An expansion algorithm for constructing axial algebras

McInroy, Justin; Shpectorov, Sergey

DOI:
10.1016/j.jalgebra.2019.12.018
https://doi.org/10.1016/j.jalgebra.2019.12.018

An expansion algorithm for constructing axial algebras

Justin McInroy, Sergey Shpectorov

PII: S0021-8693(20)30025-9
DOI: https://doi.org/10.1016/j.jalgebra.2019.12.018
Reference: YJABR 17509

To appear in: *Journal of Algebra*

Received date: 2 April 2019

An expansion algorithm for constructing axial algebras

Justin M^cInroy* Sergey Shpectorov†

Abstract

An *axial algebra* A is a commutative non-associative algebra generated by primitive idempotents, called *axes*, whose adjoint action on A is semisimple and multiplication of eigenvectors is controlled by a certain fusion law. Different fusion laws define different classes of axial algebras.

Axial algebras are inherently related to groups. Namely, when the fusion law is graded by an abelian group T, every axis a leads to a subgroup of automorphisms T_a of A. The group generated by all T_a is called the *Miyamoto group* of the algebra. We describe a new algorithm for constructing axial algebras with a given Miyamoto group. A key feature of the algorithm is the expansion step, which allows us to overcome the 2-closedness restriction of Seress's algorithm computing Majorana algebras.

At the end we provide a list of examples for the Monster fusion law, computed using a MAGMA implementation of our algorithm.

1 Introduction

Axial algebras are a new class of non-associative algebras introduced recently by Hall, Rehren and Shpectorov [6] as a broad generalization of the class of Majorana algebras of Ivanov [9]. The key features of these algebras came from the theory of vertex operator algebras (VOAs) which first arose in connection with 2D conformal field theory and they were used by Frenkel, Lepowsky and Meurman [4] in their construction of the moonshine VOA $V^\natural$ whose automorphism group is the Monster M, the largest sporadic finite simple group. The rigorous theory of VOAs was developed by Borcherds [1] as part of his proof of the monstrous moonshine conjecture.

Roughly speaking, VOAs are infinite dimensional graded vector spaces $V = \bigoplus_{i=0}^{\infty} V_i$ with infinitely many products linked in an intricate way. The

*School of Mathematics, University of Bristol, Fry Building, Woodland Road, Bristol, BS8 1UG, UK, and the Heilbronn Institute for Mathematical Research, Bristol, UK, email: justin.mcinroy@bristol.ac.uk

†School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, UK, email: S.Shpectorov@bham.ac.uk

Monster was originally constructed by Griess [5] as the automorphism group of a $196,883$-dimensional non-associative real algebra, called the Griess algebra, and the Moonshine VOA $V^\natural$ contains a unital deformation of the Griess algebra as its weight 2 part $V_2^\natural$.

One of the key properties that axial algebras axiomatise was first observed in VOAs by Miyamoto [14]. He showed that you could associate involutory automorphisms τ_a of a VOA V, called *Miyamoto involutions*, to special conformal vectors a in V_2 called *Ising vectors* [14]. Moreover, in the Moonshine VOA, $\frac{a}{2}$ is an idempotent in the Griess algebra $V_2^\natural$, called a 2A-*axis* because the corresponding involution τ_a lies in the class $2A$ of the Monster M.

The subalgebras of the Griess algebra generated by two 2A-axes, which we call *dihedral subalgebras*, were first studied by Norton [3]. He showed that the isomorphism class of the dihedral subalgebra generated by $2A$-axes a and b is determined by the conjugacy class of the product $\tau_a\tau_b$. There are nine classes in M containing products of two $2A$ involutions, labelled 1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A and 6A. Remarkably, Sakuma [17] showed that each sub VOA generated by two Ising vectors is also one of nine isomorphism types. Therefore, the above nine classes in M are used as labels for the 2-generated VOAs arising in Sakuma's theorem.

Sakuma's result was extended to Majorana algebras in [10] and later to axial algebras with the Monster fusion law and a Frobenius form[1] in [6].

Majorana algebras were introduced by Ivanov [9] to abstract the properties of $2A$-axes. Axial algebras provide a further broad generalisation removing the less essential restrictions of Majorana algebras. An *axial algebra* is a commutative non-associative algebra generated by *axes*, that is, primitive semisimple idempotents whose adjoint eigenvectors multiply according to a certain fusion law. We say that an axial algebra is of *Monster type* if its fusion law is the Monster fusion law (see Table 1). For the exact details see Section 2. A Majorana algebra is then an axial algebra of Monster type which satisfies some additional conditions.

Whenever the fusion law is T-graded, where T is an abelian group, associated to each axis a we get an automorphism $\tau_a(\chi)$ for every linear character $\chi \in T^*$. We define $T_a = \{\tau_a(\chi) : \chi \in T^*\}$, which is a subgroup of the automorphism group of the algebra of size at most $|T|$. The group generated by the T_a for all axes a is called the *Miyamoto group*. For the important motivating example of the Griess algebra, the fusion law is $\mathbb{Z}_2$-graded and so, for every axis a, there is an involutory automorphism $\tau_a := \tau_a(\chi_{-1})$ corresponding to the unique non-trivial character χ_{-1} of $\mathbb{Z}_2$. The Miyamoto group generated by all the τ_a is the Monster M and the τ_a are the whole $2A$ conjugacy class.

[1]Franchi, Mainardis and Shpectorov announced at the Axial Algebra Focused Workshop in Bristol in May 2018 that the Frobenius form condition has been removed.

Another example of a class of axial algebras with a different fusion law are algebras of Jordan type comprising Matsuo algebras, whose Miyamoto groups are 3-transposition groups, and Jordan algebras, whose Miyamoto groups include classical groups and groups of exceptional Lie type F_4 and G_2.

Problem. 1. *For a given fusion law, which groups G can occur as the Miyamoto group of an axial algebra?*

2. *For such a group G, how do we construct all axial algebras with Miyamoto group G?*

Sakuma's theorem, suitably generalised, addresses the first of the two questions for the Monster fusion law: it implies that $\tau_a \tau_b$ is of order at most 6 for all pairs of axes a and b; that is, a Miyamoto group arising from an axial algebra of Monster type (in particular, from a Majorana algebra) is a group of 6-transpositions.

Seress [18] addressed the second question for the class of Majorana algebras by developing an algorithm that computes, for a given group G, possible 2-closed Majorana algebras. (An axial algebra is 2-closed if it is spanned by axes and by products of two axes.) He also provided a GAP implementation of his algorithm. However, his code was lost when he sadly died. Pfeiffer and Whybrow [15] have recently developed an improved GAP implementation of Seress's algorithm which can now handle some m-closed algebras for $m > 2$.

In this paper we describe a new algorithm for addressing the second question for a given group G. The new algorithm is based on the concept of expansion and it differs in several key ways. Our algorithm works for a general axial algebra over an arbitrary field with an arbitrary fusion law, rather than just for the Monster fusion law over $\mathbb{R}$. Crucially, we do not assume that the algebra is m-closed for some small m. Our algorithm can complete more examples even in the class of 1-, 2- and 3-closed algebras than Seress's algorithm, but we also have found several examples which are 4- and 5-closed and these certainly seem out of reach for the previous algorithm. We also do not assume that the algebra has an associating bilinear form (a *Frobenius form*), whereas the earlier algorithm assumes this and moreover that the form is positive definite. We do not assume the so-called 2Aa, 2Ab, 3A, 4A, 5A conditions (see [18, page 314]) which restrict the configuration of the dihedral subalgebras. Last but not least, we do not require that the axes a be in bijection with the axis subgroups T_a. At the end of the paper, we present results for axial algebras of Monster type obtained using a MAGMA implementation [12, 2] of this algorithm. We choose this fusion law as it is currently the one for which we know the groups G which may occur and all the 2-generated algebras.

The first implementation of expansion was done by the second author during his visit to Seress in December 2012. It wasn't a complete program, but rather a toolbox of GAP routines to allow computing algebras of Monster type by repeated expansions. One of the early successes was the completion of all twelve possible shapes for S_4 and also the construction of the 3-closed algebra of dimension 46 for A_5. In 2016, the first author took over and implemented a much more efficient code in MAGMA, making it into a full package and introducing many further improvements including generalising the field, fusion law, gluings and implementing saving and loading of the algebras.

Let $\mathcal{F}$ be a T-graded fusion law and G be a group acting on a set X. We aim to build an axial algebra where the action on the axes by (a supergroup of) the Miyamoto group is given by the action of G on X. In Section 3 we rigorously define admissible τ-maps and the shape of an algebra. Roughly speaking, $\tau \colon X \times T^* \to G$ is an *admissible τ-map* if it has the properties that the map $(a, \chi) \mapsto \tau_a(\chi)$ in an axial algebra has. The subgroup $G_0 \trianglelefteq G$ generated by the image of this map will be our Miyamoto group. The *shape* is a choice of 2-generated subalgebra for each pair of axes $a, b \in X$. Since the isomorphism class of 2-generated subalgebras is preserved under automorphisms, in particular, under the action of the Miyamoto group, we need only make one choice for each conjugacy class of pairs of axes. In fact, there are some addition constraints on the shape given by containment of 2-generated subalgebras in one another as described in Section 3. Our algorithm takes $\mathcal{F}$, G, X, τ and the shape as its input. We show the following:

Theorem. *Suppose that the algorithm terminates and returns A. Then A is a (not necessarily primitive) axial algebra generated by axes X with automorphism group G, τ-map τ and of the given shape.*

Moreover, the algebra A is universal. That is, given any other axial algebra B with the same axes X, automorphism group G, τ-map τ and shape, B is a quotient of A.

We find several new examples of axial algebra with the Monster fusion law. Some of these are 3-closed examples (in fact we find some examples which are 5-closed), but we also find many examples that do not satisfy the so-called M8-condition. This condition severely restricts the allowable intersections of certain dihedral subalgebras in the shape. We also see in our results several shapes which do not satisfy the 2Aa, 2Ab, 3A, 4A, 5A conditions (see Section 2.2), but still produce good axial algebras.

Interestingly, all the algebras we construct have a Frobenius form which is non-zero on the axes and invariant under the action of the Miyamoto group, even though we do not require this in our algorithm. Moreover, in all our examples, the form is positive semi-definite. It is known that axial algebras of Jordan type (those with three eigenvalues, 1, 0 and η) all have Frobenius forms [8] and it has previously been observed that the other known examples also have Frobenius forms. Such a form, if it does exist, is uniquely determined by its values on the axes. So we make the following conjecture.

Conjecture. *All primitive axial algebras of Monster type admit a Frobenius form which is non-zero on the axes and invariant under the action of the Miyamoto group.*

The structure of the paper is as follows. In Section 2, we define axial algebras and discuss various properties such as Miyamoto involutions and dihedral subalgebras. We define the shape of an algebra in Section 3. Section 4 gives some lemmas and further properties of axial algebras which we will need. Our main result is the algorithm which is described in Section 5. Finally, in Section 6, we present examples computed by our MAGMA implementation of the algorithm.

We thank Simon Peacock for some useful comments on an early draft of this paper and the referee for other useful comments.

2 Background

We will review the definition and some properties of axial algebras which were first introduced by Hall, Rehren and Shpectorov in [6]. We will pay particular attention to the motivating examples coming from the Monster sporadic finite simple group and also indicate the extra conditions for such an axial algebra to be a Majorana algebra.

Definition 2.1. Let $\mathbb{F}$ be a field, $\mathcal{F} \subseteq \mathbb{F}$ a subset, and $\star \colon \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}}$ a symmetric binary operation. We call the pair $(\mathcal{F}, \star)$ a *fusion law over* $\mathbb{F}$. A single instance $\lambda \star \mu$ is called a *fusion rule*.

Abusing notation, we will often just write $\mathcal{F}$ for $(\mathcal{F}, \star)$. We can also extend the operation $\star$ to subsets $I, J \subseteq \mathcal{F}$ in the obvious way: $I \star J$ is the union of all $\mu \star \nu$ for $\mu \in I$ and $\nu \in J$. We note that after extending the operation, $(2^{\mathcal{F}}, \star)$ is closed and so is a commutative magma. We will further abuse notation and mix subsets and elements.

Let A be a commutative non-associative (i.e. not-necessarily-associative) algebra over $\mathbb{F}$. For an element $a \in A$, the *adjoint endomorphism* $\mathrm{ad}_a \colon A \to A$ is defined by $\mathrm{ad}_a(v) := av$, for all $v \in A$. Let $\mathrm{Spec}(a)$ be the set of eigenvalues of ad_a, and for $\lambda \in \mathrm{Spec}(a)$, let A^a_λ be the λ-eigenspace of ad_a. Where the context is clear, we will write A_λ for A^a_λ. We will also adopt the convention that for subsets $I \subseteq \mathcal{F}$, $A_I := \bigoplus_{\lambda \in I} A_\lambda$.

Definition 2.2. Let $(\mathcal{F}, \star)$ be a fusion law over $\mathbb{F}$. An element $a \in A$ is an $\mathcal{F}$*-axis* if the following hold:

1. a is *idempotent* (i.e. $a^2 = a$);
2. a is *semisimple* (i.e. the adjoint ad_a is diagonalisable);
3. $\mathrm{Spec}(a) \subseteq \mathcal{F}$ and $A_\lambda A_\mu \subseteq A_{\lambda\star\mu}$ for all $\lambda, \mu \in \mathrm{Spec}(a)$.

Furthermore, we say that the $\mathbb{F}$-axis a is *primitive* if $A_1 = \langle a \rangle$.

Note that, when $\mathrm{Spec}(a) \neq \mathcal{F}$, we can still talk of A^a_λ for all $\lambda \in \mathcal{F}$: if $\lambda \notin \mathrm{Spec}(a)$ then $A^a_\lambda = 0$. With this understanding, the last condition means that $A_\lambda A_\mu \subseteq A_{\lambda\star\mu}$ for all $\lambda, \mu \in \mathcal{F}$.

Definition 2.3. An $\mathcal{F}$*-axial algebra* is a pair (A, X) such that A is a commutative non-associative algebra and X is a set of $\mathcal{F}$-axes generating A. An axial algebra is *primitive* if it is generated by primitive axes.

Where the fusion law is clear from context, we will drop the $\mathcal{F}$ and simply use the term *axis* and *axial algebra*. Although an axial algebra has a distinguished generating set X, we will abuse the above notation and just write A for the pair (A, X). Note that it has been usual in the literature to drop the adjective primitive and consider only primitive axial algebras.

The fusion law over $\mathbb{R}$ associated to the Monster is given by Table 1. This fusion law is exhibited by the so-called $2A$-axes in the Griess algebra.

	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1		$\frac{1}{4}$	$\frac{1}{32}$
0		0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$1, 0$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1, 0, \frac{1}{4}$

Table 1: Monster fusion law

Indeed, noting that these generate the Griess algebra shows that it is an axial algebra. We say that an axial algebra is of *Monster type* if it is an axial algebra with the Monster fusion law.

By definition, an axial algebra A is spanned by products of the axes. We say that A is m*-closed* if A is spanned by products of length at most m in the axes.

Definition 2.4. A *Frobenius form* on an axial algebra A is a non-zero (symmetric) bilinear form $(\cdot, \cdot) \colon A \times A \to \mathbb{F}$ such that the form associates with the algebra product. That is, for all $x, y, z \in A$,

$$(x, yz) = (xy, z).$$

We will be particularly interested in Frobenius forms such that $(a, a) \neq 0$, for all $a \in X$. That is, they are non-zero on the set of axes X. Note that an associating bilinear form on an axial algebra is necessarily symmetric [6, Proposition 3.5]. Also, the eigenspaces for an axis in an axial algebra are perpendicular with respect to the Frobenius form.

Lemma 2.5. [11, Lemma 4.17] *Suppose that A is a primitive axial algebra admitting a Frobenius form. Then the form is uniquely determined by the values (a, a) on the axes $a \in X$.*

Majorana algebras were introduced by Ivanov by generalising certain properties found in subalgebras of the Griess algebra [9]. Axial algebras were developed as a generalisation of Majorana algebras, so Majorana algebras can be thought of as the precursor of axial algebras. As such, we can give a definition of them in terms of axial algebras.

Definition 2.6. A *Majorana algebra* is a primitive axial algebra A of Monster type over $\mathbb{R}$ such that

M1 A has a positive definite Frobenius form $(\cdot, \cdot)$; furthermore, $(a, a) = 1$ for every axis a.

M2 *Norton's inequality* holds. That is, for all $x, y \in A$,

$$(x \cdot y, x \cdot y) \leq (x \cdot x, y \cdot y).$$

In some papers, the M2 axiom is not assumed and in others additional axioms on the subalgebras are assumed such as the M8 axiom, which we will explain later in Section 2.2.

2.1 Gradings and automorphisms

The key property that axial algebras and Majorana algebras generalise from the Griess algebra is that there is a natural link between justinautomorphisms and axes. This link occurs precisely when we have a graded fusion law.

Definition 2.7. The fusion law $\mathcal{F}$ is *T-graded*, where T is a finite abelian group, if $\mathcal{F}$ has a partition $\mathcal{F} = \cup_{t \in T} \mathcal{F}_t$ such that

$$\mathcal{F}_s \star \mathcal{F}_t \subseteq \mathcal{F}_{st}$$

for all $s, t \in T$.

Note that, in the same way as we allow trivial eigenspaces, we also allow empty parts in the partition in the above definition.

Let A be an algebra and $a \in A$ an $\mathcal{F}$-axis (we do not require A to be an axial algebra here). If $\mathcal{F}$ is T-graded, then this induces a *T-grading* on A with respect to the axis a. The weight t subspace A_t of A is

$$A_t = A_{\mathcal{F}_t} = \bigoplus_{\lambda \in \mathcal{F}_t} A_\lambda.$$

This leads to automorphisms of the algebra. Let T^* denote the group of linear characters of T. That is, the homomorphisms from T to $\mathbb{F}^\times$. For $\chi \in T^*$, we define a map $\tau_a(\chi)\colon A \to A$ by

$$v \mapsto \chi(t)v$$

for $v \in A_t$ and extend linearly to A. Since A is T-graded, this map $\tau_a(\chi)$ is an automorphism of A, which we call a *Miyamoto automorphism*. Furthermore, the map sending χ to $\tau_a(\chi)$ is a homomorphism from T^* to $\mathrm{Aut}(A)$. The subgroup $T_a := \mathrm{Im}(\tau_a)$ of $\mathrm{Aut}(A)$ is called the *axis subgroup* corresponding to a.

We are particularly interested in $\mathbb{Z}_2$-graded fusion laws. In this case, we write $\mathbb{Z}_2$ as $\{+, -\}$ with the usual multiplication of signs. For example, the Monster fusion law $\mathcal{F}$ is $\mathbb{Z}_2$-graded where $\mathcal{F}_+ = \{1, 0, \frac{1}{4}\}$ and $\mathcal{F}_- = \{\frac{1}{32}\}$.

When the fusion law is $\mathbb{Z}_2$-graded and $\mathrm{char}(\mathbb{F}) \neq 2$, $T^* = \{\chi_1, \chi_{-1}\}$, where χ_1 is the trivial character and χ_{-1} is the alternating character of $T = \mathbb{Z}_2$. Here the axis subgroup contains just one non-trivial automorphism, $\tau_a := \tau_a(\chi_{-1})$. We call this the *Miyamoto involution* associated to a. It is given by the linear extension of

$$v^{\tau_a} = \begin{cases} v & \text{if } v \in A_+; \\ -v & \text{if } v \in A_-. \end{cases}$$

When the fusion law is $\mathbb{Z}_2$-graded, we will often consider τ to be a map on X mapping $a \in X$ to the involution τ_a.

Let $Y \subseteq X$ be a set of axes in A. We define

$$G(Y) := \langle T_a : a \in Y \rangle.$$

We call $G(X)$ the *Miyamoto group*.

For a subset $Y \subseteq X$ of axes, we define $\overline{Y} = Y^{G(Y)}$. By [11, Lemma 3.5], $G(\overline{Y}) = G(Y)$ and so $\overline{Y}^{G(\overline{Y})} = \overline{Y}$. We call $\overline{Y}$ the *closure* of Y and we say that Y is *closed* if $Y = \overline{Y}$.

2.2 Subalgebras generated by two axes

Since the defining property of axial algebras is that they are generated by a set of axes, it is natural to ask: What are the axial algebras that are generated by just two axes? We call such axial algebras 2-*generated* and, if

Type	Basis	Products & form
2A	a_0, a_1,	$a_0 \cdot a_1 = \frac{1}{8}(a_0 + a_1 - a_\rho)$
	a_ρ	$a_0 \cdot a_\rho = \frac{1}{8}(a_0 + a_\rho - a_1)$
		$(a_0, a_1) = (a_0, a_\rho) = (a_1, a_\rho) = \frac{1}{8}$
2B	a_0, a_1	$a_0 \cdot a_1 = 0$
		$(a_0, a_1) = 0$
3A	a_{-1}, a_0,	$a_0 \cdot a_1 = \frac{1}{2^5}(2a_0 + 2a_1 + a_{-1}) - \frac{3^3 \cdot 5}{2^{11}} u_\rho$
	a_1, u_ρ	$a_0 \cdot u_\rho = \frac{1}{3^2}(2a_0 - a_1 - a_{-1}) + \frac{5}{2^5} u_\rho$
		$u_\rho \cdot u_\rho = u_\rho$, $(a_0, a_1) = \frac{13}{2^8}$
		$(a_0, u_\rho) = \frac{1}{4}$, $(u_\rho, u_\rho) = \frac{2^3}{5}$
3C	a_{-1}, a_0,	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1})$
	a_1	$(a_0, a_1) = \frac{1}{2^6}$
4A	a_{-1}, a_0,	$a_0 \cdot a_1 = \frac{1}{2^6}(3a_0 + 3a_1 - a_{-1} - a_2 - 3v_\rho)$
	a_1, a_2	$a_0 \cdot v_\rho = \frac{1}{2^4}(5a_0 - 2a_1 - a_2 - 2a_{-1} + 3v_\rho)$
	v_ρ	$v_\rho \cdot v_\rho = v_\rho$, $a_0 \cdot a_2 = 0$
		$(a_0, a_1) = \frac{1}{2^5}$, $(a_0, a_2) = 0$
		$(a_0, v_\rho) = \frac{3}{2^3}$, $(v_\rho, v_\rho) = 2$
4B	a_{-1}, a_0,	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1} - a_2 + a_{\rho^2})$
	a_1, a_2	$a_0 \cdot a_2 = \frac{1}{2^3}(a_0 + a_2 - a_{\rho^2})$
	a_{ρ^2}	$(a_0, a_1) = \frac{1}{2^6}$, $(a_0, a_2) = (a_0, a_{\rho^2}) = \frac{1}{2^3}$
5A	a_{-2}, a_{-1},	$a_0 \cdot a_1 = \frac{1}{2^7}(3a_0 + 3a_1 - a_2 - a_{-1} - a_{-2}) + w_\rho$
	a_0, a_1,	$a_0 \cdot a_2 = \frac{1}{2^7}(3a_0 + 3a_2 - a_1 - a_{-1} - a_{-2}) - w_\rho$
	a_2, w_ρ	$a_0 \cdot w_\rho = \frac{7}{2^{12}}(a_1 + a_{-1} - a_2 - a_{-2}) + \frac{7}{2^5} w_\rho$
		$w_\rho \cdot w_\rho = \frac{5^2 \cdot 7}{2^{19}}(a_{-2} + a_{-1} + a_0 + a_1 + a_2)$
		$(a_0, a_1) = \frac{3}{2^7}$, $(a_0, w_\rho) = 0$, $(w_\rho, w_\rho) = \frac{5^3 \cdot 7}{2^{19}}$
6A	a_{-2}, a_{-1},	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-2} - a_{-1} - a_2 - a_3 + a_{\rho^3}) + \frac{3^2 \cdot 5}{2^{11}} u_{\rho^2}$
	a_0, a_1,	$a_0 \cdot a_2 = \frac{1}{2^5}(2a_0 + 2a_2 + a_{-2}) - \frac{3^3 \cdot 5}{2^{11}} u_{\rho^2}$
	a_2, a_3	$a_0 \cdot u_{\rho^2} = \frac{1}{3^2}(2a_0 - a_2 + a_{-2}) + \frac{5}{2^5} u_{\rho^2}$
	a_{ρ^3}, u_{ρ^2}	$a_0 \cdot a_3 = \frac{1}{2^3}(a_0 + a_3 - a_{\rho^3})$, $a_{\rho^3} \cdot u_{\rho^2} = 0$
		$(a_0, a_1) = \frac{5}{2^8}$, $(a_0, a_2) = \frac{13}{2^8}$
		$(a_0, a_3) = \frac{1}{2^3}$, $(a_{\rho^3}, u_{\rho^2}) = 0$,

Table 2: Norton-Sakuma algebras

the fusion law is $\mathbb{Z}_2$-graded, we also call them *dihedral* because the Miyamoto group in this case is dihedral.

In the Griess algebra, the dihedral subalgebras, called *Norton-Sakuma algebras*, were investigated by Norton and shown to be one of nine different types [3]. In particular, for each pair of axes a_0, a_1 in the Griess algebra, the isomorphism class of the subalgebra which they generate is determined by the conjugacy class in the Monster of the product $\tau_{a_0}\tau_{a_1}$ of the two involutions τ_{a_0} and τ_{a_1} associated to the axes. The nine different types are: 1A (when $a_0 = a_1$), 2A, 2B, 3A, 3C, 4A, 4B, 5A and 6A.

The algebra 1A is just one dimensional, but the remaining eight Norton-Sakuma algebras are given in Table 2 whose content we will now explain. The notation is from [18, Section 2]. Let nL be one of the dihedral algebras. Since its generating axes a_0 and a_1 give involutions τ_{a_0} and τ_{a_1} in the Monster, we have the dihedral group $D_{2n} \cong \langle \tau_{a_0}, \tau_{a_1} \rangle$ acting as automorphisms of nL (possibly with a kernel). In particular, let $\rho = \tau_{a_0}\tau_{a_1}$. We define

$$a_{\varepsilon+2k} = a_\varepsilon^{\rho^k}$$

for $\varepsilon = 0, 1$. It is clear that each a_i is an axis as it lies in the orbit of a_0 or a_1 under the action of ρ. In fact, the orbits of a_0 and a_1 under the action of ρ (in fact, under the action of D_{2n}) have the same size. If n is even, then these two orbits have size $\frac{n}{2}$ and are disjoint, whereas if n is odd, the orbits coincide and have size n. The map τ associates an involution to each axis a and $\tau_a^g = \tau_{a^g}$ for all $g \in \mathrm{Aut}(n\mathrm{L})$. In almost all cases, the axes a_i are not enough to span the algebra. We index the additional basis elements by powers of ρ. Using the action of D_{2n}, it is enough to just give the products in Table 2 to fully describe each algebra. The axes in each algebra are primitive and each algebra admits a Frobenius form that is non-zero on the set of axes and invariant under the Miyamoto group; the values for this are also listed in the table.

Amazingly the classification of dihedral algebras also holds, and is known as Sakuma's theorem [17], if we replace the Griess algebra by the weight two subspace V_2 of a vertex operator algebra (VOA) $V = \bigoplus_{n=0}^{\infty} V_n$ over $\mathbb{R}$ where $V_0 = \mathbb{R}1$ and $V_1 = 0$ (those of OZ-type). After Majorana algebras were defined generalising such VOAs, the result was reproved for Majorana algebras by Ivanov, Pasechnik, Seress and Shpectorov in [10]. In the paper introducing axial algebras, the result was also shown to hold in axial algebras of Monster type over a field of characteristic 0 which have a Frobenius form [6]. It is conjectured that the Frobenius form is not required.

Conjecture 2.8. *A dihedral axial algebra of Monster type over a field of characteristic* 0 *is one of the nine Norton-Sakuma algebras.*[2]

[2]A proof of this conjecture was recently announced by Franchi, Mainardis and Shpectorov at the Axial Algebra Focused Workshop in Bristol in May 2018.

For Majorana algebras, the following axiom is also often assumed.

M8 Let $a_i \in X$ be axes for $0 \leq i \leq 2$. If a_0 and a_1 generate a dihedral subalgebra of type 2A, then $a_\rho \in X$ and $\tau_{a_\rho} = \tau_{a_0}\tau_{a_1}$. Conversely, if $\tau_{a_0}\tau_{a_1}\tau_{a_2} = 1$, then a_0 and a_1 generate a dihedral subalgebra of type 2A and $a_2 = a_\rho$.

This severely restricts the possible configuration of subalgebras. We will explain this later in Section 3 once we have introduced shapes.

Seress [18] also assumed that the map τ was a bijection between the set of axes X and a union of conjugacy classes of involutions in G. Moreover the following conditions which restrict the intersections of subalgebras were also assumed. Let $a_i, b_i \in X$ and $\rho(a_0, a_1) = \tau_{a_0}\tau_{a_1}$.

2Aa If $\tau_{a_0}\tau_{a_1}\tau_{a_2} = 1$ and $\langle a_0, a_1\rangle \cong 2\text{A}$, then $a_2 \in \langle a_0, a_1\rangle$ and $a_2 = a_\rho$.

2Ab If $\langle a_0, a_1\rangle$ and $\langle b_0, b_1\rangle$ are both of type 2A and $\langle \rho(a_0, a_1)\rangle = \langle \rho(b_0, b_1)\rangle$, then the extra basis elements $a_\rho(a_0, a_1)$ and $a_\rho(b_0, b_1)$ are equal.

3A If $\langle a_0, a_1\rangle$ and $\langle b_0, b_1\rangle$ are both of type 3A and $\langle \rho(a_0, a_1)\rangle = \langle \rho(b_0, b_1)\rangle$, then the extra basis elements $u_\rho(a_0, a_1)$ and $u_\rho(b_0, b_1)$ are equal.

4A If $\langle a_0, a_1\rangle$ and $\langle b_0, b_1\rangle$ are both of type 4A and $\langle \rho(a_0, a_1)\rangle = \langle \rho(b_0, b_1)\rangle$, then the extra basis elements $v_\rho(a_0, a_1)$ and $v_\rho(b_0, b_1)$ are equal.

5A If $\langle a_0, a_1\rangle$ and $\langle b_0, b_1\rangle$ are both of type 5A and $\langle \rho(a_0, a_1)\rangle = \langle \rho(b_0, b_1)\rangle$, then the extra basis elements $w_\rho(a_0, a_1)$ and $w_\rho(b_0, b_1)$ are equal up to a change of sign.

We can also consider a wider class of axial algebras. Axial algebras of Jordan type η were considered in [7]. Here there are just three eigenvalues, 1, 0 and η. When $\eta \neq \frac{1}{2}$, all algebras were classified and they relate to 3-transposition groups. The *Ising fusion law* $\Phi(\alpha, \beta)$ is given in Table 3.

	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1, 0$	β
β	β	β	β	$1, 0, \alpha$

Table 3: Ising fusion law $\Phi(\alpha, \beta)$

In particular, note that the Monster fusion law is just $\Phi(\frac{1}{4}, \frac{1}{32})$. In [16], Rehren studies dihedral axial algebras over $\Phi(\alpha, \beta)$ with a Frobenius form and shows that the nine algebras above can be generalised and live in families which exist for values of α and β lying in certain varieties. It turns out that $(\alpha, \beta) = (\frac{1}{4}, \frac{1}{32})$ is a distinguished point.

3 Shapes

The shape of an axial algebra A specifies which 2-generated subalgebras arise in A. Clearly, a precondition for such a description is the knowledge of the possible 2-generated algebras; that is, for the class of axial algebras under consideration we either should have classified all 2-generated algebras or, minimally, we should have an explicit list of such algebras that we want to allow in A.

Note that the 2-generated algebras should be classified not up to an abstract algebra isomorphism, but rather up to the (unique possible) isomorphism sending the two generating axes of one algebra to the two generating axes of the other algebra. That is, we consider the 2-generated algebras as having marked generators and isomorphisms must respect them: if B has marked generators a and b and B' has marked generators a' and b' then $(B, (a, b))$ is isomorphic to $(B', (a', b'))$ only if there is an isomorphism $\varphi\colon B \to B'$ such that $\varphi(a) = a'$ and $\varphi(b) = b'$. In principle, an algebra may have non-equivalent pairs of generators and then this algebra must accordingly appear on the list several times. Note that for algebras of Monster type, Sakuma's theorem classifies dihedral algebras exactly in this sense: in each of the eight Norton-Sakuma algebra the marked generators are a_0 and a_1 and any other pairs of generators is equivalent to (a_0, a_1). Therefore, in order to motivate the general case, we consider first the case of an axial algebra of Monster type.

Let A be an axial algebra of Monster type and suppose that X is a set of axes which generates A. Note that by enlarging our set X, we may assume that X is closed under the action of the Miyamoto group G of A.

Lemma 3.1. *The action of G on X is faithful.*

Proof. Suppose that $g \in G$ fixes all the axes in X. However, the subspace of A fixed by g is a subalgebra and, since it contains X, it contains the whole algebra A. $\square$

As G is a group of automorphisms of A, if $a, b \in X$ generate a dihedral subalgebra B, then, for any $g \in G$, the subalgebra generated by a^g, b^g is isomorphic to B. In this way, we obtain the *shape* of the algebra which is a map S from the set of G-orbits on $X \times X$ to the set of dihedral algebras.

Given a pair of axes (a, b), let $D_{a,b}$ be the dihedral group generated by τ_a and τ_b. Define $X_{a,b} = a^D \cup b^D$, where $D := D_{a,b}$. It is clear that $D_{a,b} = D_{b,a}$ and $X_{a,b} = X_{b,a}$.

A Norton-Sakuma algebra has type nL. We wish to show that n can be determined solely from the action of the dihedral group $D_{a,b}$.

Lemma 3.2. *Let $a, b \in X$ and $D := D_{a,b}$. Then, $|a^D| = |b^D|$. If a and b are in the same orbit, then the length of this orbit is 1, 3, or 5. Otherwise, if a and b are in different orbits, then the length of each orbit is 1, 2, or 3. Moreover, the Norton-Sakuma algebra generated by a and b has type nL, where $n = |X_{a,b}|$.*

Proof. A direct proof would be long and computational. So instead we observe that each Norton-Sakuma algebra is contained in the Griess algebra and there we have a bijection between axes and 2A-involutions in the Monster M. So, we may take the dihedral subgroup $H \leq M$ generated by the involutions associated to each axis (in the Griess algebra). In particular, up to the kernel, the action of H on X is the same as the action of D on X.

Since in the Griess algebra we have a bijection between axes and 2A-involutions and $\tau_x^g = \tau_{x^g}$ for $g \in H$, we may consider the orbits of involutions in H rather than the orbits of axes. The result now follows from properties of dihedral groups and Sakuma's theorem. □

Thus, when we know the action of G on X, n is known for each orbit and the shape is determined by choices of L. Furthermore, these choices are not independent.

If $a, b, c, d \in X$ then we say (a, b) *dominates* (c, d) if $c, d \in X_{a,b}$. In particular, when this happens, $X_{c,d} \subseteq X_{a,b}$ and $D_{c,d} \leq D_{a,b}$. Note also that the subalgebra $\langle c, d\rangle$ is contained in $\langle a, b\rangle$. Hence, if (a, b) dominates (c, d), then the choice of dihedral subalgebra $\langle a, b\rangle$ determines the choice for $\langle c, d\rangle$. For the Monster fusion law, we have the following non-trivial inclusions

$\langle a, b\rangle$	$\langle c, d\rangle$
4A	2B
4B	2A
6A	2A
6A	3A

Note that here, not only does the choice of $\langle a, b\rangle$ determine the choice for $\langle c, d\rangle$, but also the choice for $\langle c, d\rangle$ uniquely determines the choice for $\langle a, b\rangle$. Additionally, note that the pair (a, b) always dominates (b, a) and vice versa, so in the next concept which describes the totality of choices, we may just work with the set $\{a, b\}$ instead of the pairs (a, b) and (b, a). Notice also that since $X_{a,b} = X_{b,a}$, the concept of domination is not affected by the switch to sets.

Let $\binom{X}{2}$ denote the set of 2-subsets of X. The orbits of G on $\binom{X}{2}$ are the vertices of a directed graph, called the *shape graph*, with the edges given by domination. By the above comment, there is at most one choice of dihedral subalgebra for each weakly connected component (i.e. a connected component of the underlying undirected graph). So, the shape of an algebra is fully described by assigning one dihedral subalgebra per weakly connected component. Sometimes there is no choice for a given component. Namely, when that component contains a 6A, or 5A.

Additionally, if the M8 axiom is assumed, then this further restricts the allowable shapes. Suppose that a and b are such that $X_{a,b} = \{a, b\}$ and τ_a and τ_b are the involutions associated to a and b. Then $\tau_a\tau_b$ has order two. If $\tau_a\tau_b$ is in the image of the τ-map, then M8 demands that the dihedral subalgebra $B = \langle a, b\rangle$ generated by a and b be a 2A. Conversely, if $\tau_a\tau_b$ is not in the image τ, then the dihedral subalgebra B must be a 2B. In both cases, this defines the shape on the connected component containing the orbit of $\{a, b\}$. However, the only connected components which don't contain any dihedral subalgebras with $n = 2$ are those which just contain a single dihedral subalgebra with $n = 3$. So, if the M8 condition is assumed the only choice over a shape is choosing whether those connected components which consist of a single 3L are 3A, or 3C.

We now turn to the general case of a fusion law $\mathcal{F}$ which is T-graded and an abstract group of permutations G acting faithfully on a set X. We are thinking of an unknown axial algebra A with fusion law $\mathcal{F}$ and the action of the Miyamoto group on the axes being the action of (a normal subgroup of) G on X. It is clear that we may just consider actions up to isomorphism. Recalling the definition of a Miyamoto automorphism from Section 2.1, we will define analogous concepts to above.

Definition 3.3. Let G be a permutation group acting on a set X and T an abelian group. A map $\tau\colon X \times T^* \to G$ is called a *τ-map* if for all $x \in X$, $\chi \in T^*$, $g \in G$

1. $\tau_x\colon T^* \to G$ is a group homomorphism;
2. $\tau_x(\chi)^g = \tau_{x^g}(\chi)$.

We call the image $G_0 := \langle \tau_x(\chi) : x \in X, \chi \in T^*\rangle \trianglelefteq G$ the *Miyamoto group* of τ.

As previously, we define $T_x := \langle \tau_x(\chi) : \chi \in T^*\rangle \leq G_0$.

Lemma 3.4. *$T_x \subseteq Z(G_x)$, where G_x is the stabiliser in G of x.*

Proof. Let $g \in G_x$. Then for $\chi \in T^*$,

$$[\tau_x(\chi), g] = \tau_x(\chi)^{-1}\tau_x(\chi)^g = \tau_x(\chi)^{-1}\tau_{x^g}(\chi) = \tau_x(\chi)^{-1}\tau_x(\chi) = 1 \qquad \square$$

We define $D = D_{a,b} := \langle T_a, T_b\rangle$ for $a, b \in X$. Unlike the Monster type case, D does not have to be a dihedral group. In an $\mathcal{F}$-axial algebra, $D_{a,b}$ acts on the subalgebra $\langle a, b\rangle$. Suppose that we know a list $\mathcal{L}$ of 2-generated subalgebras with marked generators for the fusion law $\mathcal{F}$. We wish to impose conditions on τ so that $D_{a,b}$ has an action on $X_{a,b} := a^D \cup b^D$ which is an action observed on the axes of some 2-generated algebra in our list. Otherwise, τ cannot lead to a valid $\mathcal{F}$-axial algebra.

Definition 3.5. A τ-map $\tau\colon X \times T^* \to G$ is called *$\mathcal{L}$-admissible* if for every set $\{a, b\} \in \binom{X}{2}$, the action of $D_{a,b}$ on $X_{a,b}$ agrees with at least one algebra in the list $\mathcal{L}$.

Where the list $\mathcal{L}$ is understood, we will just say admissible.

For example, let $\mathcal{F}$ be the Monster fusion law. Then the complete list $\mathcal{L}$ is just the list of Norton-Sakuma algebras given in Table 2. In particular, the orbits of a and b under D must have the properties given in Lemma 3.2. That is,

1. $k := |a^D| = |b^D|$.
2. If a and b are in the same D-orbit, then $k = 1$, 3, or 5.
3. If a and b are in different D-orbits, then $k = 1$, 2, or 3.

From now on, we only consider admissible τ-maps. The normaliser $N = N_{\mathrm{Sym}(X)}(G)$ of the action of G on X acts on the set of admissible τ-maps by

$$\tau \mapsto \tau^n \qquad \text{where } (\tau^n)_x(\chi) := \tau_{x^{n^{-1}}}(\chi)^n$$

for $n \in N$. Note that, by the definition of a τ-map, G acts trivially on each τ. So an action of N/G is induced on the set of τ-maps. Thus, we may just consider admissible τ-maps up to the action of N/G.

Next we introduce domination.

Definition 3.6. For $\{a, b\}, \{c, d\} \in \binom{X}{2}$, we say $\{a, b\}$ *dominates* $\{c, d\}$ if $c, d \in X_{a,b}$.

Definition 3.7. The *shape graph* Γ is a directed graph with vertices given by orbits of G on $\binom{X}{2}$ and edges given by domination between pairs from those orbits.

As observed above, for the Monster fusion law, any one choice of 2-generated subalgebra for a weakly connected component of the shape graph determines all other 2-generated algebras in that component. For a general fusion law, the dominated algebra may not always determine the larger algebra uniquely. However, the larger, dominating algebra always determines the smaller algebra. We will call these the *domination restrictions*.

Definition 3.8. Given an abstract group G acting faithfully on a set X and an admissible τ-map, a *shape* on X is a set of choices of 2-generated algebra for all orbits of G on $\binom{X}{2}$ which satisfy the domination restrictions.

Given a group G acting faithfully on a set X and an admissible τ-map τ, we may consider all the possible shapes. Let $K = \mathrm{Stab}_N(\tau)$. As noted above, G acts trivially on each τ, and in fact it also fixes every shape. On the other hand, K (or rather K/G) permutes the G-orbits of $\binom{X}{2}$, and so may act non-trivially on the set of shapes. So, we may consider shapes for τ up to the action of K.

In summary, given an action of a group G on a putative set of axes X, we can determine all the possible admissible τ-maps. Given a particular τ-map τ, we can further determine all the possible shapes that an axial algebra with Miyamoto group G_0 and τ-map τ could have.

4 Useful lemmas

In this section, we will discuss some properties which must hold in axial algebras. We will use these later in the algorithm to discover relations and to build up eigenspaces.

Recall that we adopt the notation that for a subset $I \subseteq \mathcal{F}$,

$$A_I = \bigoplus_{\lambda \in I} A_\lambda$$

We begin by noting that, since we allow I to be a subset, we can add and intersect the A_I.

Lemma 4.1. *Let $I, J \subseteq \mathcal{F}$, then*

1. $A_I + A_J = A_{I \cup J}$
2. $A_I \cap A_J = A_{I \cap J}$ $\square$

By an abuse of terminology, we will call the A_I eigenspaces of a.

Lemma 4.2. *Let a be an axis, $I \subseteq \mathcal{F}$, $\lambda \in I$ and $A_I = A_I^a$. Then, for all $u \in A_I$*

$$ua - \lambda u \in A_{I - \lambda}$$

Proof. We may decompose $u \in A_I$ as $u = \sum_{\mu \in I} u_\mu$, where $u_\mu \in A_\mu$. Multiplying by a and subtracting λu, we have

$$\begin{aligned} ua - \lambda u &= \sum_{\mu \in I} u_\mu a - \lambda u \\ &= \sum_{\mu \in I} (\mu - \lambda) u_\mu \end{aligned}$$

Since the coefficient of u_λ is zero, the above is in $A_{I-\lambda}$. $\square$

Recall that we extended the operation $\star$ to all subsets of $\mathcal{F}$, turning the fusion law into a magma. Moreover, the eigenspaces A_I satisfy the fusion law. However, not all fusion rules on subsets are equally useful for our algorithm. In particular, assuming that $\mathcal{F}$ is T-graded, we only need to consider I fully contained in a part $\mathcal{F}_t$ for some $t \in T$. We call such subsets *pure*.

Definition 4.3. Let $I \subseteq \mathcal{F}_s$ and $J \subseteq \mathcal{F}_t$ for $s, t \in T$. We define a fusion rule $I \star J = K$ to be *useful* if

1. $K \subsetneqq \mathcal{F}_{s\star t}$; and
2. there does not exist $I \subsetneqq I' \subseteq \mathcal{F}_s$, or $J \subsetneqq J' \subseteq \mathcal{F}_t$ such that

$$I' \star J = K \qquad \text{or} \qquad I \star J' = K$$

In particular, given a useful fusion rule $I \star J = K$, if we require it to hold, all other rules $X \star Y = K$ for subsets $X \subseteq I$ and $Y \subseteq J$ will automatically be satisfied. In this way, it is enough to impose just the useful fusion rules and the grading to capture all the information from the fusion law.

To calculate the useful fusion rules for any fusion law $\mathcal{F}$ we begin by writing out the expanded fusion table for all pure subsets of $\mathcal{F}$ with rows and columns partially ordered by inclusion. We then consider all sets K which occur as entries in the table. The useful rules are precisely those where K is not a full part $\mathcal{F}_t$, for $t \in T$, and it does not appear in the expanded table below in that column, or to the right in that row. Doing this to the Monster fusion law results in the following list.

Lemma 4.4. *The useful fusion rules for the Monster fusion table are*

$$1 \star 0 = \emptyset \qquad 1 \star \{1, 0\} = 1 \qquad 1 \star \{0, \tfrac{1}{4}\} = \tfrac{1}{4} \qquad 1 \star \{1, 0, \tfrac{1}{4}\} = \{1, \tfrac{1}{4}\}$$
$$0 \star \{1, 0\} = 0 \qquad 0 \star \{1, \tfrac{1}{4}\} = \tfrac{1}{4} \qquad 0 \star \{1, 0, \tfrac{1}{4}\} = \{0, \tfrac{1}{4}\}$$
$$\tfrac{1}{4} \star \tfrac{1}{4} = \{1, 0\} \qquad \tfrac{1}{4} \star \{1, 0\} = \tfrac{1}{4}$$
$$\{1, 0\} \star \{1, 0\} = \{1, 0\} \qquad \{1, 0\} \star \{1, \tfrac{1}{4}\} = \{1, \tfrac{1}{4}\} \qquad \{1, 0\} \star \{0, \tfrac{1}{4}\} = \{0, \tfrac{1}{4}\}$$

Note that all useful fusion rules for the Monster fusion law come from the even part. That is because the values of $\star$ involving the odd part $\{\frac{1}{32}\}$ are fully determined by the grading.

If A is primitive, then for an axis a, G_a certainly fixes every vector in A_1^a. We now describe another trick which uses this weaker condition.

Lemma 4.5. *Let $1 \in I \subset \mathcal{F}$ and $u \in A_I(a)$ for an axis a. Suppose further that G_a fixes every vector in A_1^a. Then, for all g in the stabiliser G_a,*

$$u^g - u \in A_{I-1}$$

Proof. We decompose $u = \sum_{\mu \in I} u_\mu$ with respect to the eigenspaces of a. Since g fixes a, it preserves every eigenspace of a. Furthermore, since g fixes every vector in A_1^a, we have the following

$$\begin{aligned} u^g - u &= \sum_{\mu \in I} u_\mu^g - \sum_{\mu \in I} u_\mu \\ &= u_1^g - u_1 + \sum_{\mu \in I - 1} u_\mu^g - u_\mu \\ &= \sum_{\mu \in I-1} u_\mu^g - u_\mu \in A_{I-1} \end{aligned}$$

□

5 Algorithm

In this section, we describe our main result which is an algorithm for constructing an axial algebra. A very similar algorithm can also be used to build a module for a known axial algebra. However, we don't want to complicate this paper with extra definitions and so we just deal with the task of constructing an axial algebra.

In principle, there is no reason to believe that an axial algebra which is generated by a finite set of axes is even finite dimensional. Clearly, if it is infinite dimensional, our algorithm will not finish. However, in practice, we can compute a large number of examples as we shall see in Section 6.

As described in Section 3, associated with a T-graded $\mathcal{F}$-axial algebra A we have a group G acting faithfully on a set X, an admissible τ-map $\tau\colon X \times T^* \to G_0 \trianglelefteq G$ and a shape. Given such a G, X, τ and shape, the algorithm, if successful, builds an axial algebra A with axes X and Miyamoto group G_0. It does so by defining a partial algebra and completing it step by step into a full algebra.

As input to our algorithm, we take a field $\mathbb{F}$, a T-graded fusion law $\mathcal{F}$, a group G acting faithfully on a set X, an admissible τ-map τ and a shape. These are fixed throughout the rest of this section.

5.1 Partial algebras

At the core of the algorithm is a concept which we call a partial algebra. The idea is that there is a G-module W on which we only know how to multiply vectors coming from a certain submodule V and the results of this multiplication lie in W. We stress that just because we can multiply $u, v \in V$, their product $\mu(u, v)$ does not have to lie in V. We write $S^2(V)$ for the symmetric square of V.

Definition 5.1. Given a group G, a *partial G-algebra* is a triple $W = (W, V, \mu)$ where W is a G-module over $\mathbb{F}$, $V \subseteq W$ is a G-submodule and $\mu\colon S^2(V) \to W$ is a linear map which is G-equivariant.

Where it is clear, we will abuse notation and write uv for $\mu(u,v)$.

Lemma 5.2. *Given a G-invariant set Y in W, there exists a unique smallest G-submodule $W(Y)$ of W such that*

$$W(Y) = \langle Y \rangle + \mu(S^2(W(Y) \cap V))$$

Proof. Let $U_0 := \langle Y \rangle$ and define inductively

$$U_{i+1} := U_i + \mu(S^2(U_i \cap V)).$$

Let $\tilde{U} = \bigcup_{i=0}^{\infty} U_i$; it is a G-submodule of W since every U_i is. We claim that $\tilde{U} = \langle Y \rangle + \mu(S^2(\tilde{U} \cap V))$. Clearly, $\langle Y \rangle \subseteq \tilde{U}$. Let $u, v \in \tilde{U} \cap V$, then by definition $u, v \in U_i \cap V$ for some i. Hence $\mu(u \vee v) \in U_{i+1} \subseteq \tilde{U}$ and so the right hand side is contained in $\tilde{U}$. The reverse inclusion is clear by induction.

Finally, if U is any submodule of W satisfying $U = \langle Y \rangle + \mu(S^2(U \cap V))$, we see, again by induction on i, that each $U_i \subseteq U$ and hence $\tilde{U} \subseteq U$. $\square$

We call $W(Y) = (W(Y), W(Y) \cap V, \mu|_{S^2(W(Y) \cap V)})$ the *partial subalgebra generated by Y*. If $W(Y) = W$, then we say that Y *generates* W. For example, an axial algebra A with Miyamoto group G is a partial G-algebra and the set of axes X generates A.

Definition 5.3. Let (W, V, μ) be a partial G-algebra and (W', V', μ') be a partial G'-algebra. A homomorphism of partial algebras is a pair (φ, ψ) where

1. $\varphi \colon W \to W'$ is a vector space homomorphism such that $\varphi(V) \subseteq V'$.
2. $\psi \colon G \to G'$ is a group homomorphism such that

$$\varphi(w^g) = \varphi(w)^{\psi(g)}$$

for all $w \in W$, $g \in G$.

3. $\varphi(\mu(u,v)) = \mu'(\varphi(u), \varphi(v))$ for all $u, v \in V$.

In other words, we have the following commutative diagram and additionally the action of G (sometimes acting through ψ) commutes with the diagram.

$$\begin{tikzcd}
S^2(V) \arrow[r, "\mu"] \arrow[d, "\varphi\vee\varphi"] & W \arrow[d, "\varphi"] & V \arrow[l, hook'] \arrow[d, "\varphi"] \\
S^2(V') \arrow[r, "\mu'"] & W' & V' \arrow[l, hook']
\end{tikzcd}$$

5.2 Gluings

In order to correctly build an axial algebra, we must impose the conditions coming from the shape. We do this by gluing in the 2-generated algebras appearing in the shape.

Informally, the concept of gluing can be explained as follows. Let $W(X)$ be a partial G-algebra with a τ-map τ on the axes X. Suppose that $Y \subseteq X$ is a closed set of axes. That is, Y is closed under the action of $G(Y)$ (recall from Section 2.1 that $G(Y) := \langle T_a : a \in Y \rangle$). Then $W(Y)$ is a $G(Y)$-partial algebra generated by the set Y of axes. If we want Y to generate a subalgebra B in the final algebra, then at every step of the algorithm there had better be a homomorphism of partial algebras (φ, ψ) from $W(Y)$ to B. This partial homomorphism is our gluing. In fact, we make the definition even more general by allowing our target algebra B to be a partial algebra. Note that $G(Y)$ may not act on the set Y of axes faithfully and hence ψ may have a non-trivial kernel.

Definition 5.4. Let (W, V, μ) be a G-partial algebra generated by X, $Y \subseteq X$ be a closed set of axes and suppose that (W', V', μ') is a partial H-algebra generated by a set X'. A *gluing* of W' onto Y is a homomorphism of partial algebras (φ, ψ) from the $G(Y)$-partial subalgebra $(W(Y), W(Y) \cap V, \mu|_{S^2(W(Y)\cap V)})$ to (W', V', μ') such that $\varphi(Y) = X'$.

Note that we implicitly assume in the above definition that W and W' both have the same fusion law. Typically, φ induces a bijection from Y onto X' but this is not assumed. Also, as we will see later in the algorithm, the surjection of Y onto X' identifies φ and ψ uniquely.

5.3 Relations

Throughout our algorithm, we keep track of various (sums of) eigenspaces for each axis. These are key to finding enough relations to allow our algorithm to terminate. Recall that the sum of eigenspaces is denoted by $W_I = \bigoplus_{\lambda \in I} W_\lambda$, for a subset $I \subseteq \mathcal{F}$. Note that at any given stage in our algorithm, we may not know the full λ-eigenspace and so we do not necessarily know the decomposition $W = \bigoplus_{\lambda \in \mathcal{F}} W_\lambda$. Indeed, we may know that a vector lies in W_I, for some $I \subset \mathcal{F}$, but not know how to decompose it into the sum of eigenvectors for eigenvalues $\lambda \in I$. For this reason, we keep track of sums of eigenspaces W_I. Note that relations are vectors in $W_\emptyset$. Since G acts on W, we may just consider axes and their associated decompositions up to the action of G.

It turns out that it is enough to keep track of just the W_I, for pure subsets I. That is, the W_I for $I \subseteq \mathcal{F}_t$, for $t \in T$. We show that this holds, provided we make a mild assumption on the grading group T.

Indeed, by assumption, for each axis $a \in X$, there is a decomposition $W = \bigoplus_{t \in T} W_t$. We claim that we can recover the decomposition

$W = \bigoplus_{t \in T/R} W_t$, where $R := \bigcap_{\chi \in T^*} \ker(\chi)$, from the action on T_a on W. Indeed, recall from the definition that $\tau_a(\chi) \in T_a$ acts on W_t by scalar multiplication by $\chi(t)$. Since this must hold in any axial algebra we build, we can distinguish the T-grading up to the kernel $R = \bigcap_{\chi \in T^*} \ker(\chi)$. If $T^* \cong T$, then $R = 1$. However if R is non-trivial, for example when the characteristic divides $|T|$, or when the field doesn't contain the suitable roots of unity, we can only detect a coarser grading by $T/R \cong T^*$. Since we may always consider a more coarse grading, from now on, we may assume that $T^* \cong T$ and hence T_a detects the T-grading. Note that for a $\mathbb{Z}_2$-grading, provided the field is not of characteristic two, -1 is always in the field and hence we can detect a $\mathbb{Z}_2$-grading using the axis subgroup.

Let $J \subset \mathcal{F}$. Since we know the decomposition $W = \bigoplus_{t \in T} W_t$, this induces a decomposition $W_J = \bigoplus_{t \in T} W_{J_t}$, where $J_t := J \cap \mathcal{F}_t$. Now, the only results we will use in our algorithm are those found in Section 4, namely, summation and intersection of subspaces, being an eigenvector, obeying the fusion law and, optionally, Lemma 4.5. It is easy to see that for all of these, the information gained about W_J is precisely the sum of the information gained about the W_{J_t}. For example, if $\lambda \in J$, then by Lemma 4.2, $ua - \lambda u \in A_{J-\lambda}$. But, since we may decompose $u = \sum_{t \in T} u_t$, we have

$$u_t a - \lambda u_t \in A_{J_t - \lambda} = \begin{cases} A_{J_t} & \text{if } \lambda \notin J_t \\ A_{J_t - \lambda} & \text{if } \lambda \in J_t \end{cases}$$

In particular, we recover the only non-trivial result by just considering the pure subset $J_t \subseteq \mathcal{F}_t$. This justifies our claim that it is enough to keep track of the W_I, for pure subsets I.

5.4 The algorithm

Our task is to build an algebra of the correct shape. We will do this by defining a sequence of partial algebras and at each stage 'discovering' more of the multiplication. Throughout our algorithm $W = (W, V, \mu)$ will be a partial G-algebra generated by the set X, our putative set of axes on which G acts faithfully. Our algorithm will terminate when $V = W$. That is, when we know all the multiplication. We begin with W having basis indexed by the set X. That is, W is a permutation module for the action of G on X. No products are known at this stage, so $V = 0$.

The information for the multiplication, and so also for the eigenspaces, will come from gluing in algebras to our partial algebra according to the shape. In order to fully describe our axial algebra, we must glue in enough algebras to cover all those 2-generated algebras given in the shape. However, we may glue in known algebras of the correct shape which are generated by three or more axes. These have the advantage of containing more information. (We may also glue in some partial algebras, so long as we also glue in enough known algebras to cover those given in the shape.)

Since no multiplication is known when we start and W is spanned by the axes, for each gluing of an algebra B onto a closed subset of axes Y, we have $W(Y) = \langle Y \rangle$ and φ is the corresponding injection on these axes compatible with the action.

After initialisation, the main part of the algorithm has three stages:

1. Expansion by adding the formal products of vectors we do not already know how to multiply.
2. Work to discover relations and construct the eigenspaces for the axes.
3. Reduction by factoring out by known relations.

We continue applying these three stages until $V = W$ and that is when our algorithm terminates. Again, we note that since we use the action of the group, we need only consider subalgebras and axes up to the action of G.

If our algorithm does terminate, then we have the following result, which we will prove after describing our algorithm.

Theorem 5.5. *Suppose that the algorithm terminates and returns A. Then A is a (not necessarily primitive) axial algebra generated by axes X with automorphism group G, τ-map τ and of the given shape.*

Moreover, provided we do not use the optional Lemma 4.5 *in stage* 2 *of the algorithm, the algebra is universal. That is, given any other axial algebra B with the same axes X, automorphism group G, τ-map τ and shape, B is a quotient of A.*

Note that, if we do use Lemma 4.5 in stage 2 of the above algorithm, then we have assumed that G_a fixes every vector in A_1^a for each axis a. This holds in primitive axial algebras, but not necessarily in the non-primitive case.

Initialisation

We begin with W being a permutation module for the action of G on X and $V = 0$. We glue in enough algebras to cover all 2-generated algebras in the shape.

We now give a detailed description of the steps of the algorithm.

Stage 1: Expansion

We expand W to a larger partial algebra W_{new} by adding vectors which are the formal products of elements we do not yet know how to multiply.

Step 1. We begin by finding a complement subspace C for V in W. Hence, as a vector space

$$W = V \oplus C$$

Wherever possible, we choose C to be a G-submodule. For example, in characteristic 0, this is always possible.

Since we know the multiplication on V and our multiplication is commutative, we just need to add the products of V with C and products of C with C.

Step 2. Form a new partial algebra $W_{\text{new}} = (W_{\text{new}}, V_{\text{new}}, \mu_{\text{new}})$ with

$$\begin{aligned} W_{\text{new}} &= W \oplus V \otimes C \oplus S^2(C) \\ V_{\text{new}} &= W \end{aligned}$$

and μ extended in the obvious way to μ_{new}.

Note that if C is a G-submodule, then the summands in W_{new} are all G-submodules and hence W_{new} can be seen to be a G-module in a natural way. Otherwise, we must compute the action of G on W_{new}.

Step 3. For each algebra B glued onto a closed set of axes Y, we extend the gluing as follows[3]. Since $U := W(Y) \subset W$ and $V_{\text{new}} = W$, we now know all the products of elements in U, so we expand the gluing map φ to be defined on the entire U. Specifically, let $U_V = U \cap V$ and find a complement D so that

$$U = U_V \oplus D.$$

Then

$$U_{\text{new}} := U \oplus \mu(U_V, D) \oplus \mu(D, D)$$

is the partial $G(Y)$-subalgebra of W_{new} generated by Y. Indeed, since D projects injectively into C, $\mu(D, D)$ projects injectively into $S^2(C)$. Similarly, $\mu(U_V, D)$ projects injectively into $V \otimes C$. In particular, $U_{\text{new}} \cap W = U$ and hence U_{new} is the full partial subalgebra generated by Y, as claimed.

Note that the above also means that $\mu(U_V, D) \cong U_V \otimes D$ and $\mu(D, D) \cong S^2(D)$ and so U_{new} has the same structure as W_{new}. This allows us to extend the map φ to φ_{new} in the obvious way, by mapping the new products in U_{new} to the corresponding products in B. Hence, φ_{new} preserves multiplication. Observe that U_{new} is also a $G(Y)$-submodule and so the homomorphism ψ is unchanged. Hence, $(\varphi_{\text{new}}, \psi)$ is a gluing of B onto Y in W_{new}.

Step 4. For each gluing, we add the kernel of φ_{new} to the space of relations.

[3]Note that at each stage, the image of our gluing map φ is a partial subalgebra of the algebra B we are gluing in. Only after we have done enough expansions, will we have glued in all of B.

Indeed, if $\varphi_{\text{new}}(v) = 0$, then v must be the zero vector in any final axial algebra, hence it is a relation.

Step 5. For each axis a and partial subalgebra U_{new} which contains a, we use φ_{new} to pull back the eigenspaces of $B \cap \varphi_{\text{new}}(U_{\text{new}})$ to add to the eigenspaces in W_{new}.

Since we only consider axes and gluings up to G-orbit, we must be careful as one orbit of axes may split into several orbits when intersected with the subalgebra.

We note that the above expansion step can be made to work if we do not expand to the whole of W, but just to some G-submodule U of W which contains V. That is, we choose some subspace complement C to V in U (picking it to be a G-submodule if possible) and we expand to

$$W_{\text{new}} = W \oplus V \otimes C \oplus S^2(C)$$

and have $V_{\text{new}} = U$. The gluing for the partial subalgebras and the eigenspaces are updated similarly to above. This partial expansion has the advantage that it is easier to do computationally as it is smaller and we may still be able to find relations.

Stage 2: Building up eigenspaces

We begin by recovering the grading on W_{new}, before finding further eigenvectors and relations. Recall that relations are simply elements of the eigenspace $W_{\text{new},\emptyset}$.

Step 1. For each axis a, we compute the action of T_a on W_{new} and hence find the decomposition $W_{\text{new}} = \bigoplus_{t \in T} W_{\text{new},t}$ with respect to a.

For example, in the Monster fusion law case, we have the $\mathbb{Z}_2$-decomposition $W_{\text{new}} = W_{\text{new},+} \oplus W_{\text{new},-}$, where $W_{\text{new},+}$ and $W_{\text{new},-}$ are the 1- and -1-eigenspaces of τ_a, respectively.

If C is a submodule, then the calculation can be simplified as follows

$$W_{\text{new},t} = W_t \oplus \bigoplus_{s \in T} (V_s \otimes C_{s^{-1}t}) \oplus \bigoplus_{s \in T} C_s \times C_{s^{-1}t}$$

where V_s and C_s are the T-graded parts of V and C respectively.

We no longer need the old W, so we now drop the subscript and write W for W_{new} and similarly V for V_{new}.

Step 2. We repeatedly apply the following techniques until the pure eigenspaces W_I (including the relation eigenspace $W_\emptyset$) stop growing.

1. For each $t \in T$, we sum together and take intersections of the W_I for each pure subset $I \subsetneqq \mathcal{F}_t$ as per Lemma 4.1.
2. For each $t \in T$, let $\lambda \in I \subseteq \mathcal{F}_t$. For each $u \in W_I \cap V$, we add $ua - \lambda u$ to $W_{I-\lambda}$ as per Lemma 4.2.
3. We apply each useful fusion rule $I \star J = K$. That is, for all $u \in W_I \cap V$ and $v \in W_J \cap V$, we add their product uv to W_K.

Note that in parts (2) and (3), we of course may just do these for a basis of the eigenspaces concerned.

In the case of the Monster fusion law, $\mathcal{F}_- = \{\frac{1}{32}\}$. So, for the odd subspace W_-, there are no subspaces to sum or intersect in part (1) above. Also in part (2) for W_-, since the only choice for λ is $\frac{1}{32}$, we obtain that $ua - \frac{1}{32}u \in W_\emptyset$ is a relation. Since $W_- = W_{\frac{1}{32}}$ will not grow in size, we need only apply part (2) once. Also, as noted after Lemma 4.4, all the useful fusion rules for the Monster fusion law come from the even part. Therefore, for the Monster fusion law, we only need apply part (2) once to the odd part and then just work on the even part.

Step 3. (Optional) If additionally we want to force that G_a fixes every vector in W_{1_T} (as is true for primitive algebras), then we may apply the technique from Lemma 4.5 to get $u^g - u \in W_{1_T - 1}$ for all $g \in G_a$ and $u \in W_{1_T}$.

By the assumptions in Lemma 4.5, we may only apply this lemma to subsets such that $1 \in I$. We claim that it is enough to just apply it to 1_T. By the discussion at the beginning of the section, since $1 \in 1_T$ we need just consider pure subsets $I \subset \mathcal{F}_{1_T}$ with $1 \in I$. Let $u \in W_I \subset W_{1_T}$. So, the vector $v = u^g - u$ is found in both W_{I-1} and W_{1_T-1}. Since the action of $g \in G_a$ preserves the eigenspaces, we know trivially that $v \in W_I$. So, by intersecting as in Step 2 (1), we recover that $v \in W_{I-1} = W_{1_T-1} \cap W_I$. Moreover, once we have done the expansion step, we know the decomposition given by the T-grading and this does not change until the next expansion step. Hence, we need only apply Step 3 once per expansion.

Stage 3: Reduction

If we have found some relations for our algebra (i.e. $W_\emptyset \neq 0$), we may reduce our partial algebra W by factoring out by the relations. Let R be the G-submodule generated by the $W_\emptyset$. Before forming the quotient, we search for additional relations by using the two following techniques.

First, if R intersects V non-trivially, then we may multiply $R \cap V$ by elements of V. Since elements $r \in R$ are relations and must become zero in the target algebra, so are vr, for all $r \in R \cap V$ and $v \in V$. So we repeatedly multiply by elements of V to grow R until the dimension of R stabilises.

Secondly, suppose that R intersects a subspace $U = W(Y)$ where we have glued in an algebra B. Let (φ, ψ) be the gluing map. Then $R' := \varphi(U \cap R)$ are relations in the algebra B. Since we know the multiplication in B, we can use the first technique to multiply by elements of B to grow R' (this may include multiplying by elements we do not yet know how to multiply by in W, hence giving us extra information). We then pull back R' to W using φ^{-1} to get additional relations.

Step 1. We use the above two techniques repeatedly, until we find no further relations. Let $\pi\colon W \to W/R$ be the quotient map. We define W_{new} as the image $\pi(W)$, $V_{\text{new}} = \pi(V)$ and μ_{new} is the map induced by μ.

Step 2. For each gluing, we update both the subspace and the partial subalgebra by taking $U_{\text{new}} = \pi(U)$ and $B_{\text{new}} = B/\pi(U \cap R)$ and updating the gluing maps accordingly.

Step 3. We transfer the axes and eigenspaces W_I to W_{new} by applying π.

Note that if R contains any relations of the form $a - b$ for axes a and b, then we have reduced the (potential) algebra to one generated by a smaller set of axes X'. Hence we may exit the algorithm.

Now that we have described our algorithm, we shall prove Theorem 5.5.

Proof of Theorem 5.5. It is clear from the construction of the algorithm that A is spanned by products of axes in X. Since each axis is contained in its own 1-eigenspace, they are idempotents. At stage 2 we use Lemma 4.2, so each axis must be semisimple. Also at stage 2 we impose the fusion law, therefore the multiplication must satisfy this and hence A is an axial algebra for the required fusion law. By construction, A is a G-module and the multiplication is invariant under the action of G, hence G is a group of automorphisms of A containing the Miyamoto group.

Observe that any axial algebra B with the same axes, automorphism group, τ-map and shape must satisfy the relations we have factored by in our algorithm. If we do not use Lemma 4.5 in stage 2, then we have not factored by any other relations and so B must be a quotient of A. $\square$

In practice, for reasons of efficiency, we perform some of the steps above in a different order. For example, we may perform the reduction step at any stage. In particular, it may be computationally advantageous to reduce once we find enough relations as any further calculations will be performed in a smaller space and hence may be quicker.

6 Results

In Tables 4 and 5, we present some of the results that the implementation of our algorithm [12] in MAGMA [2] has found. Our current implementation is restricted to a $\mathbb{Z}_2$-graded fusion law with one eigenvalue in the negative part and the examples given in the table are all for the Monster fusion law. All the results here are also over $\mathbb{Q}$, although our implementation works over finite fields and even function fields. Note that, although in our algorithm and implementation we do not require that the τ-map be bijective, this is the case we concentrate on in Table 4 as this is the situation considered by Seress [18, Table 3]. In Table 4, for a number of groups G_0, we list all possible shapes for the Miyamoto group G_0, action on axes and τ-map, whether completed or not, whereas in Table 5 we list some additional interesting algebras constructed without claiming we have considered all possible shapes. Note that $(S_a \times S_b)^+$ denotes the group of even elements in $S_a \times S_b$.

The columns in the tables are

- Miyamoto group G_0.
- Axes, where we give the size decomposed into the sum of orbit lengths.
- Shape. Here we omit shapes of type 5A and 6A as where these occur they are uniquely defined. If an algebra contains a 4A, or 4B, we omit to mention the 2B, or 2A, respectively, that is contained in it. Likewise, we omit the 2A and 3A that are contained in a 6A.
- Dimension of the algebra. A question mark indicates that our algorithm did not complete and a 0 indicates that the algebra collapses (that is, there is no such non-trivial algebra with the required axes, Miyamoto group, τ-map and shape).
- The minimal m for which A is m-closed. Recall that an axial algebra is m-closed if it is spanned by products of length at most m in the axes.
- Whether the algebra has a G_0-invariant Frobenius form that is non-zero on the set of axes X. If it is additionally positive definite or positive semi-definite, we mark this with a pos, or semi, respectively.

In addition to the results in the tables, we have computed many of the smaller groups acting on larger numbers of axes. For example, we have computed S_4 acting on 6, $6+6$, $6+6+6$, 12, $12+12$, $12+12+12$, $1+3+6$, $1+3+6+6$, $1+3+3+6+6$, $3+6$, $3+3+6$ and $3+6+6$ axes, but we do not present these results here. Several of these are useful for gluing in to complete examples for larger groups $G_0 \geq S_4$.

Compared to Seress [18] and Pfeiffer and Whybrow [15], we find many new algebras [4]. Importantly, we use only the fusion law and the 2-generated

[4] In the tables of both Seress [18], and Pfeiffer and Whybrow [15], some algebras appear twice. Namely, the algebras of dimension 70 and 76 arise for both A_6 and $3{}^{\cdot}A_6$ and similarly

algebras to build our algebras. Both Seress, and Pfeiffer and Whybrow, use additional axioms or extra properties. These may be assuming a positive definite Frobenius form or some extra conditions such as the so-called M8 axiom or 2Aa, 2Ab, 3A, 4A, or 5A conditions. It is perhaps the strongest argument in favour of the axial algebra approach that we do not use any of the additional conditions and yet the algebras still complete. Indeed, among our completed examples we have many algebras which satisfy these conditions but also some which do not.

Even though we do not assume it, all the examples have a G_0-invariant Frobenius form that is non-zero on the axes. This supports our conjecture given in the introduction. Moreover, in all the examples, the form is positive semi-definite. Although the vast majority are in fact positive definite, in Table 5 we give four algebras which are not. For example, there is an algebra of dimension 14 for the group 2^2 acting on $1+2+2$ axes. The radical of the form is 3-dimensional and it is an ideal in the algebra. Once we factor out by this ideal, the resulting 11-dimensional algebra has the same group, orbit structure of axes and shape and it now has a positive definite Frobenius form.

We find many new examples of m-closed algebras with $m \geq 3$. Indeed, the largest m for which we have examples is 5. In Table 5, we list three such 5-closed algebras, one of dimension 22 for the group 2^4 acting on $2+2+2+2$ axes and two of dimensions 27 and 52 for the group S_4 acting on $1+3+6$ axes. We also give three 4-closed algebras of dimensions 14, 19 and 27 for the group 2^4 acting on $2+2+2+2$ axes and a further twenty six 3-closed algebras.

In all the cases in Table 4, there is a unique admissible τ-map up to symmetry, however this is not true in general. For example, the group 2^4 acting on $2+2+2+2$ axes has four different admissible τ-maps at least three of which lead to non-trivial axial algebras. Some of these are given in Table 5.

All the examples found so far are primitive (although for some cases the optional step 3 in stage 2 using Lemma 4.5 was used to construct them).

One limitation of all available algorithms, including ours, and a possible reason why some cases cannot be completed is that there may be several different universal algebras of the same shape. Indeed, Whybrow has found an infinite family of examples for the group 2^2 acting on $2+2+2$ axes, of shape $(4\mathrm{A})^3$ [19]. It can be seen to be a 12-dimensional algebra over $\mathbb{Q}(t)$, where different specialisations of t give non-isomorphic 12-dimensional algebras. The first author together with Simon Peacock used our algorithm and code run over a function field, together with some additional code, to inves-

the algebra of dimension 196 for A_7 and $3^{\cdot}A_7$. This is because the central element of order 3 acts trivially on the algebra. Another consequence of this is that the algebras of dimension 105 for $3^{\cdot}A_6$ and of dimension 254 for $3^{\cdot}A_7$ in fact have Miyamoto groups A_6 and A_7 and this is how they appear in our Table 4.

tigate varieties of axial algebras for other shapes [13]. They reconstructed Whybrow's example and found a second case for 2^2 acting on $2+2+2$ axes of shape $(4A)^2\,(2A)^2$ where a similar parameter t could be introduced but the result in this case was quite different. Namely, the algebra collapses if $t \neq \pm\frac{1}{128}$, is 10-dimensional if $t = \frac{1}{128}$ and does not currently complete if $t = -\frac{1}{128}$.

Finally, we note that the function which takes the most time in our implementation is the construction of G-submodules in large dimensional modules. If this could be speeded up, then many larger examples could be completed.

G_0	axes	shape	dim	m	form
$S_3 \times S_3$	3+3	3A3A2A	?		
$S_3 \times S_3$	3+3	3A3A2B	8	2	pos
$S_3 \times S_3$	3+3	3A3C2A	0	0	-
$S_3 \times S_3$	3+3	3A3C2B	7	2	pos
$S_3 \times S_3$	3+3	3C3C2A	0	0	-
$S_3 \times S_3$	3+3	3C3C2B	6	1	pos
$S_3 \times S_3$	3+9	3A3A	18	2	pos
$S_3 \times S_3$	3+9	3A3C	0	0	-
$S_3 \times S_3$	3+9	3C3A	0	0	-
$S_3 \times S_3$	3+9	3C3C	0	0	-
$S_3 \times S_3$	3+3+9	3A2A	18	2	pos
$S_3 \times S_3$	3+3+9	3A2B	25	3	pos
$S_3 \times S_3$	3+3+9	3C2A	0	0	-
$S_3 \times S_3$	3+3+9	3C2B	0	0	-
S_4	6	3A2A	13	2	pos
S_4	6	3A2B	13	3	pos
S_4	6	3C2A	9	2	pos
S_4	6	3C2B	6	1	pos
S_4	3+6	4A3A2A	23	3	pos
S_4	3+6	4A3A2B	25	3	pos
S_4	3+6	4A3C2A	0	0	-
S_4	3+6	4A3C2B	12	2	pos
S_4	3+6	4B3A2A	13	2	pos
S_4	3+6	4B3A2B	16	2	pos
S_4	3+6	4B3C2A	9	1	pos
S_4	3+6	4B3C2B	12	2	pos
A_5	15	3A2A	26	2	pos
A_5	15	3A2B	46	3	pos
A_5	15	3C2A	20	2	pos
A_5	15	3C2B	21	2	pos

S_5	10	3A2A	?		
S_5	10	3A2B	?		
S_5	10	3C2A	0	0	-
S_5	10	3C2B	10	1	pos
S_5	10+15	4A	61	2	pos
S_5	10+15	4B	36	2	pos
$L_3(2)$	21	4A3A	?		
$L_3(2)$	21	4A3C	57	3	pos
$L_3(2)$	21	4B3A	49	2	pos
$L_3(2)$	21	4B3C	21	1	pos
A_6	45	4A3A3A	?		
A_6	45	4A3A3C	0	0	-
A_6	45	4A3C3C	187	3	pos
A_6	45	4B3A3A	76	2	pos
A_6	45	4B3A3C	105	2	pos
A_6	45	4B3C3C	70	2	pos
S_6	15	3A2A	?		
S_6	15	3A2B	?		
S_6	15	3C2A	0	0	-
S_6	15	3C2B	15	1	pos
S_6	15+15	4A3A3A2A	?		
S_6	15+15	4A3A3A2B	?		
S_6	15+15	4A3A3C2A	0	0	-
S_6	15+15	4A3A3C2B	0	0	-
S_6	15+15	4A3C3C2A	0	0	-
S_6	15+15	4A3C3C2B	0	0	-
S_6	15+15	4B3A3A2A	0	0	-
S_6	15+15	4B3A3A2B	?		
S_6	15+15	4B3A3C2A	0	0	-
S_6	15+15	4B3A3C2B	0	0	-
S_6	15+15	4B3C3C2A	0	0	-
S_6	15+15	4B3C3C2B	0	0	-
S_6	15+45	4A4A3A2A	151	2	pos
S_6	15+45	4A4A3A2B	0	0	-
S_6	15+45	4A4A3C2A	0	0	-
S_6	15+45	4A4A3C2B	0	0	-
S_6	15+45	4B4B3A2A	0	0	-
S_6	15+45	4B4B3A2B	91	2	pos
S_6	15+45	4B4B3C2A	0	0	-
S_6	15+45	4B4B3C2B	0	0	-

S_6	15+15+45	4A2A2A2A	151	2	pos
S_6	15+15+45	4A2A2A2B	0	0	-
S_6	15+15+45	4A2A2B2B	0	0	-
S_6	15+15+45	4A2B2A2A	151	2	pos
S_6	15+15+45	4A2B2A2B	0	0	-
S_6	15+15+45	4A2B2B2B	0	0	-
S_6	15+15+45	4B2A2A2A	0	0	-
S_6	15+15+45	4B2A2A2B	0	0	-
S_6	15+15+45	4B2A2B2B	0	0	-
S_6	15+15+45	4B2B2A2A	0	0	-
S_6	15+15+45	4B2B2A2B	0	0	-
S_6	15+15+45	4B2B2B2B	106	2	pos
$3\dot{}S_6$	45	3A	?		
$3\dot{}S_6$	45	3C	0	0	-
$3\dot{}S_6$	45+45	3A2A	0	0	-
$3\dot{}S_6$	45+45	3A2B	0	0	-
$3\dot{}S_6$	45+45	3C2A	0	0	-
$3\dot{}S_6$	45+45	3C2B	136	2	pos
$(S_4 \times S_3)^+$	18	3A3A3A	?		
$(S_4 \times S_3)^+$	18	3A3A3C	0	0	-
$(S_4 \times S_3)^+$	18	3A3C3C	?		
$(S_4 \times S_3)^+$	18	3C3C3C	?		
$(S_4 \times S_3)^+$	18+3	3A3A3A2A	?		
$(S_4 \times S_3)^+$	18+3	3A3A3A2B	?		
$(S_4 \times S_3)^+$	18+3	3A3A3C2A	0	0	-
$(S_4 \times S_3)^+$	18+3	3A3A3C2B	0	0	-
$(S_4 \times S_3)^+$	18+3	3A3C3C2A	?		
$(S_4 \times S_3)^+$	18+3	3A3C3C2B	?		
$(S_4 \times S_3)^+$	18+3	3C3C3C2A	24	2	pos
$(S_4 \times S_3)^+$	18+3	3C3C3C2B	27	2	pos
A_7	105	3A	196	2	pos
A_7	105	3C	211	2	pos
S_7	21	3A2A	?		
S_7	21	3A2B	?		
S_7	21	3C2A	0	0	-
S_7	21	3C2B	21	1	pos
S_7	21+105	3A2A	0	0	-
S_7	21+105	3A2B	217	2	pos
S_7	21+105	3C2A	0	0	-
S_7	21+105	3C2B	0	0	-

$L_2(11)$	55	6A5A5A	101	2	pos
$L_3(3)$	117	3A	0	0	-
$L_3(3)$	117	3C	144	2	pos
$(S_5 \times S_3)^+$	30	3A3A	?		
$(S_5 \times S_3)^+$	30	3A3C	0	0	-
$(S_5 \times S_3)^+$	30	3C3A	0	0	-
$(S_5 \times S_3)^+$	30	3C3C	0	0	-
$(S_5 \times S_3)^+$	15+30	3A	67	2	pos
$(S_5 \times S_3)^+$	15+30	3C	0	0	-
M_{11}	165	6A4B5A	286	2	pos

Table 4: Results for some groups

G_0	axes	shape	dim	m	form
1	1+1+1	$(2A)^3$	6,9	2,3	pos
2^2	1+2+2	$4A\,(2A)^2$	14	3	semi
2^2	1+2+2	4A 2A 2B	10	3	pos
2^2	2+2+2	$(4A)^2\,(2B)^2$	9	3	pos
2^3	2+2+4	$(4A)^2\,(2B)^3$	13	3	pos
2^3	2+2+4	$4A\,4B\,(2A)^2\,2B$	15	3	pos
2^3	2+4+4	$(4A)^2\,(4B)^2\,(2A)^2$	16	2	semi
2^4	2+2+2+2	$(4A)^2\,(2A)^2\,(2B)^2$	22	5	semi
2^4	2+2+2+2	$(4A)^3\,(2B)^3$	14	4	pos
2^4	2+2+2+2	4A4B 2A2B2A2B	27	4	semi
2^4	2+2+2+2	$4A4B\,2A\,(2B)^3$	19	4	pos
S_3	1+1+1+3	$3A\,(2A)^4\,(2B)^2$	19	3	pos
S_3	1+1+3	3A2A2A2B	13	3	pos
S_3	1+3+3	2A2B	13	3	pos
S_4	1+3+6	4A3A 2A2A2A	36	3	pos
S_4	1+3+6	4A3A 2A2A2B	36	3	pos
S_4	1+3+6	4A3A 2A2B2A	36	3	pos
S_4	1+3+6	4A3A 2A2B2B	52	5	pos
S_4	1+3+6	4A3A 2B2B2A	24	3	pos
S_4	1+3+6	4A3A 2B2B2B	26	3	pos

S_4	1+3+6	4A3C 2A2B2B	27	5	pos
S_4	3+3+6	$(4A)^2$ 3A $(2A)^3$ 2B	36	3	pos
S_4	3+6+6	$(4A)^2$ 2A2B2A	36	3	pos
$PSL(2,7)$	21+21	$(4A)^3$ $(2A)^2$	102	3	pos
$2^2 : S_4$	12+12	$(4A)^3$ 4B 3C	60	3	pos
$2^2 : S_4$	12+12	4A $(4B)^3$ 3A	59	3	pos
$2^2 : S_4$	12+12+12	4A $(4B)^4$ $(2B)^3$	72	3	pos

Table 5: Additional interesting algebras

References

[1] R.E. Borcherds, Vertex algebras, Kac-Moody algebras, and the monster, *Proc. Natl. Acad. Sci. USA* **83** (1986), 3068–3071.

[2] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language, *J. Symbolic Comput.* **24** (1997), 235–265.

[3] J.H. Conway, A simple construction for the Fischer-Griess monster group, *Invent. Math.* **79** (1985), no. 3, 513–540.

[4] I. Frenkel, J. Lepowsky and A Meurman, *Vertex operator algebras and the Monster*, Academic Press, Boston, MA, Pure and Applied Mathematics **134** (1988).

[5] R.L. Griess, The friendly giant, *Invent. Math.* **69** (1982), 1–102.

[6] J.I. Hall, F. Rehren and S. Shpectorov, Universal axial algebras and a theorem of Sakuma, *J. Algebra* **421** (2015), 394–424.

[7] J.I. Hall, F. Rehren and S. Shpectorov, Primitive axial algebras of Jordan type, *J. Algebra* **437** (2015), 79–115.

[8] J.I. Hall, Y. Segev and S. Shpectorov, On primitive axial algebras of Jordan type, *Bull. Inst. Math. Acad. Sin.* (*N.S.*) **13** (2018), no. 4, 397–409.

[9] A.A. Ivanov, *The Monster Group and Majorana Involutions*, Cambridge Univ. Press, Cambridge, Cambridge Tracts in Mathematics **176** (2009).

[10] A.A. Ivanov, D. V. Pasechnik, Á Seress and S. Shpectorov, Majorana representations of the symmetric group of degree 4, *J. Algebra* **324** (2010), no. 9, 2432–2463.

[11] S.M.S. Khasraw, J. M^{c}Inroy and S. Shpectorov, On the structure of axial algebras, *Trans. Amer. Math. Soc.* (2019), https://doi.org/10.1090/tran/7979.

[12] J. M^{c}Inroy and S. Shpectorov, Partial axial algebras – a magma package, https://github.com/JustMaths/AxialAlgebras.

[13] J. M^{c}Inroy and S. Peacock, Varieties of axial algebras, *in preparation.*

[14] M. Miyamoto, Griess algebras and conformal vectors in vertex operator algebras, *J. Algebra* **179** (1996), no. 2, 523–548.

[15] M. Pfeiffer and M. Whybrow, Constructing Majorana Representations, *arXiv*:1803.10723, 19 pages, Mar 2018.

[16] F. Rehren, Generalised dihedral subalgebras from the Monster, *Trans. Amer. Math. Soc.* **369** (2017), no. 10, 6953–6986.

[17] S. Sakuma, 6-transposition property of τ-involutions of vertex operator algebras, *Int. Math. Res. Not. IMRN*, no. 9 (2007).

[18] Á. Seress, Construction of 2-closed M-representations, *ISSAC* 2012–*Proceedings of the* 37*th International Symposium on Symbolic and Algebraic Computation*, ACM, New York, 2012, 311–318.

[19] M. Whybrow, An infinite family of axial algebras, *arXiv*:1810.00590, 31 pages, Oct 2018.