for example, each axiom of PFO is translated as a theorem of SST.

More generally, let τ be a translation from the language of one theory T_1 to that of another theory T_2 such that these three conditions are satisfied. Then τ is said to be an *interpretation* of T_1 in T_2 . Thus, we have that each of our two theories PFO and SST can be interpreted in the other, and likewise with PFO+ and SST+.

It is important to be absolutely clear about what the mutual interpretability of two theories does and does not establish. Interpretability is a purely formal notion: it concerns a translation preserving theoremhood, and it allows us to recursively turn a model of one theory into a model of another. So, two mutually interpretable theories are equivalent for the purposes of formal logic. However, there is no guarantee that the equivalence will extend beyond those purposes. Suppose the two languages are meaningful. Then there is no guarantee that the translation preserves the kinds of extra-logical properties that philosophers often care about. For example, there is no guarantee that the translation preserves features of sentences such as:

- truth value;
- meaning (perhaps understood as the set of possible worlds at which a sentence is true);
- epistemic status (such as apriority or aposteriority);
- ontological commitments.

It is often controversial whether a translation preserves these properties. The translations we consider here are no exception.

4 Should sets or pluralities be eliminated?

What is the significance of the shared structure, or mutual interpretability, that we just observed? Is this *merely* a technical result? Or does the technical result have some broader philosophical significance? When the structure of one theory can be recovered within that of another, this

raises the question of whether one of the theories can be eliminated in favor of the other. In the present context, there are three options. First, one may eliminate pluralities in favor of sets. Second, one may proceed in the opposite direction and eliminate sets in favor of pluralities. Finally, one may refrain from any elimination and retain both pluralities and sets. All three options have their defenders. Let us consider them in turn.

First, some philosophers seek to eliminate sets in favor of pluralities. That is, we can and should interpret ordinary ‘set’ talk without relying on set-theoretic resources. A classic paper by Black (1971) can be read as advocating this view.⁸ More recently, Oliver and Smiley have expressed considerable sympathy for the view, claiming to have at least shifted the burden of proof onto its opponents (2016, 316–17). Black observes that ordinary language often talks about sets: expressions such as ‘my set of chessmen’ or ‘that set of books’ feel fairly natural to English speakers. By reflecting on ordinary uses of the word ‘set’, he argues, we can come to see the intimate connection between talk about a set and about its elements. More specifically, we can come to realize that basic uses of the word ‘set’ are simply substitutes for plural expressions such as lists of terms or plural descriptions. In his example, the sentence ‘a certain set of men is running for office’ is what he calls an “indefinite surrogate” for the statement that, say, Tom, Dick, and Harry are running for office (Black 1971, 631).

Second, other philosophers hold that the plural locutions found in English and other natural languages should be eliminated in favor of talk about sets. Quine and Resnik are advocates of this view.⁹ For Quine at least, this is at root a claim about regimentation into our scientific language. It is indisputable that many natural languages contain plurals locutions. But our best scientific theory of the world has no need for such locutions. This theory is to be formulated in a singular language whose quantifiers range, among other things, over sets. When regimenting natural language into this scientific language, the plural locutions of the former should be analyzed by means of the set talk of the latter. In short, for scientific purposes, we should eschew plural resources and instead rely on set-theoretic resources. These resources also suffice to interpret “the vulgar” (as Quine once put it), that is, to regiment the plural resources indisputably found in English and other natural languages.

Finally, one may hold that neither system should be eliminated in favor of the other, because both plural logic and set theory are legitimate and earn their keep in our best scientific theory. Following Cantor and Gödel, this is the view that we will defend. Suppose we are right that both systems should be retained. This gives rise to some further questions concerning their

⁸We should note that this is not the only way to read Black. It is not entirely clear whether he proposes an eliminative reduction or favors some form of non-eliminative reductionism. An eliminative proposal is developed by Hossack (2000) who appeals to plurals and plural properties to eliminate sets.

⁹See footnote 1.

relation. We will be particularly concerned with two such questions.

- (i) Every non-empty set obviously corresponds to a plurality, namely the elements of the set. What about the other direction? Does every plurality correspond to a set? If not, under what conditions do some things form a set?¹⁰
- (ii) Suppose that some objects form a set. Can these objects be used to shed light on, or give an account of, the set that they form?

First, however, let us explain why we reject both of the eliminative proposals.

Why not follow Black and others and eliminate sets in favor of pluralities? Black recognizes that there is a gap between ordinary uses of the word ‘set’ and its uses in mathematics. For instance, ordinary speakers untrained in abstract mathematics often have misgivings about the empty set. If sets are collections of things, how can there be a collection of nothing whatsoever? Despite such misgivings, Black contends that we can rely on our ordinary understanding of plurals to make sense of “idealized” uses of the word ‘set’ as it occurs in mathematics.

However, there is an obvious difficulty for Black’s contention. Talk of *sets of sets* is ubiquitous in mathematics and, as we will see shortly, such “nested” sets are essential to the now-dominant iterative conception of set. How can we account for these uses of the word ‘set’? If talk about sets is shorthand for talk about pluralities, then sets of sets would seem to correspond to higher-order pluralities, that is, “pluralities of pluralities”.¹¹ It is controversial whether such higher-order pluralities make sense, but a putative example is given in following sentence.

- (10) My children, your children, and her children competed against each other.

The subject of this sentence appears to be a “nested” plural, that is, a plural expression formed by combining three other plural expressions. Arguably, this nesting of the subject is semantically significant. The claim is not merely that all the children in question compete against each other but that they do so in teams, each team comprising the children of each parent.¹²

While the availability of higher-order pluralities is a necessary condition for the envisaged elimination of sets, we deny that it is sufficient. As observed, the language of mathematics talks extensively about sets and appears to treat these as objects. Other things equal, it would be good to follow mathematical practice and take this language at face value. In the absence of a strong reason to deviate from the practice, this yields an independent reason not to eliminate sets. This reason is even stronger for those who accept other mathematical objects such as numbers. If numbers are accepted, why not also accept sets?

¹⁰See Hewitt 2015 for a useful overview of this issue.

¹¹For proposals along these lines, see Simons 2016 and Oliver and Smiley 2016, Chapter 15.

¹²We discuss whether there are higher-order pluralities in Florio and Linnebo forthcoming, Chapter 10.

We turn now to Quine’s and Resnik’s suggestion that pluralities be eliminated in favor of sets. It is often objected that this form of elimination would give rise to paradoxes.¹³ We do not find these arguments entirely convincing.¹⁴ Instead of entering into this debate here, however, we wish to lay out another—and, we believe, more compelling—reason why pluralities should not be eliminated in favor of sets. The reason is simply that pluralities are needed to give an account of sets. If pluralities were eliminated in favor of sets, we could not use plural reasoning to give such an account. To retain some attractive account of sets in terms of pluralities, we cannot eliminate plurals.

5 Accounting for sets in terms of pluralities

What is the promised account of sets in terms of pluralities? It is useful to recall how Cantor, the father of modern set theory, sought to explain the concept of set.¹⁵

By a ‘manifold’ or ‘set’ I understand in general every many which can be thought of as one, i.e. every totality of determinate elements which can be bound together into a whole through a law [...].¹⁶

That is, a set is a “many thought of as one”. Of course, it is far from clear how this is to be understood. But there can be no doubt that Cantor sought to understand a set in terms of the many objects that are its elements and that are somehow “thought of as one”.¹⁷

By a ‘set’ we understand every collection into a whole M of determinate, well-distinguished objects m of our intuition or our thought (which will be called the ‘elements’ of M). We write this as: $M = \{m\}$.

It is tempting to read Cantor’s variable ‘ m ’ as a plural variable (see also Oliver and Smiley 2016, 4-5). So, in line with our notation, let us replace this variable with ‘ mm ’. A set M is then said to be a collection into one of some well-distinguished objects mm , namely the elements of M . And we write $M = \{mm\}$. A closely related idea is endorsed by Gödel, who, in a passage to be

¹³See, e.g., Boolos 1984, 440-44); Lewis 1991, 68; Schein 1993, Chapter 2, Section 3.3; Higginbotham 1998, 14-17; Oliver and Smiley 2001, 303-305; Rayo 2002, 439-440; and McKay 2006, 31-32.

¹⁴See Florio and Linnebo forthcoming, Section 3.4.

¹⁵Unter einer Mannichfaltigkeit oder Menge verstehe ich nämlich allgemein jedes Viele, welches sich als Eines denken lässt, d.h. jeden Inbegriff bestimmter Elemente, welcher durch ein Gesetz zu einem Ganzen verbunden werden kann [...]. (Cantor 1883, 43)

¹⁶Since the exact choice of words is important to the point we are making here, we have chosen to provide our own translation of this passage and the next.

¹⁷Unter einer ‚Menge‘ verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unsrer Anschauung oder unseres Denkens (welche die ‚Elemente‘ von M genannt werden) zu einem Ganzen. In Zeichen drücken wir dies so aus: $M = \{m\}$. (Cantor 1895, 481)

considered shortly, discusses a “set of” operation that takes some “well-defined objects” to their set.

More generally, many philosophers and mathematicians believe that the elements of a set are somehow “prior to” the set itself and that the set is somehow “constituted” by its elements.¹⁸ Assume xx form a set $\{xx\}$. Then the objects xx can be used to give an account of $\{xx\}$. That is, properties and relations involving the set are explained in terms of properties and relations involving the plurality of its elements. Why is a an element of $\{xx\}$? An answer immediately suggests itself: because a is one of xx ! Why is $\{xx\}$ identical with $\{yy\}$? Again, the answer seems obvious: because xx are the very same objects as yy .

All of these remarks suggest to us a liberal view of mathematical definitions, which we will first sketch and then spell out and defend. This liberal view takes it to be sufficient for a mathematical object to exist that an adequate definition of it has been provided. The adequacy in question is understood as follows. Consider a “properly circumscribed” domain of objects standing in certain relations. We would like to define one or more additional objects. Suppose our definition determines the truth of any atomic statement concerned with the desired “new” objects by means of some statement concerned solely with the “old” objects with which we began. Then, according to the liberal view, the definition is permissible.

To illustrate, let us apply the view to the case of sets. Again, consider some properly circumscribed domain of objects. For every plurality of objects xx from this domain, we postulate their set $\{xx\}$, with the understanding that atomic statements concerned with any new sets should be assessed in the way mentioned above.

(i) $\{xx\} = \{yy\}$ if and only if $xx \approx yy$.

(ii) $a \in \{xx\}$ if and only if $a \prec xx$.

Notice how this account determines the truth of any atomic statement concerned with the “new” sets solely in terms of the “old” objects with which we began, as required by the liberal view.

What about the empty set? Here there is a threat of a mismatch. While standard set theory accepts an empty set, traditional plural logic does not accept an empty plurality. But we are confident that this threat can be addressed. One option is to break with traditional plural logic and accept an empty plurality, perhaps on the grounds that, although this isn’t how plurals work in English and many other natural languages, there are coherent languages where plurals do behave in this way (see Burgess and Rosen 1997, 154-155). Another option is to break with standard set theory and abandon the empty set. However, we would prefer not to deviate from successful scientific practice, in this case set theory, unless there are compelling reasons to do so.

¹⁸See, e.g., Parsons 1977 and Fine 1991.

Finally, an elegant option proposed (in a different context) by Oliver and Smiley (2016, 88-89) is to allow “co-partial functions”, that is, functions that can have a value even where the argument is undefined. Suppose the “set of” operation $xx \mapsto \{xx\}$ is such a function. Then, applied to an undefined argument, this function can have the empty set as its value.

attempt to eliminate one in favor of the other. Next, can we account for nested sets? This means going beyond the simple set theory discussed above to form a stronger set theory, where the threat of paradox re-emerges. The standard response to this threat is the so-called iterative conception of set. One of the first clear expressions of this conception is given in a famous passage by Gödel.¹⁹

The concept of set, however, according to which a set is anything obtainable from the integers (or some other well-defined objects) by iterated application of the operation “set of”, and not something obtained by dividing the totality of all existing things into two categories, has never led to any antinomy whatsoever; that is, the perfectly “naive” and uncritical working with this concept of set has so far proved completely self-consistent. (Gödel 1964, 180)

The passage calls for some explanation. First, Gödel distinguishes the iterative conception of set from a problematic conception based on the idea of “dividing the totality of all existing things into two categories.” Consider a condition that any object may or may not satisfy. One might then attempt to use this condition to divide the totality of all objects into two sets: the set of objects that satisfy the condition and the set of those that don’t. But this approach to sets is problematic: as we have seen, it gives rise to Russell’s paradox.

By contrast, the iterative conception starts with the integers or “some other well-defined objects”. We are then told to consider iterated applications of the operation “set of”. An example will help. Assume we start, at what we may call stage 0, with two objects, say a and b . The “set of” operation can be applied to any plurality of objects available at stage 0 to form their set. Thus, at stage 1, which results from the application of this operation to the objects available at stage 0, we have the following sets: \emptyset , $\{a\}$, $\{b\}$, and $\{a, b\}$. So, at stage 1, we have six objects, namely a and b together with four sets that were not available at stage 0. Now we can apply the “set of” operation again, this time to the objects available at stage 1. This yields sets such as $\{\emptyset, a\}$, $\{\{a\}, \{b\}\}$, and many others. Note that, by this procedure, the objects available at any given stage form a set at the next stage.

There is a more systematic way to describe what takes us from one stage α to the next stage

¹⁹The passage contains some footnotes that we elide.

$\alpha + 1$. For any set S , let its powerset, $\wp(S)$, be the set of all subsets of X , that is:

$$\wp(S) = \{x : x \subseteq S\}$$

Suppose the objects available at stage α are the elements of V_α . Then at stage $\alpha + 1$ we form all the subsets of V_α . So, at stage $\alpha + 1$, we have the elements of V_α as well as those of $\wp(V_\alpha)$. In symbols: $V_{\alpha+1} = V_\alpha \cup \wp(V_\alpha)$. Again, we have by this procedure that all the sets available at stage α , taken together, form a set at stage $\alpha + 1$.

In fact, we want to consider really long iterations of the “set of” operation. The first step is to define V_ω as the result of continuing in this way as many times as there are natural numbers. We do this by letting V_ω be the union of all of the collections V_n generated at a finite stage: $V_\omega = \bigcup_{n < \omega} V_n$. More generally, for any limit ordinal λ , we let V_λ be the union of all the collections of sets we have generated: $V_\lambda = \bigcup_{\gamma < \lambda} V_\gamma$. The *cumulative hierarchy of sets*, V , is the union of all of the V_α .

However, as Gödel observes in a footnote to the passage just quoted, V isn’t a set. There is no stage at which all sets are available to form a universal set. For any stage, there is a later stage containing even more sets. As a result, we ban the universal set and any other set that would lead to paradox. This raises the question of the status of the cumulative hierarchy itself, including the question of whether “it” even exists as an object.

6 Proper classes as pluralities

In fact, both the problem posed by the ontological status of the entire cumulative hierarchy and the proposed solution of invoking plurals generalize. Let us use the word ‘collection’ in an informal way for anything that has a membership structure, such as a set, class, plurality, or indeed even a Fregean concept—when the relation between instance and concept is regarded as a membership structure. We will now explain the apparent need to talk about collections that are too large to form sets, why these are sometimes regarded as problematic, and finally a brilliant proposal due to Boolos, namely that plural logic provides a way to make sense of these collections.

Let us begin with the need for a novel type of collection, in addition to sets. There are several reasons for this need. Boolos mentions two. First, collections are needed to make sense of the domain of set theory, namely the cumulative hierarchy V . For example, we would like to say that V is the subject matter of set theory and that V is well-founded.

Second, collections are needed to understand and justify two axiom schemes that are part

of ordinary Zermelo-Fraenkel set theory, ZFC, namely Replacement and Separation.²⁰ Both of these take the form of an infinite family of axioms. Consider Separation. ZFC contains an axiom

$$(Sep) \quad \forall z \exists y \forall x (x \in y \leftrightarrow x \in z \wedge \varphi)$$

for each of the infinitely many formulas φ of its language. Behind this infinite lot of axioms, however, lies a single, unified idea that can be expressed by reference to collections. For every collection C and every set x , there is a set y of all those elements of x that belong to C . Suppose we can quantify over collections. Then the infinitely many Separation axioms could be unified as the single axiom:²¹

$$(C-Sep) \quad \forall C \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge z \text{ belongs to } C)$$

In the literature, the desired collections are often known as *classes*, some of which can be shown to be “too big” to be sets. These are called *proper classes*. But what would these proper classes be? Just like sets, they are collections of many objects into one. But why, then, are proper classes not sets? As Boolos (1984, 442) nicely observes, “[s]et theory is supposed to be a theory about *all* set-like objects”.

Adding proper classes to a theory of sets is just like adding yet another layer of sets on top of the sets already recognized. In light of this, why shouldn’t the proper classes count as just more sets? William Reinhardt puts the point well:

[O]ur idea of set comes from the cumulative hierarchy, so if you are going to add a layer at the top it looks like you forgot to finish the hierarchy.²²

Plural logic seems to provide precisely what we need. There is no need for a proper class to be a *single object* that somehow collects together many things into one. Instead of referring in a singular way to a proper class, construed as an object, why not simply refer plurally to its many members? In this way, we eliminate talk about proper classes in favor of plural talk about their members. For example, there is no need for the cumulative hierarchy to be an object. It suffices to talk plurally about *all the sets*. Next, consider the axiom scheme of Separation. This can now be given a single uniform plural formulation. Given any objects pp and any set x , there is a set y of precisely those elements of x that are also among pp :

$$(P-Sep) \quad \forall pp \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge z \prec pp)$$

²⁰Analogous considerations apply to the arithmetical principle of induction.

²¹See also Kreisel 1967.

²²Reinhardt 1974, 32. For a useful elaboration of the point, see Maddy 1983, 122.

Of course, to represent all of the classes that we might be interested in, we would need an unrestricted form of plural comprehension.

Let us take stock. We have described two very attractive applications of plural logic: first, as a way of giving an account of sets; second, as a way of obtaining proper classes “for free”.

Regrettably, the two applications of plural logic appear incompatible. The first application suggests that any plurality forms a set. Consider any objects xx . Presumably, these are what Gödel calls “well-defined objects”. If so, it is permissible to apply the “set of” operation to xx , which yields the corresponding set $\{xx\}$. The second application, however, requires that there be pluralities corresponding to proper classes, which by definition are collections too big to form sets. For example, there must be a plurality of all sets whatsoever to serve as the proper class V . But, when the “set of” operation is applied to this plurality, we obtain a universal set, which is unacceptable.

Is there any way to retain both of the attractive applications of plural logic? To do so, we would have to restrict the domain of application of the “set of” operation such that the operation is *undefined* on the very large pluralities that correspond to proper classes, while it remains defined on smaller pluralities. The passage from Gödel suggests at least the possibility of such a restriction, because he requires that the “set of” operation be applied to “well-defined objects”. The obvious concern is that the needed restriction would be *ad hoc*. The “set of” operation applies to vast infinite pluralities, thus forming large sets in the cumulative hierarchy. But once we allow that these infinite pluralities form sets, why are other infinite pluralities suddenly too large to do so?

7 Towards a reasonable liberalism about definitions

To make progress, we need to take a closer look at our liberal view of definitions. What, exactly, is required for an attempted definition of a set to be permissible?

It is useful to begin with an analogy. Suppose you detest web pages that link to themselves.²³ So you wish to create a web page that links to all web pages that are innocent of this bad habit. In other words, you wish to create a web page that links to all and only the web pages that do not link to themselves. Can your wish be fulfilled? The answer depends on how your wish is analyzed. Should the scope of the crucial plural description—“the web pages that do not link to themselves”—be narrow or wide? Depending on its scope, your wish can be analyzed in either of the following two ways:

(N) You wish to design a web page y such that, for every web page x , y links to x if and only

²³This example has been independently used by Brian Rabern in teaching and on social media.

if x does not link to itself.

- (W) There are some web pages xx such that, for every web page x , x is one of xx just in case x does not link to itself, and you wish to design a web page y that links to all and only xx .

On the narrow scope reading (N), your wish is flatly incoherent. The desired web page would have to link to itself just in case it does not link to itself. On this reading, your wish is no better than the wish to bring about the existence of a Russellian barber:

- (B) You wish there to be a barber y such that, for all x , y shaves x if and only if x does not shave himself.

On the wide scope reading (W), by contrast, there is no conceptual or mathematical obstacle to the fulfillment of your wish. First, you identify all the web pages xx that refrain from the bad habit of self-linking. Then, you create a new web page that links to all and only xx .

What explains this stark difference between the two readings? The heart of the matter is how one specifies the target collection—that is, the web pages of which you wish to create a comprehensive inventory. On (N), the target is specified intensionally by means of the condition ‘ x does not link to itself’. This intensional specification means that the target shifts with the circumstances. First you find that there is no web page of the sort you wish for. So you attempt to fulfill your wish by changing the circumstances, by creating a web page of the desired sort. But since the target is specified intensionally, this new web page must itself be taken into account when assessing whether your wish has been satisfied—which of course it has not, as logic alone informs us.

By contrast, on the wide scope reading (W), the target is specified extensionally by means of the plurality xx . This extensional specification ensures that the target stays fixed when you change the circumstances. (Here we invoke the modal rigidity of pluralities, which we defend in Florio and Linnebo forthcoming, Chapter 10.) You can thus fulfill your wish by creating a new web page that links to all and only xx . Although xx are described, in the original circumstances, by means of a condition that is prone to paradox, there is no requirement that xx should remain so described in alternative circumstances. Like any other plurality, xx are tracked rigidly across alternative circumstances, not in terms of any description that these objects happen to satisfy.

With this analogy in mind, let us return to the question of what is a reasonable liberalism about mathematical definitions. Suppose you care about sets, not web pages. You wish to define a set by specifying its elements. As our web page analogy reveals, it is essential to distinguish between two different ways in which the elements of the would-be set might be specified. First, you might specify the elements *intensionally*, by means of a condition $\varphi(x)$:

- (I) You wish to define a set y such that, for every object x , x is an element of y if and only if $\varphi(x)$.

Second, you might specify the elements of the would-be set *extensionally*, by means of a plurality xx :

- (E) You wish to define a set y such that the elements of y are precisely xx .

Can either wish be fulfilled?

This is a question about what it takes for a mathematical definition to be permissible. We claim that the proposed definition is often problematic when the target is specified intensionally, but always permissible when the target is specified extensionally. Our defense of these claims will be informed by our web page analogy.

Let us begin with the negative claim that (I) is often problematic. The reason is simple. We can hardly be more liberal about mathematical definitions than we are about objects that we literally (and easily) construct, such as web pages. This means we need to be extremely cautious about which definitions of sets we deem permissible when the target is specified intensionally. To illustrate how such definitions can be problematic, observe that one instance of the intensionally specified wish (I) is an analogue of the problematic narrow-scope wish (N) concerning web pages:

- (N') You wish to define a set y such that, for every object x , x is an element of y if and only if x is not an element of itself.

Just as (N) is flatly incoherent, so, we contend, is (N').

We turn now to our positive claim, namely that the proposed definition is always permissible when the target is specified extensionally by means of a plurality xx . Here is the rough idea. The extensional specification ensures that the target won't shift with the circumstances. We therefore have no difficulty making sense of circumstances in which xx define a set—much as we have no difficulty making sense of circumstances in which some given web pages yy are precisely the ones to which some new web page links.

We can be far more specific, though. Consider a dispute between a proponent and an opponent of the proposed definition. Suppose both parties accept a domain dd . The proponent now wishes to define one or more sets of the form $\{xx\}$, where xx are drawn from dd . She does not insist that the sets to be defined be among dd ; in this sense, the sets may be “new”. To shore up the proposed definition, she provides the following account of what it takes for “new” sets to be identical or have certain elements:²⁴

- (i) $\{xx\} = \{yy\}$ if and only if $xx \approx yy$

²⁴If dd already contain some sets, then clause (i) must be understood to range over both “old” and “new” sets.

(ii) $y \in \{xx\}$ if and only if $y \prec xx$

These clauses achieve something remarkable. They provide answers to all atomic questions about the “new” sets of the form $\{xx\}$ in terms that are concerned solely with the “old” objects in dd , objects that were available before the definition. That is, all atomic questions about the “new” objects receive answers in terms of the “old” objects that both parties to the dispute accept.

Of course, this is merely an instance of the liberal view of definitions that we outlined in Section 5. According to this view, it suffices for a mathematical object to exist that an adequate definition of it can be provided—where the adequacy is understood as follows. Consider a domain dd of objects standing in certain relations. We would like to define one or more additional objects. Suppose our definition provides truth conditions for any atomic predication concerned with the desired “new” objects in the form of some statement concerned solely with the “old” objects with which we began. Thus, any atomic question about the “new” objects can be reduced to a question that is solely about the “old” objects. Then, according to our liberal view, the definition is permissible.

It is instructive to compare with the situation where the desired set is specified intensionally, by means of a membership condition for each of the desired sets. Again, we start with some objects dd accepted by both parties. A more extreme proponent of liberal definitions may wish to define sets of the form $\{x : \varphi(x)\}$, where any parameters in the membership condition $\varphi(x)$ are drawn from dd . Again, she does not insist that these sets be among dd ; they may be “new”. Her opponent will rightly challenge her to provide an account of what it takes for “new” sets to be identical or to have certain elements. Given the intensional specification of the desired sets, her answers will be as follows:

(i') $\{x : \varphi(x)\} = \{x : \psi(x)\}$ if and only if $\forall x(\varphi(x) \leftrightarrow \psi(x))$

(ii') $y \in \{x : \varphi(x)\}$ if and only if $\varphi(y)$

These answers are potentially problematic in a way that their extensional analogues, (i) and (ii), are not. An interesting example is the attempt to define a set $a = \{x : x \in x\}$. If this definition is to succeed, there must be an answer to the question of whether a is an element of itself. But the only answer we receive from clause (ii') is that $a \in a$ if and only if $a \in a$. Of course, this is useless.²⁵ More tellingly, the answer is not stated in terms of the objects accepted by both parties to the dispute. An atomic question about the “new” object a receives an answer that essentially involves *this very object*; there is no reduction to the “old” objects among dd .

²⁵It could be worse. When we ask whether the Russell set $b = \{x : x \notin x\}$ is an element of itself, we receive an inconsistent answer.

Notice that it is of no avail for the extreme liberal to allow a to lie outside of dd , that is, in our parlance, to be “new”. The set a is specified intensionally, by means of the membership condition ‘ $x \in x$ ’, and we cannot “outrun” this specification. Even in a domain that strictly extends dd , a is, by definition, the set of all and only the objects that satisfy the condition ‘ $x \in x$ ’. By contrast, when a set is specified extensionally by means of a plurality xx , it does help to consider a domain that strictly extends dd . Even if xx are, say, all the sets among dd that are not elements of themselves, xx need not satisfy this plural description in an extended domain; for xx are tracked rigidly into the extended domain, not by means of the description. This makes the world safe for the desired set $\{xx\}$, provided that the set is located outside of dd . Notice also the striking parallelism with the case of web page design. Suppose you want a web page to link to all and only the members of some collection of web pages, for example, the collection of web pages that do not link to themselves. If the target collection is specified intensionally, it is of no avail to create a *new* web page: you cannot “outrun” this problematic specification. By contrast, if the collection is specified extensionally, there is no obstacle to the creation of the desired web page.

The picture that emerges is that there is a fundamental difference between the proposed definitions of sets depending on whether the target is specified extensionally or intensionally. In the former case, every atomic question about the “new” objects is ensured an answer expressed solely in terms of the “old” objects, whereas in the latter case, this kind of reduction is often unavailable. The proposed definitions are therefore often unacceptable when the target is specified intensionally. In the case of an extensional specification, on the other hand, a proponent of liberal definitions is in a much stronger position. She has laid out certain definitions, which are mathematically fruitful and have the desirable property that all atomic questions about the “new” objects receive answers in terms that are acceptable to her opponent.

Admittedly, the proponent of liberal definitions cannot force her opponent to accept the proposed definitions: he does not contradict himself when he rejects them. But she can justifiably accuse her opponent of dogmatism that stifles scientific progress. He dogmatically clings to certain beliefs in a way that stands in the way of fruitful mathematics. By insisting that dd are all-encompassing—and thus that there can be no “new” objects outside of dd —he privileges certain metaphysical or logical dogmas to over good mathematics. We can hardly think of a better way to defend this outlook than by quoting the following passage by Cantor.

Mathematics is in its development entirely free and only bound in the self-evident respect that its concepts must both be consistent with each other and also stand in exact relationships, ordered by definitions, to those concepts which have previ-

ously been introduced and are already at hand and established. [...] the *essence* of *mathematics* lies precisely in its *freedom*.²⁶

8 The principles of critical plural logic

We have argued that any given objects can be used to define a set. Unsurprisingly, this has consequences for our choice of a plural logic. To avoid paradox, we have little choice but to restrict the plural comprehension scheme. Contrary to what has traditionally been assumed, not every condition defines a plurality. To emphasize this departure from traditional plural logic, let us call our approach *critical plural logic*.²⁷ The acceptance of this approach has implications far beyond the philosophy of mathematics, affecting views in semantics and metaphysics that rely on traditional plural logic. We provide a detailed assessment of our approach vis-à-vis the traditional one in Florio and Linnebo forthcoming (see especially Chapters 2 and 11).

How, exactly, does our critical plural logic differ from the traditional version? We accept the usual sentential and first-order logic. Furthermore, we allow the plural quantifiers to be governed by axioms and rules analogous to those governing those governing the first-order quantifiers.²⁸ Our quarrel with traditional plural logic concerns only the question of what pluralities there are, or, in other words, the question of which plural comprehension axioms to accept. It is therefore incumbent on us to clarify what pluralities we take there to be. It is insufficient merely to observe that the plural comprehension scheme needs to be restricted in some way or other to avoid a universal plurality. We need some “successor principles” to the unrestricted plural comprehension scheme that tell us what pluralities there in fact are.

How should these successor principles be chosen and motivated? When discussing this question, we believe it is useful to keep in mind the following, intuitive version of our argument for restricting the plural comprehension scheme.

To define a plurality, we need to circumscribe some objects. When we circumscribe some objects, however, we can use these objects to define yet another object, namely their set. Since yet another object can in this way be defined, it follows that the circumscribed objects cannot have included *all* objects. Thus, reality as a whole cannot be circumscribed: there is no universal plurality. Consequently, the plural comprehension scheme needs to be restricted.

²⁶Cantor 1883, 19-20, as translated in Ewald 1996, 896.

²⁷We motivate this label in Section 11.

²⁸But, of course, we should insist that the formulation of logical rules be neutral with respect to which comprehension axioms are validated.

This argument hinges on the idea that every plurality is circumscribed, or, as we will also put it, *extensionally definite*.

Can this notion of extensional definiteness be made clear enough to guide our search for successor principles and to justify, or at least to motivate, the resulting principles? Here we face a fork in the road, depending on whether or not we attempt to provide an analysis of extensional definiteness in more basic terms, and on this basis, to provide the requisite guidance and justification.

There have been several attempts to provide such an analysis. Linnebo 2013 proposes a modal analysis inspired by Cantor’s famous distinction between “consistent” and “inconsistent” multiplicities. Here is how Cantor explains the distinction in a famous letter to Dedekind of 1899:

[I]t is necessary ... to distinguish two kinds of multiplicities (by this I always mean definite multiplicities). For a multiplicity can be such that the assumption that all of its elements ‘are together’ leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as ‘one finished thing’. Such multiplicities I call *absolutely infinite* or *inconsistent multiplicities* ... If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as ‘being together’, so that they can be gathered together into ‘one thing’, I call it a *consistent multiplicity* or a ‘set’. (In Ewald 1996, 931-932)

Using the resources of modal logic, it is relatively straightforward to formalize Cantor’s notion of a multiplicity being “one finished thing”, namely that it is possible for all possible members of the multiplicity to exist or “be together”. Or, changing the idiom slightly, there is no possibility of the multiplicity gaining yet more members at more populous possible worlds. Based on this analysis, Linnebo 2013 proves various principles of extensional definiteness, which in the present context amount to principles concerning the existence of pluralities.

Another analysis of extensional definiteness is inspired by Michael Dummett’s suggestion that a domain is definite just in case quantification over this domain obeys the laws of classical logic, not just intuitionistic.²⁹ Intriguingly, it turns out that a fairly natural development of this Dummettian suggestion validates almost the same principles of extensional definiteness as the modal analysis.³⁰ Yet other analyses might be possible as well. We invite the readers to explore.

Here we wish to pursue the other fork in the road, namely to leave the notion of extensional definiteness unanalyzed and instead to use our intuitive conception of the notion, coupled with

²⁹A closely related idea is found in Solomon Feferman’s widely circulated and discussed manuscript, “Is the Continuum Hypothesis a definite mathematical problem?”.

³⁰See Linnebo 2018.

abductive considerations, to motivate principles of extensional definiteness. This strategy has both advantages and disadvantages: it is more general, as it avoids specific theoretical commitments; but it also provides less leverage and thus less of an independent check on the proposed principles of definiteness. In any case, we believe this is an option worth exploring. We thus ask what it is for a collection to be circumscribed or extensionally definite.

First, since every single object can be circumscribed, there are singleton pluralities:

$$\forall x \exists yy \forall z (z \prec yy \leftrightarrow z = x)$$

Second, because the result of adding one object to a circumscribed plurality is also circumscribed, we accept a principle of adjunction. Given any plurality xx and any object y , we can adjoin y to xx to form the plurality $xx + y$ defined by:

$$\forall u (u \prec xx + y \leftrightarrow u \prec xx \vee u = y)$$

Moreover, a plural separation principle is well motivated. Suppose you have circumscribed a collection and have formulated a sharp distinction between two ways that members of the collection can be. Then the subcollection whose members are all and only the objects that lie on one side of this distinction is in turn circumscribed. More formally, given any plurality xx and any condition $\varphi(x)$ that has an instance among xx , there is a plurality yy of those members of xx that satisfy the condition:

$$\exists x (\varphi(x) \wedge x \prec xx) \rightarrow \exists yy \forall u (u \prec yy \leftrightarrow u \prec xx \wedge \varphi(u))$$

Next, there are some plausible union principles. Let us begin with a simple case. Since two circumscribed collections can be conjoined to make a single such collection, a principle of pairwise union is plausible. Given any plurality xx and any objects yy , there is a union plurality zz defined by:

$$\forall xx \forall yy \exists zz \forall u (u \prec zz \leftrightarrow u \prec xx \vee u \prec yy)$$

A generalized union principle can also be motivated. Consider some circumscribed collections, each with its own unique tag. Suppose that the collection of tags is also circumscribed. Then the “union collection” comprising all the items that figure in at least one of the tagged collections is circumscribed. This motivates a generalized union principle to the effect that the union of an extensionally definite collection of extensionally definite collections is itself extensionally definite.

We can formulate this as follows. Suppose $\psi(x, y)$ is such that:

$$\begin{aligned} \exists xx[\forall x(x \prec xx \leftrightarrow \exists y\psi(x, y)) \wedge \\ \forall x(x \prec xx \rightarrow \exists yy\forall z(z \prec yy \leftrightarrow \psi(x, z)))] \end{aligned}$$

Then there is zz such that:

$$\forall y(y \prec zz \leftrightarrow \exists x\psi(x, y))$$

Although the generalized union principle does not, on its own, entail the pairwise one, this entailment does go through in the presence of the singleton and adjunction principles.³¹ It therefore suffices to adopt the generalized union principle.

The principles accepted so far do not entail the existence of any infinite pluralities; indeed, they have a model where every plurality is finite. Is it possible for an infinite collection to be circumscribed and thus to correspond to a plurality? This question calls to mind the ancient debate about the existence of completed infinities. Aristotle famously argued that only finite collections can be circumscribed, and that a collection can be infinite only in the potential sense that there is no finite bound on how many members the collection might have. This remained the dominant view until Cantor, who boldly defended the actual infinite and the existence of completed infinite collections. The natural numbers provide an example. Aristotle denied, whereas Cantor affirmed, the existence of a completed collection of all natural numbers.

We are interested in an analogous question concerning pluralities. Let ' $P(x, y)$ ' mean that x immediately precedes y . Following first-order arithmetic, we accept that every natural number immediately precedes another:³²

$$(11) \quad \forall x\exists y P(x, y)$$

We would like to know whether there is a circumscribed collection, or plurality, of all natural numbers. More precisely, we would like to know whether there are some objects xx containing 0 and closed under P , in the following sense:

$$(12) \quad \exists xx(0 \prec xx \wedge \forall x(x \prec xx \rightarrow \exists y(y \prec xx \wedge P(x, y)))$$

Although asserting the existence of such a plurality is a substantial step, it has also been a

³¹*Proof sketch.* Consider two pluralities xx and yy . Assume there are two distinct objects, say a and b , to tag these pluralities. (If there is only a single object, the pairwise union of xx and yy is a singleton plurality.) Now apply the generalized union principle to the formula ' $(x = a \wedge y \prec xx) \vee (x = b \wedge y \prec yy)$ ', observing that a and b form a plurality. This yields the pairwise union of xx and yy .

³²Aristotle would only accept a weaker, modal analogue of this principle, namely $\Box\forall x\Diamond\exists y P(x, y)$, where the modal operators represent metaphysical modalities.

tremendous theoretical success, as mathematics since Cantor has made amply clear. On abductive grounds, we therefore recommend accepting (12), conditional on (11), as a plural analogue of the set-theoretic axiom of Infinity.

It will be objected that this conditional principle is concerned specifically with the natural numbers and thus lacks the topic neutrality of a logical law. The objection is entirely reasonable and points to the need for a more general principle that justifies transitions such as the one from (11) to (12). There is nothing special about 0 and the functional relation P . So, for *any* plurality xx and functional relation, there should be a plurality yy containing xx and closed under that function. We therefore claim that the desired generalization is the schematic principle that every plurality can be closed under function application:

$$(13) \quad \forall x \exists! y \psi(x, y) \rightarrow \forall xx \exists yy (xx \prec yy \wedge \forall x \forall y (x \prec yy \wedge \psi(x, y) \rightarrow y \prec yy))$$

We adopt this as the official plural principle of infinity. In practice, however, it doesn't much matter whether we accept this more general schematic principle or merely (12), conditional on (11). For in the presence of first-order arithmetic, ordered pairs, and the other principles concerning pluralities, these two principles of infinity are provably equivalent.³³

A plural analogue of the axiom of Replacement is plausible as well. Consider a plurality of objects. Now you may replace any member of this plurality with any other object, or, if you prefer, leave the original object unchanged. Then the resulting collection is also circumscribed and thus defines a plurality of objects. We formalize this as follows.

$$\begin{aligned} & \forall xx [\forall x (x \prec xx \rightarrow \exists! y \psi(x, y)) \rightarrow \\ & \quad \exists yy \forall y (y \prec yy \leftrightarrow \exists x (x \prec xx \wedge \psi(x, y)))] \end{aligned}$$

It is pleasing to observe that this plural version of Replacement follows from the generalized union principle and the singleton principle. And, as in the case of sets, the plural principle of replacement entails that of separation.³⁴

To sum up, our intuitive conception of extensional definiteness motivates the following three

³³*Proof sketch.* The only hard direction is to show that the specific conditional entails the general one. Consider any xx , and assume that ψ is functional. For every member $a \prec xx$, we contend that there is a plurality zz_a containing a and closed under ψ . Given this contention, the generalized union principle enables us to define the desired plurality yy as the union of all the pluralities zz_a . To prove the contention, we observe that, using ordered pairs and plural quantification, we can produce a formula $\theta(n, y)$ which expresses that n is a natural number and that y is the n 'th successor of a in the series generated by ψ . We do this by letting $\theta(n, y)$ state that $\langle n, y \rangle$ is a member of every plurality containing $\langle 0, a \rangle$ and closed under the operation $\langle m, u \rangle \mapsto \langle m + 1, v \rangle$, where v is the unique object such that $\psi(u, v)$. Now we apply the generalized union principle to the plurality of all natural numbers and the formula θ to obtain the desired plurality zz_a .

³⁴*Proof sketch.* Consider xx and a condition $\varphi(x)$. Assume $\varphi(a)$ for some member a of xx . Now apply the principle of replacement to the condition $\psi(x, y)$ defined as $(\neg\varphi(x) \wedge y = a) \vee (\varphi(x) \wedge y = x)$. This yields the subplurality of those members of xx that satisfy $\varphi(x)$.

principles concerning pluralities:

- singleton
- adjunction
- generalized union

An additional principle receives a more theoretical justification:

- infinity

These four principles constitute the system we call *critical plural logic*.

As observed, the first three of these principles entail some other plausible principles:

- separation
- pairwise union
- replacement

Moreover, it is straightforward to verify that each principle of critical plural logic can be derived from its traditional counterpart. In essence, each of the pluralities we licence is a subplurality of the universal plurality licenced by traditional plural logic. Critical plural logic is therefore strictly weaker than the traditional system. This relative weakness is for a good cause, as will emerge clearly in Section 10, where we explore the connection between critical plural logic and set theory. This connection is far simpler and, we believe, more natural, than in the case of traditional plural logic.

9 Extensions of critical plural logic

When stronger expressive resources are accepted, various extensions of critical plural logic can be formulated and justified. Suppose there are “superpluralities”, that is, pluralities of pluralities. As customary, we let triple variables, such as ‘ xxx ’, have superplural reference. This addition enables us to express superplural analogues of the principles of critical plural logic. Here we will focus on two more interesting, novel principles.

First, we can formulate a principle of extensional definiteness that corresponds to the familiar set-theoretic axiom of Powerset. We can do this in a wholly plural way, without any mention of sets, by using superplurals. For any plurality xx , there is a superplurality yyy of all subpluralities of xx :

$$(14) \quad \forall xx \exists yyy \forall zz (zz \prec yyy \leftrightarrow zz \preceq xx)$$

The justification for this “powerplurality” principle is less straightforward than in the case of the earlier principles. The principle is certainly reasonable when the plurality xx is finite: we can then list all of its subpluralities, at least in principle. The general principle is a big and admittedly daring extrapolation of the finitary principle into the infinite. Its justification is partially abductive. Just like its set-theoretic analogue, the principle fits into a coherent and fruitful body of theory. The principle provides important information about which superpluralities there are.

Second, superplurals make it possible to formulate plural choice principles. For example, given a superplurality xxx of non-overlapping pluralities, there is a “choice plurality” whose members include one member of each plurality of xxx . That is, for each such xxx we have:

$$\exists yy \forall zz (zz \prec xxx \rightarrow \exists! y (y \prec zz \wedge y \prec yy))$$

As in the case of the powerplurality principle, plural choice principles are extrapolations from the finite into the infinite, and their justification is partially abductive.³⁵

In sum, the addition of superplural resources enables us to formulate and justify an extended critical plural logic, the two most novel principles of which are:

- powerplurality
- choice

Of course, yet stronger principles can be countenanced as ever greater expressive resources are considered.

10 Critical plural logic and set theory

The various plural principles we have discussed provide valuable information about sets. To see this, recall the correspondence we have advocated between pluralities and sets:

- (i) $\{xx\} = \{yy\}$ if and only if $xx \approx yy$
- (ii) $y \in \{xx\}$ if and only if $y \prec xx$

Using this correspondence, the plural principles entail analogous set-theoretic axioms.

However, there are two reasons to worry that the resulting theory might not be ordinary ZFC. First, since we do not ordinarily admit an empty plurality, there is a threat of losing the

³⁵See Pollard 1988 for a defense of the Axiom of Choice on the basis of a plural choice principle. If ordered pairs are available, there is less of a need for superplurals to express choice principles. For example, we can assert that for any relation coded by means of a plurality of ordered pairs, there is a functional subrelation with the same domain, again coded by means of a plurality of ordered pairs.

empty set. Some ways to address this threat were discussed in Section 5. One solution is to allow an empty plurality. Another is to allow the set-of operator $xx \mapsto \{xx\}$ to be what Oliver and Smiley (2016, 88) call a “co-partial” function, which can thus take the value \emptyset on an undefined argument. Either way, we can prove the existence of an empty set.

Second, since plural logic is applied to all sorts of objects, the mentioned correspondence introduces impure sets, that is, sets of non-sets. The relevant comparison is therefore not ZFC, but ZFCU—a modified system which accommodates urelements. This system is obtained by making explicit the quantification over sets in the axioms of ZFC. Whenever a quantifier of an axiom of ZFC is intended to range over sets even when urelements are introduced, we explicitly restrict this quantifier to sets by means of a predicate ‘ S ’ intended to be true of all and only sets. For example, the axiom of Extensionality is rewritten as:

$$\forall x \forall y [S(x) \wedge S(y) \rightarrow (\forall u (u \in x \leftrightarrow u \in y) \rightarrow x = y)]$$

Our aim, then, is to use critical plural logic and the correspondence principles (i) and (ii) to derive axioms of ZFCU. We define ‘ $S(x)$ ’ as ‘ $\exists xx(x = \{xx\})$ ’. This enables us to derive the axioms of Empty set, Pairing, Separation, Union, Infinity, and Replacement. Moreover, the axiom of Extensionality follows immediately from the correspondence between pluralities and sets, and Foundation too can be seen as explicating the way in which sets are successively formed from pluralities of elements.

To derive the axioms of Powerset and Choice, we need to go beyond critical plural logic. Choice follows naturally from the superplural choice principle discussed in the previous section. Deriving Powerset is less straightforward. Given any set a , we want to prove the existence of its powerset. To do so, we need to show that there is a plurality of all of a ’s subsets. How might this be done? One option is simply to postulate the existence of such a plurality, on the grounds that when a was formed, all its elements were available, thus giving us the ability also to form all of a ’s subsets. Another option is to utilize the powerplurality principle of the previous section, reasoning as follows. Let aa be the elements of a , and consider their superplurality bbb . For every subset x of a , if $x = \{xx\}$ for some xx , then $xx \prec bbb$. That is, bbb circumscribe all the subpluralities of aa . But if some pluralities are jointly circumscribed, so are the unique sets formed from precisely these pluralities. This gives us the desired plurality of subsets of a . (This reasoning assumes that the extended, superplural logic contains a replacement principle that allows us to replace each plurality of a superplurality with a unique object and thus arrive at a plurality.)

Our discussion shows that critical plural logic, and the plausible superplural extensions

thereof, have great explanatory power, especially in connection with the correspondence principles (i) and (ii). Still, one might worry that things are too good to be true. Do we even know that our assumptions—the mentioned plural logics and the correspondence principles—are jointly consistent? This worry can be put to rest by proving that these assumptions are consistent relative to ZFC. For critical plural logic and the correspondence principles, we do this by simply interpreting plural variables as ranging over non-empty sets. An analogous relative consistency result can be given for the described extension of critical plural logic. In that case, superplural variables are interpreted as ranging over non-empty sets of non-empty sets.

On the view we have defended, there is a close connection between the principles of critical plural logic and the axioms of set theory: the former tells us which collections can be circumscribed as pluralities; the latter adds that each of these pluralities can be used to define a single object, namely the set of the objects in question. So on this view, the principles of our critical plural logic have non-trivial mathematical content of a broadly set-theoretic character. This means that plural logic lacks one of the features commonly ascribed to pure logic, namely epistemic primacy vis-à-vis all other sciences.

Is the mathematical content of plural logic compatible with our view that pluralities can be used to explain sets? We believe it is. The explanation in question is a broadly metaphysical one: we make sense of a set $\{xx\}$ as “formed” from its elements xx . There is no conflict between this explanation and the view that plural logic has non-trivial mathematical content. Indeed, on this view, the indisputable mathematical content of set theory is in part inherited from that of plural logic.³⁶

11 Concluding remarks

Two larger questions have pervaded our entire discussion. The first of these questions concerns how we choose a “correct” logic. Some starkly different views are found in the literature. At one extreme we find Frege, who claims that logic codifies “the basic laws” of all rational thought, and the laws of logic must therefore be presupposed by all other sciences. He writes:

I take it to be a sure sign of error should logic have to rely on metaphysics and psychology, sciences which themselves require logical principles. (Frege 1893/1903, xix)

This “logic first” view has been very influential. Following Frege, logic is often regarded as epistemologically and methodologically fundamental. All disciplines, including mathematics, are answerable to logic rather than vice versa.

³⁶Thanks to Hans Robin Solberg for raising this concern.

At the opposite extreme we find Quine, whose radical holism leads him to assimilate logic and mathematics to the theoretical parts of empirical science. These disciplines, he claims, are not essentially different from theoretical physics: although they go beyond what can be observed by means of our unaided senses, they are justified by their contribution to the prediction and explanation of states of affairs that can be thus observed.

These extremes are not the only views, however. In particular, one need not be a radical holist to reject the Fregean logic-first view. What are sometimes called “critical views of logic” represent a less dramatic departure from Frege.³⁷ These views hold that the logical principles governing some subject matter may depend on the metaphysics of this subject matter or on the semantics of our discourse about it. The views thus stop short of Quine’s radical holism and emphasize instead a more local entanglement of logic with some particular discipline, such as mathematics, semantics, or some part of metaphysics. As a result of this entanglement, logic is answerable to one’s views in this other discipline.

The critical plural logic that we have defended provides a good example of such a critical view of logic. Avoiding any commitment to Quinean holism, we have argued that the principles of plural logic are entangled with our theory of correct mathematical definitions. Specifically, we have defended a liberal theory of mathematical definitions, and on the basis of that theory, we have argued that plural comprehension needs to be restricted more than has traditionally been assumed.

The second larger question on which this article bears concerns the relation between the philosophy of mathematics and mathematical practice. It is obviously a good thing when the philosophy of mathematics is informed by mathematical practice—just as the philosophy of *any* special science ought to be informed by the practice of that science. We do not regard this view as a threat to traditional philosophy of mathematics. The reason is simple. As we have argued, mathematical practice does not always speak with a single voice. In cases where it does not, there is no easy way to extract philosophical or methodological lessons from mathematical practice. Thus, even for those of us who want our philosophy of mathematics to pay close attention to mathematical practice, there remains an important role for more traditional forms of philosophical analysis.

³⁷See Parsons (2015) and a forthcoming special issue of *Inquiry* edited by M. Hartimo, F. Kjosavik, and Ø. Linnebo.

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