Abstract topological dynamics involving set-valued functions

Good, Chris; Greenwood, Sina; Uresin, Nazli

DOI:
10.1016/j.topol.2020.107240

ABSTRACT TOPOLOGICAL DYNAMICS INVOLVING SET-VALUED FUNCTIONS

CHRIS GOOD, SINA GREENWOOD, AND NAZLI URESIN

Abstract. Continuous functions over compact Hausdorff spaces have been completely characterised. We consider the more general problem: given a set-valued function T from an arbitrary set X to itself, does there exist a compact Hausdorff topology on X with respect to which T is upper semicontinuous? We give conditions that are necessary for T to be upper semicontinuous and point-closed if X is a compact Hausdorff space. We show that it is always possible to provide X with a compact T_1 topology with respect to which T is lower semicontinuous, and consequently, if $T : X \to X$ is a function, then it is always possible to provide X with a compact T_1 topology with respect to which T is continuous.

Let $T : X \to X$ be a function on a nonempty set X and let $\mathcal{P}$ be some topological property. A fundamental and natural question, tracing back to Ellis [4], asks whether one can endow X with a topology that satisfies $\mathcal{P}$ and with respect to which T is continuous.

For metric spaces the minimal conceivable conditions are required for a positive answer. De Groot and De Vries [12] show that if X is infinite, then there is always a non-discrete metrizable topology on X with respect to which T is continuous. Good and Greenwood [6] show that the existence of a separable metrizable topology on X with respect to which T is continuous depends only on the cardinality of the set X; there is such a topology precisely when the cardinality of X is no greater than $\mathfrak{c}$, the cardinality of the continuum, which is the maximum cardinality of any separable metric space [5]. In [1] continuous functions on the space of rational numbers are characterized: for countable X, there is a topology on X with respect to which f is continuous and X is homeomorphic to $\mathbb{Q}$ if and only if finite intersections of sets of the form

$$\{y : f^k(y) = x\}, \text{ where } x \in X \text{ and } 0 \leq k, \text{ or}$$
$$\{x : f^m(x) = f^n(x)\}, \text{ where } 0 \leq m, n$$

have either infinite or empty complements.

For compact, Hausdorff spaces one needs to work somewhat harder. The *full orbits of* T are the equivalence classes of the relation $\sim$, where $x \sim y$ if and only if $T^m(x) = T^n(y)$ for some $m, n \in \mathbb{N}$. A full orbit is said to be: an n-*cycle*, if it contains a point x for which $x = T^n(x)$ and $x, T(x), \dots, T^{n-1}(x)$ are distinct; a $\mathbb{Z}$-*orbit* if it contains distinct x_i, $i \in \mathbb{Z}$, such that $T(x_{i-1}) = x_i$; and an $\mathbb{N}$-*orbit* if it is neither an n-cycle nor a $\mathbb{Z}$-orbit. Good et al. [7] show that there is a compact Hausdorff topology on X with respect to which T is continuous if and only if $\bigcap_{n\in\mathbb{N}} T^n(X) = T\big(\bigcap_{n\in\mathbb{N}} T^n(X)\big) \neq \emptyset$ and one of the following holds:

(1) the total number of $\mathbb{Z}$-orbits and cycles is at least $\mathfrak{c}$; or

Date: April 23, 2020.

2010 *Mathematics Subject Classification.* Primary: 37B99, 54A10, 54B99, 54C05, 54C08, 54E99, 54H20.

Key words and phrases. dynamical system, topological dynamical system, compact, Hausdorff, upper semicontinuous, lower semicontinuous, T_1.

(2) there is at least one $\mathbb{Z}$-orbit and one cycle; or
(3) there are n_i, $i \leq k$ such that T has an n_i-cycle for each n_i and whenever T has an n-cycle, some n_i divides n,
(4) the restriction of T to $\bigcap_{n\in\mathbb{N}} T^n(X)$ is not 1-1.

In [7] they also show that if $T : X \to X$ is a bijection then there is a compact metrizable topology on X with respect to which T is a homeomorphism if and only if one of the following holds:

(1) X is finite.
(2) X is countably infinite and either:
  (a) T has both a $\mathbb{Z}$-orbit and a cycle; or
  (b) there are n_i, $i \leq k$ such that T has an n_i-cycle for each n_i and whenever T has an n-cycle, some n_i divides n.
(3) X has the cardinality of the continuum and the number of $\mathbb{Z}$-orbits and the number of n-cycles, for each $n \in \mathbb{N}$, is finite, countably infinite, or has the cardinality of the continuum.

Iwanik [16] had earlier given a characterization of continuous bijections on compact Hausdorff spaces and Sherman [23] has characterized homeomorphisms of the Cantor set.

Recently there has been considerable interest in the dynamics of set-valued functions. In 2004, Mahavier [21] introduced the notion of inverse limits of set-valued functions, or generalised inverse limits. Ingram posed a number of questions in [14] which has motivated a growing number of researchers to work in the area. A list of articles can be found in [13]; more recent examples include [11, 17, 18, 19]. Standard inverse limits in a dynamical setting have been extensively used in areas such as dynamical systems and continua theory [2, 8, 9, 10]. They have also found applications in disciplines such as economics [20, 22]. Inverse limits of set-valued functions provide greater scope for application and examples are emerging. Inverse limits of simple set-valued functions on simple spaces can be used, for example, to construct complex examples, such as Kennedy and Nall's construction of λ-dendroids from inverse limits of interval maps with graphs that are the union of two straight lines [19].

In this paper we consider Ellis's question for upper and lower semicontinuous set-valued functions. Conditions ensuring the existence of a compact Hausdorff topology with respect to which such a function is upper or lower semicontinuity seems to be a hard question. One can say something sensible in the case of compact T_1 topologies, however, and we address that question here.

Our notation and terminology are standard, as found in [5]. A space is T_1 provided singleton sets are closed. If $T : X \to X$ is a set-valued function and $A \subseteq X$, we define the image, lower preimage and upper preimage of A respectively by

$$T(A) = \bigcup\{T(x) : x \in A\},$$
$$T_-^{-1}(A) = \{x \in X : T(x) \cap A \neq \emptyset\},$$
$$T_+^{-1}(A) = \{x \in X : T(x) \subseteq A\}.$$

The proof of the following lemma is routine.

Lemma 1. *Let $T : X \to X$ be a set-valued function and let $\mathcal{A}$ be a collection of subsets of X.*

(1) $T_+^{-1}(X \setminus A) = X \setminus T_-^{-1}(A)$.
(2) $T_-^{-1}(\bigcup_{A\in\mathcal{A}} A) = \bigcup_{A\in\mathcal{A}} T_-^{-1}(A)$.

(3) $T_+^{-1}(\bigcap_{A\in\mathcal{A}} A) = \bigcap_{A\in\mathcal{A}} T_+^{-1}(A)$.

A set-valued function $T : X \to X$ is *upper semicontinuous* at the point x if for every open set V containing $T(x)$, there is an open set U containing x such that for every $y \in U$, V contains $T(y)$. T is *lower semicontinuous* at x if for each open set V that meets $T(x)$, there is an open set U containing x such that for every $y \in U$, V meets $T(y)$. T is *upper* (respectivey, *lower*) semicontinuous if it is upper (respectively, lower) semicontinuous at each $x \in X$. T is lower semicontinuous [3] if and only if, for any open set V, the set $T_-^{-1}(V) = \{x : T(x) \cap V \neq \emptyset\}$ is open if and only if, for any closed set C, the set $T_+^{-1}(C)$ is closed. Similarly, T is upper semicontinuous if and only if, for any open set V, the set $T_+^{-1}(V) = \{x : T(x) \subseteq V\}$ is open if and only if, for any closed set C, the set $T_-^{-1}(C)$ is closed. It is this difference between the lower and upper inverses and their behaviour under unions and intersections that makes it relatively easy to construct topologies making T lower semicontinuous, but hard to construct topologies making T upper semicontinuous.

We first consider lower semicontinuity.

Defining the topology τ: For a set-valued function $T : X \to X$, we define a topology τ on X by defining the collection $\mathcal{C}$ of its basic closed sets. Let $\mathcal{C}_0$ be the collection of all finite subsets of X. Given the collection $\mathcal{C}_n$, let $\mathcal{C}_{n+1}$ be the collection of all sets of the form

$$T_+^{-1}(C_1) \cup \cdots \cup T_+^{-1}(C_j) \cup C_{j+1} \cup \cdots \cup C_k,$$

where each $C_j \in \mathcal{C}_n$. Let $\mathcal{C} := \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$.

Let us say that the *rank* of a basic closed set $C \in \mathcal{C}$ is 0 if C is finite and is $n \in \mathbb{N}$ if $C \in \mathcal{C}_n \setminus \mathcal{C}_{n-1}$. Then, if C has rank $n+1$, we can write

$$C = T_+^{-1}(C_1) \cup T_+^{-1}(C_2) \cdots \cup T_+^{-1}(C_{k-1}) \cup C_k^*,$$

where:

- C_1 has rank n and C_k^* has rank 0 (so is finite, possibly empty);
- for each $i \leq k-1$, C_i has rank at most n and $T_+^{-1}(C_i)$ is infinite (otherwise it can be absorbed into C_k^*).

Note that this representation of C need not be unique. For clarity, *rank 0 sets are marked with* $*$.

Each C_i is also of this form, but of a lower rank. Therefore, associated with C, we have a finite collection of basic closed sets indexed by finite sequences of natural numbers such that the set $C_{\underline{i}}$, indexed $\underline{i} = i_1 i_2 \ldots i_{k-1}$, is either rank 0 and finite or can be written as a finite union

$$C_{\underline{i}} = T_+^{-1}(C_{\underline{i}\frown 1}) \cup T_+^{-1}(C_{\underline{i}\frown 2}) \cup \cdots \cup T_+^{-1}(C_{\underline{i}\frown(n_{\underline{i}}-1)}) \cup (C^*_{\underline{i}\frown n_{\underline{i}}})$$

for some $n_{\underline{i}} > 0$, where each $C_{\underline{i}\frown j}$ has rank less than that of $C_{\underline{i}}$ and $C^*_{\underline{i}\frown n_{\underline{j}}}$ is finite (as usual $i_1 i_2 \ldots i_{n-1} \frown i_n = i_1 \ldots i_{n-1} i_n$). There is, therefore, a finite tree Γ associated with C whose nodes are the basic closed sets $C_{\underline{i}}$. The root of Γ is C, if $C_{\underline{i}}$ is finite, then it has no successors, otherwise the successors of $C_{\underline{i}}$ are the nodes $C_{\underline{i}\frown j}$ for $1 \leq j \leq n_{\underline{i}}$. Let us refer to the finite nodes as *leaves*, so that a *branch* in Γ is a maximal chain starting at C and ending in a leaf of Γ, see Figure 2 for an example.

It is clear that, although there may be more than one tree associated with a given C in this way, the tree order together with the leaves determine C. Given a tree Γ associated with the closed set C, the number of branches is the number of leaves.

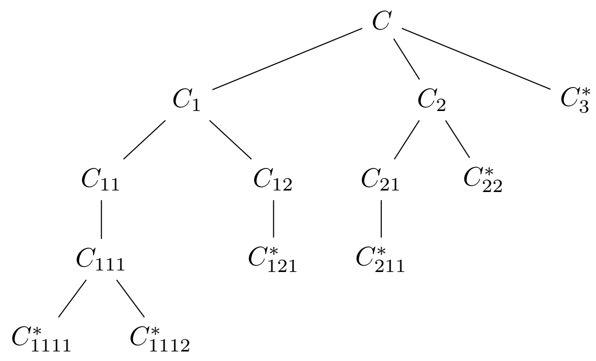

FIGURE 1. An example of a tree associated with the set $C = T_+^{-1}(C_1) \cup T_+^{-1}(C_2) \cup C_3^*$. C has rank 4 and leaves C_{1111}^*, C_{1112}^*, C_{121}^*, C_{211}^*, C_{22}^* and C_3^*.

By Lemma 1, if

$$C = T_+^{-1}(C_1) \cup \cdots \cup T_+^{-1}(C_{k-1}) \cup C_k^* \text{ and}$$
$$D = T_+^{-1}(D_1) \cup \cdots \cup T_+^{-1}(D_{m-1}) \cup D_m^*,$$

then

$$\begin{aligned} C \cap D = & T_+^{-1}(C_1 \cap D_1) \cup \cdots \cup T_+^{-1}(C_1 \cap D_{m-1}) \\ & \cup T_+^{-1}(C_2 \cap D_1) \cup \cdots \cup T_+^{-1}(C_2 \cap D_{m-1}) \\ & \vdots \\ & \cup T_+^{-1}(C_{k-1} \cap D_1) \cup \cdots \cup T_+^{-1}(C_{k-1} \cap D_{m-1}) \\ & \cup C^* \end{aligned}$$

where C^* is some subset of $C_k^* \cup D_m^*$. Therefore, $C \cap D$ can be written in the form

$$C \cap D = T_+^{-1}(E_1) \cup \cdots \cup T_+^{-1}(E_{\ell-1}) \cup E_\ell^*,$$

where each E_i, $i < \ell$, is a subset of some C_j, $T_+^{-1}(E_i)$ is infinite and E_ℓ^* is a finite set.

Now suppose that D is a subset of C and that we have fixed a tree, Γ, associated with C. It follows from the previous paragraph that there is a tree, Δ, associated with D and an order-preserving function $\phi : \Delta \to \Gamma$ such that for each node $D_{\underline{j}}$, $\underline{j} = j_1 \dots j_k$, of Δ:

- $\phi(D_{\underline{j}}) = C_{\underline{i}}$ for some $\underline{i} = i_1 \dots i_k$ with the same length as $\underline{j}$;
- $D_{\underline{j}} \subseteq C_{\underline{i}}$;
- the rank of $D_{\underline{j}}$ is at most the rank of $C_{\underline{i}}$;
- if $k' < k$, then $\phi(D_{j_1 j_2 \dots j_{k'}}) = C_{i_1 i_2 \dots i_{k'}}$.

For brevity we will say that Δ *follows* Γ and a branch

$$D, D_{i_1}, D_{i_1 i_2}, \dots, D_{i_1 \dots i_{k-1}}, D_{i_1 \dots i_k}^*$$

in Δ follows the branch $C, \phi(D_{i_1}), \phi(D_{i_1 i_2}), \dots, \phi(D_{i_1 \dots i_k}^*)$ in Γ.

In Figure 2, we give an example of a tree associated with a set $D \subsetneq C$ which follows the tree pictured in Figure 1.

Lemma 2. *Let $T : X \to X$ be a set-valued function and equip X with the topology τ defined above. Let $C, D \in \mathcal{C}$, let D be a proper subset of C, and let Γ and Δ be trees associated with C and D respectively, such that Δ follows Γ. Then there is a*

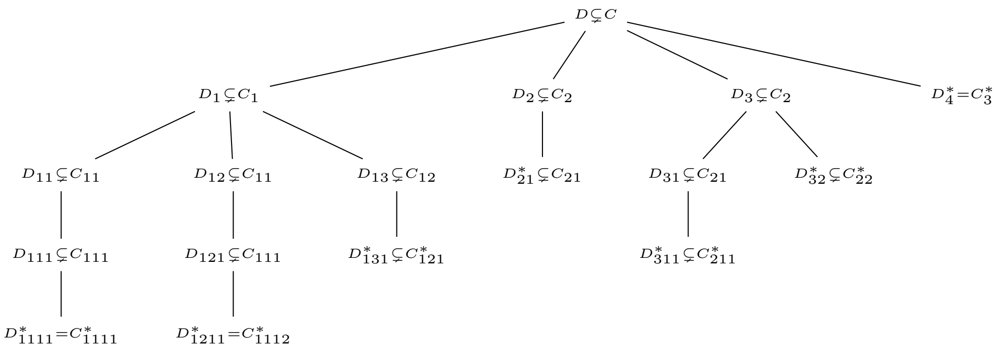

FIGURE 2. An example of a tree associated with a set D which follows the tree associated with the set C from Figure 1.

branch A in Γ such that for every branch B in Δ, if B follows A then either B has shorter length than A or the leaf of B is a proper subset of the leaf of A.

Proof. Let $x \in C \setminus D$. Recall that C and D are the roots of the trees Γ and Δ respectively. Either x is in the finite leaf successor C_k^* of C and not in the finite leaf successor $D_{k'}^*$ of D. Or there is a non-leaf successor C_1 of C such that $T(x) \subseteq C_1$, and for every infinite successor D_j of D, $T(x) \not\subseteq D_j$ and hence $C_1 \setminus D_j \neq \emptyset$. Thus by induction there is a branch $C, C_1, \ldots, C_{k-1}, C_k^*$ is Γ such that if $D, D_1, D_2, \ldots, D_{k'-1}, D_{k'}^*$ is a branch in Δ that follows $C, C_1, \ldots, C_{k-1}, C_k^*$, $k' \leq k$ and for every $j \leq k'$, D_j is a proper subset of C_j. Thus if $k = k'$ then the leaf D_k^* is a proper subset of C_k^*. □

Theorem 3. *Let X be an infinite set and let $T : X \to X$ be any set-valued function such that $T(x) \neq \emptyset$ for each $x \in X$. There is a compact T_1 topology on X with respect to which T is lower semicontinuous.*

Proof. Since $\mathcal{C} := \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$ is closed under finite unions and contains all finite sets, it is the basis for the closed sets of a T_1 topology τ on X. Moreover, if D is a closed set under the topology τ, then D is an intersection, $D = \bigcap_{C\in\mathcal{D}} C$, for some $\mathcal{D} \subseteq \mathcal{C}$. But then, by Lemma 1, $T_+^{-1}(D) = \bigcap_{C\in\mathcal{D}} T_+^{-1}(C)$. By definition, each $T_+^{-1}(C)$ is in $\mathcal{C}$, so $T_+^{-1}(D)$ is closed. Hence T is lower semicontinuous with respect to τ.

It remains to show that τ is compact. To do this we take a collection of basic closed sets with empty intersection and show that a finite subcollection also has empty intersection.

Let $\mathcal{D} = \{D_n \subseteq X : n \in \mathbb{N}\}$ be a collection of nonempty basic closed sets such that each D_{n+1} is a proper subset of D_n. For each $n \in \mathbb{N}$, let Γ_n be a tree associated with D_n. By Lemma 2, for each $n \in \mathbb{N}$ and $m > n$, there is a branch $C, C_1, \ldots, C_k^*$ in Γ_n such that if $E, E_1, E_2, \ldots, E_{k'}^*$ is any branch in Γ_m that follows $C, C_1, \ldots, C_k^*$, then $k' \leq k$ and for each $i \leq k'$, E_i is a proper subset of C_i.

Since each tree has only finitely many branches, there is a value $m_1 > 1$ such that either, for some k, $1 < k \leq m_1$ there is a branch B in Γ_1 and every branch in Γ_k that follows B is shorter in length, or the leaf of every branch in Γ_{m_1} is a singleton. If the latter holds, then for every branch B in Γ_1, every branch in Γ_{m_1+1} that follows B has shorter length than B.

By a similar argument, for any $n \in \mathbb{N}$ there exists $m_n > n$ and a branch B in Γ_n, such that every branch in Γ_{m_n} that follows B has shorter length than B. Although Γ_{m_n} may have a greater number of branches than Γ_n, the number is finite. Clearly then, there exists $n \in \mathbb{N}$ such that every branch in Γ_n is shorter than the branch in Γ_1 that it follows, and hence for some $n > 1$, every branch in D_n has only one member and hence is a finite subset of D_n, a contradiction. Thus any collection of closed subsets of X with empty intersection is finite and so τ is compact. $\square$

If $T : X \to X$ is a function, then $T_+^{-1}(A) = T_-^{-1}(A) = T^{-1}(A)$, so that the same proof yields the following.

Theorem 4. *Suppose that $T : X \to X$ is a function. There is a compact T_1 topology on X with respect to which T is continuous.*

The question of characterising the existence of a compact T_1 topology on X with respect to which a set-valued function $T : X \to X$ is upper semicontinuous seems to be harder. Certainly not every set-valued function has such a topology.

Example 5. Let X be an infinite set and let $T : X \to X$ be the set-valued function defined by $T(x) = X \setminus \{x\}$. Under any T_1 topology on X, the set $X \setminus \{x\}$ would be open. But $T_+^{-1}(X \setminus \{x\}) = \{x\}$, so if T were upper semicontinuous, X would have the discrete topology.

It is easy enough to define a T_1 topology on X with respect to which the set-valued function $T : X \to X$ is upper semicontinuous.

Defining a minimal topology σ: We define a T_1 topology σ on X with respect to which T is upper semicontinuous, and such that if σ' is a T_1 topology on X with respect to which T is upper semicontinuous, then $\sigma \subseteq \sigma'$.

Since points are closed sets, any T_1 topology must contain the cofinite topology σ_0. Given topologies σ_β for all $\beta < \alpha$ such that $\sigma_0 \subseteq \sigma_\gamma \subseteq \sigma_\beta$ for all $0 \leq \gamma \leq \beta$, let σ_α be the topology generated by the collection of sets of the form

$$T_+^{-1}(U_1) \cap \cdots \cap T_+^{-1}(U_m) \cap U_{m+1} \cap \cdots \cap U_n,$$

where $m < n$ and if $1 \leq i \leq n$, then $U_i \in \sigma_\beta$ for some $\beta < \alpha$. Since T_+^{-1} respects intersections, this is equivalent to saying that elements of σ_α are arbitrary unions of sets of the form $T_+^{-1}(U) \cap V$ for $U, V \in \bigcup_{\beta<\alpha} \sigma_\beta$. For some δ, we have $\sigma_\delta = \sigma_{\delta+1}$ and we let $\sigma = \sigma_\delta$. It is easy to see that T is upper semicontinuous with respect to σ and that any T_1 topology with respect to which T is upper semicontinuous must contain σ. Hence there is a compact T_1 topology on X with respect to which T is upper semicontinuous if and only if σ is compact. Conditions on T making σ compact, however, appear elusive.

In characterizing the existence of a compact Hausdorff topology with respect to which a function is continuous [7], two key tools are the existence of an amenable orbit structure and the fact that $T\left(\bigcap_{n\in\mathbb{N}} T^n(X)\right) = \bigcap_{n\in\mathbb{N}} T^n(X) \neq \emptyset$. For set-valued mappings the equivalence relation corresponding to the notion of a full orbit of a function is given by the connected components of the graph generated on X by placing an edge between x and y if and only if $y \in T(x)$ or $x \in T(y)$. In this case, an orbit might contain a number of cycles $x_1, \ldots, x_n$, such that for each $i < n$, $x_{i+1} \in T(x_i)$ and $x_1 \in T(x_n)$, or $\mathbb{Z}$-sequences $(x_n)_{n\in\mathbb{Z}}$, such that for each $n \in \mathbb{Z}$, $x_{n+1} \in T(x_n)$, or some combination of both, or neither. This means that there is no useful classification of orbits for set-valued mappings.

One can say something about the set $\bigcap_{n\in\mathbb{N}} T^n(X)$ in certain circumstances. A set-valued mapping $T : X \to X$ is *point-closed* if $T(x)$ is closed for each $x \in X$. It follows from [15, Theorem 2.1] that, if T is upper semicontinuous and point-closed, then $T(D)$ is closed for every closed subset $D \subseteq X$. For such mappings we have the following, which is analogous to [7, Theorem 4.1].

Theorem 6. *If X is an infinite compact Hausdorff space and $T : X \to X$ is a point-closed upper semicontinuous set-valued function, then*

$$T\Big(\bigcap_{n\in\mathbb{N}} T^n(X)\Big) = \bigcap_{n\in\mathbb{N}} T^n(X) \neq \emptyset.$$

It follows that $T \restriction_{\bigcap_{n\in\mathbb{N}} T^n(X)}$ *is onto and each of its full orbits contains a cycle or a* $\mathbb{Z}$ *sequence.*

Proof. Since T is point-closed and upper semicontinuous, for each $n \in \mathbb{N}$, $T^n(X)$ is a closed set. Thus, since $\emptyset \neq T^{n+1}(X) \subseteq T^n(X)$, if X is compact Hausdorff then $\bigcap_{n\in\mathbb{N}} T^n(X) \neq \emptyset$.

Suppose $x \in \bigcap_{n\in\mathbb{N}} T^n(X)$. Let $D_0 = T_-^{-1}(x)$, and for each $n \in \mathbb{N}$ let $C_n = T_-^{-(n+1)}(x)$ and $D_n = T^n(C_n) \cap D_{n-1}$. Thus $\langle D_n : n \in \mathbb{N}\rangle$ is a decreasing (i.e. non-increasing) sequence of closed compact nonempty subsets of the compact set D_0, and hence has nonempty intersection. If $y \in \bigcap_{n\in\mathbb{N}} D_n$, then $x \in T(y)$, and hence $x \in T\left(\bigcap_{n\in\mathbb{N}} T^n(X)\right)$. $\square$

In the following two examples the set-valued function given is not point-closed. These examples show that if $T : X \to X$ is an upper semicontinuous set-valued function but is not point-closed, then Theorem 6 does not hold in general.

Example 7. Let $X = \{1 - \frac{1}{n} : n \in \mathbb{N}\} \cup \{1\}$ with the usual topology inherited from $\mathbb{R}$ so that X is compact Hausdorff. Define a set-valued function $T : X \to X$ by $T(1) = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$, and for each $n \in \mathbb{N}$, $T(1 - \frac{1}{n}) = \{1 - \frac{1}{n+1}\}$. Then T is upper semicontinuous and $\bigcap_{n\in\mathbb{N}} T^n(X) = \emptyset$.

Example 8. Let $X = \{1 - \frac{1}{n} : n \in \mathbb{N}\} \cup \{1\} \cup \{2 - \frac{1}{n} : n \geq 2\} \cup \{2\}$ with the usual topology inherited from $\mathbb{R}$ so that X is compact Hausdorff. Let $A = \{2 - \frac{1}{2^k} : k \geq 1\}$ Define a set-valued function $T : X \to X$ by:

$$T(y) = \begin{cases} \{1 - \frac{1}{n+1}\} & \text{if} \quad y = 1 - \frac{1}{n}, n \in \mathbb{N} \\ \{1 - \frac{1}{n} : n \in \mathbb{N}\} & \text{if} \quad y = 1 \\ \{2 - \frac{1}{n-1}\} & \text{if} \quad y = 2 - \frac{1}{n}, n \geq 2 \text{ and } y \notin A \\ \{1\} & \text{if} \quad y \in A \\ \{1 - \frac{1}{n} : n \in \mathbb{N}\} \cup \{1\} \cup \{2 - \frac{1}{n} : n \geq 2\} & \text{if} \quad y = 2. \end{cases}$$

Then T is upper semicontinuous,

$$\bigcap_{n\in\mathbb{N}} T^n(X) = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\} \cup \{1\} \neq \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\} = T\left(\bigcap_{n\in\mathbb{N}} T^n(X)\right)$$

and $T \restriction_{\bigcap_{n\in\mathbb{N}} T^n(X)}$ consists of a single orbit which does not contain a cycle or a $\mathbb{Z}$ sequence.

Theorem 6 does not hold in general for lower semicontinuous set-valued functions that are not point-closed.

Example 9. Let $X = \{1 - \frac{1}{n} : n \in \mathbb{N}\} \cup [1, 2]$ with the topology inherited from $\mathbb{R}$, so that X is compact Hausdorff. Let $\{D_n : n \in \mathbb{N}\}$ be a partition of $[1, 2]$ such that each set D_n is dense in $[1, 2]$. Define $T : X \to X$ by:

$$T(x) = \begin{cases} [1, 2] \cup \{1 - \frac{1}{n+1}\} & \text{if} \quad x = 1 - \frac{1}{n}, n \in \mathbb{N} \\ D_{n+1} & \text{if} \quad x \in D_n, n \in \mathbb{N}. \end{cases}$$

Suppose $U \subseteq X$ is open. If $[1, 2] \cap U \neq \emptyset$, then $[1, 2] \subset T_-^{-1}(U)$. If $[1, 2] \cap U = \emptyset$, then $T_-^{-1}(U)$ is a subset of $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$. In either case $T_-^{-1}(U)$ is open and so T is lower semicontinuous. However,

$$\bigcap_{n\in\mathbb{N}} T^n(X) = [1, 2] \neq [1, 2] \setminus D_1 = T\left(\bigcap_{n\in\mathbb{N}} T^n(X)\right).$$

Moreover $T \restriction_{[1,2]}$ is lower-semicontinuous, has a single orbit which does not contain a cycle or a $\mathbb{Z}$ sequence and $\bigcap_{n\in\mathbb{N}} (T \restriction_{[1,2]})^n([1, 2]) = \emptyset$.

In Example 9, neither T nor $T\restriction_{[1,2]}$ are point-closed. The function T in each of the following two examples is point-closed and lower semicontinuous.

Example 10. Let X be the space from Example 7. Let $A = \left\{1 - \frac{1}{2^k} : k \geq 2\right\} \cup \{1\}$. Define a set-valued function T as follows:

$$T(x) = \begin{cases} \{\frac{1}{2}\}, & \text{if} \quad x \in A \\ \{1 - \frac{1}{n-1}, 1\}, & \text{if} \quad x = 1 - \frac{1}{n} \notin A \text{ and } n > 1 \\ \{0\}, & \text{if} \quad x = 0, \frac{1}{2}. \end{cases}$$

Then T is point-closed and lower semicontinuous, $\bigcap_{n\in\mathbb{N}} T^n(X) = \{0, \frac{1}{2}\}$ and

$$T\left(\bigcap_{n\in\mathbb{N}} T^n(X)\right) = \{0\}.$$

Example 11. Let X be the space from Example 7. Let $A = \left\{1 - \frac{1}{2^k} : k \geq 1\right\} \cup \{1\}$ and $B = \{1 - \frac{1}{2k+1} : k \geq 3\}$ Define a set-valued function T as follows:

$$T(x) = \begin{cases} \frac{6}{7}, & \text{if} \quad x \in A \cup \{1\} \\ \{1 - \frac{1}{2n+3}\}, & \text{if} \quad x = 1 - \frac{1}{2n+1} \in B \\ \{1 - \frac{1}{n-2}, 1\}, & \text{if} \quad 0 \neq x = 1 - \frac{1}{n} \notin (A \cup B). \end{cases}$$

Then T is point-closed and lower semicontinuous, $\bigcap_{n\in\mathbb{N}} T^n(X) = B$ and

$$T\left(\bigcap_{n\in\mathbb{N}} T^n(X)\right) = B \setminus \left\{\frac{6}{7}\right\},$$

and $T\restriction_{\bigcap_{n\in\mathbb{N}} T^n(X)}$ has an orbit which has no cycles or $\mathbb{Z}$ sequences.

References

[1] A. Ahmed and C. Good, *Continuity on the rational world*, preprint.

[2] R. D. Anderson and G. Croquet, *A plane continuum no two of whose non-degenerate subcontinua are homeomorphic: An application of inverse limits*, Proc. Amer. Math. Soc., 10, (1959), 347–353.

[3] Yu. G. Borisovich, B. D. Gel'man, A. D. Myshkis and V. V. Obukhovskii, *Multivalued mappings*, J. Math. Sci., (6), 24, (1984), 719–791.

[4] D. Ellis, *Orbital topologies*, Quart. J. Math. Oxford, (2), 4, (1953), 117–119.

[5] R. Engelking, *General topology*, Sigma Series in Pure Mathematics, 6, Heldermann Verlag, Berlin, 1989.

[6] C. Good and S. Greenwood, *Continuity in separable metrizable and Lindelöf spaces*, Proc. Amer. Math. Soc., (2), 138, (2010), 577–591.

[7] C. Good, S. Greenwood, R. W. Knight, D. W. McIntyre and S. Watson, *Characterizing functions on compact spaces*, Adv. Math., 206, (2006), 695–728.

[8] C. Good and J. Meddaugh, *Shifts of finite type as fundamental objects in the theory of shadowing*, Invent. Math. (2019). https://doi.org/10.1007/s00222-019-00936-8.

[9] C. Good, J. Mitchell and J. Thomas, *Preservation of shadowing in discrete dynamical systems*, J. Math. Anal. Appl., 485, (2020), https://doi.org/10.1016/j.jmaa.2019.123767.

[10] C. Good and B. E. Raines, *Continuum many tent map inverse limits with homeomorphic postcritical -limit sets*, Fund. Math. 191, (2006), 1-21.

[11] S. Greenwood, and R. Suabedissen, *2-manifolds and inverse limits of set-valued functions on intervals*, *Discrete Contin. Dyn. Syst. Ser. A*, (11), 37, 5693–5706, 2017.

[12] J. de Groot and H. de Vries, *Metrization of a set which is mapped into itself*, Quart. J. Math. Oxford, (2), 9, (1958), 144–148.

[13] W. T. Ingram, *An Introduction to Inverse Limits with set-valued Functions*, Springer, New York, 2012.

[14] W. T. Ingram, *Inverse limits with upper semicontinuous bonding functions: Problems and some partial solutions*, Topology Proc., 36, (2010), 353–373.

[15] W. T. Ingram and W. S. Mahavier, *Inverse limits of upper semicontinuous set-valued functions*, Houston Journal of Mathematics, 32, (2006), 119–130.

[16] A. Iwanik, *How restrictive is topological dynamics?*, Comment. Math. Univ. Carolin., 38, (1997), 563–569.

[17] H. Kato, *On dimension and shape of inverse limits with set-valued functions*, Fund. Math., 236, (2017), 83–99.

[18] J. P. Kelly and J. Meddaugh, *Convergence of sequences of inverse limits*, *Topology Appl.*, 184, (2015), 29–40.

[19] J. A. Kennedy and V. Nall, *Dynamical properties of shift maps on Inverse limits with a set-valued map*, Ergod. Theor. Dyn. Syst., (4), 38, (2018), 1499–1524.

[20] J. A. Kennedy, D. R. Stockman, J. A. Yorke, *The inverse limits approach to chaos*, J. Math. Econom., 44, (2008), 423-444.

[21] W. S. Mahavier, *Inverse limits with subsets of* $[0,1] \times [0,1]$, Topology Appl., 141, (2004), 225–231.

[22] A. Medio and B. E. Raines *Backward dynamics in economics. The inverse limit approach*, J. Econom. Dynam. Control, 31, (2007), 1633-1671.

[23] C. Sherman, *Characterizing orbit structures of homeomorphisms on Cantor sets*, Fund. Math., 245, (2019), 1–24.

University of Birmingham, Birmingham, UK
Email address: `c.good@bham.ac.uk`

University of Auckland, Private Bag 92019, Auckland, New Zealand
Email address: `sina@math.auckland.ac.nz`

University of Auckland, Private Bag 92019, Auckland, New Zealand
Email address: `n.uresin@math.auckland.ac.nz`