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ABSTRACT. A subgraph of an edge-coloured graph is called rainbow if all its edges have distinct colours. Our main result implies that, given any optimal colouring of a sufficiently large complete graph K_{2n}, there exists a decomposition of K_{2n} into isomorphic rainbow spanning trees. This settles conjectures of Brualdi–Hollingsworth (from 1996) and Constantine (from 2002) for large graphs.

1. INTRODUCTION

Given an edge-coloured graph G, we say a subgraph H is *rainbow* if all the edges of H have distinct colours. Moreover, we say that $H_1, \ldots, H_t$ *decompose* G if $H_1, \ldots, H_t$ are edge-disjoint subgraphs of G covering all the edges of G.

In this paper, we address the problem of decomposing an optimally edge-coloured complete graph K_{2n} into (isomorphic) rainbow spanning trees. The study of rainbow decomposition problems can be traced back to the work of Euler, who investigated for which n one can find a pair of orthogonal Latin squares of order n. That is, equivalently, for which n does there exist an optimally edge-coloured $K_{n,n}$ which can be decomposed into rainbow perfect matchings? Euler gave a construction for all $n \not\equiv 2 \mod 4$ and conjectured that these are the only admissible values. His conjecture was disproved by Parker, Bose and Shrikhande who provided constructions for the missing values, except for $n = 6$ (which corresponds to Euler's famous '36 officers problem', for which the non-existence had already been shown by Tarry in 1901).

On the other hand, given an *arbitrary* optimally edge-coloured $K_{n,n}$, a decomposition into rainbow perfect matchings need not exist. In fact, there are examples of such colourings that do not admit a single rainbow perfect matching. (An important conjecture widely attributed to Ryser–Brualdi–Stein postulates that there always exists a rainbow matching of size $n - 1$.)

Perfect matchings are, in some sense, very rigid objects, and it is natural to ask analogous questions for other types of subgraphs. In particular, several natural conjectures arose concerning decompositions into rainbow spanning trees. Here, the most notable are the Brualdi–Hollingsworth conjecture, Constantine's conjecture and the Kaneko–Kano–Suzuki conjecture. Our main result implies the first two of these.

1.1. Decompositions into rainbow spanning trees.
Note that if K_{2n} is optimally edge-coloured, then the colour classes form a *1-factorization*, that is, a decomposition of K_{2n} into perfect matchings. We will here use the term 1-factorization synonymously with an edge-colouring whose colour classes form a 1-factorization. Note that if a 1-factorization of K_n exists, then n is even. We now state the Brualdi–Hollingsworth conjecture.

Conjecture 1 (Brualdi and Hollingsworth, [6]). *For all $n > 4$ and any 1-factorization of K_n, there exists a decomposition of K_n into rainbow spanning trees.*

Note that the condition that $n > 4$ is necessary. Suppose a 1-factorization of K_n is given. Clearly, there always exists *one* rainbow spanning tree, for instance the star at any vertex. Brualdi and Hollingsworth in their original paper [6] showed that one can find two edge-disjoint rainbow spanning trees. Shortly afterwards, Krussel, Marshall and Verrall [17] were able to find three such trees. Later, Horn [13] significantly improved on this by finding $\Omega(n)$ edge-disjoint rainbow spanning trees. Very recently, Montgomery, Pokrovskiy, and Sudakov [21] proved Conjecture 1 approximately by showing that one can guarantee $(1 - o(1))n/2$ edge-disjoint rainbow spanning trees.

Several related problems have been studied, in two main directions. Firstly, we may wish to strengthen the conditions on the trees in the decomposition, most commonly by requiring the trees in the decomposition to be isomorphic. Secondly, we may wish to weaken the conditions on the colouring, most commonly by allowing non-optimal proper edge-colourings. (Naturally, those two directions can also be combined.)

A decomposition of an edge-coloured K_n into isomorphic rainbow spanning trees is also known in the literature as a *multicoloured tree parallelism (MTP)*. It turns out that the problem of finding an MTP is non-trivial even if one is allowed to choose the 1-factorization. Partial results were obtained by Constantine [9]. Akbari, Alipour, Fu and Lo [2] then proved that for all $n \in \mathbb{N}$ with $n > 2$, there exists a 1-factorization of K_{2n} which admits an MTP. Moreover, Constantine conjectured that in fact an MTP should exist for any given 1-factorization, thus generalizing the Brualdi–Hollingsworth conjecture.

Conjecture 2 (Constantine, [9, 10])**.** *For all $n > 4$, any 1-factorization of K_n admits a decomposition into isomorphic rainbow spanning trees.*

Unsurprisingly, for this conjecture, much less was known than for the Brualdi–Hollingsworth conjecture. In [12] it was shown that three isomorphic rainbow spanning trees can be guaranteed. In [23], Pokrovskiy and Sudakov showed that one can find $10^{-6}n$ edge-disjoint rainbow spanning trees all isomorphic to a so-called *t-spider* (which is even independent of the given 1-factorization). Montgomery, Pokrovskiy, and Sudakov [21], and independently, Kim, Kühn, Kupavskii and Osthus [16], proved a weak asymptotic version of the conjecture by showing that there are $(1 - o(1))n/2$ edge-disjoint rainbow paths each of length $(1 - o(1))n$.

We now discuss results on more general colourings. Intuitively it might seem that dealing with an optimal colouring is the hardest case, as having more colours should make finding rainbow subgraphs easier. However, non-optimal colourings seem genuinely harder to deal with than 1-factorizations. Kaneko, Kano and Suzuki proved that for any proper colouring of K_n, there exist three edge-disjoint rainbow spanning trees, and also generalized the Brualdi–Hollingsworth conjecture as follows.

Conjecture 3 (Kaneko, Kano, and Suzuki, 2002)**.** *For all $n > 4$, every properly edge-coloured K_n contains $\lfloor n/2 \rfloor$ edge-disjoint rainbow spanning trees.*

Note that any proper colouring is $n/2$-*bounded*, that is, every colour appears on at most $n/2$ edges. Under the weaker assumption that the colouring is $n/2$-bounded, Akbari and Alipour [1] showed that one can guarantee two edge-disjoint rainbow spanning trees, and this was significantly improved by Carraher, Hartke, and Horn [8] who showed that $\Omega(n/\log n)$ such trees exist. For proper colourings, a linear number of rainbow spanning trees was independently obtained by Pokrovskiy and Sudakov [23] and by Balogh, Liu and Montgomery [5], where in the former work, the trees are even isomorphic. Finally, the aforementioned result from [21] on Conjecture 1 also applies to proper colourings, thus proving Conjecture 3 approximately.

We now state our main theorem, which implies the Brualdi–Hollingsworth conjecture and Constantine's conjecture for large n. This is the first general exact rainbow decomposition result for spanning subgraphs, where each subgraph in the decomposition has to use all the colours.

Theorem 4. *For all sufficiently large n, there exists a tree T on n vertices such that for any 1-factorization of K_n, there exists a decomposition into rainbow subgraphs each isomorphic to T.*

Note that whereas Constantine's conjecture says that given a 1-factorization, one can decompose into isomorphic rainbow spanning trees, we actually show that one can use the same tree T for *any* 1-factorization. This tree is made up of a path of length $n - o(n)$, with $o(n)$ short paths attached to it (see Definition 5). By modifying our proof slightly, we can even ensure that $\Delta(T) \leq 3$. This is best possible in the sense that there exist 1-factorizations which do not admit a single rainbow Hamilton path [18].

Our argument relies upon the fact that the colouring is a 1-factorization. It would be very interesting to prove the result for more general colourings, in particular proper colourings.

It would also be interesting to investigate the $n/2$-bounded setting further. The best known bound is the one from [8] mentioned earlier, which provides $\Omega(n/\log n)$ edge-disjoint rainbow spanning trees. A natural question is to ask for the maximum number k of such trees that can be guaranteed. It seems unlikely that a decomposition can be obtained, but it would be interesting to see whether $k = \Omega(n)$ is possible or not. It is also natural to impose further local conditions on the colouring, e.g. that the colouring is *locally Δ-bounded*, which means that the maximum degree of each colour class is at most Δ. For instance, in [16] it is shown that for any $n/2$-bounded colouring which is locally $o(n)$-bounded, there exists an approximate decomposition into almost spanning rainbow cycles (and thus into almost spanning paths).

1.2. **Related problems.** We now discuss some related results concerning rainbow decompositions. Let us first revisit the perfect matching case. As mentioned earlier, there exist proper optimal colourings of $K_{n,n}$ which do not contain a rainbow perfect matching. However, by imposing slightly stronger boundedness conditions on the colouring, one can obtain strong results. For example, Alon, Spencer and Tetali [3] showed that if n is a power of 2 and the edge-colouring is $o(n)$-bounded (and not necessarily proper), there exists a decomposition into rainbow perfect matchings. Montgomery, Pokrovskiy and Sudakov [21] showed that any proper edge-colouring of $K_{n,n}$, where at most $(1-o(1))n$ colours appear more than $(1-o(1))n$ times, contains $(1-o(1))n$ edge-disjoint rainbow perfect matchings. This implied a conjecture of Akbari and Alipour in a strong form (which was proved independently by Keevash and Yepremyan [15]) and a conjecture of Barat and Nagy approximately, both for large n. Kim, Kühn, Kupavskii and Osthus [16] proved that for any $(1 - o(1))n$-bounded and locally $o(n/\log^2 n)$-bounded edge-colouring of $K_{n,n}$, there exist $(1 - o(1))n$ edge-disjoint rainbow perfect matchings. The authors of both [16] and [21] also obtain analogous results (in their respective settings) on approximate decompositions of K_n into rainbow Hamilton cycles. Furthermore, [16] contains results for approximate decompositions of K_n into rainbow F-factors (for any given graph F).

A further tantalizing problem concerning rainbow tree decompositions is the following special case of Rota's basis conjecture. Let $T_1, \ldots, T_{n-1}$ be spanning trees on a common vertex set of size n, each monochromatic in a different colour. Then their union (allowing multiple edges) can be decomposed into $n - 1$ rainbow spanning trees. The general version of Rota's conjecture concerns the rearrangement of bases of a matroid into disjoint transversal bases. Recently, Bucić, Kwan, Pokrovskiy and Sudakov [7] showed that $(1/2 - o(1))n$ disjoint transversal bases can be found.

2. Notation

Given a graph G with edge colouring $\phi \colon E(G) \to C$, we say a subgraph H is *D-rainbow* if H is rainbow and $\phi(E(H)) = D$. We refer to an edge $e = uv$ with colour c as a *c-edge*, and v is a *c-neighbour* of u. For each colour c, $E_c(G)$ is the set of c-edges in G. For each vertex v of G, we let $\partial_G(v)$ denote the set of all edges of G incident to v. For any $S \subseteq V(G)$, $N_G(S)$ is the common neighbourhood in G of the vertices in S. For any $x \in V(G)$ and $U \subseteq V(G)$, $d_G(x, U)$ is the number of neighbours of x in U. We denote by $G - H$ the graph obtained from G by deleting the edges of H.

For a hypergraph $\mathcal{H}$, let $\Delta^c(\mathcal{H})$ denote its maximum codegree, that is, the maximum number of edges containing any two fixed vertices.

Given a set X and $p \in [0, 1]$, a *p-random subset* is a random subset $Y \subseteq X$ which is obtained by including each element of X independently with probability p. If not otherwise stated, we always assume that such random subsets are independent. For instance, if we say that Y is a p-random subset of X and Y' is a p'-random subset of Y, we implicitly assume that these random choices are made independently. Similarly, if G is a graph, then a *p-random subgraph* is the random graph with vertex set $V(G)$ and a p-random subset of $E(G)$ as edge set.

On the other hand, we often split a random subset further into disjoint subsets. For instance, if Y is a $(p+p')$-random subset of X, we might say that we split Y into a p-random set Y_1 and a p'-random set Y_2, by which we mean that for each $y \in Y$ independently, we include y in Y_1 with probability $p/(p + p')$ and into Y_2 otherwise. Note that then Y_1 is indeed a p-random subset of X and Y_2 is a p'-random subset of X, but they are obviously not independent. To split into more sets, we use the following notation: By splitting X randomly as

$$
\begin{array}{ccccccc}
X & = & X_1 & \uplus & \ldots & \uplus & X_m \\
1 & = & p_1 & + & \ldots & + & p_m
\end{array}
$$

we mean that for every element in X independently, we choose an index $i \in [m]$ according to the probability distribution $(p_i)_{i=1}^m$, and put this element into the corresponding set X_i.

We say that a random event holds *with high probability* if the probability that it holds tends to 1 as n tends to infinity (where n is usually the number of vertices and the event depends on n).

We write $[n] := \{1, \ldots, n\}$. For $a, b, c \in \mathbb{R}$, we write $a = b \pm c$ whenever $a \in [b - c, b + c]$. For $a, b, c \in (0, 1]$, we write $a \ll b \ll c$ in our statements to mean that there are increasing functions $f, g : (0, 1] \to (0, 1]$ such that whenever $a \leq f(b)$ and $b \leq g(c)$, then the subsequent result holds.

3. Proof sketch

Our proof is based on hypergraph matching results and new absorption techniques. Suppose we are given a 1-factorization ϕ of the complete graph K_n with colour set C. We build the $t := n/2$ rainbow trees simultaneously, beginning with our absorbing structures and then gradually extending these structures to cover all the vertices and edges. For this, we further develop a recent 'distributive' form of the absorption method: we form an absorption structure along with a reservoir, such that, given any subset (of given size) from the reservoir we can distribute the elements of this subset among the different parts of the absorbing structure to always obtain a copy of the same tree. More precisely, we create a 'global' reservoir of edges, as well as 'local' reservoirs of colours and vertices (as explained below, 'local' refers to the fact that there is one such reservoir for each tree, while the 'global' reservoir is common to all trees). The structure of these absorbers and the corresponding reservoirs is described in more detail in Section 3.1.

Already, however, we can outline our proof strategy, as follows.

(1) Create an edge absorption structure and a global edge reservoir.
(2) For each tree, create a colour absorption structure and a colour reservoir.
(3) For each tree, create a vertex absorption structure and a vertex reservoir.
(4) Find $t = n/2$ edge-disjoint almost spanning rainbow paths P_i covering most of the remaining vertices.
(5) Link up the absorbers and paths to form t rainbow forests F_i and thereby cover all non-reservoir vertices.
(6) Cover non-reservoir edges by adding each such edge to one of the forests F_i.
(7) Incorporate non-reservoir colours for each forest, by adding a suitable edge from the edge reservoir.
(8) Absorb the uncovered reservoir vertices into each forest, using edges and colours from the reservoirs.

(9) Absorb the uncovered reservoir colours into each forest, using the colour absorption structure.

(10) Absorb the uncovered edges from the global edge reservoir by distributing them among the forests to complete these forests into rainbow spanning trees T_i.

To find all of the structures we use, we apply results on matchings in certain auxiliary hypergraphs, as described in Section 3.2. This allows the structures we find to look random-like, which in turn means that at each stage of the construction of the trees T_i, the currently unused sets are also random-like. In particular, this means that the leftover sets which need to be absorbed are sufficiently small and well-distributed (again, the sets we track here are vertices and edges as well as colour sets).

The main difficulty in our proof lies in obtaining a decomposition into spanning trees. The property that these trees are isomorphic (even to some T fixed in advance) can be achieved with only a little extra care. We comment more on this in Section 3.3. In Section 4, we list the tools that we use in our proof.

The above strategy is implemented in Section 5.6, following the proof of several lemmas allowing some of these tasks. In Section 5.1, we find a set of almost spanning rainbow paths. In Section 5.2, we find our colour absorption structure. In Section 5.3, we find our edge absorption structure. In Section 5.4, we show how we will connect these structures together. In Section 5.5, we find suitable rainbow matchings which we will use to absorb vertices.

3.1. **Designing absorbers.** The absorbing method has its roots in work by Erdős, Gyárfás, and Pyber, as well as Krivelevich, before its general codification by Rödl, Ruciński and Szemerédi. The key novelty in our work is to construct a 'nested' absorbing structure for the edges, colours and vertices. As the edges of a tree define its colours and vertices, we start by building an edge absorption structure and an accompanying edge reservoir (i.e. the edges in the reservoir are those which can later be absorbed).

Edge absorbers via monochromatic matchings. We create an edge absorption structure for a set of reservoir edges as follows, where η is a small constant. (Recall that $t = n/2$ is the number of trees in our decomposition.) For each $i \in [t]$, we construct a rainbow forest $\tilde{F}_i$ (where we will have $\tilde{F}_i \subseteq T_i$) and matchings $M_{i,c}$. Each $M_{i,c}$ will consist of edges of colour c, and c ranges over all elements of some colour set D_i', where $|D_i'| \sim 6\eta n$. The matchings may overlap but are edge-disjoint from $\tilde{F}_i$, and, for each matching $M_{i,c}$, any one of its edges can be added to $\tilde{F}_i$ to obtain a rainbow tree. More precisely, we have the following 'local' edge absorption property for each $i \in [t]$:

(P) If one edge $e_{i,c}$ is chosen from each matching $M_{i,c}$, then $F_i^+ := \tilde{F}_i + \sum_{c \in D_i'} e_{i,c}$ is a rainbow tree with vertex set $V(\tilde{F}_i)$.

Note that since the $M_{i,c}$ are monochromatic, the colour set of F_i^+ does not depend on the choice of $e_{i,c}$. See Figure 1 for our construction of such a subgraph $\tilde{F}_i$ and the matchings $M_{i,c}$. We think of $M_{i,c}$ as being (the essential part of) an absorber which is able to 'absorb' exactly one of the edges it contains. The chosen edge $e_{i,c}$ is then added to $\tilde{F}_i$ to become part of the tree T_i.

Since the $M_{i,c}$ will be small (of size 256) and monochromatic, the requirement that exactly one edge from each $M_{i,c}$ is to be added to $\tilde{F}_i$ is very restrictive. However, by carefully choosing how edges appear in different matchings $M_{i,c}$, we can combine these to create the following 'global' edge absorption property for two suitable subgraphs G_1 and G_2 of K_n. (Here G_1, G_2 and the forests $\tilde{F}_i$, $i \in [t]$, will be edge-disjoint.)

(Q) For any subset $E^* \subseteq E(G_1)$ which consists of precisely ηn edges of each colour $c \in C$, we can label $E^* \cup E(G_2)$ as $\{e_{i,c} : i \in [t], c \in D_i'\}$ so that $e_{i,c} \in M_{i,c}$ for each $i \in [t]$ and $c \in D_i'$.

Properties (P) and (Q) mean that, given any set $E^* \subseteq E(G_1)$ with the right number of edges of each colour, we can absorb these edges (along with those in the 'buffer set' $E(G_2)$) into
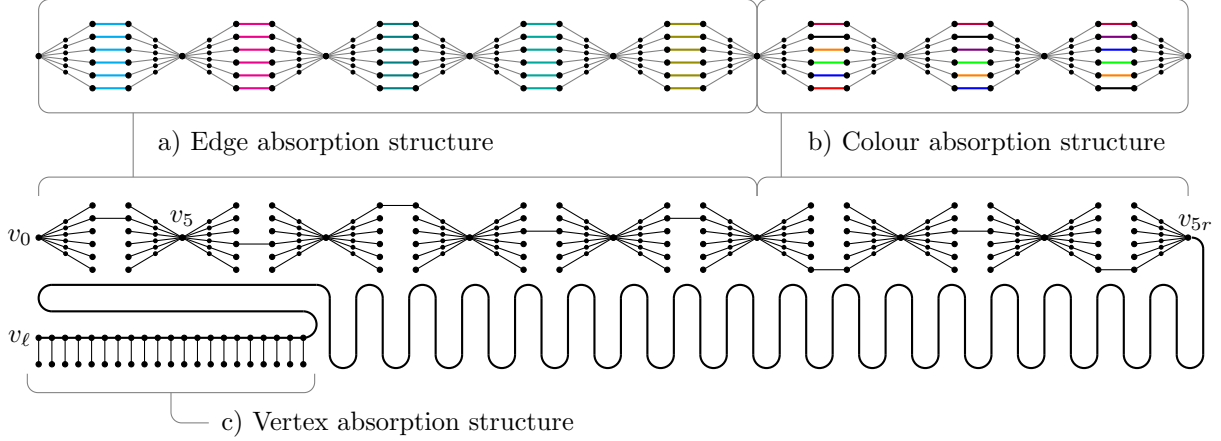
FIGURE 1. The tree $T_{n;r,b}$ which we use for our decomposition (see Definition 5), with the edge absorption structure and colour absorption structure highlighted as a) and b) above, with the vertex absorption structure marked with c). Note that, whichever our choice of one edge from each of the monochromatic matchings in a) and one edge from each of the rainbow matchings in b), the resulting tree is the same.

the forests $\tilde{F}_1, \ldots, \tilde{F}_t$ to obtain rainbow trees which span some pre-determined vertex set and colour set (these sets are different for each tree). In fact, the equidistribution condition on E^* will be naturally satisfied as the resulting trees must contain exactly one edge of each colour. Thus altogether, the local edge absorption structures give rise to a *global edge reservoir* (namely $E(G_1)$), for which we can absorb a leftover edge set E^* into the existing forests.

To choose the matchings $M_{i,c} \subseteq E(G_1 \cup G_2)$, we consider a set of auxiliary graphs (called 'robustly matchable bipartite graphs'), introduced in [20] and already a standard technique in the construction of absorbers. As the name suggests, these graphs have the property that one can find a perfect matching even after the removal of an arbitrary set of vertices (of given size) from the larger vertex class, B say. We will consider one such robustly matchable bipartite graph $H_c = H_c[A, B]$ for each colour c, where $B = E_c(G_1 \cup G_2)$. The neighbourhood in H_c of each vertex $a \in A$ will correspond to some matching $M_{i,c}$, where i is such that $c \in D'_i$. Thus adjacencies in H_c encode the possible absorber matchings $M_{i,c}$ (and thus the possible trees) that a reservoir edge $e \in B$ can be assigned to. A matching in H_c saturating $E_c^* \cup E_c(G_2)$ (where E_c^* is the set of c-edges in E^*) gives an assignment of these 'leftover' edges of colour c to absorbers and thus to the trees T_i. Carrying this out for all $c \in C$ allows us to absorb all the leftover edges E^* from the edge reservoir G_1 and the buffer edges (i.e. those in G_2).

The robustly matchable graphs are discussed in more detail in Section 4.2 and the properties of the edge absorption structure are described in Lemma 19.

Colour absorbers via rainbow matchings. The above properties allow us to use part of the edge reservoir G_1 to create separate *colour absorbers* for each tree. This means that for the ith tree we have a reservoir $C'_{i,1}$ of colours with the property that any 'leftover' (i.e. so far unused) set of colours $C^* \subseteq C'_{i,1}$ of given size can be absorbed into the ith rainbow forest so that the result is still a (larger) rainbow forest.

More precisely, for the ith tree, we find a rainbow forest $\tilde{F}'_i$ which is vertex- and colour-disjoint from $\tilde{F}_i$, along with small rainbow matchings $M'_{i,1}, \ldots, M'_{i,3s}$ which are edge-disjoint from $\tilde{F}'_i$, as well as colour sets $C'_{i,1}$ (the 'colour reservoir') and $C'_{i,2}$ (the 'buffer set'), such that the following 'local colour absorption' property holds for each $i \in [t]$ (where $s = \eta n/768$, and $|C'_{i,1}| = |C'_{i,2}| = 2s$).

(P') Given any set $C^* \subseteq C'_{i,1}$ of s colours, we can choose one edge $f_{i,j}$ from each $M'_{i,j}$ so that $\tilde{F}'_i + f_{i,1} + \ldots + f_{i,3s}$ is a $(\phi(E(\tilde{F}'_i)) \cup C^* \cup C'_{i,2})$-rainbow tree with vertex set $V(\tilde{F}'_i)$.

For each colour c appearing on an edge in $M'_{i,j}$, we think of $M'_{i,j}$ as (part of) an absorber which can absorb colour c into the ith tree (and for each c, we will provide several of these absorbers). The edges of the $M'_{i,j}$ will lie in the edge reservoir G_1. Crucially, this means that when absorbing a colour c, it does not matter which edges/absorbers are actually involved in this colour absorption step – we can absorb any unused ones later. This means that the colour absorption step is less delicate than the edge absorption step. See Figure 1 for our construction of such an $\tilde{F}'_i$. (In the main proof, we actually construct the forests $\tilde{F}_i$ and $\tilde{F}'_i$ simultaneously, and denote them $\tilde{F}_i$.)

The matchings $M'_{i,j}$ will be small edge-disjoint rainbow matchings, where the colours of each matching $M'_{i,j}$ are chosen according to some auxiliary robustly matchable bipartite graph. We will consider one such auxiliary graph H_i for each tree T_i, with the larger vertex class consisting of the colour reservoir $C'_{i,1}$ together with the buffer set $C'_{i,2}$. The edges of H_i connect each colour c to some indices $j \in [3s]$. The colour set of $M'_{i,j}$ will consist of precisely those colours in $N_{H_i}(j)$. For any set $C^* \subseteq C'_{i,1}$ of size s, a matching saturating $C^* \cup C'_{i,2}$ absorbs all the 'leftover' colours, as required. The details are given in Lemma 18.

Vertex absorbers. We then use part of both the edge reservoir and the colour reservoir to create *vertex absorbers*. This construction is relatively simple, and the resulting vertex reservoir consists of some vertices unused by the ith tree so far. For each $i \in [t]$, we take a small random set A_i of vertices and connect them into a rainbow *vertex absorbing path*, while reserving a further random set of vertices B_i that is slightly smaller than A_i. When we reach Step (8), the set of uncovered vertices will be a subset of B_i and contain almost all vertices of B_i. (So one can view B_i as a vertex reservoir.) We will match those vertices in B_i which are still uncovered onto the vertex absorbing path. The randomness of A_i and B_i allows us to do this with a rainbow matching between A_i and B_i.

Covering outside the reservoirs. By construction, the edge and colour absorbing structures can only deal with edges/colours within the respective reservoirs. Thus, after we construct the ith forest F_i which covers almost all the colours, we must extend it slightly so that it now uses every colour outside its reservoir, and that collectively the resulting forests use all the edges outside of the global edge reservoir. We achieve this as follows: To cover an edge e outside the global edge reservoir (in Step (6)), we include e as an edge between A_i and B_i for some suitable i. Similarly, to cover a colour c outside the ith colour reservoir (in Step (7)), we choose a suitable c-edge e between A_i and B_i, again from the edge reservoir. We can carry this out in such a way that these edges form a relatively small A_iB_i-matching, thus enabling us to carry out the vertex absorption procedure described above with only minor modifications.

3.2. **Almost-packing random subgraphs.** We will find the different structures for the strategy outlined above by defining (for each of these structures) an auxiliary hypergraph in which a large matching corresponds to the desired structure. The hypergraph will be roughly regular, with small codegrees, and thus the existence of this matching will follow from standard results (see Theorem 7 in Section 4.1). In each case, the hypergraph is defined in a similar way, but to give a concrete example we will sketch how to find $t = n/2$ almost-spanning rainbow paths in any optimally coloured K_n. (Note that this construction as described below is already present in [16]. We repeat it informally here, as it forms a template for several more involved applications in this paper.)

To simplify further, we note that by randomly reserving edges, colours and vertices, we can greedily connect a given set of long disjoint rainbow paths together via very short paths (which use their own set of reserved edges, colours and vertices) into a single rainbow path. Thus, to cut to the main part of the argument, let us suppose we want to find the following, where ℓ is a large constant, and $r\ell \leq (1 - \varepsilon)n$, for some small $\varepsilon > 0$.

Aim: To find in K_n, for each $i \in [t]$, a set $\mathcal{F}_i$ of r vertex-disjoint colour-disjoint rainbow cycles of length ℓ, so that all the cycles in $\bigcup_{i \in [t]} \mathcal{F}_i$ are edge-disjoint.

The key is to construct a hypergraph $\mathcal{H}$ in which a large matching corresponds to the required cycles (where each matching edge directs us to include some cycle into some set $\mathcal{F}_i$). We take vertices for $\mathcal{H}$ as follows. We need all the cycles we find to be edge-disjoint, so each edge in G will appear as a vertex of $\mathcal{H}$. All the cycles in $\mathcal{F}_i$ must be vertex-disjoint, so we wish to represent the vertices of $V = V(K_n)$ as vertices in $\mathcal{H}$. However, different cycles in different sets $\mathcal{F}_i$ are permitted to share vertices. Thus, for each $i \in [t]$, we include a copy of V by including the vertices in $\{i\} \times V$ in $V(\mathcal{H})$. Similarly, we represent the colours for cycles by including $\{i\} \times C$ for each $i \in [t]$. We define the hyperedges of $\mathcal{H}$ as follows. For each rainbow cycle $F \subseteq K_n$ of length ℓ and $i \in [t]$, we include the hyperedge

$$f_{i,F} := E(F) \cup (\{i\} \times V(F)) \cup (\{i\} \times \phi(E(F))).$$

Suppose then we had a matching $\mathcal{M}$ in $\mathcal{H}$. Then, for each $i \in [t]$, let $\mathcal{F}_i$ be the set of cycles F with $f_{i,F} \in \mathcal{M}$. Note that we have the following properties.

- If $F, L \in \mathcal{F}_i$ are distinct, we have the following.
  - As $\{i\} \times V(F) \subseteq f_{i,F}$, $\{i\} \times V(L) \subseteq f_{i,L}$ and $f_{i,F}, f_{i,L} \in \mathcal{M}$, we have that F and L are vertex-disjoint.
  - As $\{i\} \times \phi(E(F)) \subseteq f_{i,F}$, $\{i\} \times \phi(E(L)) \subseteq f_{i,L}$ and $f_{i,F}, f_{i,L} \in \mathcal{M}$, we have that F and L are colour-disjoint.
- For any $F \in \mathcal{F}_i$ and $L \in \mathcal{F}_j$ with $i \neq j$, we have $E(F) \subseteq f_{i,F}$, $E(L) \subseteq f_{j,L}$, and $f_{i,F}, f_{j,L} \in \mathcal{M}$, so F and L are edge-disjoint.

That is, each $\mathcal{F}_i$ is a set of vertex- and colour-disjoint rainbow cycles, and the cycles in $\bigcup_{i \in [t]} \mathcal{F}_i$ are edge-disjoint, as required in the above aim.

In the actual proof we will find the required structures within prescribed (randomly chosen) vertex, edge and colour sets, with parameters carefully chosen so that the construction uses almost all of the available sets each time. Together, this has the advantage that the overall leftover after removing these structures is also randomly distributed and sufficiently small so that it can be absorbed.

3.3. **Isomorphic trees.** The main achievement of our techniques is to find a decomposition into (any) spanning rainbow trees. However, by taking care at several points in our argument, the trees we construct can be kept isomorphic. The key point here is to observe that in Figure 1 the resulting structure from the absorber is the same regardless of which edges are used from the reservoir.

In fact, we not only find isomorphic trees, but we find copies of the same fixed tree, regardless of the 1-factorization of K_n. We define this tree as follows (see Figure 1).

Definition 5. Given $n, r, b \in \mathbb{N}$ such that $\ell := n - 1020r - b - 1 > r + b$, we define the tree $T_{n;r,b}$ as follows: Take a path $v_0 \ldots v_\ell$ of length ℓ. For all $k \in [r-1]$, add 510 paths of length 2 to v_{5k} (i.e. v_{5k} will become an endvertex of these 510 paths), and add 255 paths of length 2 to each of v_0 and v_{5r}. Moreover, take a set B of b new vertices and add a perfect matching between B and $\{v_{\ell-b+1}, \ldots, v_\ell\}$.

The set B corresponds to the set B_i in the vertex absorption structure. For each $i \in [t]$ there will be an integer $r_i \leq r$ so that for each $k \in [r_i]$, the 'middle' edge on the path between $v_{5(k-1)}$ and v_{5k} will be an edge of some 'absorber-matching' $M_{i,c}$ or $M'_{i,j}$. Note that $|T_{n;r,b}| = n$ and $\Delta(T_{n;r,b}) \leq 512$. We will prove Theorem 4 with $T = T_{n;r,b}$, where r and b are small but linear in n. So T contains an almost spanning path. After proving Theorem 4 in Section 5.6, we describe how this construction can be slightly modified to achieve that $\Delta(T) = 3$.

4. Tools

4.1. **Hypergraph matchings.** We make frequent use of the existence of large matchings in almost regular hypergraphs with small codegrees (such matchings are constructed via semi-random nibble methods pioneered by Rödl [25]). Moreover, we wish to have a matching which is

'well-distributed' across a number of vertex subsets. To make this precise, we use the following definition.

Definition 6. Given a hypergraph $\mathcal{H}$ and a collection $\mathcal{F}$ of subsets of $V(\mathcal{H})$, we say a matching $\mathcal{M}$ in $\mathcal{H}$ is $(\gamma, \mathcal{F})$-*perfect* if for each $F \in \mathcal{F}$, at most $\gamma \cdot \max\{|F|, |V(\mathcal{H})|^{2/5}\}$ vertices of F are left uncovered by $\mathcal{M}$.

Pippenger and Spencer [22] showed that in an almost regular hypergraph $\mathcal{H}$ with small co-degrees there are many large edge-disjoint matchings. Alon and Yuster [4] observed that by randomly splitting $V(\mathcal{H})$ into many parts, and applying the Pippenger–Spencer theorem to each induced subhypergraph and then selecting a matching in each of these subhypergraphs at random, one can obtain an almost perfect matching $\mathcal{M}$ of $\mathcal{H}$ that is 'well-distributed' in the sense of Definition 6. We will use the following consequence of Theorem 1.2 in [4].

Theorem 7 ([4]). *Suppose $1/n \ll \varepsilon \ll \gamma, 1/r$. Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices such that for some $D \in \mathbb{N}$, we have $d_{\mathcal{H}}(x) = (1 \pm \varepsilon)D$ for all $x \in V(\mathcal{H})$, and $\Delta^c(\mathcal{H}) \leq D/\log^{9r} n$. Suppose that $\mathcal{F}$ is a collection of subsets of $V(\mathcal{H})$ such that $|\mathcal{F}| \leq n^{\log n}$. Then there exists a $(\gamma, \mathcal{F})$-perfect matching in $\mathcal{H}$.*

We apply Theorem 7 to several different hypergraphs in our proof, each time checking the appropriate degree and codegree bounds. We comment here generally why our hypergraphs are almost regular with small codegree. Indeed, roughly speaking, in each hypergraph $\mathcal{H}$ we define (see Section 3.2), estimating vertex degrees will correspond to counting the number of rainbow copies of a certain graph in K_n with one fixed characteristic (e.g. one fixed vertex/edge/colour). The symmetry in our choice of random subsets and subgraphs will mean that for each characteristic, the vertex degrees in $\mathcal{H}$ are roughly the same. Our choice of edge, colour and vertex probabilities then results in an almost regular hypergraph. (Here, it is also useful that we consider 1-factorizations rather than proper colourings.) Counting codegrees corresponds roughly to counting the number of copies of the same subgraph but with two characteristics fixed. This means that the codegrees are small in comparison to the degrees, giving the additional condition we need to apply Theorem 7.

4.2. **Robustly matchable bipartite graphs.** As noted in Section 3.1, we use robustly matchable bipartite graphs as auxiliary graphs to tell us how to distribute edges during the absorbing steps. These graphs are defined as follows.

Definition 8. Given pairwise disjoint vertex sets X, Y, Z, an $\mathrm{RMBG}(X, Y, Z)$ is a bipartite graph H with bipartition $(X, Y \cup Z)$ and the following crucial property: for any set $Y' \subseteq Y$ with $|Y'| = |X| - |Z|$, the subgraph $H[X, Y' \cup Z]$ has a perfect matching.
   We also refer to H as an $\mathrm{RMBG}(|X|, |Y|, |Z|)$ with parts X, Y, Z.

Such graphs were introduced in [20] in order to find given spanning trees in random graphs.

Lemma 9 ([20, Lemma 10.7]). *For all sufficiently large m, there exists an $\mathrm{RMBG}(3m, 2m, 2m)$ with maximum degree at most 100.*

We say that a bipartite graph H with bipartition (X, Y) is (ℓ, r)-*regular* if all the vertices in X have degree ℓ and all the vertices in Y have degree r. Using the Max-Flow-Min-Cut-theorem, it is straightforward to find a supergraph of an RMBG from Lemma 9 which is appropriately regular.

Corollary 10. *For all fixed $d \geq 59$ and sufficiently large m, there exists a $(4d, 3d)$-regular $\mathrm{RMBG}(3m, 2m, 2m)$.*

Proof. Let H be an $\mathrm{RMBG}(3m, 2m, 2m)$ with parts X, Y and Z and maximum degree at most 100, as in Lemma 9. Take new vertices s, t and let G be the directed graph obtained from the complete bipartite graph between X and $Y \cup Z$ (with all edges directed towards $Y \cup Z$) by removing the edges of H and adding all edges from s to X and from $Y \cup Z$ to t. An

edge sx receives capacity $4d - d_H(x)$, and an edge yt receives capacity $3d - d_H(y)$. All edges in $G[X, Y \cup Z]$ receive capacity 1. We claim that $(\{s\}, V(G) \setminus \{s\})$ and $(\{t\}, V(G) \setminus \{t\})$ are minimal (s, t)-cuts. Indeed, first note that the capacity of these cuts is $12dm - e(H)$. Now, let (S, T) be any (s, t)-cut. Let $S_1 := S \cap X$, $S_2 := S \cap (Y \cup Z)$, $T_1 := T \cap X$ and $T_2 := T \cap (Y \cup Z)$. The capacity $c(S, T)$ of the cut (S, T) satisfies

$$
\begin{aligned}
(1) \qquad c(S, T) &= \sum_{x \in T_1} (4d - d_H(x)) + |S_1||T_2| - e_H(S_1, T_2) + \sum_{y \in S_2} (3d - d_H(y)) \\
&= 12dm + |S_1|(|T_2| - 4d) + 3d|S_2| - e(H) - e_H(T_1, S_2) \\
&= 12dm + 4d|T_1| + |T_2|(|S_1| - 3d) - e(H) - e_H(T_1, S_2).
\end{aligned}
$$

Thus, if $|S_1| \geq 3d$ or $|T_2| \geq 4d$, then $c(S, T) \geq 12dm - e(H)$, as desired. Therefore, assume that $|S_1| < 3d$ and $|T_2| < 4d$. Then, (1) implies that $c(S, T) \geq (3m - 3d)(4d - 100) + (4m - 4d)(3d - 100) \geq 12md$, where the last inequality uses $d \geq 59$. This proves the claim.

By the Max-Flow-Min-Cut-theorem, there exists an (s, t)-flow in G with value $12dm - e(H)$. This yields a subgraph $H' \subseteq G \setminus \{s, t\}$ such that $d_{H'}(x) = 4d - d_H(x)$ for all $x \in X$ and $d_{H'}(y) = 3d - d_H(y)$ for all $y \in Y \cup Z$. Thus, $H \cup H'$ is the desired $(4d, 3d)$-regular RMBG$(3m, 2m, 2m)$. $\qquad \square$

4.3. **Probabilistic tools.** In order to show various properties of random subgraphs and subsets, we will use common concentration inequalities, as follows.

Lemma 11 (see [14, Corollary 2.3, Corollary 2.4 and Theorem 2.8])**.** *Let X be the sum of n independent Bernoulli random variables. Then the following hold.*

   (i) *For all $0 \leq \varepsilon \leq 3/2$, $\mathbb{P}(|X - \mathbb{E}(X)| \geq \varepsilon \mathbb{E}(X)) \leq 2\mathrm{e}^{-\varepsilon^2 \mathbb{E}(X)/3}$.*
   (ii) *If $t \geq 7\mathbb{E}(X)$, then $\mathbb{P}(X \geq t) \leq \mathrm{e}^{-t}$.*

Throughout, we will refer to (i) as 'Chernoff's bound'. Often, we will use this in conjunction with an implicit union bound to show that several properties hold altogether with high probability.

Fact 12 (cf. [24, Lemma 8])**.** *Let $X_1, \ldots, X_n$ be Bernoulli random variables such that for all $i \in [n]$, we have $\mathbb{P}(X_i = 1 \mid X_1, \ldots, X_{i-1}) \leq p$. Let $B \sim Bin(n, p)$ and $X := \sum_{i=1}^n X_i$. Then $\mathbb{P}(X \geq t) \leq \mathbb{P}(B \geq t)$ for all $t \geq 0$.*

One important tool to prove concentration of our random variables is McDiarmid's inequality. Let $X_1, \ldots, X_m$ be independent random variables taking values in $\mathcal{X}$. Let $f \colon \mathcal{X}^m \to \mathbb{R}$ be a function of $X_1, \ldots, X_m$ such that

$$
|f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m) - f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_m)| \leq c_i
$$

for all $i \in [m]$, $x_i', x_1, \ldots, x_m \in \mathcal{X}$. If this holds, we say that X_i *affects* f by at most c_i.

Theorem 13 (McDiarmid's inequality, see [19, Lemma 1.2])**.** *Let $X_1, \ldots, X_m$, $c_1, \ldots, c_m$ and f be as stated above. Then, for all $t > 0$,*

$$
\mathbb{P}(|f(X_1, \ldots, X_m) - \mathbb{E}(f(X_1, \ldots, X_m))| \geq t) \leq 2\mathrm{e}^{-\frac{2t^2}{\sum_{i=1}^m c_i^2}}.
$$

For our purposes, we will have $\mathcal{X} = \{0, 1\}$, and the X_i will be indicator variables of certain events. We will often use different indicator variables (which are not necessarily independent) to compute $\mathbb{E}(f)$, and then use McDiarmid's inequality to prove concentration.

5. Proof

5.1. **Approximate decomposition.** The main result in this Section is Lemma 16, which implies the existence of an approximate decomposition into rainbow almost spanning paths for any given 1-factorization of K_n. As noted earlier, this result was already proved in [16, 21].

However, we need to strengthen the result somewhat – in particular, we need to constrain the paths to use given (randomly chosen) vertex and colour sets, and we need the paths to be well-behaved towards given subsets of these sets (the latter is encapsulated in the concept of 'boundedness' defined below).

The proof of Lemma 16 relies on ideas from Theorem 1.5 and Lemma 2.14 in [16] (and simplifies some aspects of that argument). As described in Section 4.1, the strategy is to first find for each $i \in [t]$ an almost spanning collection of vertex-disjoint long rainbow cycles. Then we delete an edge from each such cycle and connect them into a long rainbow path via Lemma 15.

Definition 14 (m-bounded). Let ϕ be a 1-factorization of the complete graph K_n with vertex set V and colour set C. Given a subgraph $G \subseteq K_n$, vertex sets $\{V_i\}_{i \in [t]} \subseteq V$ and colour sets $\{C_i\}_{i \in [t]} \subseteq C$, we say that $(G, \{V_i\}_{i \in [t]}, \{C_i\}_{i \in [t]})$ is m-*bounded* if the following hold:

  (B1) for all $i \in [t]$, $|V_i|, |C_i| \le m$;
  (B2) for all $v \in V$, we have $|\{i \in [t] : v \in V_i\}| \le m$ and $d_G(v) \le m$;
  (B3) for all $c \in C$, $|\{i \in [t] : c \in C_i\}| \le m$ and $|E_c(G)| \le m$.

Here, we think of $(G, \{V_i\}_{i \in [t]}, \{C_i\}_{i \in [t]})$ as being 'leftovers' that we want to be 'well-behaved' in the above sense.

The following lemma allows us to embed rooted graphs in a rainbow fashion. We will often apply it to find the desired rainbow subgraphs that were missed by an application of Theorem 7.

Lemma 15. *Suppose $1/n \ll \gamma \ll 1/\Delta$ and $t \le n$. Let G be a graph on n vertices and $\phi \colon E(G) \to C$ a proper edge colouring. Moreover, let $V_1, \ldots, V_t \subseteq V(G)$ and $C_1, \ldots, C_t \subseteq C$ be such that for each $i \in [t]$ and any set $S \subseteq V(G)$ with $|S| \le \Delta$, we have that*

$$(2) \qquad |\{v \in N_G(S) \cap V_i : \phi(uv) \in C_i \text{ for each } u \in S\}| \ge \gamma^{1/3} n.$$

For each $i \in [t]$, let H_i be a graph with $|V(H_i)|, |E(H_i)| \le \gamma n$ and $\Delta(H_i) \le \Delta$, and let $\Lambda_i \colon X_i \to V(G)$ be an injection, where $X_i \subseteq V(H_i)$ is independent. Assume that for all $u \in V(G)$, there are at most γn indices $i \in [t]$ for which $u \in \operatorname{Im}(\Lambda_i)$.

Then, there exist embeddings $\psi_i \colon H_i \to G$, $i \in [t]$, such that, for each $i \in [t]$, $\psi_i(H_i)$ is rainbow with colours in C_i, $\psi_i(x) = \Lambda_i(x)$ for all $x \in X_i$ and $\psi_i(x) \in V_i$ for all $x \in V(H_i) \setminus X_i$, and such that $\psi_1(H_1), \ldots, \psi_t(H_t)$ are edge-disjoint.

Proof. We find the embeddings $\psi_1, \ldots, \psi_t$ successively and greedily. For $s \in [t]$ and a vertex $u \in V(G)$, let $r(u, s)$ be the number of indices $i \in [s]$ for which $u \in \operatorname{Im}(\Lambda_i)$. By assumption, $r(u, s) \le \gamma n$.

Suppose that, for some $s \in [t]$, we have already found suitable embeddings $\psi_1, \ldots, \psi_{s-1}$ such that, additionally,

$$(3) \qquad \text{the degree of each } u \in V \text{ in } \bigcup_{i \in [s-1]} \psi_i(H_i) \text{ is at most } 2\sqrt{\gamma} n + r(u, s-1)\Delta.$$

Now, we find a suitable embedding ψ_s such that (3) holds with s replaced by $s+1$. Let B be the set of all vertices whose degree in $\bigcup_{i \in [s-1]} \psi_i(H_i)$ is larger than $\sqrt{\gamma} n$. Since $\sum_{i \in [s-1]} |E(H_i)| \le \gamma n^2$, observe that $|B| \le 2\sqrt{\gamma} n$.

We can now greedily embed H_s while avoiding the vertices in $B \setminus \Lambda_s(X_s)$. For all $x \in X_s$, define $\psi_s(x) = \Lambda_s(x)$. Order the remaining vertices of H_s arbitrarily and embed them one by one into $V(G) \setminus B$ as follows. When we consider $x \in V(H_s) \setminus X_s$, let S be the set of images of the neighbours of x which have already been embedded. We would like to choose an image for x from $Y := \{v \in N_G(S) \cap V_s : \phi(uv) \in C_s \text{ for each } u \in S\} \setminus B$.

First, note that by (2) with $i = s$, and as $|B| \le 2\sqrt{\gamma}n$, we have $|Y| \ge \gamma^{1/3}n - 2\sqrt{\gamma}n$. At most γn vertices of Y are blocked because they have already been chosen as an image for H_s, and at most $|S|\gamma n \le \Delta\gamma n$ vertices $v \in Y$ are blocked because $\{\phi(uv) : u \in S\}$ contains a colour that has already been used. Moreover, invoking (3), at most $\Delta(2\sqrt{\gamma} + \Delta\gamma)n$ vertices v are blocked because $uv \in E(\bigcup_{i \in [s-1]} \psi_i(H_i))$ for some $u \in S$. Hence, there exists a suitable image for x. Thus, we can finish the embedding of H_s in this way.

Clearly, $\psi_s(H_s)$ is rainbow with colours in C_s, and edge-disjoint from $\psi_1(H_1), \dots, \psi_{s-1}(H_{s-1})$. Moreover, for any vertex $u \in V(G)$, the degree of u in $\psi_s(H_s)$ is at most Δ. In particular, if $u \notin B$, then the degree of u in $\bigcup_{i \in [s]} \psi_i(H_i)$ is at most $\sqrt{\gamma}n + \Delta \le 2\sqrt{\gamma}n$. Moreover, if $u \in B$, then $u \in V(\psi_s(H_s))$ if and only if $u \in \mathrm{Im}(\Lambda_s)$, in which case the degree of u in $\psi_s(H_s)$ is at most $\Delta = (r(u, s) - r(u, s-1))\Delta$. Thus, (3) holds with s replaced by $s+1$, which completes the proof.   □

Lemma 16. *Suppose* $1/n \ll \gamma, \kappa \ll p$ *and let* $q := \beta := p$ *and* $t := n/2$. *Let* ϕ *be a 1-factorization of the complete graph* K_n *with vertex set* V *and colour set* C. *For every* $i \in [t]$, *let* V_i *be a* p-*random subset of* V, *and let* C_i *be a* q-*random subset of* C. *Moreover, let* G *be a* β-*random subgraph of* K_n.

Then with high probability, there exist edge-disjoint rainbow paths $P_1, \dots, P_t$ *in* G *such that*

(P1) $V(P_i) \subseteq V_i$, $\phi(E(P_i)) \subseteq C_i$;
(P2) $(G - \bigcup_{i \in [t]} P_i, \{V_i \setminus V(P_i)\}_{i \in [t]}, \{C_i \setminus \phi(E(P_i))\}_{i \in [t]})$ *is* $5\gamma n$-*bounded;*
(P3) *for all* $v \in V$, *the number of* $i \in [t]$ *for which* $v \in V(P_i)$ *and the subpath from* v *to one of the endvertices of* P_i *has length at most* κn, *is at most* $21\kappa p^{-1}n$.

Proof. Choose new constants $\varepsilon > 0$ and $\ell \in \mathbb{N}$ such that $1/n \ll \varepsilon \ll 1/\ell \ll \gamma, \kappa \ll p$.

For each $i \in [t]$, split V_i into a $p(1-\gamma)$-random subset $V_{i,1}$ and a $p\gamma$-random subset $V_{i,2}$, and split C_i into a $q(1-\gamma)$-random subset $C_{i,1}$ and a $q\gamma$-random subset $C_{i,2}$. We also split G into a $\beta(1-\gamma)$-random subgraph G_1 and a $\beta\gamma$-random subgraph G_2. We claim that with high probability, we have the following:

(a) for each $i \in [t]$ and $S \subseteq V$ with $|S| \le 2$,

$$|\{v \in N_{G_1}(S) \cap V_{i,1} : \phi(uv) \in C_{i,1} \text{ for each } u \in S\}| = (1 \pm \varepsilon)(\beta q(1-\gamma)^2)^{|S|}p(1-\gamma)n;$$

(b) for each $i \in [t]$ and $S \subseteq V$ with $|S| \le 2$,

$$|\{v \in N_{G_2}(S) \cap V_{i,2} : \phi(uv) \in C_{i,2} \text{ for each } u \in S\}| \ge \gamma^6 n;$$

(c) for each $e \in E(K_n)$, $|\{i \in [t] : e \subseteq V_{i,1}, \phi(e) \in C_{i,1}\}| = (1 \pm \varepsilon)p^2 q(1-\gamma)^3 n/2$;
(d) for each $i \in [t]$ and $c \in C$, we have that $|E_c(G_1[V_{i,1}])| = (1 \pm \varepsilon)p^2\beta(1-\gamma)^3 n/2$;
(e) for each $i \in [t]$, $|V_i| = (1 \pm \varepsilon)pn$;
(f) $(G_2, \{V_{i,2}\}_{i \in [t]}, \{C_{i,2}\}_{i \in [t]})$ is $2\gamma n$-bounded.

Indeed, using Chernoff's bound, it is straightforward to check that (c)–(f) hold with high probability. For (a) and (b), we can apply McDiarmid's inequality, since of the at most $|S|n$ edges incident with S, each has an effect of at most 1, each vertex has an effect of at most 1, and each colour has an effect of at most $|S|$.

Henceforth, assume that these random choices have been made and satisfy the above properties.

For all $i \in [t]$, let $\mathcal{R}_i$ be the collection of all rainbow cycles R of length ℓ in G_1 for which $V(R) \subseteq V_{i,1}$ and $\phi(E(R)) \subseteq C_{i,1}$. (Note that the $\mathcal{R}_i$'s are not necessarily disjoint.) For $v \in V$, $c \in C$ and $e \in E(K_n)$, we let $\mathcal{R}_i(v)$, $\mathcal{R}_i(c)$ and $\mathcal{R}_i(e)$ denote the set of all $R \in \mathcal{R}_i$ with $v \in V(R)$,

$c \in \phi(E(R))$ and $e \in E(R)$, respectively. Using (a), we can now count that, for all $v \in V_{i,1}$,

$$|\mathcal{R}_i(v)| = \frac{1}{2} \cdot ((1 \pm 2\varepsilon)(1-\gamma)^3 \beta qpn)^{\ell-2} \cdot (1 \pm 2\varepsilon)(1-\gamma)^5 (\beta q)^2 pn$$

(4)
$$= (1 \pm \sqrt{\varepsilon}) \frac{1}{2} (1-\gamma)^{3\ell-1} \beta^\ell q^\ell p^{\ell-1} n^{\ell-1},$$

and, for all $e \in E(G_1[V_{i,1}])$ with $\phi(e) \in C_{i,1}$,

$$|\mathcal{R}_i(e)| = ((1 \pm 2\varepsilon)(1-\gamma)^3 \beta qpn)^{\ell-3} \cdot (1 \pm 2\varepsilon)(1-\gamma)^5 (\beta q)^2 pn$$

(5)
$$= (1 \pm \sqrt{\varepsilon})(1-\gamma)^{3\ell-4} \beta^{\ell-1} q^{\ell-1} p^{\ell-2} n^{\ell-2}.$$

We define an auxiliary hypergraph $\mathcal{H}$ as follows. The vertex set of $\mathcal{H}$ consists of three parts. The first part is simply $E(G_1)$. The second part is the set $\mathcal{V}$ of all pairs (i,v) with $i \in [t]$ and $v \in V_{i,1}$. The third part is the set $\mathcal{C}$ of all pairs (i,c) with $i \in [t]$ and $c \in C_{i,1}$.

Now, we define the edge set of $\mathcal{H}$. For each $i \in [t]$ and $R \in \mathcal{R}_i$, we add the hyperedge

(6)
$$f_{i,R} := E(R) \cup (\{i\} \times V(R)) \cup (\{i\} \times \phi(E(R))).$$

Hence, $\mathcal{H}$ is 3ℓ-uniform.

Clearly, using (4), we have for each $(i,v) \in \mathcal{V}$ that

(7)
$$d_\mathcal{H}((i,v)) = |\mathcal{R}_i(v)| = (1 \pm \sqrt{\varepsilon}) \frac{1}{2} (1-\gamma)^{3\ell-1} \beta^\ell q^\ell p^{\ell-1} n^{\ell-1}.$$

Moreover, we have for each $e \in E(G_1)$ that

(8)
$$d_\mathcal{H}(e) = \sum_{i \in [t] \,:\, e \subseteq V_{i,1}, \phi(e) \in C_{i,1}} |\mathcal{R}_i(e)| \overset{(c),(5)}{=} (1 \pm 2\sqrt{\varepsilon}) \frac{1}{2} (1-\gamma)^{3\ell-1} \beta^{\ell-1} q^\ell p^\ell n^{\ell-1}$$

and for all $(i,c) \in \mathcal{C}$ that

(9)
$$d_\mathcal{H}((i,c)) = \sum_{e \in E_c(G_1[V_{i,1}])} |\mathcal{R}_i(e)| \overset{(d),(5)}{=} (1 \pm 2\sqrt{\varepsilon}) \frac{1}{2} (1-\gamma)^{3\ell-1} \beta^\ell q^{\ell-1} p^\ell n^{\ell-1}.$$

(Note that no hyperedge is counted more than once since each rainbow cycle contains at most one c-edge.)

Claim 1: $\Delta^c(\mathcal{H}) \leq \ell^4 n^{\ell-2}$.

Proof of claim: Recall, from (6), that each hyperedge of $\mathcal{H}$ is uniquely fixed by some $i \in [t]$ and $R \in \mathcal{R}_i$. Note that for a set S of vertices and $i \in [t]$, the number of $R \in \mathcal{R}_i$ with $S \subseteq V(R)$ is at most $\ell^{|S|} n^{\ell-|S|}$. This easily implies that codegrees of pairs in $E(G_1) \times \mathcal{V}$, $E(G_1) \times \mathcal{C}$ and $\mathcal{V} \times \mathcal{V}$ are at most $\ell^2 n^{\ell-2}$.

Next, consider distinct $e, e' \in E(G_1)$. For each $i \in [t]$, the number of $R \in \mathcal{R}_i$ with $e \cup e' \subseteq V(R)$ is at most $\ell^3 n^{\ell-3}$. Summing over all $i \in [t]$ yields the desired bound.

Now, take $(i,v) \in \mathcal{V}$ and $(i,c) \in \mathcal{C}$. Each $R \in \mathcal{R}_i$ with $\{(i,v),(i,c)\} \subseteq f_{i,R}$ will contain some c-edge e. We distinguish two cases for e. If e is incident to v, there is only one choice for e, and then at most $\ell^2 n^{\ell-2}$ choices left. If e is not incident to v, there are at most $n/2$ choices for e, and then at most $\ell^3 n^{\ell-3}$ choices left. Thus, in total, there are at most $2\ell^3 n^{\ell-2}$ choices.

Similarly, we check that the codegree of (i,c) and (i,c') for distinct $c, c' \in C$ is at most $\ell^4 n^{\ell-2}$. We have to choose a c-edge e and a c'-edge e' and again distinguish two cases. If e and e' share a vertex v, there are at most n choices for v (which determines e and e'), and then at most $\ell^3 n^{\ell-3}$ choices left. If e and e' form a matching, there are at most $(n/2)^2$ ways to choose e and e', and then at most $\ell^4 n^{\ell-4}$ choices left. —

For each $v \in V$, let $\mathcal{V}_v$ be the set of all pairs (i, v) with $i \in [t]$ and $v \in V_{i,1}$. For each colour $c \in C$, let $\mathcal{C}_c$ be the set of all pairs (i, c) with $i \in [t]$ and $c \in C_{i,1}$. Let

$$\mathcal{F} := \{\{i\} \times V_{i,1}, \{i\} \times C_{i,1} \ : \ i \in [t]\} \cup \{\mathcal{V}_v, \partial_{G_1}(v) \ : \ v \in V\} \cup \{\mathcal{C}_c, E_c(G_1) \ : \ c \in C\}.$$

Using (7), (8), (9), and Claim 1, we now apply Theorem 7 to obtain a $(\gamma, \mathcal{F})$-perfect matching $\mathcal{M}$ in $\mathcal{H}$. For $i \in [t]$, let $\mathcal{R}'_i$ be the collection of all $R \in \mathcal{R}_i$ for which $f_{i,R} \in \mathcal{M}$. For distinct $R, R' \in \mathcal{R}_i$, $\{i\} \times V(R) \subseteq f_{i,R}$ and $\{i\} \times V(R') \subseteq f_{i,R'}$, and therefore, as $\mathcal{M}$ is a matching, R and R' are vertex-disjoint. Similarly, R and R' are colour-disjoint. Thus, $\mathcal{R}'_i \subseteq \mathcal{R}_i$ is a collection of vertex-disjoint ℓ-cycles in $G_1[V_{i,1}]$ whose union is rainbow with colours in $C_{i,1}$. Moreover, as for each $i \in [t]$ and $R \in \mathcal{R}_i$, $E(R) \subseteq f_{i,R}$, all these cycles are edge-disjoint.

For each $i \in [t]$, we will now randomly break each cycle in $\mathcal{R}'_i$ into a path, before joining all these paths together into a single cycle. For each $i \in [t]$ and all $R \in \mathcal{R}'_i$, choose an edge $e_{i,R} \in E(R)$ uniformly at random. For each $i \in [t]$, let $X_i := \bigcup_{R \in \mathcal{R}'_i} e_{i,R} \subseteq V_{i,1}$ and $D_i := \{\phi(e_{i,R}) \ : \ R \in \mathcal{R}'_i\} \subseteq C_{i,1}$. We claim that, with high probability, we have

$$(10) \qquad |\{i \in [t] \ : \ v \in X_i\}|, |\{i \in [t] \ : \ c \in D_i\}| \leq 7n/\ell \quad \text{for all } v \in V, c \in C.$$

Indeed, fix a vertex $v \in V$. We have $\mathbb{P}(v \in X_i) \leq 2/\ell$ for all $i \in [t]$, and those events are independent. Similarly, for a fixed colour $c \in C$, we have $\mathbb{P}(c \in D_i) \leq 1/\ell$ for all $i \in [t]$, and those events are independent too. Thus, the claim follows with Lemma 11(ii) and a union bound.

Now, assume that (10) holds. For each $i \in [t]$, let H_i be the graph obtained as follows: Give every edge $\{e_{i,R}\}_{R \in \mathcal{R}'_i}$ an orientation, and (cyclically) enumerate these edges. Now, for each edge, add a path of length 2 between its head and the tail of the next edge, using a new vertex as the internal vertex each time. (So H_i consists of the union of all these $|\mathcal{R}'_i|$ paths of length 2, but does not contain the edges $e_{i,R}$. In particular, X_i is independent in H_i.) Observe that $|V(H_i)|, |E(H_i)| \leq 3|\mathcal{R}'_i| \leq 3n/\ell$.

By (b) and (10), we can apply Lemma 15 (with $G_2, \{V_{i,2}\}_{i \in [t]}, \{C_{i,2}\}_{i \in [t]}$ taking the place of $G, \{V_i\}_{i \in [t]}, \{C_i\}_{i \in [t]}$) to find for each $i \in [t]$ an embedding $\psi_i \colon H_i \to G_2$ such that $\psi_i(H_i)$ is rainbow with colours in $C_{i,2}$ and $\psi_i(x) = x$ for all $x \in X_i$ and $\psi_i(x) \in V_{i,2}$ for all $x \in V(H_i) \setminus X_i$, and such that $\psi_1(H_1), \ldots, \psi_t(H_t)$ are edge-disjoint.

For $i \in [t]$, let

$$\tilde{R}_i := \bigcup_{R \in \mathcal{R}'_i} (R - e_{i,R}) \cup \psi_i(H_i).$$

We have that $\tilde{R}_1, \ldots, \tilde{R}_t$ are edge-disjoint rainbow cycles in G, where $\tilde{R}_i$ is rainbow with colours in C_i and $V(\tilde{R}_i) \subseteq V_i$. Moreover, by the definition of $\mathcal{F}$, (f) and (10) and the fact that $|D_i| = |\mathcal{R}'_i| \leq n/\ell$ for all $i \in [t]$, we have that $(G - \bigcup_{i \in [t]} \tilde{R}_i, \{V_i \setminus V(\tilde{R}_i)\}_{i \in [t]}, \{C_i \setminus \phi(E(\tilde{R}_i))\}_{i \in [t]})$ is $4\gamma n$-bounded.

Finally, choose for each $i \in [t]$ an edge $e_i \in E(\tilde{R}_i)$ uniformly at random and let $P_i := \tilde{R}_i - e_i$. For a vertex $v \in V$, let I_v be the set of indices $i \in [t]$ for which $v \in V(\tilde{R}_i)$ and the subpath from v to one of the endvertices of P_i has length at most κn. Note that, for each $i \in [t]$, as $|V_i \setminus V(\tilde{R}_i)| \leq 4\gamma n$, the cycle $\tilde{R}_i$ has length at least $pn/2$ by (e), implying $\mathbb{P}(i \in I_v) \leq \frac{3\kappa n}{pn/2} = 6\kappa p^{-1}$, and these events are independent. Thus, with Lemma 11(ii), we conclude that (P3) holds with high probability. Similarly, we can deduce that with high probability, for every $v \in V$, the number of $i \in [t]$ for which v is incident with e_i, is at most $\log^2 n$, and for every $c \in C$, the number of $i \in [t]$ for which $\phi(e_i) = c$, is at most $\log^2 n$. This implies that (P2) still holds with high probability. $\square$

5.2. Matchings for colour absorption.
In this subsection, we find the rainbow matchings which form the crucial part of the colour absorption structure (see Lemma 18). The following lemma prepares the ground for this.

Lemma 17. *Suppose $1/n \ll \gamma \ll \eta \ll 1$ and let $p := 2\eta$, $q := \eta/192$ and $t := n/2$. Let ϕ be a 1-factorization of K_n with vertex set V and colour set C. For every $i \in [t]$, let V_i be a $p(1 + \gamma)$-random subset of V, and let C_i be a q-random subset of C. Moreover, let G be an $\eta(1+\gamma)$-random subgraph of K_n. Then, with high probability, there exist edge-disjoint matchings $M_1, \ldots, M_t$ in G such that the following hold:*

    (i) *$V(M_i) \subseteq V_i$ for all $i \in [t]$;*
    (ii) *for all $i \in [t]$, M_i consists of 192 c-edges for each $c \in C_i$;*
    (iii) *for every vertex $v \in V$, the number of $i \in [t]$ for which v is covered by M_i is $(1 \pm 3\gamma)pt$.*

Later on some edges of M_i will be used to construct the ith tree T_i of the decomposition.

Proof. Choose a new constant $\varepsilon > 0$ such that $1/n \ll \varepsilon \ll \gamma \ll \eta \ll 1$. For each $i \in [t]$, we randomly split V_i into a p-random set $V_{i,1}$ and a $p\gamma$-random set $V_{i,2}$. Similarly, we split G into an η-random subgraph G_1 and an $\eta\gamma$-random subgraph G_2. For $c \in C$ and $i \in [t]$, let $Y_{c,i}$ denote the number of c-edges in $G_2[V_{i,2}]$.

We define a (random) auxiliary hypergraph $\mathcal{H}$ as follows. The vertex set of $\mathcal{H}$ consists of three parts: The first part is simply $E(G_1)$. The second part is the set $\mathcal{V}$ of all pairs (i,v) with $i \in [t]$ and $v \in V_{i,1}$. The third part of $V(\mathcal{H})$ is the set $\mathcal{C}$ of all triples (i,c,ℓ) with $i \in [t]$, $c \in C_i$ and $\ell \in [192]$. For all $e = uv \in E(K_n)$, $i \in [t]$ and $\ell \in [192]$, we add the hyperedge

$$(11) \qquad\qquad f_{e,i,\ell} := \{e, (i,u), (i,v), (i,\phi(e),\ell)\}$$

if and only if $e \in E(G_1)$, $u,v \in V_{i,1}$ and $\phi(e) \in C_i$. Thus, $\mathcal{H}$ is a 4-uniform hypergraph.

Claim 1: With high probability, for each $e \in E(G_1)$, $d_{\mathcal{H}}(e) = (1 \pm \varepsilon)192tp^2q$.

Proof of claim: Fix an edge $e = uv$ and assume $e \in E(G_1)$. For $i \in [t]$, let X_i be the indicator variable of the event that $u,v \in V_{i,1}$ and $\phi(e) \in C_i$. Note that $d_{\mathcal{H}}(e) = 192\sum_{i\in[t]} X_i$. Since $\mathbb{P}(X_i = 1) = p^2q$ for each i and the X_i's are independent, we can easily deduce from Chernoff's bound that the claim holds. $-$

Claim 2: With high probability, for each $(i,v) \in \mathcal{V}$, $d_{\mathcal{H}}((i,v)) = (1 \pm \varepsilon)192npq\eta$.

Proof of claim: Fix $i \in [t]$ and $v \in V$ and assume $(i,v) \in \mathcal{V}$. For every vertex $u \neq v$, let X_u be the indicator variable of the event that $u \in V_{i,1}$, $uv \in E(G_1)$ and $\phi(uv) \in C_i$. Note that $d_{\mathcal{H}}((i,v)) = 192\sum_{u\in V\setminus\{v\}} X_u$. Since $\mathbb{P}(X_u = 1) = p\eta q$ for each u and the X_u's are independent, we can easily deduce from Chernoff's bound that the claim holds. $-$

Claim 3: With high probability, for each $(i,c,\ell) \in \mathcal{C}$, $d_{\mathcal{H}}((i,c,\ell)) = (1 \pm \varepsilon)p^2\eta n/2$, and, for each $c \in C$ and $i \in [t]$, $Y_{c,i} = (1 \pm \varepsilon)(p\gamma)^2(\eta\gamma)n/2$.

Proof of claim: Fix $i \in [t]$, $c \in C$ and $\ell \in [192]$ and assume that $(i,c,\ell) \in \mathcal{C}$. For every c-edge $e \in E(K_n)$, let X_e be the indicator variable of the event that $e \in E(G_1)$ and $e \subseteq V_{i,1}$. Note that $d_{\mathcal{H}}((i,c,\ell)) = \sum_{e\in E_c(K_n)} X_e$. Since $\mathbb{P}(X_e = 1) = p^2\eta$ for each $e \in E(K_n)$ and the X_e's are independent, we can easily deduce from Chernoff's bound that the claim holds. (A similar argument works for $Y_{c,i}$.) $-$

Claim 4: With high probability, we have $|\{i \in [t] : v \in V_{i,1}\}| = (1 \pm \varepsilon)pt$ and $|\{i \in [t] : v \in V_{i,2}\}| = (1 \pm \varepsilon)p\gamma t$ for all $v \in V$.

Proof of claim: This is an easy consequence of Chernoff's bound. $-$

Claim 5: $\Delta^c(\mathcal{H}) \leq 192$.

Proof of claim: Clearly, the codegree of pairs in $E(G_1) \times E(G_1)$ and $\mathcal{C} \times \mathcal{C}$ is 0. Moreover, the codegree of any pair in $E(G_1) \times \mathcal{C}$ and $\mathcal{V} \times \mathcal{C}$ is at most 1, and the codegree of any pair in $E(G_1) \times \mathcal{V}$ is at most 192. Finally, consider a pair in $\mathcal{V} \times \mathcal{V}$, say (i,u) and (i',v). If $i \neq i'$, then the codegree is 0, so assume $i = i'$. If $uv \notin E(G_1)$, then the codegree is also zero, so assume otherwise and let c be the colour of uv. Then the codegree is at most 192. $-$

We now assume that the properties stated in Claims 1–5 are satisfied. By our choice of p, q, η, we have that $d_{\mathcal{H}}(x) = (1 \pm \varepsilon)192pq\eta\eta n$ for all $x \in V(\mathcal{H})$. For every vertex $v \in V$, let $\mathcal{V}_v$ be the set of all pairs $(i, v) \in \mathcal{V}$ with $i \in [t]$ and $v \in V_{i,1}$. For every $c \in C$, let $\mathcal{C}_c$ be the set of all (i, c, ℓ) with $c \in C_i$ and $\ell \in [192]$. Let

$$\mathcal{F} := \{\mathcal{V}_v \ : \ v \in V\} \cup \{\{i\} \times C_i \times [192] \ : \ i \in [t]\} \cup \{\mathcal{C}_c \ : \ c \in C\}$$

Thus, we can apply Theorem 7 to find a $(\gamma^5, \mathcal{F})$-perfect matching $\mathcal{M}$ in $\mathcal{H}$. For $i \in [t]$, let M_i' be the set of all edges $e \in E(G_1)$ such that $f_{e,i,\ell} \in \mathcal{M}$ for some $\ell \in [192]$.

Hence, by definition of $\mathcal{H}$, we have that $M_1', \ldots, M_t'$ are edge-disjoint matchings in G_1, and, for each $i \in [t]$, we have $V(M_i') \subseteq V_i$ and M_i' consists of at most 192 edges with colour c, for each $c \in C_i$.

For each $i \in [t]$ and $c \in C_i$, let $r_{i,c} := 192 - |M_i' \cap E_c(K_n)|$. Thus, $r_{i,c}$ is the number of c-edges that are missing in M_i' in order to satisfy (ii). Since $\mathcal{M}$ is $(\gamma^5, \mathcal{F})$-perfect, we have for each $i \in [t]$ and $c \in C$ that

$$(12) \qquad \sum_{c' \in C_i} r_{i,c'} \leq \gamma^5 |\{i\} \times C_i \times [192]| \leq 192\gamma^5 n \quad \text{and} \quad \sum_{i' \in [t]: c \in C_{i'}} r_{i',c} \leq \gamma^5 |\mathcal{C}_c| \leq 192\gamma^5 n.$$

Moreover, for each vertex $v \in V$, the number of $i \in [t]$ for which $v \in V_{i,1}$ but v is not covered by M_i', is at most $\gamma^5 n$. Since $|\{i \in [t] \ : \ v \in V_{i,1}\}| = (1 \pm \varepsilon)pt$ by Claim 4, this implies that the number of $i \in [t]$ for which v is covered by M_i' is $(1 \pm \gamma^4)pt$.

Now, we wish to find edge-disjoint matchings $M_1'', \ldots, M_t''$ in G_2 such that, for each $i \in [t]$, $V(M_i'') \subseteq V_{i,2}$ and M_i'' contains precisely $r_{i,c}$ c-edges for each $c \in C$. This can be done in order greedily using Claim 3 and (12). Indeed, suppose we want to add c-edges to M_i''. By (12), we added at most $192\gamma^5 n$ c-edges to previous matchings M_j'', $j < i$, and at most $192\gamma^5 n$ edges to M_i''. Thus, at most $3 \cdot 192\gamma^5 n$ c-edges are blocked (since every edge in M_i'' might block 2 c-edges). Since $Y_{c,i} \geq \gamma^4 n$ by Claim 3, we can find $r_{i,c}$ suitable c-edges in $G_2[V_{i,2}]$ and add them to M_i''.

Note that, by Claim 4, for every vertex $v \in V$, the number of $i \in [t]$ for which v is covered by M_i'', is at most $2\gamma pt$. Finally, for each $i \in [t]$, let $M_i := M_i' \cup M_i''$. It is easy to see that $M_1, \ldots, M_t$ are the desired matchings. $\qquad \square$

Lemma 18. *Suppose $1/n \ll \gamma \ll \eta \ll 1$ and let $p := 2\eta$, $q := \eta/192$ and $t := n/2$. Suppose $s \in \mathbb{N}$ with $s = (q/4 - 2\gamma/5 \pm \gamma^2)n$ and $0 \leq \alpha \leq q/2 - \gamma$. Let ϕ be a 1-factorization of K_n with vertex set V and colour set C. Let G be an $\eta(1 + \gamma)$-random subgraph of K_n. For every $i \in [t]$, let V_i be a $p(1 + \gamma)$-random subset of V, and let $C_{i,1}$, $C_{i,2}$ be disjoint $q/2$-random subsets of C. Split $C_{i,1}$ further into an α-random set $C_{i,1,1}$ and a $(q/2 - \alpha)$-random subset $C_{i,1,2}$.*

Then with high probability, for each $i \in [t]$, there exist $C_{i,1}', C_{i,2}'$ such that $C_{i,1,1} \subseteq C_{i,1}' \subseteq C_{i,1}$ and $C_{i,2}' \subseteq C_{i,2}$ and vertex-disjoint rainbow matchings $\{M_{i,j} \ : \ j \in [3s]\}$ in $G[V_i]$, such that altogether the following hold:

  (i) *for each $i \in [t]$, $|C_{i,1}'| = |C_{i,2}'| = 2s$;*
  (ii) *for all $c \in C$, $|\{i \in [t] \ : \ c \in C_{i,2} \setminus C_{i,2}'\}| \leq \sqrt{\gamma}n$;*
  (iii) *for each $i \in [t]$, $M_i := \bigcup_{j \in [3s]} M_{i,j}$ consists of 192 c-edges for each $c \in C_{i,1}' \cup C_{i,2}'$;*
  (iv) *for each $i \in [t]$ and any subset $C_i^* \subseteq C_{i,1}'$ of size s, there exists $J_i \subseteq M_i$ such that J_i is $(C_i^* \cup C_{i,2}')$-rainbow and contains exactly one edge from each of $\{M_{i,j} \ : \ j \in [3s]\}$;*
  (v) *the matchings $M_1, \ldots, M_t$ are edge-disjoint, and $|M_{i,j}| = 256$ for all $(i, j) \in [t] \times [3s]$;*
  (vi) *for every vertex $v \in V$, the number of $i \in [t]$ for which v is covered by M_i is $(1 \pm \sqrt{\gamma})pt$.*

Here, the crucial property is (iv), which will allow us to use some colours of $C_{i,1}'$ flexibly before assigning the remaining colours (i.e. those in C_i^*) together with the 'buffer' $C_{i,2}'$ in such a way

that each matching $\{M_{i,j} \ : \ i \in [t], j \in [3s]\}$ contributes exactly one edge to J_i which will be part of T_i.

Proof. Clearly, we may assume that $\alpha = q/2 - \gamma$. We choose the random colour sets according to the following procedure: For each $i \in [t]$, let C_i be a q-random subset of C, and let $\tau_i \colon C \to [4]$ be a random function such that $\mathbb{P}\left(\tau_i(c) = 1\right) = \mathbb{P}\left(\tau_i(c) = 2\right) = 1/2 - \gamma/q$ and $\mathbb{P}\left(\tau_i(c) = 3\right) = \mathbb{P}\left(\tau_i(c) = 4\right) = \gamma/q$. For $i \in [t]$ and $k \in [2]$, let $C_{i,k,1} = \{c \in C_i \ : \ \tau_i(c) = k\}$ and $C_{i,k,2} = \{c \in C_i \ : \ \tau_i(c) = k + 2\}$, and let $C_{i,k} := C_{i,k,1} \cup C_{i,k,2}$. Then $C_{i,1}, C_{i,2}, C_{i,1,1}, C_{i,1,2}$ are as in the statement.

Now, we first expose all random choices except the functions $\{\tau_i\}_{i \in [t]}$. By Lemma 17, with high probability, there exist edge-disjoint matchings $M'_1, \ldots, M'_t$ in G such that the following hold:

    (a) $V(M'_i) \subseteq V_i$ for all $i \in [t]$;
    (b) for all $i \in [t]$, M'_i consists of 192 c-edges for each $c \in C_i$;
    (c) for every vertex $v \in V$, the number of $i \in [t]$ for which v is covered by M'_i is $(1 \pm 3\gamma)pt$.

Henceforth, assume that these random choices have been made and satisfy the above properties. It remains to expose the functions τ_i.

With high probability, we have for all $i \in [t]$ and $k \in [2]$ that

$$(13) \qquad |C_{i,k,1}| = (1 \pm \gamma^2)(q/2 - \gamma)n \quad \text{and} \quad |C_{i,k,2}| = (1 \pm \gamma^2)\gamma n.$$

With high probability, we also have for all $c \in C$ that

$$(14) \qquad |\{i \in [t] \ : \ \tau_i(c) = 4\}| \leq \sqrt{\gamma}n.$$

For $v \in V$, let us call $i \in [t]$ *unreliable for v* if v is covered by M'_i via an edge whose colour is in $C_{i,1,2} \cup C_{i,2,2}$. Then, also with high probability, for all $v \in V$,

$$(15) \qquad \text{at most } 2\gamma q^{-1}n \text{ indices } i \in [t] \text{ are unreliable for } v.$$

From now on, assume that (13)–(15) hold. For each $i \in [t]$ and $k \in [2]$, note that by (13) we have $2s - 2\gamma n/5 \leq |C_{i,k,1}| \leq 2s$ and thus, again by (13), we can choose $C'_{i,k,2} \subseteq C_{i,k,2}$ such that $|C'_{i,k,2}| = 2s - |C_{i,k,1}|$, and define

$$C'_{i,k} := C_{i,k,1} \cup C'_{i,k,2}.$$

Then (i) clearly holds and (ii) follows from (14). Moreover, let

$$M_i := \{e \in M'_i \ : \ \phi(e) \in C'_{i,1} \cup C'_{i,2}\}$$

for each $i \in [t]$. Observe that (c) and (15) imply (vi).

We now use RMBG's to break each M_i into small rainbow matchings. For each $i \in [t]$, let H_i be a $(256, 192)$-regular RMBG$(3s, 2s, 2s)$ with parts $[3s], C'_{i,1}$ and $C'_{i,2}$, which exist by Corollary 10. For each $i \in [t]$, partition M_i into matchings $M_{i,1}, \ldots, M_{i,3s}$, such that, for each $j \in [3s]$, $M_{i,j}$ is an $N_{H_i}(j)$-rainbow matching. This can be done greedily since to do so we need precisely 192 c-edges of each colour $c \in C'_{i,1} \cup C'_{i,2}$, which M_i contains. Clearly, $|M_{i,j}| = |N_{H_i}(j)| = 256$, and thus (iii) and (v) hold.

Finally, we check that the crucial property (iv) holds. Consider $i \in [t]$ and suppose $C^*_i \subseteq C'_{i,1}$ has size s. Since H_i is an RMBG with parts $[3s], C'_{i,1}$ and $C'_{i,2}$, there exists a perfect matching τ in H_i between $[3s]$ and $C^*_i \cup C'_{i,2}$. Now, for each $j \in [3s]$, we select the $\tau(j)$-edge from $M_{i,j}$ and include it in J_i. (Here we view $\tau(j)$ as the colour matched to j in the matching τ, and we use that $M_{i,j}$ is $N_{H_i}(j)$-rainbow.) Clearly, J_i is as desired. $\qquad\square$

5.3. **Matchings for edge absorption.** We now find the monochromatic matchings which form the crucial ingredients for the edge absorption process.

Lemma 19. *Suppose $1/n \ll \varepsilon \ll \gamma \ll \eta \ll 1$ and let $p := 3072\eta$, $q := 6\eta$, $t := n/2$, and suppose $m \in \mathbb{N}$ with $m = (\eta - \varepsilon/5 \pm \varepsilon^2)n$ and $0 \le \alpha \le 4\eta - \varepsilon$. Let ϕ be a 1-factorization of K_n with vertex set V and colour set C. Let G_1', G_2' be edge-disjoint 4η-random subgraphs of K_n, and split G_1' further into an α-random subgraph $G_{1,1}'$ and a $(4\eta - \alpha)$-random subgraph $G_{1,2}'$. For each $i \in [t]$, let V_i be a $p(1 + \gamma)$-random subset of V and let D_i be a $q(1 + \gamma)$-random subset of C.*

Then, with high probability, there exist G_1 and G_2 such that $G_{1,1}' \subseteq G_1 \subseteq G_1'$ and $G_2 \subseteq G_2'$ with $\Delta(G_2' - G_2) \le 2\varepsilon n$ and, for each $i \in [t]$, there exists $D_i' \subseteq D_i$ of size $(1 \pm 2\gamma)qn$ and vertex-disjoint matchings $\{M_{i,c}\}_{c \in D_i'}$ in $(G_1 \cup G_2)[V_i]$, where $M_{i,c}$ consists of 256 c-edges, such that altogether the following hold:

- (i) *for each $c \in C$, $|E_c(G_1)| = |E_c(G_2)| = 2m$ and $|\{i \in [t] : c \in D_i'\}| = 3m$;*
- (ii) *for any subset $E^* \subseteq E(G_1)$ which consists of precisely m edges of each colour $c \in C$, there exists a partition of $E^* \cup E(G_2)$ into sets $J_1, \ldots, J_t$, such that, for each $i \in [t]$, J_i contains exactly one edge from each of $\{M_{i,c}\}_{c \in D_i'}$;*
- (iii) *every vertex $v \in V$ is covered by $(1 \pm \gamma)pt$ of the matchings $\{M_{i,c} : i \in [t], c \in D_i'\}$.*

Here, the crucial property is (ii), which will allow us to use some edges of the global edge reservoir G_1 flexibly before assigning the remaining edges (i.e. those in E^*) together with the 'buffer' $E(G_2)$ in such a way that each matching $\{M_{i,c} : i \in [t], c \in D_i'\}$ contributes exactly one edge to J_i. J_i will then be assigned to the ith tree T_i.

Proof. We may clearly assume that $\alpha = 4\eta - \varepsilon$. We also split G_2' further into a $(4\eta - \varepsilon)$-random subgraph $G_{2,1}'$ and an ε-random subgraph $G_{2,2}'$. We first expose $G_{1,1}', G_{1,2}', G_{2,1}', G_{2,2}'$. Using Chernoff's bound, it is easy to see that, with high probability, we have for all $j \in [2]$, $c \in C$ and $v \in V$ that

$$(16) \qquad |E_c(G_{j,1}')| = (1 \pm \varepsilon^2)(4\eta - \varepsilon)n/2 \quad \text{and} \quad |E_c(G_{j,2}')| = (1 \pm \varepsilon^2)\varepsilon n/2,$$

$$(17) \qquad d_{G_{j,1}'}(v) = (1 \pm \varepsilon^2)(4\eta - \varepsilon)n \quad \text{and} \quad d_{G_{j,2}'}(v) = (1 \pm \varepsilon^2)\varepsilon n.$$

Henceforth, we assume that $G_{1,1}', G_{1,2}', G_{2,1}', G_{2,2}'$ are fixed with the above properties, and expose the other random sets.

By (16), we have for $j \in [2]$ and $c \in C$ that $2m - \varepsilon n/5 \le |E_c(G_{j,1}')| \le 2m$. Therefore, by (16) again, for each $j \in [2]$, we can choose $G_{j,2}'' \subseteq G_{j,2}'$ such that $|E_c(G_{j,2}'')| = 2m - |E_c(G_{j,1}')|$ for all $c \in C$, and define

$$G_j := G_{j,1}' \cup G_{j,2}''.$$

Clearly, this choice of G_1 and G_2 satisfies the first part of (i). Moreover, from (17), we can infer that $\Delta(G_2' - G_2) \le 2\varepsilon n$, as desired, and that

$$(18) \qquad d_{G_1 \cup G_2}(v) = (1 \pm \sqrt{\varepsilon})8\eta n \text{ for all } v \in V.$$

As indicated in the proof sketch, the key to obtaining (ii) is to use an RMBG for each colour which matches the $3m$ c-edges of $E^* \cup E(G_2)$ to $3m$ 'absorbers'. Let $\hat{H}$ be a $(256, 192)$-regular RMBG$(3m, 2m, 2m)$ with parts $[3m], \hat{Y}_1, \hat{Y}_2$, which exists by Corollary 10. We identify $\hat{Y}_1$ and $\hat{Y}_2$ with $E_c(G_1)$ and $E_c(G_2)$. We carry out this identification randomly in order to obtain a codegree condition in some hypergraph $\mathcal{H}$ which we will define later. (This codegree condition will be needed when applying Theorem 7 to $\mathcal{H}$.) For each colour $c \in C$, pick random bijections $\pi_{c,1} : E_c(G_1) \to \hat{Y}_1$ and $\pi_{c,2} : E_c(G_2) \to \hat{Y}_2$, all independently. Obtain a copy H_c of $\hat{H}$ by identifying $E_c(G_1)$ with $\hat{Y}_1$ according to $\pi_{c,1}$ and $E_c(G_2)$ with $\hat{Y}_2$ according to $\pi_{c,2}$.

For two vertices v, v', we define $r_{v,v'}$ as the number of colours $c \in C$ for which $N_{H_c}(e) \cap N_{H_c}(e') \ne \emptyset$, where e and e' are the unique c-edges at v and v', respectively. (In particular, if e or e' is not contained in $E_c(G_1 \cup G_2)$, then c contributes 0 to $r_{v,v'}$.)

Claim 1: With positive probability, $r_{v,v'} \le 3 \log n$ for all distinct vertices $v, v' \in V$.

Proof of claim: Fix two distinct vertices $v, v' \in V$. For $c \in C$, let X_c be the indicator variable of the event that there exist c-edges e, e' at v and v', respectively (which are unique if existent), and $N_{H_c}(e) \cap N_{H_c}(e') \neq \emptyset$. Fix $c \in C \setminus \{\phi(vv')\}$ and let e, e' be as above. We claim that $\mathbb{P}(X_c = 1) \leq 10^5 m^{-1}$. Note that e, e' are distinct. Let $k, k' \in \{1, 2\}$ be such that $e \in E(G_k)$ and $e' \in E(G_{k'})$. Thus,

$$\mathbb{P}(X_c = 1) \leq \sum_{j \in [3m]} \mathbb{P}\left(\pi_{c,k}(e), \pi_{c,k'}(e') \in N_{\hat{H}}(j)\right) \leq 3m \cdot \frac{256 \cdot 255}{2m \cdot (2m - 1)} \leq 10^5 m^{-1}.$$

Hence, $\mathbb{E}(r_{v,v'}) \leq \eta^{-2}$, and since the X_c's are independent, Chernoff's bound implies that the probability that $r_{v,v'} > 3 \log n$ is smaller than n^{-2}. A union bound then implies the claim. —

From now on, fix RMBG's $\{H_c\}_{c \in C}$ for which the conclusion of Claim 1 holds. Let $\mathcal{A} := C \times [3m]$. For each $(c, j) \in \mathcal{A}$, we define $A_{c,j} := N_{H_c}(j)$. We refer to $A_{c,j}$ as an *absorber* and will sometimes identify $A_{c,j}$ with $(c, j) \in \mathcal{A}$. Note that $A_{c,j}$ is a matching consisting of 256 c-edges.

By our choice of RMBG's, we have that for any two distinct vertices $v, v' \in V$,

(19)   there are at most $192 \cdot 3 \log n$ absorbers $(c, j) \in \mathcal{A}$ with $v, v' \in V(A_{c,j})$.

We will now assign to each absorber an index $i \in [t]$. The assignment will be obtained as follows: We first define an auxiliary hypergraph $\mathcal{H}$, in which we will find an almost perfect matching that provides an almost complete assignment. For the remaining absorbers not yet assigned, we will greedily pick images from a reserve.

In order to set aside this 'reserve', we randomly split V_i and D_i further as follows. For each $i \in [t]$, split V_i into a p-random set $V_{i,1}$ and a $p\gamma$-random set $V_{i,2}$, and split D_i into a q-random set $D_{i,1}$ and a $q\gamma$-random set $D_{i,2}$.

We can now define the (random) auxiliary hypergraph $\mathcal{H}$ as follows. The vertex set of $\mathcal{H}$ consists of three different parts: The first part is simply the set $\mathcal{A}$ which represents all the absorbers. The second part is the set $\mathcal{V}$ of all pairs (i, v) with $i \in [t]$ and $v \in V_{i,1}$. The third part is the set $\mathcal{C}$ of all pairs (i, c) with $i \in [t]$ and $c \in D_{i,1}$.

Now, we define the edge set of $\mathcal{H}$. For every $i \in [t]$ and every absorber $(c, j) \in \mathcal{A}$, we add the hyperedge

(20) $$f_{c,j,i} := \{(c, j), (i, c)\} \cup (\{i\} \times V(A_{c,j}))$$

if and only if $c \in D_{i,1}$ and $V(A_{c,j}) \subseteq V_{i,1}$. Hence, $\mathcal{H}$ is 514-uniform. (Recall that $A_{c,j}$ is a matching consisting of 256 c-edges.)

Moreover, for each absorber $(c, j) \in \mathcal{A}$, we define the random set $Y_{c,j}$ of indices $i \in [t]$ for which $c \in D_{i,2}$ and $V(A_{c,j}) \subseteq V_{i,2}$. We aim to apply Theorem 7 to $\mathcal{H}$. For this, we first establish the following properties.

Claim 2: With high probability, for each $(c, j) \in \mathcal{A}$, $d_{\mathcal{H}}((c, j)) = (1 \pm \varepsilon)tp^{512}q$ and $|Y_{c,j}| = (1 \pm \varepsilon)t(p\gamma)^{512}q\gamma$.

Proof of claim: Fix an absorber $(c, j) \in \mathcal{A}$. For $i \in [t]$, let X_i be the indicator variable of the event that $c \in D_{i,1}$ and $V(A_{c,j}) \subseteq V_{i,1}$ and let Y_i be the indicator variable of the event that $c \in D_{i,2}$ and $V(A_{c,j}) \subseteq V_{i,2}$. Note that $d_{\mathcal{H}}((c, j)) = \sum_{i \in [t]} X_i$ and $|Y_{c,j}| = \sum_{i \in [t]} Y_i$. For each $i \in [t]$, we have that $\mathbb{P}(X_i = 1) = p^{512}q$ and $\mathbb{P}(Y_i = 1) = (p\gamma)^{512}q\gamma$. Thus, $\mathbb{E}(d_{\mathcal{H}}((c, j))) = tp^{512}q$ and $\mathbb{E}(|Y_{c,j}|) = t(p\gamma)^{512}q\gamma$. Since the X_i's are independent, and similarly, the Y_i's are independent, we can deduce with Chernoff's bound that the claim holds. —

Claim 3: With high probability, for each $(i, c) \in \mathcal{C}$, $d_{\mathcal{H}}((i, c)) = (1 \pm \varepsilon)3mp^{512}$.

Proof of claim: Fix $(i, c) \in \mathcal{C}$. For $j \in [3m]$, let X_j be the indicator variable of the event that $V(A_{c,j}) \subseteq V_{i,1}$. Note that $d_{\mathcal{H}}((i, c)) = \sum_{j \in [3m]} X_j$. For each $j \in [3m]$ we have $\mathbb{P}(X_j = 1) = p^{512}$. Thus, $\mathbb{E}(d_{\mathcal{H}}((i, c))) = 3mp^{512}$.

Moreover, $d_{\mathcal{H}}((i,c))$ is determined by the independent random variables $\{\mathbb{1}_{\{v\in V_{i,1}\}} : v \in V\}$. Since $\mathbb{1}_{\{v\in V_{i,1}\}}$ affects $d_{\mathcal{H}}((i,c))$ by at most 192, the claim follows by an application of McDiarmid's inequality. $-$

Claim 4: With high probability, for all $(i,v) \in \mathcal{V}$, $d_{\mathcal{H}}((i,v)) = (1 \pm 2\sqrt{\varepsilon})1536\eta p^{511}qn$.

Proof of claim: Fix $(i,v) \in \mathcal{V}$. For each edge e at v in $G_1 \cup G_2$, say with colour c, e has 192 neighbours $j \in [3m]$ in H_c, and for each of those we have $f_{c,j,i} \in E(\mathcal{H})$ iff $c \in D_{i,1}$ and the 511 other vertices of $A_{c,j}$ are contained in $V_{i,1}$. Thus, $\mathbb{E}(d_{\mathcal{H}}((i,v))) = d_{G_1\cup G_2}(v) \cdot 192 \cdot p^{511}q = (1 \pm \sqrt{\varepsilon}) \cdot 8 \cdot 192\eta p^{511}qn$ by (18).

Moreover, $d_{\mathcal{H}}((i,v))$ is determined by the independent random variables $\{\mathbb{1}_{\{u\in V_{i,1}\}} : u \in V \setminus \{v\}\} \cup \{\mathbb{1}_{\{c\in D_{i,1}\}} : c \in C\}$. The effect of $\mathbb{1}_{\{c\in D_{i,1}\}}$ on $d_{\mathcal{H}}((i,v))$ is at most 192. Moreover, for each $u \in V \setminus \{v\}$, by (19), $\mathbb{1}_{\{u\in V_{i,1}\}}$ affects $d_{\mathcal{H}}((i,v))$ by at most $192 \cdot 3\log n$. The claim now follows from an application of McDiarmid's inequality. $-$

Claim 5: $\Delta^c(\mathcal{H}) \leq 192 \cdot 3\log n$.

Proof of claim: Clearly, the codegree of pairs in $\mathcal{A} \times \mathcal{A}$ and $\mathcal{C} \times \mathcal{C}$ is 0. Moreover, the codegree of pairs in $\mathcal{A} \times \mathcal{V}$ and $\mathcal{A} \times \mathcal{C}$ is at most 1. It is also easy to see that the codegree of a pair in $\mathcal{V} \times \mathcal{C}$ is at most 192.

Finally, consider a pair in $\mathcal{V} \times \mathcal{V}$, say (i,u) and (i',v). If $i \neq i'$, then the codegree is 0, so assume $i = i'$. Crucially, by (19), the codegree of (i,u) and (i,v) is at most $192 \cdot 3\log n$. $-$

We now assume that the properties stated in Claims 2–5 are satisfied. Using Chernoff's bound, we can assume that the following simple properties hold as well:

$$(21) \qquad |D_{i,1}| = (1 \pm \varepsilon)qn \quad \text{and} \quad |D_{i,2}| = (1 \pm \varepsilon)q\gamma n.$$

By our choice of p, q, η, t, m, we have that $d_{\mathcal{H}}(x) = (1 \pm 2\sqrt{\varepsilon})3\eta p^{512}n$ for all $x \in V(\mathcal{H})$. In combination with Claim 5, we can thus apply Theorem 7 to find an almost perfect matching in $\mathcal{H}$. In order to gain control over the leftover vertices in $\mathcal{H}$, we define the following vertex sets. For each vertex $v \in V$, let $\mathcal{A}_v$ be the set of all absorbers $(c,j) \in \mathcal{A}$ for which $v \in V(A_{c,j})$. Note that

$$(22) \qquad |\mathcal{A}_v| = 192d_{G_1\cup G_2}(v) \overset{(18)}{=} (1 \pm \sqrt{\varepsilon}) \cdot 8 \cdot 192\eta n = (1 \pm \sqrt{\varepsilon})pt.$$

Define

$$\mathcal{F} := \{\mathcal{A}_v : v \in V\} \cup \{\{c\} \times [3m] : c \in C\} \cup \{\{i\} \times D_{i,1} : i \in [t]\}.$$

Now, apply Theorem 7 to find a

$$(23) \qquad\qquad (\gamma^{515}, \mathcal{F})\text{-perfect matching } \mathcal{M} \text{ in } \mathcal{H}.$$

Our goal is to define a map $\sigma \colon \mathcal{A} \to [t]$. Let $\mathcal{A}'$ be the set of absorbers $(c,j) \in \mathcal{A}$ which are not covered by $\mathcal{M}$. For each $(c,j) \in \mathcal{A} \setminus \mathcal{A}'$, the absorber (c,j) is covered by a (unique) hyperedge $f_{c,j,i} \in \mathcal{M}$, and we define $\sigma(c,j) := i$. For all uncovered absorbers, we now use the 'reserve' sets $V_{i,2}$ and $D_{i,2}$ to pick suitable images.

For all $(c,j) \in \mathcal{A}'$, we successively define $\sigma(c,j)$ as follows: when we consider $(c,j) \in \mathcal{A}'$, let $\mathcal{A}''$ be the set of all previously considered $(c',j') \in \mathcal{A}'$ with $c' = c$ or $V(A_{c,j}) \cap V(A_{c',j'}) \neq \emptyset$. By (23), we have that

$$|\mathcal{A}''| \leq \gamma^{515} \cdot |\{c\} \times [3m]| + \sum_{v\in V(A_{c,j})} \gamma^{515}|\mathcal{A}_v| \overset{(22)}{\leq} \gamma^{515} \cdot 3m + 512\gamma^{515} \cdot 2pt < \gamma^{514}n/2.$$

Recall from Claim 2 that $|Y_{c,j}| \geq \gamma^{514}n$. Thus, there is $i \in Y_{c,j} \setminus \sigma(\mathcal{A}'')$ and we define $\sigma(c,j) := i$.

Altogether, we have found a map $\sigma \colon \mathcal{A} \to [t]$, which we show has the following properties:

(a) $V(A_{c,j}) \subseteq V_{\sigma(c,j)}$ and $c \in D_{\sigma(c,j)}$ for all $(c,j) \in \mathcal{A}$;
(b) $V(A_{c,j}) \cap V(A_{c',j'}) = \emptyset$ whenever $\sigma(c,j) = \sigma(c',j')$;
(c) for all $c \in C$ and $i \in [t]$, there is at most one $j \in [3m]$ with $\sigma(c,j) = i$.

Here, (a) clearly holds by the definitions of $\mathcal{H}$, $Y_{c,j}$ and σ. To see (b), suppose $\sigma(c,j) = \sigma(c',j') = i$. If $(c,j),(c',j') \in \mathcal{A} \setminus \mathcal{A}'$, then we have $V(A_{c,j}) \cap V(A_{c',j'}) = \emptyset$ since $\mathcal{M}$ is a matching and as such covers every vertex $(i,v) \in \mathcal{V}$ at most once. If $(c,j) \in \mathcal{A}$ and $(c',j') \in \mathcal{A}'$, then $V(A_{c,j}) \subseteq V_{i,1}$ and $V(A_{c',j'}) \subseteq V_{i,2}$. Finally, suppose $(c,j),(c',j') \in \mathcal{A}'$ and assume that we defined $\sigma(c,j)$ after $\sigma(c',j')$. If $V(A_{c,j}) \cap V(A_{c',j'}) \neq \emptyset$, then $(c',j') \in \mathcal{A}''$ (with notation as above) and hence $i \in \sigma(\mathcal{A}'')$, a contradiction.

For (c), fix $c \in C$ and $i \in [t]$. Suppose $\sigma(c,j) = i$ for some $j \in [3m]$. We consider two cases. In the first case, we have $(c,j) \in \mathcal{A} \setminus \mathcal{A}'$ and $f_{c,j,i} \in \mathcal{M}$. In particular, there is at most one j which satisfies this and we must have $c \in D_{i,1}$. In the second case, we must have $(c,j) \in \mathcal{A}'$ and $c \in D_{i,2}$, and there can only be one j which satisfies this by definition of $\mathcal{A}''$ above. Since $D_{i,1}$ and $D_{i,2}$ are disjoint, (c) follows.

Now, for every $c \in C$, define $\sigma_c := \sigma(c, \cdot)$ and $I_c := \text{Im}(\sigma_c)$. By (c), $\sigma_c \colon [3m] \to I_c$ is a bijection. For all $i \in [t]$, define
$$D_i' := \{c \in C \,:\, i \in I_c\}.$$
For all $c \in C$, we have $|\{i \in [t] \,:\, c \in D_i'\}| = |I_c| = 3m$, so the second part of (i) holds too. Observe that if $c \in D_i'$, then $i \in I_c$ and hence there exists some $j \in [3m]$ for which $\sigma(c,j) = i$. By (a), we have $c \in D_i$. Thus, $D_i' \subseteq D_i$. In particular, we have $|D_i'| \leq |D_i| \leq (1+2\gamma)qn$ by (21). Moreover, since $\{i\} \times D_{i,1} \in \mathcal{F}$, at least $(1-\gamma^{515})|D_{i,1}|$ elements of $\{i\} \times D_{i,1}$ are covered by $\mathcal{M}$, which means that for at least $(1-\gamma^{515})|D_{i,1}|$ colours c in $D_{i,1}$, we have $i \in I_c$ and therefore $c \in D_i'$. Thus, $|D_i'| \geq (1-\gamma^{515})|D_{i,1}| \geq (1-2\gamma)qn$ by (21). Hence, $|D_i'| = (1\pm 2\gamma)qn$, as required.

Furthermore, for all $i \in [t]$ and $c \in D_i'$, let
$$M_{i,c} := A_{c,\sigma_c^{-1}(i)}.$$

Clearly, $M_{i,c}$ is a matching consisting of 256 c-edges in $G_1 \cup G_2$. Using (a), we can also see that $V(M_{i,c}) \subseteq V_i$. Moreover, for fixed $i \in [t]$, all the matchings $\{M_{i,c}\}_{c \in D_i'}$ are vertex-disjoint by (b). To check (iii), consider any vertex $v \in V$. Clearly, the number of matchings $\{M_{i,c} \,:\, i \in [t], c \in D_i'\}$ covering v is at most $|\mathcal{A}_v| \leq (1+\sqrt{\varepsilon})pt$ by (22). Moreover, since $\mathcal{M}$ covers all but at most $\gamma^{515}|\mathcal{A}_v|$ absorbers in $\mathcal{A}_v$, we obtain a lower bound of $(1-\gamma^{515})|\mathcal{A}_v| \geq (1-\gamma)pt$, as desired.

It remains to show the crucial property (ii). Suppose $E^* \subseteq E(G_1)$ consists of precisely m edges of each colour $c \in C$. For each $c \in C$, let E_c^* be the set of c-edges in E^*. Since H_c is an RMBG with parts $[3m]$, $E_c(G_1)$ and $E_c(G_2)$, there exists a bijection $\tau_c \colon [3m] \to E_c^* \cup E_c(G_2)$ such that $\tau_c(j) \in N_{H_c}(j)$ for all $j \in [3m]$.

We can now define the desired partition of $E^* \cup E(G_2)$ as follows. Let $e \in E^* \cup E(G_2)$. Let c be the colour of e and $j := \tau_c^{-1}(e)$. Thus, we have $e \in N_{H_c}(j) = A_{c,j} = M_{i,c}$, where $i := \sigma_c(j)$. Note that $i \in I_c$ and hence $c \in D_i'$. Assign e to J_i. Clearly, this defines a partition of $E^* \cup E(G_2)$ into $J_1, \ldots, J_t$. Consider $i \in [t]$. By construction, every edge $e \in J_i$ belongs to some $M_{i,c}$ with $c \in D_i'$. Moreover, for fixed $c \in D_i'$, only one edge of $M_{i,c}$ is included in J_i because σ_c and τ_c are bijective. $\qquad\square$

5.4. Connecting lemma.

The following lemma will be used to efficiently connect up the (edges from the) matchings produced by Lemmas 18 and 19 of the trees T_i.

Given a k-uniform matching $\mathcal{R}$, we say that a graph F is an $\mathcal{R}$-connector if F is obtained from the empty graph on $V(\mathcal{R})$ by adding, for every $R \in \mathcal{R}$, new vertices $v_{R,1}, \ldots, v_{R,k+1}$, a perfect matching between R and $\{v_{R,1}, \ldots, v_{R,k}\}$ and all edges from $\{v_{R,1}, \ldots, v_{R,k}\}$ to $v_{R,k+1}$.

Lemma 20. *Suppose $1/n \ll \varepsilon \ll \gamma \ll p' \ll 1/k$ and let $p := p'/k$ and $\beta := q := 2p'$ and $t := n/2$ and suppose $p'' = (1 \pm \varepsilon)p'$. Let ϕ be a 1-factorization of the complete graph K_n with vertex set V and colour set C. Let $\tilde{G}$ be a $\beta(1+\gamma)$-random subgraph of K_n. For every $i \in [t]$,*

let $U_i, \tilde{V}_i$ be disjoint subsets of V that are p''-random and $(p' + p)(1 + \gamma)$-random, respectively, and let $\tilde{C}_i$ be a $q(1 + \gamma)$-random subset of C.

Then, the following holds with high probability: Let $\mathcal{R}$ be any k-uniform (multi-)hypergraph which is the union of t matchings $\mathcal{R}_1, \ldots, \mathcal{R}_t$ such that $V(\mathcal{R}_i) \subseteq U_i$ and $|\mathcal{R}_i| = (1 \pm \varepsilon)pn$ for all $i \in [t]$, and such that $d_{\mathcal{R}}(x) = (1 \pm \varepsilon)p't$ for all $x \in V$. Then, for each $i \in [t]$, there exists an $\mathcal{R}_i$-connector $\tilde{F}_i$ in $\tilde{G}[U_i \cup \tilde{V}_i]$ such that the following hold:

  (i) *$\tilde{F}_1, \ldots, \tilde{F}_t$ are edge-disjoint;*
  (ii) *for each $i \in [t]$, $\tilde{F}_i$ is rainbow with colours in $\tilde{C}_i$;*
  (iii) *$(\tilde{G} - \bigcup_{i \in [t]} \tilde{F}_i, \{(U_i \cup \tilde{V}_i) \setminus V(\tilde{F}_i)\}_{i \in [t]}, \{\tilde{C}_i \setminus \phi(E(\tilde{F}_i))\}_{i \in [t]})$ is $2\gamma n$-bounded.*

In the proof, we will find most of the required connections via Theorem 7 (which allows us to do this 'efficiently') and the remaining ones via Lemma 15.

Proof. Choose a new constant $\xi > 0$ such that $\varepsilon \ll \xi \ll \gamma$. Split $\tilde{G}$ further into a β-random subgraph G and a $\beta\gamma$-random subgraph G'. Moreover, for each $i \in [t]$, split $\tilde{V}_i$ into a p'-random subset V_i, a p-random subset W_i and a $(p' + p)\gamma$-random subset V_i'. Split $\tilde{C}_i$ into a q-random subset C_i and a $q\gamma$-random subset C_i'. We will now establish a few properties concerning the random sets which hold with high probability. From these properties, we can then (deterministically) find the desired connections for any admissible $\mathcal{R}$.

For $i \in [t]$, let G_i be the spanning subgraph of G with all C_i-edges, and let G_i' be the spanning subgraph of G' with all C_i'-edges.

For each edge $e \in E(K_n)$, let $\tilde{I}_{e,1}$ be the set of $i \in [t]$ for which $\phi(e) \in C_i$ and e intersects both U_i, V_i, and let $I_{e,2}$ be the set of $i \in [t]$ for which $\phi(e) \in C_i$ and e intersects both V_i, W_i.

For $i \in [t]$ and $c \in C$, let $\tilde{E}_{i,c,1}$ be the set of c-edges in $E_G(U_i, V_i)$, and let $E_{i,c,2}$ be the set of c-edges in $E_G(V_i, W_i)$.

We claim that the following hold with high probability:

  (a) *for all $i \in [t]$, $|U_i| = (1 \pm 3\varepsilon)p'n$, $|V_i| = (1 \pm \varepsilon)p'n$, and $|W_i| = (1 \pm \varepsilon)pn$;*
  (b) *for all $u \in V$, we have $|\{i \in [t] : u \in U_i\}| = (1 \pm 3\varepsilon)p't$;*
  (c) *for all $i \in [t]$ and $x \in V$, $d_{G_i}(x, U_i) = (1 \pm 3\varepsilon)p'q\beta n$, $d_{G_i}(x, V_i) = (1 \pm \varepsilon)p'q\beta n$ and $d_{G_i}(x, W_i) = (1 \pm \varepsilon)pq\beta n$;*
  (d) *for all $i \in [t]$ and distinct $x, y \in V$, $|N_{G_i}(\{x, y\}) \cap V_i| = (1 \pm \varepsilon)p'q^2\beta^2 n$;*
  (e) *for all $i \in [t]$ and $S \subseteq V$ with $1 \leq |S| \leq k$, we have $|N_{G_i'}(S) \cap V_i'| \geq \gamma^{2k+2}n$;*
  (f) *for all $e \in E(K_n)$, $|\tilde{I}_{e,1}| = (1 \pm 2\varepsilon)p'^2qn$ and $|I_{e,2}| = (1 \pm \varepsilon)p'pqn$;*
  (g) *for all $i \in [t]$ and $c \in C$, $|\tilde{E}_{i,c,1}| = (1 \pm 2\varepsilon)p'^2\beta n$ and $|E_{i,c,2}| = (1 \pm \varepsilon)p'p\beta n$;*
  (h) *$(G', \{V_i'\}_{i \in [t]}, \{C_i'\}_{i \in [t]})$ is γn-bounded.*

Indeed, (a), (b), (c), (f), (g) and (h) follow easily from Chernoff's bound. For (d) and (e), we use McDiarmid's inequality, as follows. Consider $i \in [t]$ and distinct $x, y \in V$. Clearly, $\mathbb{E}(|N_{G_i}(\{x, y\}) \cap V_i|) = p'\beta^2 q^2(n - 2)$. Moreover, of the at most $2n$ edges incident with either x or y, each has an effect of at most 1. Each vertex has an effect of at most 1, and each colour has an effect of at most 2, and so McDiarmid's inequality applies. A similar argument works for (e).

Now assume that (a)–(h) hold. Let $\mathcal{R}$ be given arbitrarily as in the lemma statement. Let $U_i' := V(\mathcal{R}_i)$. By (a) and since $|V(\mathcal{R}_i)| = k|\mathcal{R}_i| = (1 \pm \varepsilon)p'n$, we have that

$$|U_i \setminus U_i'| \leq 4\varepsilon p'n \leq \varepsilon n. \tag{24}$$

Moreover, for every vertex $u \in V$, it follows from (b) and since $d_{\mathcal{R}}(u) = (1 \pm \varepsilon)p't$ that

$$|\{i \in [t] : u \in U_i \setminus U_i'\}| \leq 4\varepsilon p't \leq \varepsilon n. \tag{25}$$

From (c) and (24) we infer that

$$\text{for all } i \in [t] \text{ and } x \in V, d_{G_i}(x, U_i') = (1 \pm \sqrt{\varepsilon})p'q\beta n. \tag{26}$$

For an edge $e \in E(K_n)$, let $I_{e,1}$ be the set of $i \in \tilde{I}_{e,1}$ for which e intersects U_i'. From (f) and (25), we deduce that

$$(27) \qquad |I_{e,1}| = (1 \pm \sqrt{\varepsilon})p'^2 qn.$$

For all $i \in [t]$ and $c \in C$, let $E_{i,c,1}$ be the set of c-edges in $E_G(U_i', V_i)$. By (g) and (24), we have

$$(28) \qquad |E_{i,c,1}| = (1 \pm \sqrt{\varepsilon})p'^2 \beta n.$$

We now define an auxiliary hypergraph $\mathcal{H}$ whose vertex set is the union of five parts. The first part is simply $E(G)$. The second part is the set $\mathcal{R}^*$ of all pairs (i, R) such that $R \in \mathcal{R}_i$. The third part is the set $\mathcal{V}$ of all pairs (i, v) with $v \in V_i$. The fourth part is the set $\mathcal{W}$ of all pairs (i, w) with $w \in W_i$. The fifth part is the set $\mathcal{C}$ of all pairs (i, c) with $c \in C_i$.

We now define the edge set of $\mathcal{H}$. For disjoint $R, T, \{w\} \subseteq V$ and a bijection $\pi \colon T \to R$, let $S_{R,T,w,\pi}$ denote the graph on $R \cup T \cup \{w\}$ with edge set $\{\pi(v)v, vw \; : \; v \in T\}$. Note that $S_{R,T,w,\pi}$ is an $\{R\}$-connector.

For all $(i, R) \in \mathcal{R}^*$, $T \subseteq V_i$, $w \in W_i$ and bijections $\pi \colon T \to R$, we add the hyperedge

$$f_{i,R,T,w,\pi} := E(S_{R,T,w,\pi}) \cup \{(i, R)\} \cup (\{i\} \times T) \cup \{(i, w)\} \cup (\{i\} \times \phi(E(S_{R,T,w,\pi})))$$

to $\mathcal{H}$ if and only if $S_{R,T,w,\pi}$ is a rainbow subgraph of G_i. Note that $\mathcal{H}$ is $(5k + 2)$-uniform since $|T| = |R| = k$ and hence $E(S_{R,T,w,\pi}) = 2k$. We will apply Theorem 7 to $\mathcal{H}$. For this, we first check that $\mathcal{H}$ is roughly regular.

For each $i \in [t]$ and $e \in E(G_i)$, let

$$d_{i,e} := |\{f_{i,R,T,w,\pi} \in E(\mathcal{H}) \; : \; e \in E(S_{R,T,w,\pi})\}|,$$

and let $d_{i,e} := 0$ for each $i \in [t]$ and $e \notin E(G_i)$.

Claim 1: For each $i \in [t]$ and $e \in E(G_i)$, we have

$$d_{i,e} = \begin{cases} (1 \pm \sqrt{\varepsilon})pq\beta n(p'\beta^2 q^2 n)^{k-1} & \text{if } e \text{ intersects both } U_i' \text{ and } V_i, \\ (1 \pm 3\sqrt{\varepsilon})p'q\beta n(p'\beta^2 q^2 n)^{k-1} & \text{if } e \text{ intersects both } V_i \text{ and } W_i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of claim: First, assume $e = uv$ with $u \in U_i'$ and $v \in V_i$. There is a unique $R \in \mathcal{R}_i$ with $u \in R$. By (c), there are $(1 \pm \varepsilon)pq\beta n$ choices for $w \in N_{G_i}(v) \cap W_i$. For each $u' \in R \setminus \{u\}$ in turn, by (d), we have $(1 \pm 2\varepsilon)p'q^2\beta^2 n$ choices for $\pi^{-1}(u') \in N_{G_i}(u') \cap N_{G_i}(w) \cap V_i$ while avoiding previously chosen vertices and previously used colours. We deduce that $d_{i,e} = (1 \pm \sqrt{\varepsilon})pq\beta n(p'\beta^2 q^2 n)^{k-1}$.

Next, assume $e = vw$ with $v \in V_i$ and $w \in W_i$. By (26), there are $(1 \pm \sqrt{\varepsilon})p'q\beta n$ choices for $\pi(v) \in N_{G_i}(v) \cap U_i'$, which yields a unique $R \in \mathcal{R}_i$ with $\pi(v) \in R$. Using (d) as above, we conclude that $d_{i,e} = (1 \pm 3\sqrt{\varepsilon})p'q\beta n(p'\beta^2 q^2 n)^{k-1}$.

Clearly, in any other case, we have $d_{i,e} = 0$. $\qquad -$

We will use Claim 1 below without explicit reference.

Claim 2: For all $x \in V(\mathcal{H})$, we have $d_{\mathcal{H}}(x) = (1 \pm \varepsilon^{1/3})pp'^k \beta^{2k} q^{2k} n^{k+1}$.

Proof of claim: First, consider $e \in E(G)$. We have

$$d_{\mathcal{H}}(e) \quad = \quad \sum_{i \in [t]} d_{i,e} = |I_{e,1}| \cdot (1 \pm \sqrt{\varepsilon})pq\beta n(p'\beta^2 q^2 n)^{k-1} + |I_{e,2}| \cdot (1 \pm 3\sqrt{\varepsilon})p'q\beta n(p'\beta^2 q^2 n)^{k-1}$$

$$\overset{\text{(f),(27)}}{=} \quad (1 \pm \varepsilon^{1/3})2pp'^{k+1}\beta^{2k-1}q^{2k}n^{k+1} = (1 \pm \varepsilon^{1/3})pp'^k\beta^{2k}q^{2k}n^{k+1},$$

as $p' = \beta/2$.

Next, consider $(i, R) \in \mathcal{R}^*$. By (a), there are $(1 \pm \varepsilon)pn$ choices for $w \in W_i$. For each $u \in R$ in turn, by (d), we have $(1 \pm 2\varepsilon)p'\beta^2 q^2 n$ choices for $\pi^{-1}(u) \in N_{G_i}(u) \cap N_{G_i}(w) \cap V_i$ while avoiding previously chosen vertices and previously used colours. We deduce that

$$d_{\mathcal{H}}((i, R)) = (1 \pm \sqrt{\varepsilon})pn(p'\beta^2 q^2 n)^k = (1 \pm \sqrt{\varepsilon})pp'^k\beta^{2k}q^{2k}n^{k+1}.$$

Now, consider $(i, v) \in \mathcal{V}$. We have

$$d_{\mathcal{H}}((i,v)) = \sum_{w \in N_{G_i}(v) \cap W_i} d_{i,vw} \overset{(c)}{=} (1 \pm \varepsilon)pq\beta n \cdot (1 \pm 3\sqrt{\varepsilon})p'q\beta n(p'\beta^2 q^2 n)^{k-1}$$

$$= (1 \pm \varepsilon^{1/3})pp'^k\beta^{2k}q^{2k}n^{k+1}.$$

Next, consider $(i, w) \in \mathcal{W}$. By assumption, we have $|\mathcal{R}_i| = (1 \pm \varepsilon)pn$ choices for $R \in \mathcal{R}_i$. For each $u \in R$ in turn, by (d), we have $(1 \pm 2\varepsilon)p'\beta^2 q^2 n$ choices for $\pi^{-1}(u) \in N_{G_i}(u) \cap N_{G_i}(w) \cap V_i$ while avoiding previously chosen vertices and previously used colours. We deduce that

$$d_{\mathcal{H}}(i,w) = (1 \pm \varepsilon^{1/3})pn(p'\beta^2 q^2 n)^k = (1 \pm \varepsilon^{1/3})pp'^k\beta^{2k}q^{2k}n^{k+1}.$$

Finally, consider $(i, c) \in \mathcal{C}$. Note that $d_{\mathcal{H}}((i,c)) = \sum_{e \in E_c(G)} d_{i,e}$. Hence,

$$\begin{aligned}
d_{\mathcal{H}}((i,c)) &= |E_{i,c,1}| \cdot (1 \pm \sqrt{\varepsilon})pq\beta n(p'\beta^2 q^2 n)^{k-1} + |E_{i,c,2}| \cdot (1 \pm 3\sqrt{\varepsilon})p'q\beta n(p'\beta^2 q^2 n)^{k-1} \\
&\overset{(g),(28)}{=} (1 \pm \varepsilon^{1/3})((p'^2\beta n)(pq\beta n)(p'\beta^2 q^2 n)^{k-1} + (p'p\beta n)(p'q\beta n)(p'\beta^2 q^2 n)^{k-1}) \\
&= (1 \pm \varepsilon^{1/3})2pp'^{k+1}\beta^{2k}q^{2k-1}n^{k+1} = (1 \pm \varepsilon^{1/3})pp'^k\beta^{2k}q^{2k}n^{k+1},
\end{aligned}$$

since $p' = q/2$. $-$

Claim 3: $\Delta^c(\mathcal{H}) \leq 4k^2 n^k$.

Proof of claim: Clearly, the codegrees of pairs in $\mathcal{R}^* \times \mathcal{R}^*$ and $\mathcal{W} \times \mathcal{W}$ are 0. Moreover, by Claim 1, we have $d_{i,e} \leq n^k$ for all $i \in [t]$ and $e \in E(G_i)$. This implies that the codegrees of pairs in $E(G) \times \mathcal{R}^*$, $E(G) \times \mathcal{V}$, $E(G) \times \mathcal{W}$ and $E(G) \times \mathcal{C}$ are at most n^k, as required. It is also easy to see that the codegrees of pairs in $\mathcal{R}^* \times \mathcal{V}$, $\mathcal{R}^* \times \mathcal{W}$, $\mathcal{V} \times \mathcal{V}$ and $\mathcal{V} \times \mathcal{W}$ are at most n^k, since for fixed i, we always have at most $|\mathcal{R}_i| \leq n$ choices for R and at most n choices for each remaining vertex.

Consider distinct $e, e' \in E(G)$. There are $t \leq n$ choices for i. If $e = vw$ and $e' = v'w$ with $v, v' \in V_i$ and $w \in W_i$, then there are at most n choices for R and at most n^{k-2} choices for $T \setminus \{v, v'\}$. Otherwise, we may assume that $e = uv$ for $u \in U_i', v \in V_i$ and e' is incident to a vertex $x \in (V_i \cup W_i) \setminus \{v\}$. Now u determines R and there are at most n^{k-1} choices for $(T \cup \{w\}) \setminus \{v, x\}$. Altogether, we conclude that the codegree of e, e' is at most n^k.

Next, consider $(i, c), (i, c') \in \mathcal{C}$ with $c \neq c'$. We have to provide an upper bound for the number of R, T, w, π for which $f_{i,R,T,w,\pi} \in E(\mathcal{H})$ and $S_{R,T,w,\pi}$ contains a c-edge e and a c'-edge e'. To count these possibilities, we distinguish some cases regarding how e, e' intersect U_i', V_i, W_i. First, assume that $e, e' \in E_G(U_i', V_i)$. In this case, there are at most n choices for R and then at most $k(k-1)$ choices for e, e'. Moreover, since e, e' must form a matching, two vertices of T are determined. This leaves at most n^{k-1} choices for the remaining vertices, which yields a total of $k^2 n^k$ choices in this case. Next, assume that $e, e' \in E_G(V_i, W_i)$. In this case, there are at most n choices for w, which then determines e and e' and thus two vertices from T. There are at most n choices for R and at most n^{k-2} choices for the remaining vertices of T. Finally, assume that $e \in E_G(U_i', V_i), e' \in E_G(V_i, W_i)$. We divide this case into two subcases. First, assume that e, e' share their endpoint v in V_i. Then we have at most n choices for v, which determines e, e', which in turn determines R and w, and leaves at most n^{k-1} choices for the vertices in $T \setminus \{v\}$. On the other hand, if e, e' form a matching, then we have at most n^2 choices for e, e', which determines R and w as before and leaves at most n^{k-2} choices for the remaining vertices in T. Thus, altogether, the codegree of $(i, c), (i, c')$ is at most $k^2 n^k + n^k + 2n^k \leq 4k^2 n^k$.

Next, consider $(i, R) \in \mathcal{R}^*$ and $(i, c) \in \mathcal{C}$. We have to choose a c-edge e. If $e \in E_G(U_i', V_i)$, there are at most k choices for e, which also fixes one vertex of T, and leaves at most n^k choices for the remaining vertices. If $e \in E_G(V_i, W_i)$, then there are at most n choices for e, which fixes w and one vertex from T, and leaves at most n^{k-1} choices for the remaining vertices. Thus, (i, R) and (i, c) have codegree at most $(k+1)n^k$.

Finally, consider $(i, x) \in \mathcal{V} \cup \mathcal{W}$ and $(i, c) \in \mathcal{C}$. We have to choose a c-edge e. If e is incident with x, then there is only one choice for e. This either fixes R, in which case at most n^k choices are left for the remaining vertices, or it fixes another vertex from $T \cup \{w\}$, in which case there are at most n choices for R and at most n^{k-1} choices for the remaining vertices. If e is not incident with x, then there are at most n choices for e. However, this either fixes R and leaves at most n^{k-1} choices for the remaining vertices, or it fixes two more vertices, which leaves at most n choices for R and at most n^{k-2} choices for the remaining vertices. Thus, (i, x) and (i, c) have codegree at most $2n^k$. $-$

For $v \in V$, let $\mathcal{R}_v$ be the set of all $(i, R) \in \mathcal{R}^*$ with $v \in R$, and let $\mathcal{V}_v$ be the set of all pairs (i, v) with $i \in [t]$ and $v \in V_i \cup W_i$. For a colour $c \in C$, let $\mathcal{C}_c$ be the set of all pairs (i, c) with $i \in [t]$ and $c \in C_i$. Let

$$\mathcal{F} := \{\{i\} \times \mathcal{R}_i, \{i\} \times (V_i \cup W_i), \{i\} \times C_i \ : \ i \in [t]\} \cup \{\mathcal{R}_v, \mathcal{V}_v, \partial_G(v) \ : \ v \in V\} \cup \{\mathcal{C}_c, E_c(G) \ : \ c \in C\}.$$

Now, apply Theorem 7 to obtain a $(\xi, \mathcal{F})$-perfect matching $\mathcal{M}$ in $\mathcal{H}$. For each $i \in [t]$, let $\mathcal{R}_i'$ be the set of all $R \in \mathcal{R}_i$ with $(i, R) \notin V(\mathcal{M})$. Since $\{i\} \times \mathcal{R}_i \in \mathcal{F}$, we have $|\mathcal{R}_i'| \leq \xi|\mathcal{R}_i| \leq \xi n$. For each $R \in \mathcal{R}_i \setminus \mathcal{R}_i'$, there is a unique edge $f_{i,R,T,w,\pi}$ in $\mathcal{M}$ which covers (i, R). Let $S_{i,R} := S_{R,T,w,\pi}$ and define $F_i := \bigcup_{R \in \mathcal{R}_i \setminus \mathcal{R}_i'} S_{i,R}$. By construction of $\mathcal{H}$, $F_1, \ldots, F_t$ are edge-disjoint subgraphs of G, and, for each $i \in [t]$, we have that F_i is a rainbow $(\mathcal{R}_i \setminus \mathcal{R}_i')$-connector with colours in C_i, and $V(F_i) \cap V(\mathcal{R}_i') = \emptyset$ and $V(F_i) \setminus U_i' \subseteq V_i \cup W_i$. Moreover, observe that

$$(29) \qquad (G - \bigcup_{i \in [t]} F_i, \{(V_i \cup W_i) \setminus V(F_i)\}_{i \in [t]}, \{C_i \setminus \phi(E(F_i))\}_{i \in [t]}) \text{ is } \xi n\text{-bounded.}$$

Indeed, (B1) holds since $\{i\} \times (V_i \cup W_i), \{i\} \times C_i \in \mathcal{F}$ for every $i \in [t]$. Similarly, (B2) holds since $\mathcal{V}_v, \partial_G(v) \in \mathcal{F}$ for every $v \in V$. Finally, (B3) holds since $\mathcal{C}_c, E_c(G) \in \mathcal{F}$ for every $c \in C$.

We will find the missing connectors using Lemma 15. Let H_i be an $\mathcal{R}_i'$-connector. Clearly, $V(\mathcal{R}_i')$ is an independent set in H_i, $\Delta(H_i) \leq k$ and $|V(H_i)|, |E(H_i)| \leq (2k + 1)|\mathcal{R}_i'| \leq 3k\xi n$. Moreover, for every vertex $x \in V$, the number of $i \in [t]$ for which $x \in V(\mathcal{R}_i')$, is at most $|\mathcal{R}_x| \leq \xi n$.

Using (e), we can thus apply Lemma 15 (with $G', \{V_i'\}_{i \in [t]}, \{C_i'\}_{i \in [t]}$ playing the roles of $G, \{V_i\}_{i \in [t]}, \{C_i\}_{i \in [t]}$) to find for each $i \in [t]$, an embedding $\psi_i \colon H_i \to G'$ such that $\psi_i(H_i)$ is rainbow with colours in C_i' and $\psi_i(x) = x$ for all $x \in V(\mathcal{R}_i')$ and $\psi_i(x) \in V_i'$ for all $x \in V(H_i) \setminus V(\mathcal{R}_i')$, and such that $\psi_1(H_1), \ldots, \psi_t(H_t)$ are edge-disjoint.

Finally, let $\tilde{F}_i := F_i \cup \psi_i(H_i)$. Clearly, $\tilde{F}_i$ is a rainbow $\mathcal{R}_i$-connector in $\tilde{G}[U_i \cup \tilde{V}_i]$ with colours in $\tilde{C}_i$, and $\tilde{F}_1, \ldots, \tilde{F}_t$ are edge-disjoint. Moreover, (iii) follows from (h), (29), (24) and (25).   □

5.5. **Rainbow perfect matchings.** Given a bipartite graph G with vertex classes V_1, V_2, we say that G is (ε, d)-*quasirandom* if for all $j \in [2]$ and distinct $v, v' \in V_j$, we have $d_G(v, V_{3-j}) = (1 \pm \varepsilon)d|V_{3-j}|$ and $|N_G(\{v, v'\}) \cap V_{3-j}| = (1 \pm \varepsilon)d^2|V_{3-j}|$.

The next lemma follows easily from a result of Coulson and Perarnau [11, Lemma 6]. Indeed, it is well known that the above notion of quasirandomness implies super-regularity, which is sufficient for the existence of a perfect matching. Moreover, using regularity, it is straightforward to count the number of 'switchable edges', as required in the general statement in [11].

Lemma 21. *Suppose $1/n \ll \varepsilon \ll d$. Let G be a bipartite graph with vertex classes A, B such that $|A| = |B| = n$ and G is (ε, d)-quasirandom. Then, given any edge-colouring of G where each colour appears at most εn times, there exists a rainbow perfect matching of G.*

We now use Lemma 21 to obtain several edge-disjoint rainbow perfect matchings.

Lemma 22. *Suppose $1/n \ll \mu \ll d$ and let $t \leq n$. Let V be a vertex set of size n and assume that $U_1, \ldots, U_t$ are subsets of V such that $|U_i| \geq \mu n$ for all $i \in [t]$, $|U_i \cap U_j| \leq 5\mu^2 n$ for all distinct $i, j \in [t]$, and, for every $v \in V$, the number of $i \in [t]$ for which $v \in U_i$ is at most $3\mu t$.*

For each $i \in [t]$, suppose U_i is partitioned into equal-sized sets A_i and B_i, and G_i is a $(\mu^{1/3}, d)$-quasirandom bipartite graph with vertex classes A_i, B_i. Assume that G_i is edge-coloured and each colour appears at most $2\mu^2 n$ times in G_i.

Then there exist edge-disjoint $M_1, \ldots, M_t$ such that M_i is a rainbow perfect matching of G_i for each $i \in [t]$.

We find $M_1, \ldots, M_t$ using a randomised greedy algorithm.

Proof. Let $r := \lceil 105\mu^{3/2} n \rceil$. Suppose that we have already found $M_1, \ldots, M_{s-1}$ for some $s \in [t]$. We now define M_s as follows. Let $H_{s-1} := \bigcup_{i=1}^{s-1} M_i$ and let $G'_s := G_s - H_{s-1}$. If $\Delta(H_{s-1}[U_s]) \leq \mu^{3/2} n$, then G'_s is $(\mu^{1/4}, d)$-quasirandom. Thus, by Lemma 21 used repeatedly, we can find edge-disjoint rainbow perfect matchings $M_{s,1}, \ldots, M_{s,r}$ of G'_s. Otherwise, if $\Delta(H_{s-1}[U_s]) > \mu^{3/2} n$, let $M_{s,1}, \ldots, M_{s,r}$ be empty graphs on U_s. In either case, pick $j \in [r]$ uniformly at random and let $M_s := M_{s,j}$. The lemma clearly follows if the following holds with positive probability:

$$(30) \qquad\qquad \Delta(H_{s-1}[U_s]) \leq \mu^{3/2} n \text{ for all } s \in [t].$$

For $s \in [t]$ and $u \in U_s$, let $J^{s,u}$ be the set of indices $i \in [s-1]$ such that $u \in U_i$, so that $|J^{s,u}| \leq 3\mu t$, and for $i \in J^{s,u}$, let $Y_i^{s,u}$ be the indicator variable of the event that $uu' \in E(M_i)$ for some $u' \in U_s$. Observe that

$$d_{H_{s-1}[U_s]}(u) = \sum_{i \in J^{s,u}} Y_i^{s,u}.$$

Now, fix $s \in [t]$ and $u \in U_s$. Crucially, for any $i \in J^{s,u}$, since $|U_s \cap U_i| \leq 5\mu^2 n$, at most $5\mu^2 n$ of the matchings $M_{i,1}, \ldots, M_{i,r}$ that we picked in G'_i contain an edge incident to u in G_s (regardless of the previous choices). Let $i_1, \ldots, i_{|J^{s,u}|}$ be the enumeration of $J^{s,u}$ in increasing order. By the above, for all $\ell \in [|J^{s,u}|]$, we have

$$\mathbb{P}\left(Y_{i_\ell}^{s,u} = 1 \mid Y_{i_1}^{s,u}, \ldots, Y_{i_{\ell-1}}^{s,u}\right) \leq \frac{5\mu^2 n}{r} \leq \frac{\mu^{1/2}}{21}.$$

Let $B \sim Bin(|J^{s,u}|, \mu^{1/2}/21)$. Since $|J^{s,u}| \leq 3\mu n$, we have $\mathbb{E}(B) \leq \mu^{3/2} n / 7$. Using Fact 12 and Lemma 11(ii), we infer that

$$\mathbb{P}\left(\sum_{i \in J^{s,u}} Y_i^{s,u} > \mu^{3/2} n\right) \leq \mathbb{P}\left(B > \mu^{3/2} n\right) \leq e^{-\mu^{3/2} n}.$$

Finally, a union bound implies that (30) holds with high probability. $\qquad\square$

5.6. Proof of Theorem 4. We are now ready to prove our main theorem.

Proof of Theorem 4. Choose new constants $\varepsilon, \gamma, \xi, \mu, \eta > 0$ such that

$$1/n \ll \varepsilon \ll \gamma \ll \xi \ll \mu \ll \eta \ll 1,$$

and let

$$t := n/2 \qquad r := \lceil (\eta/256 + 6\eta + 3\gamma) n \rceil \qquad b := \lceil (\mu - \xi^{1/3}) n \rceil.$$

Let ϕ be a 1-factorization of K_n with vertex set V and colour set C. We will obtain a decomposition into t rainbow copies of $T_{n;r,b}$ (cf. Definition 5). Hence, r and b are essentially determined by η and μ, respectively, and ε, γ, ξ are best thought of as error parameters.

In order to apply the lemmas that we have proven in this section without interference, we will split $E(K_n)$, V and C into random subsets each reserved for the application of the relevant lemma. For convenience, we now define the relevant constants in one place (where the letters p, q, β represent vertex, colour and edge probabilities, respectively).

$$p_{\mathrm{rb}} := 2\eta \qquad q_{\mathrm{rb}} := \eta/192 \qquad p_{\mathrm{mc}} := 3072\eta \qquad q_{\mathrm{mc}} := 6\eta$$

$$m := \lceil (\eta - \varepsilon/5)n \rceil \qquad s := \lceil (q_{\mathrm{rb}}/4 - 2\gamma/5)n \rceil$$

$$\tilde{p}' := p_{\mathrm{rb}} + p_{\mathrm{mc}} \qquad \tilde{p} := \eta/256 + 6\eta \qquad \tilde{\beta} := \tilde{q} := 2\tilde{p}'$$

$$p_\circ := 1 - \tilde{p}'(1 + \gamma) - (\tilde{p}' + \tilde{p})(1 + \xi) - 2\mu$$

$$q_{\circ,1} := 1 - q_{\mathrm{rb}} - q_{\mathrm{mc}}(1 + \gamma) - \tilde{q}(1 + \xi) - \mu \qquad q_{\circ,2} := p_\circ - q_{\circ,1}$$

$$\beta_{\circ,1} := 1 - 8\eta - \tilde{\beta}(1 + \xi) - \mu \qquad \beta_{\circ,2} := p_\circ - \beta_{\circ,1}$$

$$q_\triangle := (q_{\mathrm{rb}}/2 - q_{\circ,2})/3 \qquad \beta_\triangle := (4\eta - \eta(1 + \gamma) - \beta_{\circ,2})/3.$$

Note that, as $\gamma \ll \xi$, $q_{\circ,2} = \eta/768 - \mu \pm \xi$ and $\beta_{\circ,2} = 2\eta - \eta/256 - \mu \pm \xi$ and hence $q_\triangle \geq \eta/2304$ and $\beta_\triangle \geq \eta/3$.

Step 1: Random splitting

Split vertices. For each $i \in [t]$, we split V randomly as follows:

$$
\begin{array}{ccccccccccc}
V & = & U_i & \uplus & \tilde{V}_i & \uplus & V_i^\circ & \uplus & A_i & \uplus & B_i \\
1 & = & \tilde{p}'(1 + \gamma) & + & (\tilde{p}' + \tilde{p})(1 + \xi) & + & p_\circ & + & \mu & + & \mu.
\end{array}
$$

We split U_i and B_i further as follows:

$$
\begin{array}{ccccc}
U_i & = & V_i^{\mathrm{rb}} & \uplus & V_i^{\mathrm{mc}} \\
\tilde{p}'(1 + \gamma) & = & p_{\mathrm{rb}}(1 + \gamma) & + & p_{\mathrm{mc}}(1 + \gamma)
\end{array}
\quad \text{and} \quad
\begin{array}{ccccc}
B_i & = & B_{i,1} & \uplus & B_{i,2} \\
\mu & = & \mu/2 & + & \mu/2.
\end{array}
$$

Split colours. Moreover, for each $i \in [t]$, we split C randomly as follows:

$$
\begin{array}{ccccccccccccc}
C & = & C_{i,1} & \uplus & C_{i,2} & \uplus & D_i & \uplus & \tilde{C}_i & \uplus & C_i^\bullet & \uplus & C_i^{\circ,1} \\
1 & = & q_{\mathrm{rb}}/2 & + & q_{\mathrm{rb}}/2 & + & q_{\mathrm{mc}}(1 + \gamma) & + & \tilde{q}(1 + \xi) & + & \mu & + & q_{\circ,1}.
\end{array}
$$

We split $C_{i,1}$ further as follows:

$$
\begin{array}{ccccccccc}
C_{i,1} & = & C_{i,1}^\triangle & \uplus & C_{i,2}^\triangle & \uplus & C_{i,3}^\triangle & \uplus & C_i^{\circ,2} \\
q_{\mathrm{rb}}/2 & = & q_\triangle & + & q_\triangle & + & q_\triangle & + & q_{\circ,2}.
\end{array}
$$

Let $C_i^\circ := C_i^{\circ,1} \cup C_i^{\circ,2}$. Hence, C_i° is a $p_\circ$-random set. Moreover, let $C_i^\triangle := C_{i,1}^\triangle \cup C_{i,2}^\triangle \cup C_{i,3}^\triangle$.

Split edges. We split K_n randomly as follows:

$$
\begin{array}{ccccccccccccc}
K_n & = & G_1 & \uplus & G_2 & \uplus & \tilde{G} & \uplus & G^\bullet & \uplus & G^{\circ,1} \\
1 & = & 4\eta & + & 4\eta & + & \tilde{\beta}(1 + \xi) & + & \mu & + & \beta_{\circ,1}.
\end{array}
$$

Split G_1 further as follows:

$$
\begin{array}{ccccccccccc}
G_1 & = & G^{\mathrm{rb}} & \uplus & G_1^\triangle & \uplus & G_2^\triangle & \uplus & G_3^\triangle & \uplus & G^{\circ,2} \\
4\eta & = & \eta(1 + \gamma) & + & \beta_\triangle & + & \beta_\triangle & + & \beta_\triangle & + & \beta_{\circ,2}.
\end{array}
$$

Let $G^\circ := G^{\circ,1} \cup G^{\circ,2}$. Thus, G° is a $p_\circ$-random subgraph. Moreover, let $G^\triangle := G_1^\triangle \cup G_2^\triangle \cup G_3^\triangle$.

Create the edge reservoir. By Lemma 19 (with $G_1, G_2, G^{\mathrm{rb}} \cup G^\triangle, G^{\circ,2}, \{V_i^{\mathrm{mc}}, D_i\}_{i \in [t]}$ in place of $G_1', G_2', G_{1,1}', G_{1,2}', \{V_i, D_i\}_{i \in [t]}$), with high probability, there exist G_1', G_2' such that

$$G^{\mathrm{rb}} \cup G^\triangle \subseteq G_1' \subseteq G_1 \text{ and } G_2' \subseteq G_2$$

with $\Delta(G_2 - G_2') \leq 2\varepsilon n$ and, for each $i \in [t]$, there exists $D_i' \subseteq D_i$ of size $(1 \pm 2\gamma)q_{\mathrm{mc}}n$ and vertex-disjoint matchings $\{M_{i,c}'\}_{c \in D_i'}$ in $(G_1' \cup G_2')[V_i^{\mathrm{mc}}]$, where $M_{i,c}'$ consists of 256 c-edges, such that altogether the following hold:

(M1) $|E_c(G_1')| = |E_c(G_2')| = 2m$ and $|\{i \in [t] : c \in D_i'\}| = 3m$ for all $c \in C$;

(M2) for any subset $E^* \subseteq E(G_1')$ which consists of precisely m edges of each colour $c \in C$, there exists a partition of $E^* \cup E(G_2')$ into sets $J_1', \ldots, J_t'$, such that for each $i \in [t]$, J_i' contains exactly one edge from each of $\{M_{i,c}'\}_{c \in D_i'}$;

(M3) every vertex $v \in V$ is covered by $(1 \pm \gamma)p_{\mathrm{mc}}t$ of the matchings $\{M'_{i,c} : i \in [t], c \in D'_i\}$.

Create colour reservoirs. We apply Lemma 18 (with $G^{\mathrm{rb}}, \{V_i^{\mathrm{rb}}, C_{i,1}, C_{i,2}, C_i^{\triangle}, C_i^{\circ,2}\}_{i \in [t]}$ in place of $G, \{V_i, C_{i,1}, C_{i,2}, C_{i,1,1}, C_{i,1,2}\}_{i \in [t]}$) to see that with high probability, for each $i \in [t]$, there exist
$$C_i^{\triangle} \subseteq C'_{i,1} \subseteq C_{i,1} \text{ and } C'_{i,2} \subseteq C_{i,2}$$
and vertex-disjoint rainbow matchings $\{M_{i,j} : j \in [3s]\}$ in $G^{\mathrm{rb}}[V_i^{\mathrm{rb}}]$, such that altogether the following hold:

(R1) for each $i \in [t]$, $|C'_{i,1}| = |C'_{i,2}| = 2s$;
(R2) for all $c \in C$, $|\{i \in [t] : c \in C_{i,2} \setminus C'_{i,2}\}| \leq \sqrt{\gamma}n$;
(R3) for each $i \in [t]$, $M_i := \bigcup_{j \in [3s]} M_{i,j}$ consists of 192 c-edges for each $c \in C'_{i,1} \cup C'_{i,2}$;
(R4) for each $i \in [t]$ and any subset $C_i^* \subseteq C'_{i,1}$ of size s, there exists $J_i \subseteq M_i$ such that J_i is $(C_i^* \cup C'_{i,2})$-rainbow and contains exactly one edge from each of $\{M_{i,j} : j \in [3s]\}$;
(R5) the matchings $\{M_{i,j} : (i,j) \in [t] \times [3s]\}$ are edge-disjoint, and $|M_{i,j}| = 256$ for all $(i,j) \in [t] \times [3s]$;
(R6) for every vertex $v \in V$, the number of $i \in [t]$ for which v is covered by M_i is $(1 \pm \sqrt{\gamma})p_{\mathrm{rb}}t$.

Create short paths for the vertex absorption. By Lemma 16 (with $\varepsilon/5, 2\xi^{1/3}, \mu$ playing the roles of γ, κ, p), with high probability, there exist edge-disjoint rainbow paths $Q_1, \ldots, Q_t$ in $G^{\bullet}$ such that

(Q1) for each $i \in [t]$, we have $V(Q_i) \subseteq A_i$ and $\phi(E(Q_i)) \subseteq C_i^{\bullet}$;
(Q2) $(G^{\bullet} - \bigcup_{i \in [t]} Q_i, \{A_i \setminus V(Q_i)\}_{i \in [t]}, \{C_i^{\bullet} \setminus \phi(E(Q_i))\}_{i \in [t]})$ is εn-bounded;
(Q3) for each $v \in V$, the number of $i \in [t]$ for which $v \in V(Q_i)$ and the subpath from v to one of the endvertices of Q_i has length at most $2\xi^{1/3}n$, is at most $\xi^{1/4}n$.

Properties for vertex absorption and covering non-reservoir edges/colours/vertices. For $i \in [t]$, let $G^{(i)}$ be the subgraph of $G_3^{\triangle}[A_i, B_i]$ containing precisely the $C_{i,3}^{\triangle}$-edges. In addition to the above, with high probability, the following hold:

(A1) for all $i \in [t]$ and $c \in C$, there are at most $2\mu^2 n$ c-edges between A_i and B_i;
(A2) for all distinct $i, i' \in [t]$, we have $|(A_i \cup B_i) \cap (A_{i'} \cup B_{i'})| \leq 5\mu^2 n$;
(A3) for every $v \in V$, the number of $i \in [t]$ for which $v \in A_i \cup B_i$, is at most $3\mu t$;
(A4) for all $i \in [t]$, $|A_i|, |B_i| = (1 \pm \varepsilon)\mu n$ and $G^{(i)}$ is $(\varepsilon, \beta_{\triangle}q_{\triangle})$-quasirandom;
(A5) for all $i \in [t]$ and all $S \subseteq V$ with $|S| \leq 512$, we have that
$$|\{v \in N_{G_1^{\triangle}}(S) \cap B_{i,1} : \phi(uv) \in C_{i,1}^{\triangle} \text{ for each } u \in S\}| \geq \mu^2 n;$$

(A6) for all $e \in E(K_n)$, the number of $i \in [t]$ for which e intersects A_i and $B_{i,2}$, and $\phi(e) \in C_{i,2}^{\triangle}$, is at least $q_{\triangle}\mu^2 n/3$;
(A7) for all $i \in [t]$ and $c \in C$, the number of c-edges in $E_{G_2^{\triangle}}(A_i, B_{i,2})$, is at least $\beta_{\triangle}\mu^2 n/3$;
(A8) for all $i \in [t]$, $|C_{i,2}| = (1 \pm \varepsilon)q_{\mathrm{rb}}n/2$ and $|D_i| = (1 \pm 2\gamma)q_{\mathrm{mc}}n$;
(A9) for all $c \in C$, $|E_c(G_2)| = (1 \pm \varepsilon)2\eta n$ and $|\{i \in [t] : c \in D_i\}| = (1 \pm \sqrt{\gamma})q_{\mathrm{mc}}t$.

Here, to deal with the sizes of the common neighbourhoods in (A4) and (A5), we use McDiarmid's inequality. For all other claims, Chernoff's bound suffices.

Find almost-spanning paths. We now find an approximate decomposition of G° (and thus K_n) into $t = n/2$ almost spanning rainbow paths. By Lemma 16 (with $\varepsilon/5, \varepsilon^2, p_{\circ}$ playing the roles of γ, κ, p), with high probability, there exist edge-disjoint rainbow paths $P_1, \ldots, P_t$ in G° such that

(P1) for each $i \in [t]$, we have $V(P_i) \subseteq V_i^{\circ}$ and $\phi(E(P_i)) \subseteq C_i^{\circ}$;
(P2) $(G^{\circ} - \bigcup_{i \in [t]} P_i, \{V_i^{\circ} \setminus V(P_i)\}_{i \in [t]}, \{C_i^{\circ} \setminus \phi(E(P_i))\}_{i \in [t]})$ is εn-bounded;
(P3) every vertex $v \in V$ is an endvertex of at most εn paths.

Establish connection properties. By Lemma 20 (with $\gamma^{1/3}, \xi, \tilde{p}', \tilde{p}, 512$ playing the roles of $\varepsilon, \gamma, p', p, k$), with high probability, the following is true:

(C) Let $\mathcal{R}$ be any 512-uniform (multi-)hypergraph which is the union of t matchings $\mathcal{R}_1, \ldots, \mathcal{R}_t$ such that $V(\mathcal{R}_i) \subseteq U_i$ and $|\mathcal{R}_i| = (1 \pm \gamma^{1/3})\tilde{p}n$ for all $i \in [t]$, and such that $d_{\mathcal{R}}(x) = (1 \pm \gamma^{1/3})\tilde{p}'t$ for all $x \in V$. Then, for each $i \in [t]$, there exists an $\mathcal{R}_i$-connector $\tilde{F}_i$ in $\tilde{G}[U_i \cup \tilde{V}_i]$ such that the following hold:
   (C1) $\tilde{F}_1, \ldots, \tilde{F}_t$ are edge-disjoint;
   (C2) for each $i \in [t]$, $\tilde{F}_i$ is rainbow with colours in $\tilde{C}_i$;
   (C3) $(\tilde{G} - \bigcup_{i \in [t]} \tilde{F}_i, \{(U_i \cup \tilde{V}_i) \setminus V(\tilde{F}_i)\}_{i \in [t]}, \{\tilde{C}_i \setminus \phi(E(\tilde{F}_i))\}_{i \in [t]})$ is $2\xi n$-bounded.

Henceforth, we assume that all random choices have been made and satisfy the above properties.

Step 2: Connecting the pieces

For each $i \in [t]$, we now aim to connect the matchings $\{M_{i,j} : j \in [3s]\}$ and $\{M'_{i,c} : c \in D'_i\}$. For this, we will define a 512-uniform matching $\mathcal{R}_i$, which consists of 256 vertices of one matching and 256 vertices of the next matching, and then apply (C).

To make this more precise, for each $i \in [t]$, let

$$(31) \qquad r_i := 3s + |D'_i| \quad \text{and} \quad \mathcal{M}_i := \{M_{i,j} : j \in [3s]\} \cup \{M'_{i,c} : c \in D'_i\}.$$

So $r_i = |\mathcal{M}_i|$. Note that $r_i = \tilde{p}n \pm 3\gamma n$ and hence $0 \leq r - r_i \leq 7\gamma n$. Also note that since V_i^{rb} and V_i^{mc} are disjoint, all the matchings in $\mathcal{M}_i$ are vertex-disjoint, and recall that each matching consists of 256 edges. For each $i \in [t]$, find two distinct $M_i^-, M_i^+ \in \mathcal{M}_i$ such that altogether,

$$(32) \qquad \text{each vertex } v \in V \text{ is contained in } V(M_i^- \cup M_i^+) \text{ for at most } \eta^{-2} \text{ indices } i \in [t].$$

This can clearly be done greedily. Now, for each $i \in [t]$, choose an arbitrary bijection $\sigma_i \colon [r_i] \to \mathcal{M}_i$ such that $\sigma_i(1) = M_i^-$ and $\sigma_i(r_i) = M_i^+$, and partition for each matching $M \in \mathcal{M}_i$ the vertices $V(M)$ arbitrarily into a 'tail set' $T(M)$ and a 'head set' $H(M)$ such that M is a perfect matching between $T(M)$ and $H(M)$. Define

$$\mathcal{R}_i := \{H(\sigma_i(k)) \cup T(\sigma_i(k+1)) : k \in [r_i - 1]\}.$$

Hence, $\mathcal{R}_i$ is a 512-uniform matching in U_i. Note that

$$|\mathcal{R}_i| = r_i - 1 = (1 \pm \sqrt{\gamma})\tilde{p}n.$$

Let $\mathcal{R} := \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_t$. By (R6), (M3) and (32), we have that $d_{\mathcal{R}}(x) = (1 \pm 2\sqrt{\gamma})(p_{\mathrm{rb}} + p_{\mathrm{mc}})t = (1 \pm 2\sqrt{\gamma})\tilde{p}'t$ for all $x \in V$. Hence, applying (C), for each $i \in [t]$, there exists an $\mathcal{R}_i$-connector $\tilde{F}_i$ in $\tilde{G}[U_i \cup \tilde{V}_i]$ such that altogether the following hold:

(C1′) $\tilde{F}_1, \ldots, \tilde{F}_t$ are edge-disjoint;
(C2′) for each $i \in [t]$, $\tilde{F}_i$ is rainbow with colours in $\tilde{C}_i$;
(C3′) $(\tilde{G} - \bigcup_{i \in [t]} \tilde{F}_i, \{(U_i \cup \tilde{V}_i) \setminus V(\tilde{F}_i)\}_{i \in [t]}, \{\tilde{C}_i \setminus \phi(E(\tilde{F}_i))\}_{i \in [t]})$ is $2\xi n$-bounded.

For each $i \in [t]$, let

$$F_i := P_i \cup Q_i \cup \tilde{F}_i.$$

We will eventually have $F_i \subseteq T_i$ for all $i \in [t]$. Note that $F_1, \ldots, F_t$ are edge-disjoint rainbow forests in $K_n - (G^{\mathrm{rb}} \cup G^\Delta \cup G_2)$. Moreover, for each $i \in [t]$, $V(F_i) \subseteq V \setminus B_i$ and $\phi(E(F_i)) \subseteq C \setminus (C_i^\Delta \cup C_{i,2} \cup D_i)$. Let

$$(33) \qquad \hat{V}_i := V \setminus (V(F_i) \cup B_i),$$

$$(34) \qquad \hat{C}_i := C \setminus (\phi(E(F_i)) \cup C'_{i,1} \cup C'_{i,2} \cup D'_i),$$

$$(35) \qquad \hat{G} := K_n - \bigcup_{i \in [t]} F_i - (G'_1 \cup G'_2).$$

We think of the above as leftover sets. The following claim asserts that this leftover is well-behaved.

Claim 1: $(\hat{G}, \{\hat{V}_i\}_{i\in[t]}, \{\hat{C}_i\}_{i\in[t]})$ *is* $\sqrt{\xi}n$*-bounded.*

Proof of claim: Observe that

$$\hat{V}_i = ((U_i \cup \tilde{V}_i) \setminus V(\tilde{F}_i)) \cup (V_i^{\circ} \setminus V(P_i)) \cup (A_i \setminus V(Q_i)),$$

$$\hat{C}_i \subseteq ((C_{i,2} \cup D_i) \setminus (C'_{i,2} \cup D'_i)) \cup (\tilde{C}_i \setminus \phi(E(\tilde{F}_i))) \cup (C_i^{\circ} \setminus \phi(E(P_i))) \cup (C_i^{\bullet} \setminus \phi(E(Q_i))),$$

$$\hat{G} \subseteq (G_2 - G'_2) \cup \left(\tilde{G} - \bigcup_{i\in[t]} \tilde{F}_i\right) \cup \left(G^{\circ} - \bigcup_{i\in[t]} P_i\right) \cup \left(G^{\bullet} - \bigcup_{i\in[t]} Q_i\right).$$

Recall that $\Delta(G_2 - G'_2) \leq 2\varepsilon n$ and $|D'_i| = (1 \pm 2\gamma)q_{\mathrm{mc}}n$ for all $i \in [t]$. Thus, (R1) and (A8) imply that, for all $i \in [t]$,

$$|(C_{i,2} \cup D_i) \setminus (C'_{i,2} \cup D'_i)| \leq \gamma^{1/3}n$$

and (R2), (M1) and (A9) imply that, for all $c \in C$,

$$|E_c(G_2 - G'_2)|, |\{i \in [t] \,:\, c \in (C_{i,2} \cup D_i) \setminus (C'_{i,2} \cup D'_i)\}| \leq \gamma^{1/3}n.$$

Hence, the claim follows together with (P2), (Q2) and (C3'). $-$

We now use Lemma 15 to join the pieces of each F_i together. Moreover, since the sets $\mathcal{R}_i$ have different sizes, we artificially add some structure that will ensure that ultimately, all trees are isomorphic to T (cf. (†) below). In this process we can cover all remaining vertices outside the vertex reservoir B_i.

For $i \in [t]$, let v_i^-, v_i^+ be the endvertices of P_i, and let w_i^-, w_i^+ be the endvertices of Q_i. Let

$$X_i := T(M_i^-) \cup H(M_i^+) \cup \{v_i^-, v_i^+, w_i^-\} \cup \hat{V}_i.$$

We now define a graph H_i in which X_i is independent and all other vertices are new vertices. Take new vertices $z_0, z_{r_i}, \ldots, z_r$. For each $x \in T(M_i^-)$, add a path of length 2 between x and z_0, and for each $x \in H(M_i^+)$, add a path of length 2 between x and z_{r_i}. For each $k \in \{r_i, \ldots, r-1\}$, add a path of length 5 between z_k and z_{k+1}. For each $k \in \{r_i+1, \ldots, r-1\}$, add 510 further paths of length 2 onto z_k (so z_k will be an endvertex of those paths of length 2), and add 255 paths of length 2 onto each of z_{r_i} and z_r. Connect z_r and v_i^- by a path which contains $\hat{V}_i$ such that $\hat{V}_i$ is an independent set. Finally, add a path of length 2 between v_i^+ and w_i^-. Clearly, $\Delta(H_i) \leq 512$. Note that since $|\hat{V}_i| \leq \sqrt{\xi}n$, $r - r_i \leq 7\gamma n$ and $|B_i| - b = \xi^{1/3}n \pm \varepsilon n$ by (A4), we can choose H_i in such a way that

(36) $|V(H_i) \setminus X_i| = |B_i| - b$ and $|V(H_i)|, |E(H_i)| \leq 2\xi^{1/3}n.$

Also, for every $v \in V$, by (32), (P3), (Q3) and Claim 1, the number of $i \in [t]$ for which $v \in X_i$, is at most $2\xi^{1/4}n$.

By (A5), we can now apply Lemma 15 (with $G_1^{\triangle}, \{C_{i,1}^{\triangle}\}_{i\in[t]}, \{B_{i,1}\}_{i\in[t]}$ taking the place of $G, \{C_i\}_{i\in[t]}, \{V_i\}_{i\in[t]}$) to obtain, for each $i \in [t]$, an embedding $\psi_i \colon H_i \to G_1^{\triangle}$ such that $\psi_i(H_i)$ is rainbow with colours in $C_{i,1}^{\triangle}$, $\psi_i(x) = x$ for all $x \in X_i$, $\psi_i(x) \in B_{i,1}$ for all $x \in V(H_i) \setminus X_i$ and such that $\psi_1(H_1), \ldots, \psi_t(H_t)$ are edge-disjoint.

For each $i \in [t]$, let

$$F_i^* := F_i \cup \psi_i(H_i) \text{ and } B_i^* := V \setminus V(F_i^*) \subseteq B_i.$$

Note that F_i^* is rainbow as $C_{i,1}^{\triangle} \subseteq C_i^{\triangle}$. By (36), we have that $|B_i^*| = b$. Moreover, $B_{i,2} \subseteq B_i^*$. Let A_i^* be the set of the last b vertices on Q_i, containing w_i^+, so that $A_i^* \subseteq A_i$. From (Q2), (Q3) and (A4), we deduce that, for each $i \in [t]$,

(37) $|A_i \setminus A_i^*| \leq |A_i| - b \leq 2\xi^{1/3}n,$

and, for each $v \in V$,

$$(38) \qquad |\{i \in [t] \,:\, v \in A_i \setminus A_i^*\}| \leq 2\xi^{1/4}n.$$

Crucially, observe that for each $i \in [t]$,

$(\dagger)$ any graph obtained from F_i^* by adding a perfect matching between A_i^* and B_i^* and exactly one edge from each of the matchings in $\mathcal{M}_i$, is isomorphic to $T_{n;r,b}$.

In particular,

$$(39) \qquad |E(F_i^*)| = n - 1 - b - r_i.$$

Step 3: Final absorption

We will find the perfect matchings between A_i^* and B_i^* using Lemma 22, and then select exactly one edge from each of the matchings in $\mathcal{M}_i$ using (R4) and (M2). For the last step to work, we need to ensure that all leftover colours are in $C_{i,1}' \cup C_{i,2}' \cup D_i'$ and all leftover edges are in $G_1' \cup G_2'$. Thus, prior to applying Lemma 22, we greedily deal with the colours in $\hat{C}_i$ and the edges in $\hat{\hat{G}}$.

Cover the remaining non-reservoir edges. First, find a partition of $E(\hat{G})$ into rainbow matchings $\hat{M}_1, \ldots, \hat{M}_t$ such that $\hat{M}_i \subseteq E_{\hat{G}}(A_i^*, B_{i,2})$, $\phi(\hat{M}_i) \subseteq C_{i,2}^{\triangle}$ and $|\hat{M}_i| \leq \xi^{1/3}n$. This can be done greedily. Indeed, suppose we want to assign to $e \in E(\hat{G})$ an index $i \in [t]$. Let $e = xy$ and $c := \phi(e)$. By (A6) and (38), the number of $i \in [t]$ for which $e \in E_{\hat{G}}(A_i^*, B_{i,2})$ and $c \in C_{i,2}^{\triangle}$, is at least $\mu^3 n$. By Claim 1, we have that $|E_c(\hat{G})|, d_{\hat{G}}(x), d_{\hat{G}}(y) \leq \sqrt{\xi}n$ and $|E(\hat{G})| \leq \sqrt{\xi}n^2$. Thus, there exists a suitable $i \in [t]$ such that no other c-edge of $\hat{G}$ has been assigned to i, and $\hat{M}_i$ does not yet cover x or y and contains at most $\xi^{1/3}n/2$ edges so far. Finally, by (35) we have that

$$(40) \qquad E(G_2') \subseteq E(G_2' \cup G^{\mathrm{rb}} \cup G_2^{\triangle} \cup G_3^{\triangle}) \subseteq E\left(K_n - \bigcup_{i \in [t]}(F_i^* \cup \hat{M}_i)\right) \subseteq E(G_1' \cup G_2').$$

Cover the remaining non-reservoir colours. Next, find edge-disjoint matchings $\hat{M}_1', \ldots, \hat{M}_t'$ in $G_2^{\triangle}$ such that, for each $i \in [t]$, $V(\hat{M}_i') \cap V(\hat{M}_i) = \emptyset$, $\hat{M}_i' \subseteq E_{G_2^{\triangle}}(A_i^*, B_{i,2})$, and $\hat{M}_i'$ consists of exactly one c-edge for each $c \in \hat{C}_i$. (Hence, $|\hat{M}_i'| = |\hat{C}_i|$.) This can also be done greedily. Indeed, suppose we want to add a c-edge to $\hat{M}_i'$. By (A7) and (37), there are at least $\mu^3 n$ c-edges in $E_{G_2^{\triangle}}(A_i^*, B_{i,2})$. By Claim 1, we have that $|\hat{C}_i| \leq \sqrt{\xi}n$ and $|\{i \in [t] \,:\, c \in \hat{C}_i\}| \leq \sqrt{\xi}n$. Also recall that $|\hat{M}_i| \leq \xi^{1/3}n$. Hence, there exists a suitable c-edge which has not been used by another matching $\hat{M}_{i'}'$, and whose endvertices are not covered by $\hat{M}_i$ or yet by $\hat{M}_i'$. Hence, for each $i \in [t]$, we have by (34) that

$$(41) \qquad C_{i,2}' \cup D_i' \subseteq C_{i,2}' \cup D_i' \cup C_{i,3}^{\triangle} \subseteq C \setminus \phi(E(F_i^*) \cup \hat{M}_i \cup \hat{M}_i') \subseteq C_{i,1}' \cup C_{i,2}' \cup D_i'.$$

Absorb the uncovered vertices. We now extend $F_i^* \cup \hat{M}_i \cup \hat{M}_i'$ into a spanning forest by adding a matching $M_i^{\triangle}$. For each $i \in [t]$, let $A_i' := A_i^* \setminus V(\hat{M}_i \cup \hat{M}_i')$ and $B_i' := B_i^* \setminus V(\hat{M}_i \cup \hat{M}_i')$. We aim to apply Lemma 22 (with $\mu, \beta_{\triangle}q_{\triangle}, \{A_i', B_i', G^{(i)}[A_i', B_i']\}_{i \in [t]}$ playing the roles of $\mu, d, \{A_i, B_i, G_i\}_{i \in [t]}$). (Recall that $G^{(i)}$ was defined just before (A1).) Clearly, $|A_i'| = |B_i'| = b - |\hat{M}_i \cup \hat{M}_i'| \geq (\mu - 3\xi^{1/3})n$. In particular, by (A4), $|A_i \setminus A_i'|, |B_i \setminus B_i'| \leq 4\xi^{1/3}n$ and thus $G^{(i)}[A_i', B_i']$ is $(\mu^{1/3}, \beta_{\triangle}q_{\triangle})$-quasirandom. Finally, since $A_i' \subseteq A_i$ and $B_i' \subseteq B_i$, the remaining conditions for Lemma 22 follow immediately from (A1), (A2) and (A3). Therefore, we can find edge-disjoint $M_1^{\triangle}, \ldots, M_t^{\triangle}$ such that $M_i^{\triangle}$ is a rainbow perfect matching of $G^{(i)}[A_i', B_i']$ for each $i \in [t]$.

Absorb the uncovered colours. Now, for each $i \in [t]$, let

$$(42) \qquad M_i^* := M_i^{\triangle} \cup \hat{M}_i \cup \hat{M}_i' \quad \text{and} \quad C_i^* := C_{i,1}' \setminus \phi(E(F_i^*) \cup M_i^*).$$

Note that by (41), M_i^* is a rainbow perfect matching between A_i^* and B_i^*. Similarly, by (41) and (42), $F_i^* \cup M_i^*$ is rainbow and $\phi(E(F_i^*) \cup M_i^*) = C \setminus (C_i^* \cup C_{i,2}' \cup D_i')$. Also, $\bigcup_{i \in [t]} F_i^*$ is edge-disjoint from $\bigcup_{i \in [t]} M_i^*$ (since $G_1^\triangle \uplus G_2^\triangle \uplus G_3^\triangle = G^\triangle$). Since $F_i^* \cup M_i^*$ has $(n-1) - r_i$ edges by (39), we deduce that $|C_i^* \cup C_{i,2}' \cup D_i'| = r_i$, implying that $|C_i^*| = s$ by (31) and (R1). Therefore, using (R4), there exists $J_i \subseteq M_i$ such that J_i is $(C_i^* \cup C_{i,2}')$-rainbow and contains exactly one edge from each of $\{M_{i,j} : j \in [3s]\}$. Note that the J_i are edge-disjoint from each other by (R5) and also edge-disjoint from $\bigcup_{i \in [t]}(F_i^* \cup M_i^*)$ by (40).

Absorb the uncovered edges. Finally, let

$$E^* := E(G_1') \setminus \bigcup_{i \in [t]} (F_i^* \cup M_i^* \cup J_i).$$

We claim that E^* contains precisely m c-edges for every $c \in C$. Note that, by (40), we have $\bigcup_{i \in [t]}(F_i^* \cup M_i^* \cup J_i) = K_n - (E^* \cup E(G_2'))$. Moreover, for each $i \in [t]$, $F_i^* \cup M_i^* \cup J_i$ is $(C \setminus D_i')$-rainbow, implying that for each $c \in C$ the number of c-edges in $\bigcup_{i \in [t]}(F_i^* \cup M_i^* \cup J_i)$ is $|\{i \in [t] : c \in C \setminus D_i'\}| = t - 3m$ by (M1). Thus, the number of c-edges in $E^* \cup E(G_2')$ is $3m$, which implies the claim, using (M1) again.

Thus, by (M2), there exists a partition of $E^* \cup E(G_2')$ into sets $J_1', \ldots, J_t'$, such that for each $i \in [t]$, J_i' contains exactly one edge from each of $\{M_{i,c}'\}_{c \in D_i'}$. In particular, J_i' is D_i'-rainbow.

Let $T_i := F_i^* \cup M_i^* \cup J_i \cup J_i'$. By (†), T_i is a rainbow spanning tree isomorphic to $T_{n;r,b}$, and $T_1, \ldots, T_t$ decompose K_n, as desired. $\qquad\square$

Finally, we briefly mention how the proof can be adapted to prove Theorem 4 with $\Delta(T) = 3$. The only necessary change is in how we connect the matchings in $\mathcal{M}_i$ by using Lemma 20. Suppose that in Step 2 in the proof of Theorem 4, we want to connect the 'head set' $H(M)$ with the 'tail set' $T(M')$ for two consecutive $M, M' \in \mathcal{M}_i$. In the current proof, we find a vertex w and internally disjoint paths of length 2 from w to each vertex in $H(M) \cup T(M')$. Instead, we could also connect $H(M) \cup T(M')$ as follows: let B be a binary tree with root b and leaves $H(M)$, and let B' be a binary tree with root b' and leaves $T(M')$, and such that $V(B) \cap V(B') = \emptyset$. (Recall that $|H(M)| = |T(M')| = 2^8$.) Let R be the graph obtained from $B \cup B'$ by adding a path of length 2 between b and b', and then subdividing every edge once. Clearly, $\Delta(R) \leq 3$, and this construction ensures that still, the tree T_i is always the same, independent of which edge of M is ultimately selected for T_i. To find all the required connections R, one could still employ Lemma 20, here repeatedly, with $k = 2$. However, this necessitates to split V, C and $E(K_n)$ into even more subsets, so for clarity, we omitted this from the proof.

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