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Pareto or Non-Pareto: Bi-Criterion Evolution in Multi-Objective Optimization

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***Abstract*—It is known that Pareto dominance has its own weaknesses as the selection criterion in evolutionary multi-objective optimization. Algorithms based on Pareto criterion (PC) can suffer from slow convergence to the optimal front, inferior performance on problems with many objectives, etc. Non-Pareto criterion (NPC), such as decomposition-based criterion and indicator-based criterion, has already shown promising results in this regard, but its high selection pressure may lead to the algorithm to prefer some specific areas of the problem's Pareto front, especially when the front is highly irregular. In this paper, we propose a bi-criterion evolution framework of PC and NPC, which attempts to make use of their strengths and compensates for each other's weaknesses. The proposed framework consists of two parts, PC evolution and NPC evolution. The two parts work collaboratively, with an abundant exchange of information to facilitate each other's evolution. Specifically, the NPC evolution leads the PC evolution forward and the PC evolution compensates the possible diversity loss of the NPC evolution. The proposed framework keeps the freedom on the implementation of the NPC evolution part, thus making it applicable for any non-Pareto-based algorithm. In the PC evolution, two operations, population maintenance and individual exploration, are presented. The former is to maintain a set of representative nondominated individuals, and the latter is to explore some promising areas which are undeveloped (or not well-developed) in the NPC evolution. Experimental results have shown the effectiveness of the proposed framework. The BCE works well on seven groups of 42 test problems with various characteristics, including those where Pareto-based algorithms or non-Pareto-based algorithms struggle.**

***Index Terms*—Evolutionary multi-objective optimization, Pareto criterion, non-Pareto criterion, bi-criterion evolution.**

I. Introduction

THE area of multi-objective optimization has developed rapidly over the past few decades, reflecting the need of simultaneously dealing with multiple objectives in real-world problems. Unlike global optimization in which there is often a single optimal solution, multi-objective optimization involves a set of Pareto optimal solutions. Evolutionary algorithms (EAs) have shown high practicability in solving such multi-objective optimization problems (MOPs). Their population-based search aims at finding a finite-size set of well-converged, well-distributed solutions, each representing a unique trade-off among the objectives.

In evolutionary multi-objective optimization (EMO), the selection criterion of individuals in the population plays a key role. Since the output of an EMO algorithm for an MOP is a set of Pareto non-dominated solutions, Pareto dominance naturally becomes a viable criterion to select individuals during the evolutionary process. Pareto dominance reflects the weakest assumption about the preference of a decision maker; an individual x is said to Pareto dominate an individual y if it is as good as y in all objectives and better in at least one objective. This criterion, however, fails to distinguish between individuals when they have their own advantage in different objectives of an MOP. In this case, most Pareto-based EMO algorithms, such as the nondominated sorting genetic algorithm II (NSGA-II) [14], introduce the density information of individuals in the population to further rank them, serving the purpose of evolving towards different parts of the problem's Pareto front.

Despite its popularity in the EMO community, the Pareto criterion (PC) or a Pareto-based algorithm is known to suffer from some drawbacks, such as slow convergence to the optimal front [63], no information of quantitative difference between two individuals [5], [71], and inferior performance on MOPs with a complex Pareto set (PS) [44] or a high-dimensional objective space [31], [49], [68]. Recently, some non-Pareto selection criteria have been shown to be promising in tackling MOPs. Typically, they convert an objective vector into a scalar value, thus providing a totally-ordered set of individuals in the population. Compared with the PC, such criteria have clear advantages, e.g., providing higher selection pressure towards the Pareto front [7], [35], [39] and being easier to work with local search techniques stemming from global optimization [5], [43].

The indicator-based EA (IBEA) [82] and decomposition-based multi-objective EA (MOEA/D) [75] are two representative examples in using the non-Pareto criterion (NPC) to deal with MOPs. IBEA adopts a performance indicator to optimize a desired property of the evolutionary population, and MOEA/D decomposes an MOP into a set of scalar subproblems and handles them collaboratively with the aid of the information from their neighbors. These two algorithms have laid the foundation for much state-of-the-art work to date, leading to indicator-based and decomposition-based criteria, along with the PC, to have become three mainstream selection criteria in the EMO area [12], [77].

However, an NPC also comes with some shortcomings. Ideally, the outcome of an EMO algorithm is a set of uniformly-distributed solutions on the whole Pareto front. These solutions are Pareto optimal and are supposed to be incomparable in terms of proximity (convergence). But, most non-Pareto criteria, which typically provide higher selection pressure than the PC, make Pareto optimal solutions comparable, completely

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[13], [82] or partly [17], [42]. In such criteria, different parts of the Pareto front are treated differently, e.g., the knee and border of the front being usually preferred by the *hypervolume*-based criterion [3], [59]. In this way, even some Pareto optimal solutions may be eliminated during the evolutionary process as they are not in favor with the criterion used, which can result in the final solutions distributed "regularly", but not uniformly along the Pareto front.

Note that the decomposition-based criterion seems to be exempt from the above problem since it, by using a set of weight vectors, specifies multiple search directions towards different parts of the Pareto front. One key issue in decomposition-based EMO techniques, however, is how to maintain the uniformity of intersection points of the specified search directions and the problem's Pareto front. Uniformly-distributed weight vectors cannot guarantee the uniformity of the intersection points. In fact, it is very challenging for decomposition-based algorithms to access a set of the well-distributed intersection points for any MOP, in particular in real-world scenarios where the information of a problem's Pareto front is often unknown. Although much effort has been made on this issue recently [2], [16], [21]–[24], [37], [58], [72], it is still far from being resolved completely, especially when facing an MOP with a highly irregular optimal front (e.g., a discontinuous or degenerate front).

Given the above, one question could arise: is it possible to develop an algorithm of synthesizing the Pareto and non-Pareto selection criteria, which makes full use of their advantages as well as effectively avoiding their disadvantages? In this paper, we make an attempt along this line and present a bi-criterion evolution (BCE) framework for MOPs. In BCE, the PC and NPC work collaboratively, trying to guide the population evolving fast towards the optimal front and simultaneously maintain individuals' diversity during the evolutionary process.

BCE manipulates two evolutionary populations, called the NPC population and the PC population, respectively, each of which is associated with one criterion. The NPC population steers the PC population searching towards the optimal front, while the PC population compensates the possible diversity loss of the NPC population by exploring some undeveloped (or not well-developed), but potentially promising regions in the objective space. The two populations communicate with each other in a generational manner; once one population produces good individuals, the other is able to apply them directly within its search process.

BCE keeps it free on the design of the NPC evolution part, thus making the framework applicable for any non-Pareto-based EMO algorithm in the area. Effort of BCE is primarily on the PC evolution part. In the PC evolution, an individual exploration operation, coupled with a novel population maintenance strategy, is proposed to adaptively allocate resources (search effort), based on the information contrast between the current states of the two evolution parts.

The rest of this paper is organized as follows. Section II explains the motivation of the proposed approach. Section III is devoted to the description of BCE, including the basic algorithmic framework, the population maintenance and individual exploration operations, and the analysis of the algorithm's time complexity. Section IV experimentally verifies the proposed BCE framework, based on its implementation with three representative non-Pareto-based algorithms. Further investigation and discussion of BCE's behavior are given in Section V and Section VI, respectively. Finally, Section VII draws the conclusions of the paper.

II. Motivation and Related Work

Over the past few years, non-Pareto criteria have demonstrated their success in dealing with many challenging MOPs, such as an MOP with a huge number of local Pareto fronts [17], with a complex PS [44], [74], or with a high-dimensional objective space [32], [51], [68]. They typically provide higher selection pressure than the Pareto criterion, by either modifying the traditional Pareto dominance relation (such as the ϵ-dominance [17], [42], [67], fuzzy-based dominance [26], and dominance area control [60]) or introducing a quantitative individual comparison criterion (such as the distance-based criterion [54], [70], indicator-based criterion [8], [39], [82], and decomposition-based criterion [55], [75]).

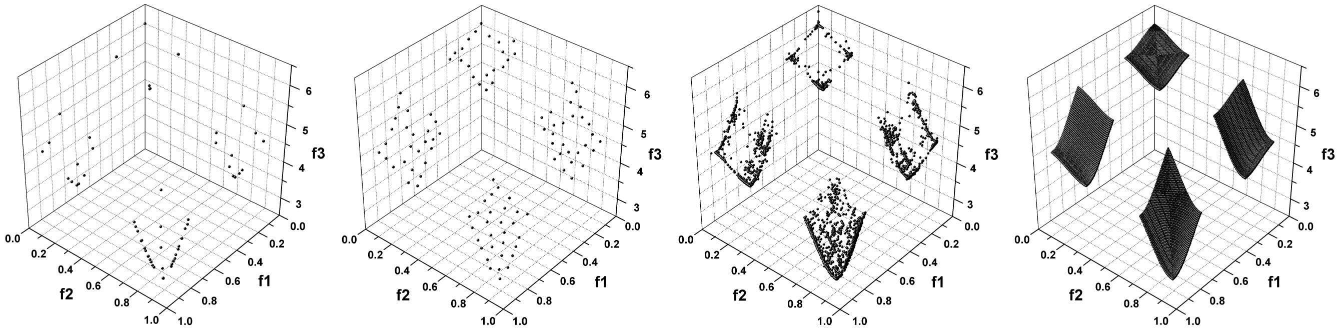
However, non-Pareto criteria also suffer from problems, e.g., in terms of maintaining individuals' diversity (especially uniformity) in the population. In general, the ideal output of an EMO algorithm, in the absence of any preference information, is a set of uniformly-distributed nondominated solutions over the whole Pareto front. This means that the comparison between the Pareto optimal solutions should be solely based on their density information. But, this is not the case in non-Pareto criteria where the Pareto optimal solutions could be ranked, depending not only on their density but also on their position in the population as well as the shape of the Pareto front. For example, the ϵ-dominance criterion [42] is likely to eliminate boundary individuals of the population [27], [51]. Some indicator-based criteria, like the hypervolume [79] and $R2$ [9], prefer the knee region of the Pareto front [19], [59]. The algorithms based on the decomposition criterion search towards a set of points intersected by the specified search directions and the Pareto front, but struggle to maintain the uniformity of these intersection points when the front is highly irregular [24], [37], [58].

Next, we give an empirical example to show the failure of a non-Pareto-based algorithm in providing a set of representative solutions. Fig. 1 shows the results with respect to one typical run[1] of a popular decomposition-based algorithm, MOEA/D [75] with the Tchebycheff scalarizing function[2] (denoted as MOEA/D+TCH), on a discontinuous test problem DTLZ7 [18]. The final solutions obtained by MOEA/D+TCH are plotted in Fig. 1(a). For contrast, Fig. 1(b) gives the final result of the solution set maintained by the criteria of Pareto-based algorithms (i.e., solutions being tested first by their Pareto dominance relation and then by their density[3]) in this

[1]The parameter setting in the run is the same as in the experimental studies, described in Section IV.

[2]In order to obtain more uniform solutions, in the Tchebycheff scalarizing function, "multiplying the weight vector w_i" in the original MOEA/D+TCH [75] is replaced by "dividing w_i", as suggested and practiced in recent studies [16], [45].

[3]Here individuals' density is estimated by the method in BCE, described in Section III-B.



(a) (b) (c) (d)

Fig. 1. An empirical example of the failure of an NPC in both diversity maintenance and search, where the results are obtained with respect to one run of MOEA/D+TCH on the problem DTLZ7. (a) Solution set maintained by the original criterion of MOEA/D+TCH; (b) Solution set maintained by the criteria of Pareto dominance and density; (c) Nondominated set of all solutions produced in the run; (d) Pareto front.

run of MOEA/D+TCH. That is, an external archive set is added in the algorithm to store well-distributed nondominated solutions produced throughout the whole evolutionary process. In addition, the nondominated set of all solutions produced in this run is given in Fig. 1(c).

As can be seen from Figs. 1(a) and (c), MOEA/D+TCH fails to select a set of diverse solutions from all the solutions produced in the whole evolutionary process. In contrast, the selection criteria of Pareto-based algorithms, which consider the Pareto dominance relation and density of candidate solutions, can make the algorithm's output representative, as shown in Fig. 1(b). On the other hand, the deficiency of the algorithm in diversity maintenance also has a detrimental effect on its search ability. To explain this, the Pareto front of the problem is added in Fig. 1(d) for the comparison between the real optimal solutions and the solutions produced during the evolutionary process. From Figs. 1(c) and (d), it can be observed that there exist several large pieces of unexplored regions in MOEA/D+TCH's search process. This occurrence can be attributed to the fact that the selection operation in this non-Pareto algorithm is always around some particular points (cf. Fig. 1(a)) at each generation, thus leading individuals' exploration to concentrate only on some specific regions of the objective space.

The above problems of non-Pareto criteria are precisely the underlying motivation of our study. In this paper, we introduce a BCE framework of Pareto and non-Pareto criteria, in order to use their strengths and compensate for each other's weaknesses.

It is worth pointing out that the combination of NPC and PC is not uncommon in EMO. For example, Ishibuchi *et al.* [33] combined the Pareto-based algorithm NSGA-II with the weighted sum criterion to probabilistically pick out solutions in both mating and environmental selection processes. Al Moubayed *et al.* [1] used a decomposition-based criterion to select the leaders in multi-objective particle swarm optimization (PSO) and introduced the crowding distance to maintain the diversity of nondominated solutions in the decision and objective spaces. Deb and Jain [16] proposed a hybrid EMO algorithm, NSGA-III, which uses the Pareto nondominated sorting to develop convergence and the decomposition-based criterion to maintain diversity during the evolutionary process.

On the other hand, some studies in the literature adopted multiple archives (or populations) to separately promote convergence and diversity during the evolutionary process. Wang *et al.* [69] developed a two-archive many-objective algorithm, with one archive being driven by an indicator-based criterion and the other being maintained by an L_p-norm-based distance criterion. Zăvoianu *et al.* [73] presented a hybrid coevolutionary algorithm with three populations, each one associated with a classic algorithm, i.e., SPEA2 [78], differential evolution [41], and a decomposition-based algorithm. Cai *et al.* [10] proposed a hybrid EMO algorithm for combinatorial MOPs, by using a decomposition-based strategy to guide its internal population and a domination-based sorting technique to maintain the external archive. In addition, the idea of having separate archives has also been used in multi-objective scatter search, where the reference set is split into two subsets that promote convergence and diversity, respectively. In multi-objective scatter search algorithms, Pareto dominance and decomposition criteria are often used in the convergence-promoting subset, and distance-based criteria in the diversity-promoting subset [6], [53], [56].

An important difference between the proposed BCE and existing hybrid EMO algorithms with multiple criteria and/or multiple archives is that BCE takes advantage of the information contrast between the evolutionary populations based on distinct selection criteria, thus making the search focused on promising regions in terms of both Pareto and non-Pareto criteria. Another clear difference is that BCE is a general framework rather than a specific algorithm, and it can work with any non-Pareto EMO algorithm.

III. Bi-Criterion Evolution

Fig. 2 gives the overall framework of BCE. As shown, BCE consists of two evolution parts, NPC evolution and PC evolution. BCE keeps the freedom on the implementation of the NPC evolution part – any non-Pareto EMO algorithm can be directly embedded, with all components (population setting, individual initialization, fitness assignment, selection, variation, etc.) remaining unchanged. The only newly-introduced operation is that the population for the next-generation evolution (the bottom box) is comprised of individuals which are

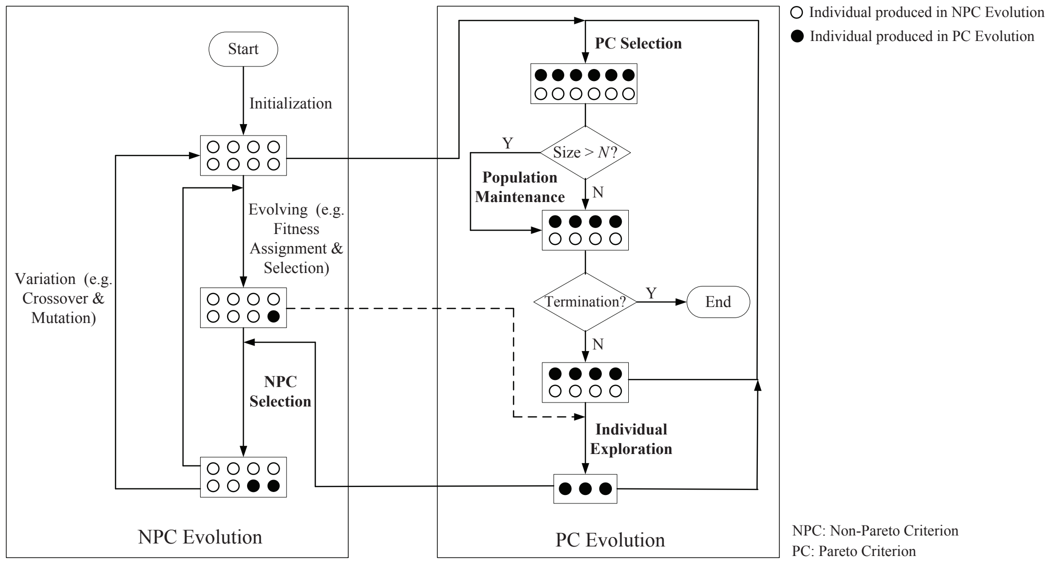


Fig. 2. Overall framework of bi-criterion evolution.

selected from itself and newly produced individuals in the PC evolution part (called the NPC selection).

For the PC evolution part, the manipulated population (i.e., the PC population) only preserves the Pareto nondominated solutions produced in both NPC and PC evolution, thereby having a varying size. When the size of the PC population is larger than a predefined threshold, a population maintenance operation will be implemented to eliminate some poorly distributed individuals. If here the termination condition is satisfied (e.g., reaching a preset number of evaluations), the evolution ends with the PC population as the final output. Otherwise, an individual exploration operation is implemented to explore some promising individuals in the PC population, bearing the evolutionary information from the NPC population.

In BCE, the two populations share and exchange information frequently, but evolve based on their own criterion. Any new individual (wherever it is produced) will be considered in both sides of BCE to see if it could be preserved in their own population. In general, the PC population can be regarded as a good complement to the NPC population. It is able to not only preserve representative Pareto nondominated solutions which could be eliminated in the NPC evolution, but also reflect the current status of the NPC evolution by the information contrast between the two populations.

Next, we describe key operations of BCE. They are the PC and NPC selection, population maintenance, and individual exploration.

A. PC Selection and NPC Selection

As their names suggest, the PC and NPC selections are to select individuals (from the considered population and the newly produced individuals) according to the PC and NPC, respectively. The PC selection is implemented by directly picking out the Pareto nondominated individuals from the mixed set of the PC population and new individuals produced in both the NPC and PC evolutions .

The NPC selection, in general, can be simply implemented by the environmental selection operation of the embedded non-Pareto-based algorithm. For example, in the NPC selection of BCE-IBEA (i.e., BCE with IBEA embedded into its NPC evolution part), the resulting NPC population is comprised of the individuals with the highest fitness with respect to the considered criterion (indicator) in the mixed set of the NPC population and the new individuals from the PC evolution. But, this is impracticable for some algorithms where the survival of a newly produced individual is relevant to the information from its parent(s), such as MOEA/D. This is because the candidate individuals in the NPC selection are from different evolution parts, without being in the parent-child relationship. For these algorithms, the NPC selection compares each individual from the PC evolution with all the members of the NPC population. If an individual from the PC evolution performs better than one or more population members with respect to the considered criterion, then it replaces one of them (chosen at random); otherwise, it is discarded. Algorithm 1 gives the procedure of the NPC selection.

B. Population Maintenance

In the PC evolution part, the population preserves the Pareto nondominated individuals produced during the whole search process and has a varying size. When the size of the population exceeds a predefined capacity, population maintenance will be activated to truncate some of its individuals with poor distribution. It is known that an effective population maintenance operation can maintain a set of representative individuals, which is independent of the properties of the problem (e.g., the number of objectives and the shape of the Pareto front). In this paper, we present a niche-based approach, attempting to preserve a set of representative individuals for any MOP.

Algorithm 1 $NPCselection(Q, T)$

Require: Q (non-Pareto criterion population), T (newly produced individual set in the Pareto criterion evolution part), $label$ (type of the environmental selection in the embedded non-Pareto algorithm)

$label \leftarrow SelectionType()$ /* Return 1 if the survival of a newly produced individual is irrelevant to its parent(s) in the selection process of the non-Pareto algorithm; otherwise, return 0 */
if $label = 1$ **then**
  $Q \leftarrow EnvironmentalSelection(Q, T)$
    /* Select $|Q|$ individuals with the highest fitness with respect to the considered non-Pareto criterion from $Q \cup T$ */
else
  for all $t \in T$ **do**
    $Z \leftarrow \emptyset$
    for all $q \in Q$ **do**
      if t is better than q with respect to the considered criterion **then**
        $Z \leftarrow Z \cup q$
      end if
    end for
    if $Z \neq \emptyset$ **then**
      $z \leftarrow Random(Z)$
        /* Select one individual from Z at random */
      $Q \leftarrow Q \setminus z$
      $Q \leftarrow Q \cup t$
    end if
  end for
end if
return Q

Niching is a class of popular diversity maintenance techniques in the EA field. Originating from the idea of sharing resources, niching can be used to measure individuals' crowding degree (density) in the population. Here, we estimate the crowding degree of an individual by considering both the number and location of the individuals in its niche. Specifically, the crowding degree of an individual p in the population P is defined as follows:

$$D(p) = 1 - \prod_{q \in P, q \neq p} R(p, q) \tag{1}$$

$$R(p, q) = \begin{cases} d(p, q)/r \ , & \text{if } d(p, q) \leq r \\ 1, & \text{otherwise} \end{cases} \tag{2}$$

where $d(p, q)$ denotes the Euclidean distance between individuals p and q, and r is the radius of the niche (its setting will be explained later). Note that the scale of the problem's objectives could be highly different and this will affect the estimation of individuals' crowding degree here. To avoid this kind of problem, in BCE all the objectives will be normalized (with respect to their minimum and maximum values in the population) when the considered operation involves the integration of multiple objectives.

Next, we give some explanations of the proposed crowding degree estimation method.

- The crowding degree of an individual is in the range $[0, 1]$, with a lower value being preferable. An individual having the crowding degree 0 means that there is no other individual in its niche. On the other hand, duplicate individuals have the highest crowding degree 1, regardless of the distribution of other individuals in their niche.
- The crowding degree of an individual is determined by the number of its neighbors (i.e., the individuals in its niche) and the distance between it and these neighbors. Individuals having more neighbors or closer distance to their neighbors are likely to obtain a higher (worse) crowding degree.
- The crowding degree of an individual is influenced more by its closer neighbor(s). For example, considering two individuals p and q, let both of them have two neighbors and the sum of the distance to their own neighbors be the same (say 0.2 and 0.8 for p and 0.4 and 0.6 for q). According to the definition, p, which has a shorter distance (0.2) to its closer neighbor, will have a higher crowding degree than q ($1 - 0.16/r^2 = 0.84 > 1 - 0.24/r^2 = 0.76$, assuming $r = 1.0$ here). Actually, even if p has only one neighbor (closer one), its crowding degree is still higher than that of q ($1 - 0.2/r = 0.8 > 1 - 0.24/r^2 = 0.76$). This means that an individual which has very close neighbor(s) will be assigned a high crowding degree no matter how far it is from other individuals in the population. This is in line with the target of developing the diversity of individuals.

One crucial issue in the proposed crowding degree estimator is the setting of the niche radius, which determines the number of neighbors as well as their location in the niche. Unlike some niching techniques where it is fixed and/or set by the user, the niche radius in the proposed estimator is determined by the evolutionary population. We consider the average of the distance from all the individuals to their kth nearest individual in the population as the radius, attempting to enable most of the individuals to have one or several neighbors in their niche. Here, k is set to 3. The reason of this setting will be explained in detail in the discussion section of the paper (Section VI)

Based on the crowding degree of individuals in the population, the truncation operation can be simply implemented. First, the individual which has the highest crowding degree is removed; if there are several individuals with the highest crowding degree, the tie will be split randomly. Then, the crowding degree of the individuals who are neighbors of the removed individual (i.e., in its niche) is renewed, and again the current most crowded individual is found and removed. This process is repeated until a predefined population size is achieved. Overall, the proposed method iteratively removes crowded individuals and thus leaves a representative population, and this can also be observed in the example of Fig. 1 (see Figs. 1(b) and (c)).

C. Individual Exploration

In BCE, the NPC evolution generally has higher selection pressure than the PC evolution and may prefer partial area(s) of the Pareto front sometimes. This may cause repeating search on some particular regions of the objective space. The individual exploration operation in this section aims to cover this issue. It attempts to explore some *promising* individuals in the PC population which have been eliminated, are not well-developed, or are even unvisited in the NPC evolution. This exploration is adaptive, based on the information comparison

Algorithm 2 $Exploration(P, Q)$

Require: P (Pareto criterion population), Q (non-Pareto criterion population), S (set of the individuals to be explored), T (set of newly produced individuals)

1: $S \leftarrow \emptyset$
2: $r \leftarrow Radius()$ /* Determine the size of the niche */
3: **for all** $p \in P$ **do**
4: $\quad count \leftarrow 0$ /* For recording the number of the NPC individuals in the niche of p */
5: $\quad$ **for all** $q \in Q$ **do**
6: $\quad\quad$ **if** $d(p, q) \leq r$ **then**
7: $\quad\quad\quad count \leftarrow count + 1$ /* When q is in the niche of p */
8: $\quad\quad$ **end if**
9: $\quad$ **end for**
10: $\quad$ **if** $count = 0$ **or** $count = 1$ **then**
11: $\quad\quad S \leftarrow S \cup p$
12: $\quad$ **end if**
13: **end for**
14: $T \leftarrow \emptyset$
15: **for all** $s \in S$ **do**
16: $\quad s' \leftarrow Variation(s)$
17: $\quad T \leftarrow T \cup s'$
18: **end for**
19: **return** T

between the two evolutionary populations. If the NPC population has been found to be well distributed, little exploration will be made; otherwise, much exploration will be made around those promising individuals.

Now, a key question may arise – what individuals are promising and need to be explored? Since the PC population is comprised of a set of representative nondominated individuals, it generally performs well in both convergence and diversity. Nevertheless, it is unnecessary to explore the whole PC population as some of its individuals may already be well explored in the NPC evolution. Such individuals are preferred by the considered NPC and there may be many individuals in the NPC population located around the regions where such individuals reside.

In view of this, we consider two kinds of individuals out of the whole PC population: 1) individuals whose niche has no NPC individual[4] and 2) individuals whose niche has only one NPC individual. The first kind of individuals is clearly not preferred by the considered NPC. Exploring them means to probe into undeveloped regions in the NPC evolution. The niches in which the second kind of individuals resides correspond to low density regions of the NPC population. Exploring them means to probe into the regions which are not well developed in the NPC evolution, but may be potentially promising since they still have individual(s) existing in both the NPC and PC populations after (iterative) selection based on the non-Pareto and Pareto criteria, respectively.

Algorithm 2 gives the main procedure of individual exploration. As shown, the algorithm can primarily be divided into two parts. One is to determine which individuals in the PC population will be explored (Steps 3–13) and the other is to carry out the exploration on those individuals (Steps 15–18). In the proposed framework, the variation operation (Step 16) is not fixed and can be freely specified by users. It can be the same with what is in the NPC evolution (as done in our experimental studies), be chosen from other existing variation operators, or even be directly designed for the exploration here. In addition, note that in different variation operators the number of parent individuals may be different. For a variation operator with only one parent (like mutation), the explored individual is applied directly. For a variation operator with two or more parents (like crossover), the explored individual is considered as one parent (or the primary parent in the operator, e.g., in differential evolution) and the remaining parent(s) will be selected randomly from the PC population.

In Step 2 of Algorithm 2, the radius of the considered niche is calculated. The niche range is an important factor in individual exploration, which, together with the distribution of NPC individuals, determines how many individuals will be explored in the PC population. A small enough niche is likely to lead all PC individuals to be explored, and a large enough niche can cause none of them to be done. Here, we introduce a variable niche, whose range varies with the size of the PC population.

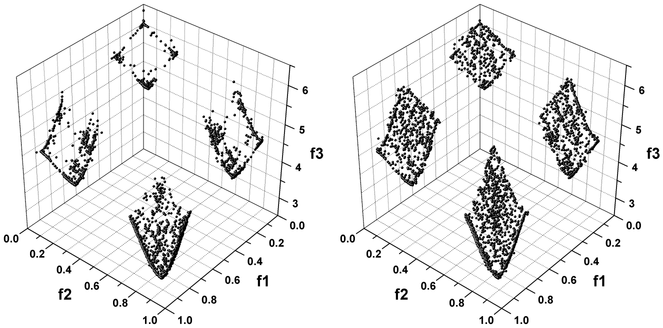
The PC population only preserves nondominated individuals, and its size can reflect the role of the Pareto dominance criterion during the evolutionary process. A small population size means that Pareto dominance can provide sufficient selection pressure to eliminate poorly-performed individuals. This usually happens in the initial stage of the evolution. At this time, the population maintenance operation is not activated, and the PC population which stores all nondominated individuals produced in both the NPC and PC evolutions represents the best individuals found so far. Therefore, it is desirable to put more effort to explore it. With the progress of the evolution, more and more individuals are produced and Pareto dominance may gradually fail to provide sufficient selection pressure. When newly produced nondominated individuals significantly exceed the remaining slots of the population capacity, the PC evolution will slow down. At this time, it is beneficial to make relatively less exploration on the PC population, thus leading to more resources possessed by the NPC evolution which generally has high selection pressure. Given the above, the radius of the niche is determined as follows:

$$r = (N'/N) \times r_0 \tag{3}$$

where N denotes the capacity of the PC population, N' denotes the actual size of the PC population before the truncation, and r_0 is the basic niche radius, calculated in the same way as in the population maintenance operation.

In BCE, individual exploration in the PC evolution inevitably competes with the variation operation in the NPC evolution for limited computational resources (i.e., function evaluations). Given a fixed computational budget, the number of individuals explored directly affects the evolutionary level of the NPC population. However, it is worth noting that individual exploration here is adaptive, depending on the current evolutionary status of the NPC population. When the NPC population has diversity loss (like the case in Fig. 1(a) where the decomposition-based criterion struggles to maintain

[4] For brevity, individuals in the NPC and PC populations are denoted as NPC and PC individuals, respectively.



(a) MOEA/D+TCH (b) BCE-MOEA/D+TCH

Fig. 3. The nondominated set of all the solutions produced in one run of MOEA/D+TCH and BCE-MOEA/D+TCH on DTLZ7, respectively.

diversity), intensive exploration will be made. When the population has been found to be well distributed, little or even no exploration will be done; for instance, for the test function DTLZ2 [18] the decomposition-based evolutionary population can work very well and thus no individual is explored in the PC population (this will also be empirically presented in Section V-B).

Finally, Fig. 3 gives the comparative results of the original MOEA/D+TCH (i.e., Fig. 1(c)) and BCE-MOEA/D+TCH (BCE with MOEA/D+TCH embedded into its NPC evolution part) by plotting all of their nondominated individuals produced in one run. In contrast to MOEA/D+TCH's solutions which are located around some specific regions, the solutions produced by BCE-MOEA/D+TCH nearly cover the whole optimal space. This difference can be fully attributed to the individual exploration operation in the PC evolution part of the algorithm, which conducts the search on some undeveloped (or not well-developed) regions in the NPC evolution.

D. Computational Complexity of One Generation of BCE

The computational cost of BCE comes from two parts, the NPC evolution and the PC evolution. For simplicity, let both parts have the population size (capacity) N. For the time complexity of the NPC evolution, there are two possible situations, depending on the selection operation in the embedded non-Pareto algorithm. When the survival of an individual is determined by its fitness in the population (like in IBEA), the NPC selection is implemented in the same way as the individual selection in the embedded algorithm (cf. Section III-A). In this case, the NPC evolution has the same time complexity as the embedded algorithm (denoted as C). On the other hand, when the survival of an individual is relevant to the information from its parents (like in MOEA/D), the NPC selection is implemented by comparing the individuals produced in the PC evolution with the members of the NPC population. This requires $O(N^2)$ comparisons at most. Hence, the time complexity of the NPC evolution in this situation is C or $O(N^2)$, whichever is larger.

The computational cost of the PC evolution part is determined by three operations, the PC selection, population maintenance, and individual exploration. The PC selection, which identifies nondominated individuals from a population with $3N$ members at most, requires $O(mN^2)$ comparisons [14], where m is the number of objectives. In the population maintenance, the Euclidean distance between each pair of individuals in the population is first calculated, which requires $O(mN^2)$ computations. Then, determining the niche radius requires $O(N^2)$ computations, in which finding the kth smallest distance ($k = 3$) for an individual needs $O(N)$ comparisons. Thereafter, the crowding degree estimation and the population truncation are sequentially implemented. Both requires $O(N^2)$ computations (or comparisons). It is worth mentioning that in the population truncation we only need to update the crowding degree of the neighbors of the removed individual (i.e., the individuals which is in the same niche of the removed individual). In general, the niche of an individual only has a few individuals (independent of N) due to the setting of the radius (namely, the average distance from all individuals in the population to their 3rd nearest individual). In the individual exploration, the Euclidean distance between the individuals and the radius of the niche are also calculated first, which require $O(mN^2)$ and $O(N^2)$ computations, respectively. Then, determining which individuals in the population will be explored requires $O(N^2)$ comparisons (Steps 3–13 in Algorithm 2). Finally, carrying out the exploration operation on the selected individuals requires $O(N)$ computations at most. Therefore, the total time complexity of the PC evolution is $O(mN^2)$.

To sum up, the overall computational complexity of one generation of BCE is bounded by C or $O(mN^2)$, whichever is larger, where C is the computational complexity of the embedded non-Pareto algorithm.

IV. Performance Verification of BCE

The proposed framework is verified by embedding non-Pareto EMO algorithms into its NPC evolution part and comparing these non-Pareto algorithms with the resulting BCE algorithms. We consider two representative non-Pareto-based algorithms, IBEA [82] and MOEA/D [44], which lead the evolution via the indicator-based criterion and decomposition-based criterion, respectively. In MOEA/D, two scalarizing functions Tchebycheff (TCH) and penalty-based boundary intersection (PBI) are commonly used in the literature, and both are included in our experiments in view of their good performance for different MOPs [16], [32], [44], [48]. Note that MOEA/D used here is sourced from [44] rather than from its original paper [75]. This improved version can largely enhance the diversity of the population, by allowing parent individuals to be selected from the whole population as well as setting a limit of the maximal number of individuals replaced by a newly produced child individual. In addition, in the TCH scalarizing function, we replace "multiplying the weight vector" with "dividing it" for obtaining more uniform individuals, as pointed out in [16], [37]. Overall, the intention that we consider this version of MOEA/D is to verify the effectiveness of BCE even when the considered non-Pareto algorithms already work fairly well in terms of diversity maintenance on most MOPs.

A comprehensive set of 42 MOPs are introduced in the experiments. These test problems, which are widely used in

TABLE I
SETTINGS AND PROPERTIES OF TEST PROBLEMS. m AND d DENOTE THE NUMBER OF OBJECTIVES AND DECISION VARIABLES, RESPECTIVELY

Problem	m	d	Properties	Problem	m	d	Properties	Problem	m	d	Properties
SCH1	2	1	Convex	WFG7	2	22	Concave, Biased	UF2	2	30	Convex, Complex PS
SCH2	2	1	Discontinuous	WFG8	2	22	Concave, Nonseparable, Biased	UF3	2	30	Convex, Complex PS
KUR	2	3	Discontinuous	WFG9	2	22	Concave, Nonseparable, Deceptive, Biased	UF4	2	30	Concave, Complex PS
ZDT1	2	30	Convex	VNT1	3	2	Convex	UF5	2	30	Linear, Discrete, Complex PS
ZDT2	2	30	Concave	VNT2	3	2	Mixed	UF6	2	30	Linear, Discontinuous, Complex PS
ZDT3	2	30	Discontinuous	VNT3	3	2	Mixed, Degenerate	UF7	2	30	Linear, Complex PS
ZDT4	2	10	Convex, Multimodal	DTLZ1	3	7	Linear, Multimodal	UF8	3	30	Concave, Complex PS
ZDT6	2	10	Concave, Multimodal, Biased	DTLZ2	3	12	Concave	UF9	3	30	Linear, Discontinuous, Complex PS
WFG1	2	22	Mixed, Biased	DTLZ3	3	12	Concave, Multimodal	UF10	3	30	Concave, Complex PS
WFG2	2	22	Convex, Discontinuous, Nonseparable	DTLZ4	3	12	Concave, Biased	DTLZ2(4)	4	13	Concave
WFG3	2	22	Linear, Degenerate, Nonseparable	DTLZ5	3	12	Concave, Degenerate	DTLZ2(6)	6	15	Concave
WFG4	2	22	Concave, Multimodal	DTLZ6	3	12	Concave, Degenerate, Biased	DTLZ2(10)	10	19	Concave
WFG5	2	22	Concave, Deceptive	DTLZ7	3	22	Mixed, Discontinuous, Multimodal	DTLZ5(2,10)	10	19	Concave, Degenerate
WFG6	2	22	Concave, Nonseparable	UF1	2	30	Convex, Complex PS	DTLZ5(3,10)	10	19	Concave, Degenerate

TABLE II
POPULATION SIZE AND FUNCTION EVALUATIONS IN THE EXPERIMENTS

Test Problems	Population Size	Function Evaluations
2-Obj. UF	600	300 000
3-Obj. UF	1 000	300 000
General 2-Obj. MOPs	100	25 000
General 3-Obj. MOPs	105	30 000
4-Obj. MOPs	220	100 000
6-Obj. MOPs	252	100 000
10-Obj. MOPs	220	100 000

the area, have various properties, such as having a convex, concave, mixed, discontinuous or degenerate Pareto front, having a multimodal, biased or deceptive search space, and/or having strong-linkage decision variables. They certainly include some MOPs where non-Pareto algorithms generally work well, like an MOP with a linear (or fairly regular) Pareto front, and also have some where the algorithms may encounter difficulties, like an MOP with a discontinuous (or highly irregular) Pareto front. Table I summarizes the properties and configuration of these MOPs. All the problems are configured as described in their original papers [18], [28], [61], [66], [74], [80].

To compare the performance of the algorithms, two widely-used quality indicators, the inverted generational distance (IGD) [16], [75] and hypervolume (HV) [79], are considered as they can provide a combined information of convergence and diversity of a solution set. IGD measures the average Euclidean distance from uniformly distributed points along the whole Pareto front to their closest solution in the obtained solution set, and a smaller value is preferable. HV calculates the volume of the objective space between the obtained solution set and a specified reference point, and a larger value is preferable.

In the calculation of HV, two crucial issues are the scaling of the search space [20] and the choice of the reference point [3]. Since the objectives in the considered test problems take different ranges of values, we standardize the objective value of the obtained solutions according to the range of the problem's Pareto front. Following the recommendation in [34], the reference point is set to 1.1 times the upper bound of the Pareto front (i.e., $r = 1.1^m$) to emphasize the balance between proximity and diversity of the obtained solution set. Note that solutions that do not dominate the reference point are discarded (i.e., solutions that are worse than the reference point in at least one objective contribute zero to HV).

All the results presented in this study are obtained by executing 30 independent runs for each algorithm. For a fair comparison, all the algorithms have the same size (or capacity) of the population (for BCE, this refers to both the NPC and PC populations) and the same number of function evaluations on each problem. Table II lists the settings of the population size and function evaluations for all the test problems in the experiments. For the UF functions from the CEC2009 competition [74], the population size and function evaluations are specified the same as in their original report [76]. For other MOPs, we used a smaller population size and fewer function evaluations as they are generally easier than the UF functions. Like some existing studies [52], the number of function evaluations is set to 25,000 and 30,000 for 2- and 3-objective MOPs, respectively. Note that in MOEA/D the population size corresponds to the number of weight vectors and the algorithm cannot generate uniformly distributed weight vectors at an arbitrary number. So, we set the population size consistent with the number of the uniformly generated weight vectors in MOEA/D. That is 100, 105, 220, 252 and 220 for the 2-, 3-, 4-, 6- and 10-objective MOPs, respectively. In addition, given that many-objective problems often bring bigger challenges for EMO algorithms than MOPs with 2 or 3 objectives [57], we assign them a larger population size and more function evaluations, following the practice in [47].

Parameters need to be set in the considered algorithms. According to [74], the size of the neighborhood, the probability of parent individuals selected from the neighborhood, and the maximum number of replaced individuals in MOEA/D were specified as 10% of the population size, 0.9, and 1% of the population size, respectively. As suggested in [75], [82], the penalty parameter θ in MOEA/D+PBI was set to 5 and the scaling factor κ in IBEA to 0.05. In BCE, the embedded non-Pareto algorithms used the same setting of parameters as in their original versions.

All the considered algorithms were given real-valued variables. Two widely-used crossover and mutation operators, simulated binary crossover (SBX) and polynomial mutation (with distribution indexes 20 [15]), were used on all the MOPs except UF. The crossover probability was set to $p_c = 1.0$ and mutation probability to $p_m = 1/d$, where d denotes the number of decision variables. For the UF problems which have a strong linkage in variables, the use of variable-independent SBX may not be adequate [16], [44]. Following the study in

TABLE III

IGD RESULTS (MEAN AND SD) OF THE THREE GROUPS OF PAIRED ALGORITHMS. THE BETTER MEAN FOR EACH CASE IS HIGHLIGHTED IN BOLDFACE

Property	Problem	IBEA	BCE-IBEA	MOEA/D+TCH	BCE-MOEA/D+TCH	MOEA/D+PBI	BCE-MOEA/D+PBI
Convex	SCH1	1.9093E–2(4.1E–4)	**1.6732E–2(1.1E–4)**†	4.9139E–2(1.5E–3)	**1.6761E–2(1.4E–4)**†	1.7765E–1(5.2E–3)	**1.6729E–2(9.3E–5)**†
	ZDT1	4.0773E–3(6.8E–5)	**3.9475E–3(5.0E–5)**†	4.3626E–3(3.4E–4)	**4.2756E–3(9.4E–5)**†	8.2654E–3(1.7E–3)	**5.9401E–3(4.3E–4)**†
	ZDT4	6.2464E–1(1.1E–1)	**1.2270E–1(5.9E–2)**†	1.0001E–2(3.4E–3)	**8.2586E–3(3.2E–3)**†	3.7084E–2(2.8E–2)	**1.3458E–2(6.5E–3)**†
	VNT1	1.6688E–1(2.7E–2)	**1.2731E–1(3.3E–3)**†	2.2398E–1(1.3E–3)	**1.2240E–1(2.7E–3)**†	1.7364E–1(5.6E–4)	**1.2544E–1(2.7E–3)**†
Concave	ZDT2	9.1909E–3(3.7E–4)	**4.0420E–3(3.3E–4)**†	4.3342E–3(1.8E–4)	**4.1688E–3(9.3E–5)**†	1.0540E–2(1.2E–3)	**7.2119E–3(7.8E–4)**†
	ZDT6	5.4849E–3(1.8E–4)	**3.4449E–3(8.6E–5)**†	1.5027E–2(1.7E–3)	**1.0328E–2(1.3E–3)**†	3.9299E–2(4.6E–3)	**2.1787E–2(2.7E–3)**†
	WFG4	1.8370E–2(8.3E–4)	**1.3005E–2(4.6E–4)**†	2.2551E–2(1.8E–3)	**1.9664E–2(1.5E–3)**†	5.0721E–2(5.2E–3)	**3.8183E–2(3.9E–3)**†
	WFG5	7.2029E–2(7.6E–4)	**6.7165E–2(1.7E–4)**†	**6.8459E–2(5.9E–4)**	6.8558E–2(6.3E–4)	8.2498E–2(2.9E–3)	**7.7588E–2(1.8E–3)**†
	WFG6	6.5849E–2(9.4E–3)	**6.1760E–2(9.2E–3)**†	6.8500E–2(1.0E–2)	**6.4056E–2(8.2E–3)**	1.0092E–1(1.4E–2)	**8.6534E–2(1.2E–2)**†
	WFG7	2.1525E–2(1.0E–3)	**1.4264E–2(3.1E–4)**†	1.6770E–2(4.7E–4)	**1.5242E–2(3.3E–4)**†	3.7482E–2(3.3E–3)	**2.9628E–2(2.2E–3)**†
	WFG8	9.3529E–2(8.0E–3)	**8.0848E–2(6.1E–3)**†	8.7532E–2(5.6E–3)	**8.5839E–2(5.6E–3)**	1.1848E–1(7.9E–3)	**1.0653E–1(8.1E–3)**†
	WFG9	7.8794E–2(5.2E–2)	**7.4934E–2(5.2E–2)**	**3.5633E–2(2.4E–2)**	5.6954E–2(4.5E–2)†	**5.6432E–2(1.6E–2)**	7.3108E–2(3.9E–2)†
	DTLZ2	1.1894E–1(2.2E–3)	**5.3051E–2(1.0E–3)**†	**5.1151E–2(4.2E–4)**	5.1708E–2(7.7E–4)†	**5.0151E–2(3.1E–6)**	5.1369E–2(8.3E–4)†
	DTLZ3	**5.1829E–1(1.5E–1)**	5.3253E–1(4.3E–1)	**5.0856E–1(6.5E–1)**	1.6158E+0(1.3E+0)†	**1.0267E+0(9.1E–1)**	1.4062E+0(1.1E+0)
	DTLZ4	4.6424E–1(3.4E–1)	**2.3409E–1(3.4E–1)**†	1.7346E–1(2.3E–1)	**5.2207E–2(9.4E–4)**†	1.1261E–1(1.7E–1)	**5.1944E–2(9.8E–4)**†
	DTLZ5	2.5627E–2(1.1E–3)	**4.1949E–3(2.4E–4)**†	1.8107E–2(1.5E–5)	**4.4611E–3(3.2E–4)**†	3.1920E–2(1.0E–4)	**4.5173E–3(3.4E–4)**†
	DTLZ6	1.0148E–1(2.0E–2)	**7.3724E–2(2.5E–2)**†	**2.4922E–1(3.7E–2)**	3.9000E–1(3.3E–2)†	**6.4541E–1(5.7E–2)**	7.6720E–1(4.8E–2)†
Linear	WFG3	1.5944E–2(1.2E–3)	**1.5917E–2(1.8E–3)**	2.5079E–2(4.3E–3)	**2.3228E–2(3.3E–3)**†	4.7442E–2(8.1E–3)	**3.9358E–2(7.4E–3)**†
	DTLZ1	1.8267E–1(1.7E–2)	**2.2154E–2(2.0E–3)**†	**1.9571E–2(1.1E–3)**	2.1149E–2(1.8E–3)†	**1.9130E–2(5.7E–4)**	2.2134E–2(2.4E–2)†
Mixed	WFG1	**7.5883E–1(6.2E–2)**	7.7944E–1(6.3E–2)	1.0487E+0(7.2E–2)	**9.6279E–1(5.4E–2)**†	1.1983E+0(5.9E–2)	**1.0490E+0(5.3E–2)**†
	VNT2	4.2602E–2(9.4E–3)	**1.2219E–2(3.2E–4)**†	4.6425E–2(2.9E–4)	**1.2121E–2(2.8E–4)**†	5.8432E–2(2.4E–4)	**1.2436E–2(2.5E–4)**†
	VNT3	2.5369E+0(8.7E–2)	**3.8870E–2(1.4E–3)**†	2.3174E+0(1.3E–1)	**3.8536E–2(1.2E–3)**†	2.5622E+0(8.2E–2)	**3.8244E–2(1.1E–3)**†
Discontinuous	SCH2	1.2321E–1(4.0E–2)	**2.0902E–2(2.3E–4)**†	1.0491E–1(2.8E–4)	**2.0832E–2(2.7E–4)**†	5.0208E+0(5.0E–3)	**2.0861E–2(2.1E–4)**†
	KUR	1.9336E–1(2.3E–2)	**3.4693E–2(6.7E–4)**†	4.1089E–2(2.3E–4)	**3.4142E–2(6.2E–4)**†	4.3389E–2(8.7E–4)	**3.4960E–2(9.2E–4)**†
	ZDT3	3.0837E–2(8.0E–4)	**4.6028E–3(6.5E–5)**†	1.0929E–2(7.9E–5)	**4.7997E–3(6.0E–5)**†	1.3874E–2(5.3E–3)	**5.4836E–3(1.5E–4)**†
	WFG2	7.0639E–2(8.6E–3)	**2.5660E–2(1.1E–2)**†	3.9081E–2(3.7E–3)	**2.0238E–2(4.1E–3)**†	1.1083E–1(1.1E–2)	**3.0526E–2(6.9E–3)**†
	DTLZ7	4.2838E–1(2.9E–1)	**2.7161E–1(2.5E–1)**†	1.2946E–1(9.6E–4)	**5.6166E–2(1.1E–3)**†	1.2840E–1(9.3E–4)	**5.8486E–2(9.7E–4)**†
Complex PS	UF1	4.6600E–2(3.4E–3)	**3.5263E–2(5.4E–3)**†	2.1201E–2(3.3E–2)	**1.6430E–3(1.6E–4)**†	7.4430E–2(6.9E–2)	**9.3327E–3(2.1E–2)**†
	UF2	4.6187E–2(1.9E–3)	**2.2549E–2(2.8E–3)**†	1.4578E–2(2.2E–2)	**6.5645E–3(1.4E–3)**†	5.2665E–2(5.2E–2)	**1.1159E–2(2.7E–3)**†
	UF3	7.1289E–2(2.8E–2)	**4.2563E–2(2.3E–2)**†	1.1098E–2(1.3E–2)	**9.5736E–3(1.0E–2)**	3.8405E–2(3.1E–2)	**1.3594E–2(1.5E–2)**†
	UF4	5.2290E–2(2.0E–3)	**5.0860E–2(2.6E–3)**†	**6.4407E–2(4.0E–3)**	6.6063E–2(5.7E–3)	6.7517E–2(5.7E–3)	**6.6512E–2(5.9E–3)**
	UF5	1.1661E+0(1.2E–1)	**5.1121E–1(1.4E–1)**†	4.9510E–1(1.7E–1)	**4.0341E–1(1.4E–1)**†	**4.6553E–1(1.3E–1)**	4.7409E–1(1.5E–1)
	UF6	3.5670E–1(2.1E–2)	**2.8400E–1(1.1E–2)**†	5.0766E–1(1.5E–1)	**4.2500E–1(1.4E–1)**	5.2782E–1(1.4E–1)	**4.4913E–1(1.5E–1)**†
	UF7	2.3288E–2(1.9E–3)	**1.5350E–2(1.1E–3)**†	2.0330E–2(1.5E–2)	**1.2116E–2(5.0E–3)**†	4.8084E–2(1.2E–1)	**1.4598E–2(3.8E–3)**†
	UF8	3.8577E–1(9.3E–3)	**1.3722E–1(3.5E–2)**†	**5.4927E–2(1.5E–2)**	5.6104E–2(1.1E–2)	1.1147E–1(4.6E–2)	**8.9914E–2(3.3E–2)**†
	UF9	**9.9491E–2(3.0E–3)**	1.1081E–1(4.1E–2)	**1.1732E–1(4.8E–2)**	1.3153E–1(3.4E–2)†	1.3193E–1(3.7E–2)	**1.2588E–1(4.1E–2)**
	UF10	**2.2497E+0(2.8E–1)**	2.3864E+0(1.8E–1)†	4.6931E–1(6.6E–2)	**4.6496E–1(7.0E–2)**	4.8214E–1(1.1E–1)	**4.2631E–1(6.1E–2)**†
Many Objectives	DTLZ2(4)	1.7281E–1(1.1E–3)	**9.8155E–2(4.8E–3)**†	**8.9093E–2(7.0E–4)**	9.3172E–2(3.2E–3)†	**8.7399E–2(5.1E–6)**	9.3679E–2(4.6E–3)†
	DTLZ2(6)	3.7149E–1(2.4E–3)	**2.3931E–1(2.7E–3)**†	2.8757E–1(1.4E–2)	**2.3766E–1(1.9E–3)**†	2.5665E–1(2.4E–5)	**2.4064E–1(7.6E–4)**†
	DTLZ2(10)	8.6901E–1(3.4E–2)	**4.8722E–1(4.8E–3)**†	5.6890E–1(1.9E–2)	**4.6966E–1(7.4E–3)**†	4.9222E–1(2.4E–5)	**4.7015E–1(2.7E–3)**†
	DTLZ5(2,10)	7.4584E–1(0.0E+0)	**2.1183E–3(3.2E–5)**†	1.6977E–1(1.8E–3)	**2.1620E–3(7.2E–5)**†	6.4940E–2(5.3E–5)	**2.1492E–3(6.8E–5)**†
	DTLZ5(3,10)	6.1236E–1(7.6E–2)	**3.7996E–2(3.1E–3)**†	2.5109E–1(3.2E–4)	**3.7136E–2(1.1E–3)**†	1.6867E–1(3.1E–3)	**3.8918E–2(2.1E–3)**†

"†" indicates that the value of the BCE algorithm is significantly different from that of its corresponding non-Pareto algorithm at a 0.05 level by the Wilcoxon's rank sum test.

[44], [74], we adopted the differential evolution (DE) operation for these problems, with the two control parameters $CR = 1.0$ and $F = 0.5$.

Tables III and IV give the HV and IGD results (mean and standard deviation), respectively, for the three groups of paired algorithms, IBEA vs BCE-IBEA, MOEA/D+TCH vs BCE-MOEA/D+TCH, and MOEA/D+PBI vs BCE-MOEA/D+PBI, on all the 42 MOPs. The better mean for each problem is highlighted in boldface. To have statistically sound conclusions, the Wilcoxon's rank sum test [81] at a 0.05 significance level is adopted to test the significance of the differences between the results obtained by paired algorithms.

As stated before, the advantage of NPC is primarily on addressing challenging MOPs (such as with a complex PS or with a high-dimensional objective space), whereas the advantage of PC lies in dealing with MOPs with an irregular Pareto front. Here, we divide the test problems into seven categories, to systematically investigate the effectiveness of BCE for problems with distinct preference of NPC or PC. They are convex, concave, linear, mixed, discontinuous, complex-PS, and high-dimensional problem categories.

A. Test Problems with a Convex Pareto Front

In this category, we consider four problems, SCH1, ZDT1, ZDT4, and VNT1. As can be seen from Tables III and IV, the BCE algorithms show a clear advantage over the non-Pareto algorithms on these problems. The three BCE algorithms outperform their corresponding competitors for both IGD and HV on all the four problems, and the difference in all of these comparisons is statistically significant.

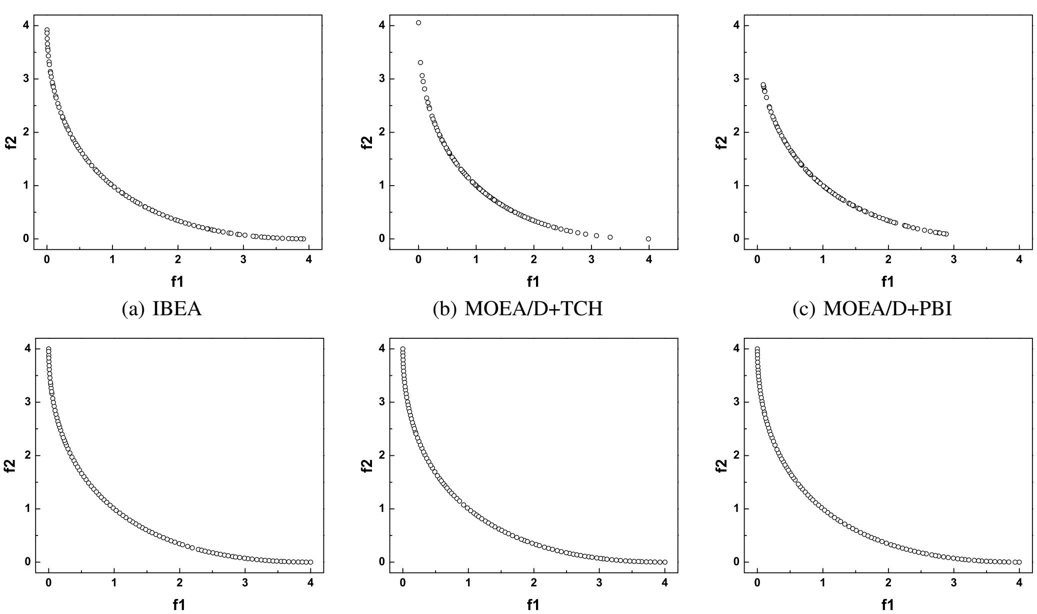
Fig. 4 plots the final solutions of the six algorithms in a single run on SCH1. This particular run, along with others for visual demonstration in the paper, is associated with the result which is the closest to the mean IGD value. SCH1 has a convex Pareto optimal curve in the range $f_1, f_2 \in [0, 4]$. As shown, IBEA and MOEA/D+TCH struggle to maintain the uniformity of the solutions, especially around the edges of the Pareto front. MOEA/D+PBI fails to find boundary points of the Pareto front, with their solutions concentrating in the range $[0, 3]$. On the other hand, the three BCE algorithms perform well. Their performance appears similar and all of their solutions are uniformly distributed along the whole Pareto front. This

TABLE IV

HV RESULTS (MEAN AND SD) OF THE THREE GROUPS OF PAIRED ALGORITHMS. THE BETTER MEAN FOR EACH CASE IS HIGHLIGHTED IN BOLDFACE

Property	Problem	IBEA	BCE-IBEA	MOEA/D+TCH	BCE-MOEA/D+TCH	MOEA/D+PBI	BCE-MOEA/D+PBI
Convex	SCH1	1.0395E+0(8.0E–5)	**1.0396E+0(5.0E–5)**†	1.0375E+0(1.7E–4)	**1.0396E+0(5.4E–5)**†	1.0275E+0(5.6E–4)	**1.0396E+0(3.5E–5)**†
	ZDT1	8.7140E–1(1.2E–4)	**8.7156E–1(1.0E–4)**†	8.6969E–1(5.5E–4)	**8.6998E–1(3.1E–4)**†	8.6149E–1(2.6E–3)	**8.6639E–1(8.4E–4)**†
	ZDT4	3.5648E–1(1.1E–1)	**7.8067E–1(4.1E–2)**†	8.5850E–1(6.1E–3)	**8.6174E–1(5.8E–3)**†	8.1319E–1(4.4E–2)	**8.5370E–1(9.8E–3)**†
	VNT1	9.8473E–1(1.2E–2)	**1.0134E+0(1.2E–3)**†	9.7156E–1(2.5E–4)	**1.0144E+0(1.3E–3)**†	9.9419E–1(3.1E–4)	**1.0138E+0(9.2E–4)**†
Concave	ZDT2	5.3694E–1(1.1E–4)	**5.3793E–3(1.0E–4)**†	5.3587E–1(6.3E–4)	**5.3654E–1(4.1E–4)**†	5.2098E–1(2.6E–3)	**5.2856E–1(1.8E–3)**†
	ZDT6	6.0575E–1(4.7E–4)	**6.0900E–1(3.4E–4)**†	5.8120E–1(3.7E–2)	**5.9186E–1(2.8E–3)**†	5.2924E–1(9.4E–3)	**5.6742E–1(5.6E–3)**†
	WFG4	4.1692E–1(4.6E–4)	**4.1786E–1(4.9E–4)**†	4.1132E–1(1.3E–3)	**4.1254E–1(9.7E–4)**†	3.9106E–1(3.5E–3)	**3.9880E–1(2.1E–3)**†
	WFG5	3.7429E–1(1.6E–3)	**3.7556E–1(1.8E–3)**†	3.7089E–1(3.6E–4)	**3.7175E–1(6.7E–4)**†	3.6200E–1(1.9E–3)	**3.6589E–1(1.1E–3)**†
	WFG6	3.8234E–1(6.4E–3)	**3.8235E–1(6.1E–3)**	3.7690E–1(7.1E–3)	**3.8085E–1(5.3E–3)**†	3.5462E–1(8.8E–3)	**3.6529E–1(7.8E–3)**†
	WFG7	4.1791E–1(2.4E–4)	**4.1933E–1(2.4E–4)**†	4.1642E–1(7.6E–4)	**4.1729E–1(4.8E–4)**†	3.9833E–1(3.1E–3)	**4.0522E–1(1.6E–3)**†
	WFG8	3.5069E–1(3.5E–3)	**3.5539E–1(3.3E–3)**†	3.5389E–1(3.2E–3)	**3.5561E–1(3.2E–3)**†	3.3363E–1(4.9E–3)	**3.4172E–1(5.1E–3)**†
	WFG9	3.6836E–1(3.4E–2)	**3.6929E–1(3.4E–2)**	**3.9231E–1(1.5E–2)**	3.8015E–1(2.8E–2)†	**3.7718E–1(1.1E–2)**	3.6887E–1(2.4E–2)
	DTLZ2	**7.4369E–1(5.5E–4)**	7.4216E–1(2.0E–3)†	**7.4893E–1(1.3E–4)**	7.4760E–1(1.1E–3)†	**7.4897E–1(7.9E–5)**	7.4745E–1(1.1E–3)†
	DTLZ3	2.4094E–1(8.1E–2)	**2.4504E–1(1.7E–1)**	**3.8094E–1(2.4E–1)**	9.4924E–2(1.8E–1)†	**2.2132E–1(2.7E–1)**	1.0137E–1(1.8E–1)
	DTLZ4	5.1710E–1(2.2E–1)	**6.4365E–1(2.0E–1)**†	6.8054E–1(1.2E–1)	**7.4702E–1(1.3E–3)**†	7.0945E–1(1.0E–1)	**7.4745E–1(1.3E–3)**†
	DTLZ5	2.6296E–1(2.5E–4)	**2.6596E–1(2.7E–4)**†	2.5645E–1(8.0E–6)	**2.6593E–1(3.1E–4)**†	2.4415E–1(7.1E–5)	**2.6576E–1(1.8E–4)**†
	DTLZ6	1.5127E–1(3.9E–2)	**1.7524E–1(3.3E–2)**†	**3.5183E–2(1.5E–2)**	3.0630E–3(3.7E–3)†	0.0000E+0(0.0E+0)	0.0000E+0(0.0E+0)
Linear	WFG3	6.9936E–1(1.3E–3)	**6.9938E–1(1.5E–3)**	6.9099E–1(3.1E–3)	**6.9282E–1(2.5E–3)**†	6.7584E–1(5.6E–3)	**6.8170E–1(4.8E–3)**†
	DTLZ1	5.0330E–1(9.1E–2)	**1.1126E+0(8.1E–3)**†	**1.1193E+0(4.9E–3)**	1.1150E+0(8.0E–3)†	**1.1194E+0(3.3E–3)**	1.1025E+0(3.7E–2)†
Mixed	WFG1	**3.5198E–1(4.0E–2)**	3.4025E–1(4.2E–2)	2.1352E–1(3.8E–2)	**2.5791E–1(3.9E–2)**†	1.5041E–1(2.3E–2)	**2.2037E–1(3.8E–2)**†
	VNT2	1.2258E+0(9.5E–3)	**1.2481E+0(4.4E–4)**†	1.2179E+0(1.2E–4)	**1.2483E+0(3.4E–4)**†	1.2053E+0(4.8E–4)	**1.2479E+0(3.4E–4)**†
	VNT3	1.1427E+0(4.5E–3)	**1.1476E+0(3.7E–4)**†	1.1255E+0(9.1E–4)	**1.4756E+0(3.2E–4)**†	1.1296E+0(5.0E–4)	**1.1476E+0(4.1E–4)**†
Discontinuous	SCH2	8.0570E–1(2.4E–3)	**8.1170E–1(6.4E–5)**†	8.0177E–1(6.0E–5)	**8.1168E–1(1.6E–4)**†	6.4919E–1(3.6E–5)	**8.1163E–1(4.0E–4)**†
	KUR	6.0476E–1(6.8E–4)	**6.1110E–1(1.5E–4)**†	6.0987E–1(1.8E–4)	**6.1108E–1(1.9E–4)**†	6.0806E–1(4.1E–4)	**6.1042E–1(3.1E–4)**†
	ZDT3	7.2021E–1(4.9E–4)	**7.2542E–1(3.1E–4)**†	7.2236E–1(2.9E–4)	**7.2448E–1(2.4E–4)**†	7.1218E–1(4.3E–3)	**7.2140E–1(4.9E–4)**†
	WFG2	7.3840E–1(8.1E–3)	**7.3850E–1(1.4E–2)**	7.4146E–1(1.2E–2)	**7.4707E–1(1.2E–2)**†	7.1395E–1(8.5E–3)	**7.3599E–1(1.1E–2)**†
	DTLZ7	4.5444E–1(8.3E–2)	**5.0576E–1(6.2E–2)**†	5.3340E–1(4.1E–4)	**5.6217E–1(1.6E–3)**†	5.1590E–1(2.7E–3)	**5.5925E–1(2.0E–3)**†
Complex PS	UF1	8.0299E–1(4.9E–3)	**8.1707E–1(9.3E–3)**†	8.1707E–1(9.3E–3)	**8.7067E–1(1.5E–2)**†	8.1090E–1(5.4E–2)	**8.6548E–1(2.0E–2)**†
	UF2	8.0179E–1(3.0E–3)	**8.4197E–1(3.4E–3)**†	8.5721E–1(2.2E–2)	**8.6395E–1(1.2E–2)**†	8.2907E–1(3.5E–2)	**8.6149E–1(2.9E–3)**†
	UF3	7.6131E–1(4.2E–2)	**8.0534E–1(3.6E–2)**†	8.5673E–1(2.5E–2)	**8.5955E–1(1.2E–2)**	8.0639E–1(5.3E–2)	**8.3706E–1(3.9E–2)**†
	UF4	4.4840E–1(4.3E–3)	**4.5528E–1(5.8E–3)**†	**4.3154E–1(6.6E–3)**	4.3115E–1(9.4E–3)	4.2700E–1(9.6E–3)	**4.3010E–1(9.3E–3)**
	UF5	0.0000E+0(0.0E+0)	**2.9497E–2(4.3E–2)**†	1.9400E–1(9.7E–2)	**2.5115E–1(9.8E–2)**†	1.9166E–1(1.1E–1)	**1.9518E–1(1.0E–1)**
	UF6	1.2521E–1(1.7E–2)	**2.2287E–1(4.8E–2)**†	2.6455E–1(7.3E–2)	**2.9623E–1(6.0E–2)**†	2.3798E–1(7.0E–2)	**2.6663E–1(6.2E–2)**†
	UF7	6.7217E–1(2.9E–3)	**6.8245E–1(1.9E–3)**†	6.7607E–1(2.1E–2)	**6.8884E–1(1.5E–2)**†	6.5252E–1(9.8E–2)	**6.7787E–1(1.3E–2)**†
	UF8	2.8638E–1(8.7E–3)	**5.5758E–1(4.6E–2)**†	**6.8837E–1(3.2E–2)**	6.7743E–1(2.8E–2)	5.6021E–1(8.9E–2)	**6.0717E–1(6.3E–2)**†
	UF9	**9.0647E–1(8.9E–3)**	8.9953E–1(5.6E–2)	**9.1761E–1(7.7E–2)**	8.9569E–1(5.7E–2)	8.9252E–1(5.9E–2)	**9.0663E–1(6.8E–2)**
	UF10	0.0000E+0(0.0E+0)	0.0000E+0(0.0E+0)	1.1111E–1(3.5E–2)	**1.1990E–1(3.3E–2)**	1.7454E–1(4.4E–2)	**1.7519E–1(4.3E–2)**
Many Objectives	DTLZ2(4)	**1.0506E+0(6.0E–4)**	1.0428E+0(2.7E–3)†	**1.0585E+0(2.0E–4)**	1.0538E+0(1.2E–3)†	**1.0588E+0(3.2E–5)**	1.0536E+0(1.2E–3)†
	DTLZ2(6)	**1.5551E+0(9.7E–4)**	1.5324E+0(3.4E–3)†	1.5232E+0(1.1E–2)	**1.5311E+0(4.0E–3)**†	**1.5504E+0(8.5E–4)**	1.5423E+0(2.4E–3)†
	DTLZ2(10)	1.9101E+0(1.3E–1)	**2.5049E+0(3.8E–3)**†	2.4452E+0(2.5E–2)	**2.4701E+0(7.4E–3)**†	**2.5092E+0(3.7E–4)**	2.4954E+0(3.7E–3)†
	DTLZ5(2,10)	2.3597E–1(8.8E–4)	**2.6133E–1(1.2E–3)**†	2.3582E–1(5.9E–4)	**2.6095E–1(1.4E–3)**†	2.5850E–1(7.8E–4)	**2.6078E–1(1.6E–3)**†
	DTLZ5(3,10)	4.1017E–1(9.7E–2)	**8.7522E–1(5.3E–3)**†	7.5172E–1(1.1E–3)	**8.7512E–1(7.3E–3)**†	7.5719E–1(8.9E–3)	**8.7723E–1(6.6E–3)**†

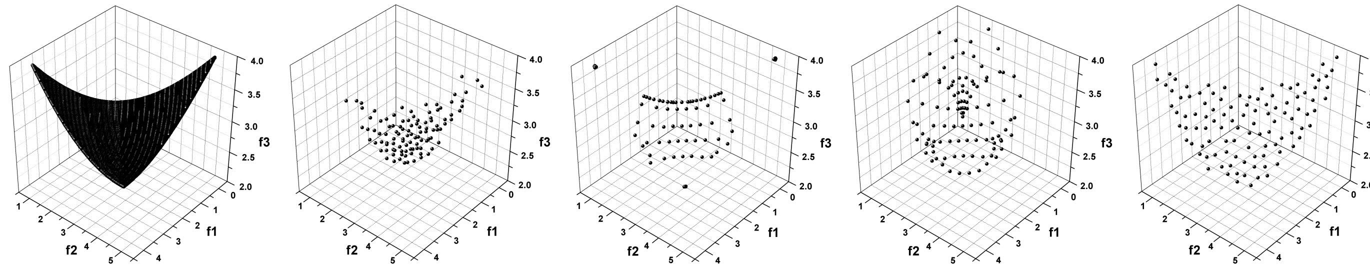
"†" indicates that the value of the BCE algorithm is significantly different from that of its corresponding non-Pareto algorithm at a 0.05 level by the Wilcoxon's rank sum test.



(a) IBEA (b) MOEA/D+TCH (c) MOEA/D+PBI

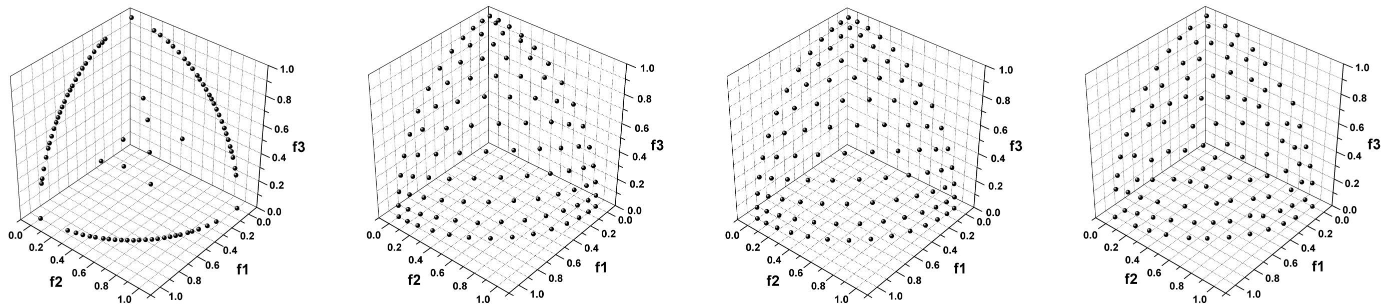
(d) BCE-IBEA (e) BCE-MOEA/D+TCH (f) BCE-MOEA/D+PBI

Fig. 4. The final solution set of the six algorithms on SCH1.



(a) Pareto front (b) IBEA (c) MOEA/D+TCH (d) MOEA/D+PBI (e) BCE-MOEA/D+TCH

Fig. 5. Pareto front and the final solution set on VNT1, where the solutions of the three BCE algorithms have similar distribution.



(a) IBEA (b) MOEA/D+TCH (c) MOEA/D+PBI (d) BCE-MOEA/D+TCH

Fig. 6. The final solution set on DTLZ2, where the solutions of the three BCE algorithms have similar distribution.

happens mainly due to the population maintenance operation in BCE, which can effectively eliminate poorly distributed solutions in the evolutionary process. In the rest of the paper, for brevity, we only plot the solutions of one of the BCE algorithms if they perform visually similarly.

In addition, Fig. 5 shows the final solutions on VNT1. Clearly, for this 3-objective problem, only the BCE algorithms have good diversity. The solutions obtained by IBEA are solely located in the middle of the Pareto front. The solutions of the two MOEA/D algorithms, which correspond to uniformly distributed weight vectors, exhibit a specific structure but do not have a good distribution over the desired front.

B. Test Problems with a Concave Pareto Front

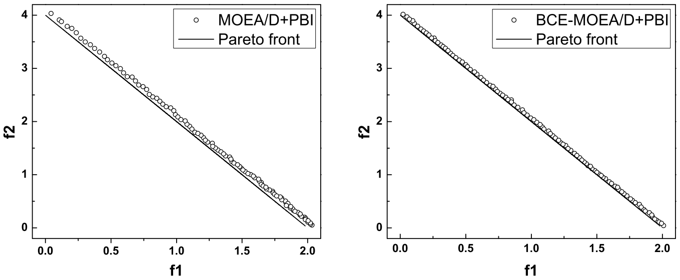
In this category, we consider 13 problems from the ZDT, WFG, and DTLZ problem suites. As can be seen from Tables III and IV, the three BCE algorithms generally perform better than their competitors. Specifically, BCE-IBEA, BCE-MOEA/D+TCH, and BCE-MOEA/D+PBI obtain a better IGD value in 12, 8, and 9 out of the 13 test instances, respectively. For HV, BCE-IBEA, BCE-MOEA/D+TCH, and BCE-MOEA/D+PBI outperform their corresponding non-Pareto algorithms in 12, 9, and 9 out of the 13 instances, respectively.

In fact, for some MOPs (such as DTLZ2), some non-Pareto algorithms already work quite well. In this case, the exploration operation in the PC evolution of BCE can hardly further improve individuals' performance, but can lead to the decrease of the computational resources (i.e., function evaluations) occupied by the NPC evolution. Fig. 6 gives the final solutions obtained by IBEA, MOEA/D+TCH, MOEA/D+PBI, and BCE-MOEA/D+TCH on DTLZ2. Clearly, for this problem, IBEA is unable to maintain uniformity of the solutions but MOEA/D+TCH and MOEA/D+PBI have a set of excellently distributed solutions over the Pareto front. This is consistent with the result in Table III, where IBEA performs worse than BCE-IBEA but the two MOEA/D algorithms perform better than their competitors.

As to the statistical results, it can be observed from the tables that the difference between the paired algorithms is significant for most of the test instances. Specifically, the proportion of the test instances where the three BCE algorithms BCE-IBEA, BCE-MOEA/D+TCH and BCE-MOEA/D+PBI outperform their competitors with statistical significance is 11/13, 6/13 and 9/13 for IGD and 9/13, 9/13 and 9/13 for HV, respectively. Conversely, the proportion of the instances where the three non-Pareto algorithms IBEA, MOEA/D+TCH and MOEA/D+PBI are superior with statistical significance is 0/13, 4/13 and 3/13 for IGD and 1/13, 4/13 and 1/13 for HV, respectively.

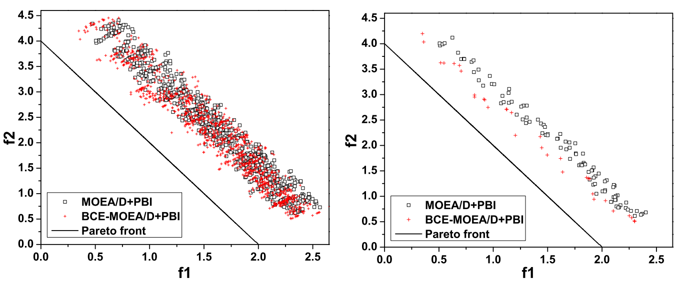
C. Test Problems with a Linear Pareto Front

Non-Pareto EMO algorithms in general work well on this kind of problems as their NPC is not likely to prefer specific areas of a plane Pareto front. Despite that, the proposed approach is still competitive, as can be seen from the results on test problems WFG3 and DTLZ1 in Tables III and IV. For WFG3, the three BCE algorithms all outperform their competitors. For a visual comparison, Fig. 7 plots the final solutions of MOEA/D+PBI and BCE-MOEA/D+PBI as well as the problem's Pareto front. As shown, BCE-MOEA/D+PBI has a better performance than MOEA/D+PBI in terms of both diversity and convergence. This observation is interesting because it is commonly believed that the solutions guided by an NPC have a better convergence than those by the



(a) MOEA/D+PBI (b) BCE-MOEA/D+PBI

Fig. 7. The final solution set obtained by MOEA/D+PBI and BCE-MOEA/D+PBI on WFG3.



(a) All the individuals produced (b) Population at the 1,000 evaluations

Fig. 8. Results of MOEA/D+PBI and BCE-MOEA/D+PBI during initial 1,000 function evaluations on WFG3: (a) All the individuals produced during the 1,000 evaluations; (b) Evolutionary population at the 1,000 evaluations.

PC. One important reason for this occurrence is that the exploration around the nondominated solutions in BCE can effectively drive the population evolving towards the Pareto front, especially at the initial stage of evolution.

To take a closer look, Fig. 8 gives the results of MOEA/D+PBI and BCE-MOEA/D+PBI during the initial 1,000 function evaluations, where Fig. 8(a) plots all 1,000 individuals produced by the two algorithms and Fig. 8(b) plots their evolutionary population at the 1,000 evaluations. As can be seen from Fig. 8(a), there exist some individuals of BCE-MOEA/D+PBI apparently closer to the optimal front. This is the result of effective exploration of the nondominated individuals (i.e., the PC population) in BCE. These nondominated individuals, whose number is smaller than the population capacity at that time, can represent the best individuals found so far, as seen from the comparison between Figs. 8(a) and (b).

The test function DTLZ1 has a huge number of local optimal fronts (11^5-1). For this problem, BCE-IBEA outperforms IBEA, but BCE-MOEA/D+TCH and BCE-MOEA/D+PBI perform worse than the two MOEA/D algorithms. In Section V-C, we will provide a detailed explanation for why BCE may be outperformed by some non-Pareto algorithms on such MOPs with a number of local optima.

D. Test Problems with a Mixed Pareto Front

The results of three of this kind of problems, WFG1, VNT2 and VNT3, are shown in Tables III and IV, where the BCE algorithms significantly outperform their competitors. They are superior with statistical significance in 8 out of all the 9 comparisons for both IGD and HV indicators. For a visual observation, Fig. 9 plots the final solutions obtained by IBEA, MOEA/D+TCH, MOEA/D+PBI, and BCE-MOEA/D+TCH on VNT3. Clearly, only BCE-MOEA/D+TCH has well-distributed solutions over the whole Pareto front. The solutions obtained by the three non-Pareto algorithms concentrate mainly in the middle segment and fail to extend to the left part of the optimal front.

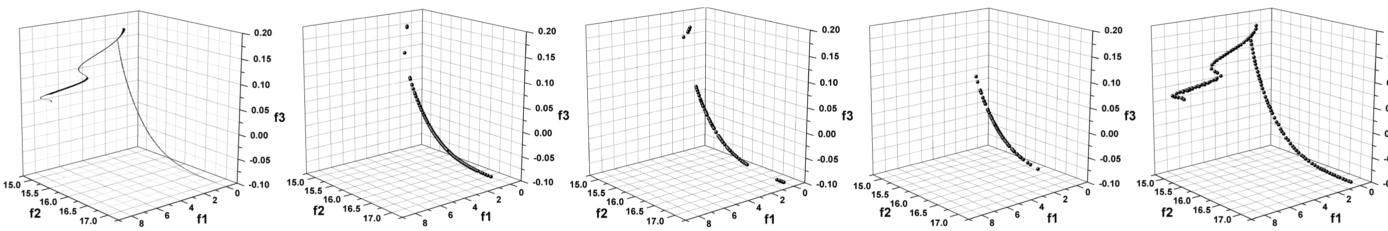
E. Test Problems with a Discontinuous Pareto Front

As can be seen from the two tables, for MOPs with a discontinuous Pareto front, the proposed approaches have a clear advantage over the non-Pareto algorithms. The three BCE algorithms significantly outperform their competitors for all the instances, and on most of these instances they even have an order of magnitude smaller IGD values.

In fact, non-Pareto algorithms commonly struggle to maintain the diversity of solutions on this kind of MOPs. This happens mainly due to the fact that the imaginary parts of the discontinuous Pareto front largely affect the accuracy of the fitness estimation based on an NPC. For example, in MOEA/D, the breakpoints of the discontinuous Pareto front may correspond to the optimal solution of multiple scalar subproblems [58]. This is likely to cause the failure of uniformity maintenance of solutions, further leading to the search of the algorithm only on some specific regions of the objective space. Fig. 10 plots the final solutions obtained by IBEA, MOEA/D+TCH, MOEA/D+PBI, and BCE-MOEA/D+TCH on SCH2. It can be observed that IBEA and MOEA/D+TCH are unable to maintain the uniformity of solutions, and MOEA/D+PBI fails to find the upper part of the Pareto front. Note that there exist some dominated solutions in the set of solutions obtained by MOEA/D+PBI. This is because a dominated solution may have a closer distance than a nondominated one to the corresponding reference point in MOEA/D+PBI, as pointed out in [16].

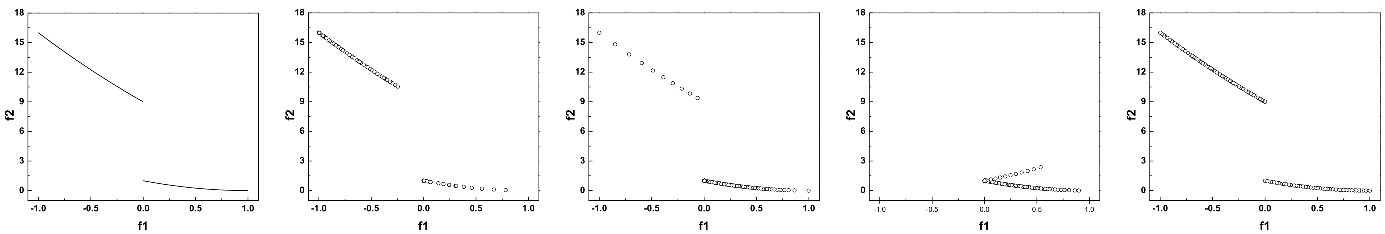
F. Test Problems with a Complex PS

In this section, we consider the UF problem suite from the CEC2009 competition [74]. These MOPs involve a strong linkage in variables among the Pareto optimal solutions, thereby posing a big challenge for EMO algorithms [44], [76]. In spite of that, the BCE algorithms outperform their corresponding non-Pareto algorithms on the majority of the test instances, as shown in Tables III and IV. Specifically, BCE-IBEA, BCE-MOEA/D+TCH, and BCE-MOEA/D+PBI achieve a better IGD value than their competitors in 8, 7, and 9 out of the 10 test instances, respectively. For HV, BCE-IBEA, BCE-MOEA/D+TCH, and BCE-MOEA/D+PBI outperform their competitors in 8, 7, and 10 out of the 10 instances, respectively. Also, the difference in most of these comparisons is statistically significant, with the winning ratio of the BCE algorithms against their competitors being 19 to 2 for IGD and 19 to 0 for HV in the 30 comparisons, respectively. One reason for this occurrence is likely due to the population maintenance operation in BCE, which is able to maintain the diversity of solutions effectively.



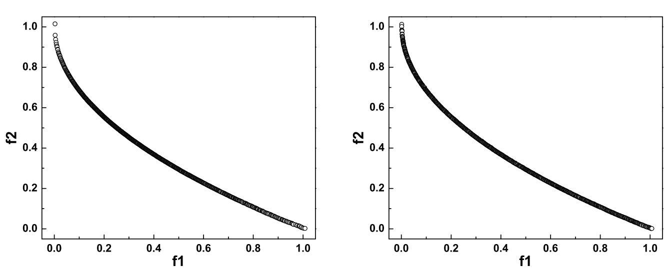
(a) Pareto front (b) IBEA (c) MOEA/D+TCH (d) MOEA/D+PBI (e) BCE-MOEA/D+TCH

Fig. 9. Pareto front and the final solution set on VNT3, where the solutions of the three BCE algorithms have similar distribution.



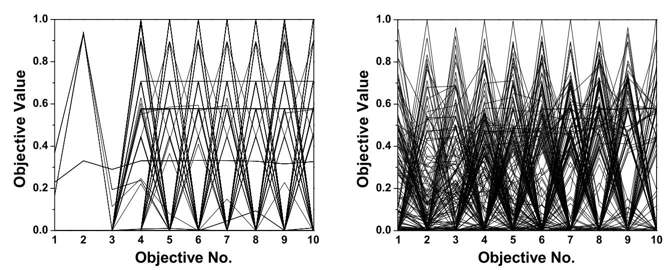
(a) Pareto front (b) IBEA (c) MOEA/D+TCH (d) MOEA/D+PBI (e) BCE-MOEA/D+TCH

Fig. 10. Pareto front and the final solution set on SCH2, where the solutions of the three BCE algorithms have similar distribution.



(a) MOEA/D+TCH (b) BCE-MOEA/D+TCH

Fig. 11. The final solution set obtained by MOEA/D+TCH and BCE-MOEA/D+TCH on UF1.



(a) MOEA/D+TCH (b) BCE-MOEA/D+TCH

Fig. 12. The final solution set obtained by MOEA/D+TCH and BCE-MOEA/D+TCH on the 10-objective DTLZ2, shown by parallel coordinates.

Fig. 11 plots the final solutions obtained by MOEA/D+TCH and BCE-MOEA/D+TCH on UF1. As shown, although a good distribution of solutions is obtained in most parts of the Pareto front by MOEA/D+TCH, there is a clear interval between the upper bound and the other solutions. In contrast, the solutions obtained by BCE-MOEA/D+TCH have a good distribution uniformity along the whole front.

G. Test Problems with Many Objectives

In this section, test problems DTLZ2 [18] and DTLZ5(I, m) [62], [64] are used to verify the performance of BCE on many-objective problems. DTLZ2 has a spherical Pareto front in the range $f_1, f_2, ..., f_m \in [0, 1]$, and DTLZ5($I, m$) has a degenerate Pareto front, with its dimensionality I lower than that of the objective space m.

Tables III and IV show the results of the six algorithms on five instances of DTLZ2 and DTLZ5(I, m). These instances are three DTLZ2 functions with 4, 6 and 10 objectives, and two 10-objective DTLZ5(I, m) functions with 2- and 3-dimensional Pareto front, i.e., DTLZ5(2,10) and DTLZ5(3,10). As can be seen from the tables, the compared algorithms perform similarly on the 4- and 6-objective instances, where the BCE algorithms often have a better IGD result while the non-Pareto algorithms generally obtain a higher HV value. For three 10-objective instances, the BCE algorithms significantly outperform their competitors. This suggests that the advantage of the proposed approach becomes clearer when a higher-dimensional space is involved. Fig. 12 shows the final solutions of MOEA/D+TCH and BCE-MOEA/D+TCH on the 10-objective DTLZ2 by parallel coordinates. As shown, MOEA/D+TCH fails to find widely distributed solutions on objectives f_1 to f_3, which is in contrast to the result of BCE-MOEA/D+TCH, where a spread of solutions over $f_i \in [0, 1]$ is obtained.

In DTLZ5(I, m), all objectives within $\{f_1, ..., f_{m-I+1}\}$ are positively correlated, while the objectives in $\{f_{m-I+1}, ..., f_m\}$ conflict with each other. Fig. 13 plots the final solutions of IBEA, MOEA/D+TCH, MOEA/D+PBI, and BCE-MOEA/D+TCH in the last three objectives (f_8, f_9, f_{10}) of DTLZ5(3,10). The Pareto optimal solutions of the problem with respect to these objectives satisfy $2f_8^2 + f_9^2 + f_{10}^2 = 1$. As shown, despite several solutions of BCE-MOEA/D+TCH not located on the optimal front, the rest has good uniformity and coverage over the whole front. In contrast, the three non-Pareto algorithms struggle to maintain diversity, with their solutions

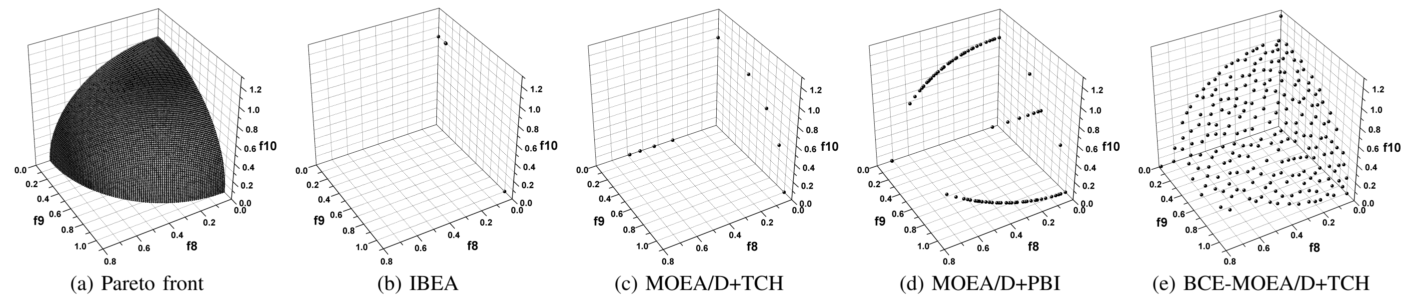


Fig. 13. Pareto front and the final solution set in the subspace (f_8, f_9, f_{10}) of DTLZ5(3,10), where the solutions of the three BCE algorithms have similar distribution.

concentrating in some tiny parts (or even several points) of the Pareto front.

Finally, it is worth mentioning that Pareto-based algorithms are often seen to fail to deal with many-objective problems. Their Pareto dominance and density-based selection criteria even could push the population against the optimal front in a high-dimensional space [31], [47], [68]. Interestingly, the solution set of BCE, which is the PC population maintained by Pareto dominance and density, performs well in many-objective problems. This occurrence can be attributed to the role of the NPC evolution in BCE, which leads the PC population to evolve towards the desired direction.

H. Result Summary

To summarize, the BCE algorithms generally outperform their corresponding non-Pareto algorithms. BCE-IBEA, BCE-MOEA/D+TCH, and BCE-MOEA/D+PBI have a better IGD value in 38, 32, and 35 out of all the 42 test problems. For HV, BCE-IBEA, BCE-MOEA/D+TCH, and BCE-MOEA/D+PBI perform better in 36, 33, and 34 out of all the 42 test problems. Note that for a couple of test problems, some paired algorithms obtain different comparison results with respect to IGD and HV, although both indicators involve comprehensive performance of convergence and diversity. For example, for the 4- and 6-objective DTLZ2, BCE-IBEA has a better IGD, but worse HV than the original IBEA. This contradiction between HV and IGD happens more on the problems with a concave Pareto front, as reported in [38]. The reason for this occurrence may be due to the different preference of the two indicators [50]. IGD, which is based on uniformly-distributed points along the entire Pareto front, prefers the distribution uniformity of the solution set, while HV, which is typically influenced more by the boundary solutions, has a bias towards the extensity of the solution set.

V. Further Investigations of BCE

Having demonstrated its competitiveness on various test problems above, BCE is further investigated in this section for a deeper understanding of its behavior. Due to the space limit and similar comparative results obtained by the IGD and HV indicators, we only present the IGD results in this and the following sections.

A. Performance Verification of the NPC Evolution

The previous experimental results have shown the effectiveness of the PC evolution in maintaining the individual diversity and approaching the optimal front. Now a question may arise – how about the NPC evolution? Does the NPC evolution benefit from the information exchange with the PC evolution? In other words, can non-Pareto EMO algorithms themselves benefit when working under this BCE framework?

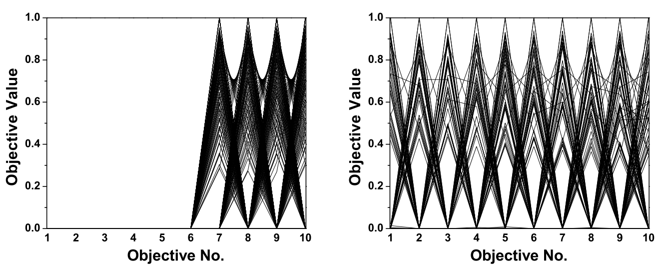
To answer this question, we give the IGD results between the solution set of the original non-Pareto algorithms and that of the embedded ones (i.e., the NPC population) on the 42 test problems in Table V. As shown, for most of the problems, the performance of the three non-Pareto algorithms is improved when working under the BCE framework. The NPC population of BCE-IBEA, BCE-MOEA/D+TCH, and BCE-MOEA/D+TCH has a better IGD value than the solution set of the corresponding non-Pareto algorithms in 31, 31, and 32 out of all the 42 instances, respectively.

In addition, it is worth mentioning that unlike the PC evolution which can preserve any nondominated individual located in a sparse region, the NPC evolution may not preserve such individuals due to its own selection criterion. That is, even when the PC evolution produces plenty of promising individuals which clearly help enhance the population diversity, they may still not enter the NPC population if not preferred by the considered NPC. That is why for some MOPs in which the PC population significantly outperforms that of the non-Pareto algorithm, the NPC population yet performs similarly to (or even slightly worse than) the latter, such as the results of BCE-MOEA/D+TCH on DTLZ5(2,10). But, interestingly, there are also some exceptions that the population of non-Pareto algorithms has a clear improvement when working collaboratively with the PC population in BCE. Fig. 14 gives such an example, where the solution set of the original IBEA and the one working under the BCE framework (i.e., the NPC population of BCE-IBEA) are plotted by parallel coordinates. Clearly, in contrast to IBEA which fails to find diverse solutions on objectives f_1 to f_6, the NPC evolution in BCE-IBEA maintains a good diversity of population for all the objectives of the Pareto front.

In summary, despite evolving based on their own selection criterion, the non-Pareto algorithms generally show a performance improvement when embedded into the NPC evolution part of BCE. This, along with the experimental results in

TABLE V
IGD COMPARISON BETWEEN THE ORIGINAL NON-PARETO ALGORITHMS AND THE NPC EVOLUTION IN THEIR CORRESPONDING BCE ALGORITHMS. THE BETTER MEAN FOR EACH CASE IS HIGHLIGHTED IN BOLDFACE

Problem	IBEA	BCE-IBEA	MOEA/D+TCH	BCE-MOEA/D+TCH	MOEA/D+PBI	BCE-MOEA/D+PBI
SCH1	**1.9093E–2(4.1E–4)**	2.0162E–2(1.1E–3)	4.9139E–2(1.5E–3)	**4.6963E–2(7.7E–4)**	**1.7765E–1(5.2E–3)**	1.7910E–1(2.0E–3)
SCH2	1.2321E–1(4.0E–2)	**7.5079E–2(2.0E–2)**	1.0491E–1(2.8E–4)	**1.0488E–1(1.1E–4)**	5.0208E+0(5.0E–3)	**5.0155E+0(8.7E–4)**
KUR	1.9336E–1(2.3E–2)	**1.6513E–1(3.0E–2)**	4.1089E–2(2.3E–4)	**4.1073E–2(3.3E–4)**	4.3389E–2(8.7E–4)	**4.2867E–2(9.1E–4)**
ZDT1	**4.0773E–3(6.8E–5)**	4.1000E–3(6.7E–5)	4.3626E–3(3.4E–4)	**4.1924E–3(1.1E–4)**	8.2654E–3(1.7E–3)	**6.6341E–3(5.8E–4)**
ZDT2	9.1909E–3(3.7E–4)	**9.0121E–1(9.8E–4)**	4.3342E–3(1.8E–4)	**4.1933E–3(1.2E–4)**	1.0540E–2(1.2E–3)	**8.2292E–3(1.0E–3)**
ZDT3	3.0837E–2(8.0E–4)	**3.0414E–2(1.6E–3)**	1.0929E–2(7.9E–5)	**1.0923E–2(1.1E–4)**	1.3874E–2(5.3E–3)	**1.1675E–2(4.5E–4)**
ZDT4	6.2464E–1(1.1E–1)	**1.3199E–1(9.5E–4)**	1.0001E–2(3.4E–3)	**8.9033E–3(3.3E–3)**	3.7084E–2(2.8E–2)	**1.4819E–2(6.7E–3)**
ZDT6	**5.4849E–3(1.8E–4)**	5.5631E–3(1.6E–4)	1.5027E–2(1.7E–3)	**1.0515E–2(1.3E–3)**	3.9299E–2(4.6E–3)	**2.3791E–2(2.9E–3)**
WFG1	**7.5883E–1(6.2E–2)**	7.8579E–1(6.0E–2)	1.0487E+0(7.2E–2)	**9.6474E–1(5.3E–2)**	1.1983E+0(5.9E–2)	**1.0520E+0(5.2E–2)**
WFG2	7.0639E–2(8.6E–3)	**6.9895E–2(8.5E–3)**	3.9081E–2(3.7E–3)	**3.8934E–2(3.8E–3)**	1.1083E–1(1.1E–2)	**9.3369E–2(6.8E–3)**
WFG3	**1.5944E–2(1.2E–3)**	1.7071E–2(2.7E–3)	2.5079E–2(4.3E–3)	**2.3297E–2(3.2E–3)**	4.7442E–2(8.1E–3)	**4.2171E–2(8.1E–3)**
WFG4	**1.8370E–2(8.3E–4)**	1.8880E–2(1.0E–3)	2.2551E–2(1.8E–3)	**2.1534E–2(1.5E–3)**	5.0721E–2(5.2E–3)	**4.5001E–2(4.4E–3)**
WFG5	7.2029E–2(7.6E–4)	**7.1733E–2(9.4E–4)**	**6.8459E–2(5.9E–4)**	6.8496E–2(5.2E–4)	8.2498E–2(2.9E–3)	**7.9373E–2(2.4E–3)**
WFG6	**6.5849E–2(9.4E–3)**	6.5980E–2(7.9E–3)	6.8500E–2(1.0E–2)	**6.3961E–2(8.1E–3)**	1.0092E–1(1.4E–2)	**9.0327E–2(1.2E–2)**
WFG7	2.1525E–2(1.0E–3)	**2.1004E–2(9.9E–4)**	1.6770E–2(4.7E–4)	**1.6614E–2(2.5E–4)**	3.7482E–2(3.3E–3)	**3.3662E–2(2.4E–3)**
WFG8	9.3529E–2(8.0E–3)	**3.3918E–2(2.4E–3)**	8.7532E–2(5.6E–3)	**8.5901E–2(5.8E–3)**	1.1848E–1(7.9E–3)	**1.1040E–1(8.9E–3)**
WFG9	7.8794E–2(5.2E–2)	**7.8164E–2(5.2E–2)**	**3.5633E–2(2.4E–2)**	5.7653E–2(4.4E–2)	**5.6432E–2(1.6E–2)**	7.6906E–2(3.7E–2)
VNT1	1.6688E–1(2.7E–2)	**1.3937E–1(6.9E–3)**	2.2398E–1(1.3E–3)	**2.2369E–1(1.1E–3)**	1.7364E–1(5.6E–4)	**1.7361E–1(6.4E–4)**
VNT2	4.2602E–2(9.4E–3)	**2.7724E–2(8.5E–3)**	4.6425E–2(2.9E–4)	**4.6404E–2(3.4E–4)**	5.8432E–2(2.4E–4)	**5.8373E–2(3.4E–4)**
VNT3	2.5369E+0(8.7E–2)	**2.2216E+0(4.4E–1)**	2.3174E+0(1.3E–1)	**2.2537E+0(1.0E–1)**	2.5622E+0(8.2E–2)	**2.4384E+0(5.5E–3)**
DTLZ1	1.8267E–1(1.7E–2)	**1.0071E–1(2.3E–2)**	1.9571E–2(1.1E–3)	**1.9465E–2(6.8E–4)**	**1.9130E–2(5.7E–4)**	1.9507E–2(1.1E–3)
DTLZ2	1.1894E–1(2.2E–3)	**1.1820E–1(2.5E–3)**	**5.1151E–2(4.2E–4)**	5.1421E–2(5.1E–4)	5.0151E–2(3.1E–6)	**5.0150E–2(3.4E–6)**
DTLZ3	**5.1829E–1(1.5E–1)**	6.5375E–1(3.8E–1)	**5.0856E–1(6.5E–1)**	1.6290E+0(1.3E+0)	**1.0267E+0(9.1E–1)**	1.4211E+0(1.0E+0)
DTLZ4	4.6424E–1(3.4E–1)	**2.8527E–1(3.2E–1)**	1.7346E–1(2.3E–1)	**5.1825E–2(1.1E–3)**	1.1261E–1(1.7E–1)	**5.0157E–2(1.3E–5)**
DTLZ5	2.5627E–2(1.1E–3)	**2.4306E–2(1.6E–3)**	1.8107E–2(1.5E–5)	**1.8101E–2(9.7E–6)**	3.1920E–2(1.0E–4)	**3.1786E–2(1.2E–4)**
DTLZ6	1.0148E–1(2.0E–2)	**8.2460E–2(2.4E–2)**	**2.4922E–1(3.7E–2)**	3.3996E–1(3.2E–2)	**6.4541E–1(5.7E–2)**	7.7428E–1(5.1E–2)
DTLZ7	4.2838E–1(2.9E–1)	**3.4369E–1(2.3E–1)**	1.2946E–1(9.6E–4)	**1.2907E–1(1.2E–3)**	1.2840E–1(9.3E–4)	**1.2773E–1(4.7E–4)**
UF1	4.6600E–2(3.4E–3)	**4.1616E–2(6.1E–3)**	2.1201E–2(3.3E–2)	**1.6545E–3(2.2E–4)**	7.4430E–2(6.9E–2)	**1.1961E–2(2.6E–2)**
UF2	4.6187E–2(1.9E–3)	**2.3027E–2(2.8E–3)**	1.4578E–2(2.2E–2)	**6.6129E–3(1.4E–2)**	5.2665E–2(5.2E–2)	**1.2393E–2(2.8E–3)**
UF3	7.1289E–2(2.8E–2)	**4.2602E–2(2.3E–2)**	1.1098E–2(1.3E–2)	**9.8210E–3(7.6E–3)**	3.8405E–2(3.1E–2)	**2.4841E–2(2.7E–2)**
UF4	5.2290E–2(2.0E–3)	**5.1971E–2(2.7E–3)**	**6.4407E–2(4.0E–3)**	6.5795E–2(5.7E–3)	6.7517E–2(5.7E–3)	**6.6706E–2(6.1E–3)**
UF5	1.1661E+0(1.2E–1)	**5.1095E–1(1.4E–1)**	4.9510E–1(1.7E–1)	**4.3255E–1(1.6E–1)**	**4.6553E–1(1.3E–1)**	5.0649E–1(1.7E–1)
UF6	3.5670E–1(2.1E–2)	**2.8411E–1(1.1E–1)**	5.0766E–1(1.5E–1)	**4.7432E–1(1.7E–1)**	5.2782E–1(1.4E–1)	**5.1456E–1(1.7E–1)**
UF7	2.3288E–2(1.9E–3)	**1.9814E–2(1.5E–3)**	2.0330E–2(1.5E–2)	**1.2150E–2(9.9E–3)**	4.8084E–2(1.2E–1)	**1.9857E–2(1.2E–2)**
UF8	3.8577E–1(9.3E–3)	**3.6610E–1(2.5E–2)**	**5.4927E–2(1.5E–2)**	6.0981E–2(1.4E–2)	1.1147E–1(4.6E–2)	**9.2471E–2(3.2E–2)**
UF9	**9.9491E–2(3.0E–3)**	1.2012E–1(4.3E–2)	**1.1732E–1(4.8E–2)**	1.3119E–1(3.4E–2)	1.3193E–1(3.7E–2)	**1.2819E–1(4.1E–2)**
UF10	**2.2497E+0(2.8E–1)**	2.3864E+0(1.8E–1)	4.6931E–1(6.6E–2)	**4.6540E–1(8.1E–2)**	**4.8214E–1(1.1E–1)**	4.8285E–1(8.9E–2)
DTLZ2(4)	1.7281E–1(1.1E–3)	**1.7250E–1(9.1E–4)**	**8.9093E–2(7.0E–4)**	8.9121E–2(7.9E–4)	8.7399E–2(5.1E–6)	8.7399E–2(4.8E–6)
DTLZ2(6)	**3.7149E–1(2.4E–3)**	3.7347E–1(2.7E–3)	2.8757E–1(1.4E–2)	**2.8751E–1(1.2E–2)**	2.5665E–1(2.4E–5)	2.5665E–1(1.9E–5)
DTLZ2(10)	8.6901E–1(3.4E–2)	**5.6541E–1(2.1E–3)**	**5.6890E–1(1.9E–2)**	5.7893E–1(1.6E–2)	4.9222E–1(2.4E–5)	4.9222E–1(2.5E–5)
DTLZ5(2,10)	7.4584E–1(0.0E+0)	**1.0511E–2(2.3E–3)**	**1.6977E–1(1.8E–3)**	1.7116E–1(1.0E–3)	6.4940E–2(5.3E–5)	**6.4930E–2(4.2E–5)**
DTLZ5(3,10)	6.1236E–1(7.6E–2)	**2.7412E–1(1.2E–1)**	2.5113E–1(3.2E–4)	**2.5110E–1(3.7E–4)**	1.6867E–1(3.1E–3)	**1.6739E–1(3.9E–3)**



(a) IBEA (b) BCE-IBEA

Fig. 14. The final solution set of IBEA and the one working under the BCE framework (i.e., the NPC population of BCE-IBEA) on the 10-objective DTLZ2, shown by parallel coordinates.

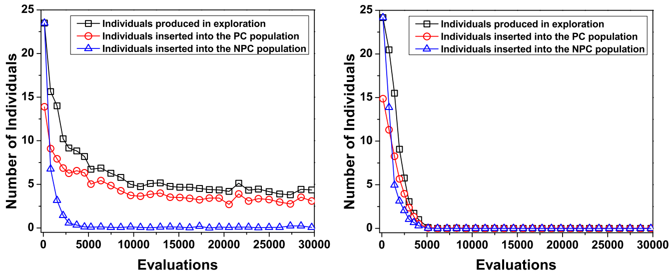
the previous section, indicates that both the NPC and PC evolutions benefit from the information share and exchange under the BCE framework.

B. Individual Exploration in the PC Evolution

In BCE, the individual exploration operation plays a key role. It is designed to compensate for the possible diversity loss of the NPC evolution, by exploring some promising individuals in the PC population which are not well developed or have already been eliminated in the NPC population. In this section, we take a closer look at this operation, investigating its role during the evolutionary process. That is, we record how many individuals are produced in the operation and from them how many individuals enter the PC and NPC populations via the PC and NPC selection, respectively.

For a comparison observation, we consider two situations where the embedded non-Pareto algorithm performs poorly and well, separately. They are BCE-IBEA and BCE-MOEA/D+PBI on DTLZ2 (cf. Figs. 6(a) and (c)). Fig. 15 gives the evolutionary trajectories of the average number of those individuals that are produced in the exploration operation and enter the PC and NPC populations across the 30 runs. As can be seen from Fig. 15, the exploration appears to be adaptive, based on the performance of the embedded non-Pareto algorithm. If the embedded algorithm performs poorly, constant exploration is being made throughout the whole evolutionary process; if the algorithm works well, the exploration stops at certain evaluations, giving the NPC evolution more computational resources. This adaptive operation leads to a good balance between the NPC and PC evolutions during the search process, and enables BCE to be always competitive no matter whether the embedded non-Pareto algorithm performs well or not.

Next, we consider the number of individuals that enter the PC and NPC populations. In both situations, most of the



(a) BCE-IBEA (b) BCE-MOEA/D+PBI

Fig. 15. Evolutionary trajectories of the average number of individuals that are produced in individual exploration and enter the PC and NPC populations across the 30 runs of BCE-IBEA and BCE-MOEA/D+PBI on DTLZ2. Black square denotes the number of individuals produced in the individual exploration operation, and red circle and blue triangle denote the number of the individuals entering the PC and NPC populations, respectively.

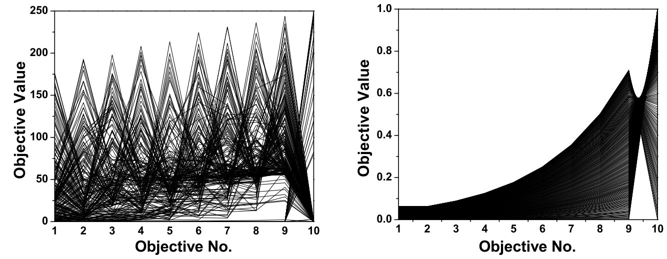
individuals produced in the exploration operation are preserved in the PC population. This shows the effectiveness of the exploration in producing competitive individuals in terms of the PC. On the other hand, very few individuals produced in the exploration operation can be selected into the NPC population after around 3,000 evaluations for BCE-IBEA. This is because the NPC evolution of the algorithm already performs "well" based on its own criterion. In spite of that, there do exist a number of individuals successfully entering the NPC population in the initial stage of the evolution for both algorithms, especially at the first generation where all the individuals produced in the exploration operation are preserved in the NPC population. This indicates the effectiveness of exploring promising nondominated individuals on accelerating the evolution of the NPC population during the initial stage of the search.

C. *Population Maintenance in the PC Evolution*

Like in Pareto-based algorithms, the population in the PC evolution is maintained by the Pareto dominance relation and individual density. They prefer nondominated individuals and individuals with a lower crowding degree. Now a concern may arise – does the PC evolution suffer from what Pareto-based algorithms commonly suffer, such as inferior performance on MOPs with a complex PS [44] or with a high-dimensional objective space [68]?

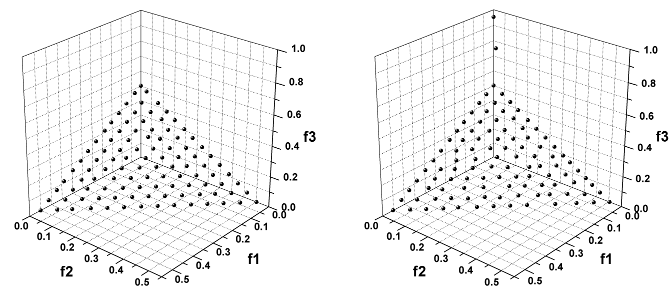
In fact, the answer to the above question can be found from the results in the previous section (Sections IV-F and IV-G). As shown in Table III, for most of the variable-linkage and many-objective problems, the PC population outperforms the solution set obtained by the indicator-based and decomposition-based algorithms. And these non-Pareto algorithms have already been demonstrated to have a clear advantage over Pareto-based algorithms on such MOPs [25], [32], [44], [51], [68], [76]. This suggests a fundamental difference of performance between the PC evolution and Pareto-based algorithms.

For a visual comparison, we give the results of BCE-MOEA/D+TCH and a well-known Pareto-based algorithm, NSGA-II, on the 10-objective problem DTLZ5(2,10). Fig. 16 plots the solution set obtained by the two algorithms via parallel coordinates. It is clear that in contrast to NSGA-II whose solutions are far away from the optimal front (the objective value being up to around 200), BCE-MOEA/D+TCH performs superiorly, with its solutions fully covering the whole Pareto front. This contrast indicates the interplay between the NPC evolution and the PC evolution in the BCE framework. Not only does the PC evolution maintain a set of well-distributed individuals to compensate for possible diversity loss of the NPC population, but the NPC evolution also guides the PC population forward – it produces sufficient well-converged individuals, which can "pull" the PC population moving towards the Pareto front.



(a) NSGA-II (b) BCE-MOEA/D+TCH

Fig. 16. The final solution set obtained by NSGA-II and BCE-MOEA/D+TCH on DTLZ5(2,10), shown by parallel coordinates.



(a) MOEA/D+TCH (b) BCE-MOEA/D+TCH

Fig. 17. The final solution set of MOEA/D+TCH and BCE-MOEA/D+TCH on DTLZ1.

Finally, it is necessary to point out that although BCE generally works well on the MOPs where Pareto-based algorithms have struggled, its Pareto dominance and density-based population maintenance strategy, in some cases, still has an impact on the algorithm's performance. This maintenance strategy can cause the existence of some dominance resistant solutions[5] (DRSs) [30] in the PC population. This has often been observed in problems with many local optimal fronts, such as DTLZ1 and DTLZ3. Fig. 17 shows the final solution set obtained by MOEA/D+TCH and BCE-MOEA/D+TCH in one typical run on DTLZ1. Clearly, in contrast to MOEA/D+TCH whose solutions all converge into the Pareto front, there exist two solutions far away from the optimal front in BCE-MOEA/D+TCH. Such solutions typically have a low crowding degree and will be preferred since no individual in the population dominates them.

The existence of DRSs in the PC population is detrimental

[5]Dominance resistant solutions are the solutions with a quite poor value in at least one of the objectives but with (near) optimal values in the others, which Pareto-based algorithms have difficulty in getting rid of [18], [30], [36].

not only to population maintenance but also to individual exploration. In the individual exploration operation, DRSs are always considered since there is no NPC individual located in their niche. Exploring them could have very little contribution to the algorithm's performance in view of their poor performance in terms of convergence. A straightforward approach to remove DRSs is to increase the selection pressure of Pareto dominance; however, this will lead to nondominated individuals to be treated differently, and thus will probably affect their distribution uniformity over the Pareto front. We leave this for our future study.

D. Comparison with State of the Art Algorithms

The previous experimental results have demonstrated the effectiveness of the BCE framework in improving three non-Pareto algorithms. In this section, we further investigate the competitiveness of the BCE framework by comparing the three BCE algorithms with two state-of-the-art EMO algorithms, NSGA-III [16] and SMS-EMOA [7].

NSGA-III and SMS-EMOA use both PC and NPC in their selection mechanism. NSGA-III combines the Pareto nondominated sorting with a decomposition-based niching technique to balance solutions' convergence and diversity in the evolutionary process. NSGA-III has shown its advantage over the two decomposition-based algorithms MOEA/D-TCH and MOEA/D-PBI in its original paper [16] and has been found to significantly outperform IBEA in a recent study [69]. Working with the Pareto nondominated sorting, SMS-EMOA maximizes the hypervolume contribution of nondominated solutions during the evolutionary process. SMS-EMOA has also been demonstrated to generally outperform IBEA and MOEA/D in a very recent study [39].

The intention that we introduce NSGA-III and SMS-EMOA as peer algorithms is to 1) verify the competitiveness of BCE in comparison with hybrid algorithms based on both PC and NPC and 2) see how the three BCE algorithms would perform against state-of-the-art algorithms that outperform their original non-Pareto versions.

Note that the execution of SMS-EMOA with a large population size and a large number of objectives can take unacceptable time. Therefore, for some MOPs (i.e., UF8–UF10, many-objective DTLZ2 and DTLZ5(I, m)), we approximately estimate the hypervolume indicator in SMS-EMOA by the Monte Carlo sampling method used in [4]. Following the practice in HypE [4], 10,000 sampling points are used here. In addition, all configurations in this experiment were kept the same as in previous studies.

Table VI gives the experimental results of the three BCE algorithms against NSGA-III and SMS-EMOA. As can be seen, the three BCE algorithms generally outperform NSGA-III and SMS-EMOA. Specifically, BCE-IBEA, BCE-MOEA/D+TCH and BCE-MOEA/D+PBI perform statistically better than/equally to/worse than NSGA-III on 28/8/6, 27/6/9 and 21/5/16 problems, respectively. BCE-IBEA, BCE-MOEA/D+TCH and BCE-MOEA/D+PBI perform statistically better than/equally to/worse than SMS-EMOA on 21/6/15, 23/3/16 and 21/3/18 problems, respectively.

It is worth mentioning that actually the original versions of the three non-Pareto algorithms (i.e., IBEA, MOEA/D+TCH, and MOEA/D+PBI) are significantly outperformed by NSGA-III and SMS-EMOA. From the comparison of the IGD results, IBEA, MOEA/D+TCH and MOEA/D+PBI perform statistically better than/equally to/worse than NSGA-III on 11/4/27, 15/3/24 and 13/4/25 problems, respectively; IBEA, MOEA/D+TCH and MOEA/D+PBI perform statistically better than/equally to/worse than SMS-EMOA on 5/3/34, 13/4/25 and 11/8/23 problems, respectively. This contrast clearly indicates the effectiveness of the BCE framework – when working under the BCE framework, all the three non-Pareto algorithms have a significant performance improvement and now are very competitive with or even generally outperform the state-of-the-art NSGA-III and SMS-EMOA.

VI. Discussions

One important issue in the proposed BCE framework is the setting of the niche radius, since both the population maintenance and individual exploration operations involve the niche-based density estimation. BCE considers the average of the Euclidean distance from all the individuals to their kth nearest individual in the population as the niche radius. A large k will result in a large radius. However, how to set k cannot be treated as trivial.

In population maintenance, the crowding degree estimation of an individual is affected by the number of other individuals in its niche (called its neighbors). A large k would make outer individuals of the population to be preferred since the number of their neighbors is generally fewer than that of inner ones. A too small k would make many individuals have no neighbor residing in their niche, thereby leading to the failure of differentiating them. In fact, the BCE algorithms can work well in terms of diversity maintenance when $k \in [3, 6]$. Fig. 18 plots the solution sets obtained by BCE-MOEA/D+TCH with different k values on DTLZ2. As shown, BCE-MOEA/D+TCH with $k = 3, 4, 6$ performs well, while the algorithm with $k = 2$ struggles to maintain uniformity and more boundary solutions are obtained when k is set to 10.

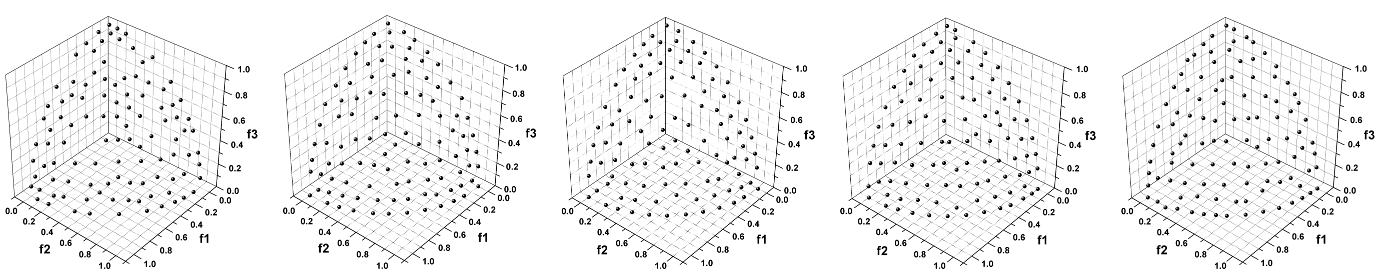
In individual exploration, the niche size affects the number of individuals to be explored. A large niche can lead to very few (or even none of) individuals in the PC population to be explored. Table VII gives the experimental results of BCE-MOEA/D+TCH with different k values on the nine WFG problems. Similar results can also be observed on other problems. As can be seen from the table, setting a small k can generally lead to a better result of the algorithm. BCE-MOEA/D+TCH with $k = 2$ performs best or second best in 7 out of the 9 problems, and the algorithm with $k = 3$ in 8 out of the 9 problems. This indicates that setting k to 2 or 3 is suitable for the individual exploration operation.

From the above observations, the BCE algorithm with k set to 3 can work well in both the population maintenance and individual exploration operations.

Finally, it is worth mentioning that the previous experiments are all about the test of comprehensive performance of BCE (i.e., combined performance of convergence and diversity).

TABLE VI
IGD RESULTS (MEAN AND SD) OF THE THREE BCE ALGORITHMS, NSGA-III AND SMS-EMOA. THE TWO MARKS ASSOCIATED WITH EACH BCE ALGORITHM INDICATE ITS STATISTICAL COMPARISON (ACROSS 30 RUNS) AGAINST NSGA-III AND SMS-EMOA, RESPECTIVELY. "<", "≈" AND ">" INDICATE THAT THE BCE ALGORITHM STATISTICALLY PERFORMS BETTER, EQUALLY AND WORSE, RESPECTIVELY, AT A 0.05 LEVEL BY THE WILCOXON'S RANK SUM TEST.

Problem	BCE-IBEA	BCE-MOEA/D+TCH	BCE-MOEA/D+PBI	NSGA-III	SMS-EMOA
SCH1	1.6732E–2(1.1E–4) < <	1.6761E–2(1.4E–4) < <	1.6729E–2(9.3E–5) < <	4.8446E–2(2.0E–4)	1.8529E–2(3.2E–4)
SCH2	2.0902E–2(2.3E–4) < >	2.0832E–2(2.7E–4) < >	2.0861E–2(2.1E–4) < >	3.0851E–2(2.4E–3)	1.9640E–2(1.3E–4)
KUR	3.4693E–2(6.7E–4) < >	3.4142E–2(6.2E–4) < >	3.4960E–2(9.2E–4) < >	4.0147E–2(6.6E–4)	3.2104E–2(3.4E–4)
ZDT1	3.9475E–3(5.0E–5) < >	4.2756E–3(9.4E–5) > >	5.9401E–3(4.3E–4) > >	4.0427E–3(5.7E–5)	3.6445E–3(1.9E–5)
ZDT2	4.0420E–3(3.3E–4) ≈ <	4.1688E–3(9.3E–5) > <	7.2119E–3(7.8E–4) > >	4.0919E–3(1.2E–4)	4.3563E–3(1.8E–4)
ZDT3	4.6028E–3(6.5E–5) < ≈	4.7997E–3(6.0E–5) < ≈	5.4836E–3(1.5E–4) < ≈	5.5924E–3(1.4E–4)	1.0180E–2(1.2E–2)
ZDT4	1.2270E–1(5.9E–2) > >	8.2586E–3(3.2E–3) < <	1.3458E–2(6.5E–3) < <	4.7561E–2(4.0E–2)	2.3873E–2(3.9E–2)
ZDT6	3.4449E–3(8.6E–5) < <	1.0328E–2(1.3E–3) < >	2.1787E–2(2.7E–3) > >	1.2670E–2(1.2E–3)	3.8964E–3(1.5E–4)
WFG1	7.7944E–1(6.3E–2) < <	9.6279E–1(5.4E–2) < <	1.0490E+0(5.3E–2) < <	1.1094E+0(6.4E–2)	1.1023E+0(1.3E–1)
WFG2	2.5660E–2(1.1E–2) ≈ >	2.0238E–2(4.1E–3) < ≈	3.0526E–2(6.9E–3) > >	2.5672E–2(3.7E–3)	1.8368E–2(3.9E–3)
WFG3	1.6083E–2(2.0E–3) < >	2.3228E–2(3.3E–3) > >	3.9358E–2(7.4E–3) > >	2.1876E–2(3.7E–3)	1.4330E–2(1.2E–3)
WFG4	1.3005E–2(4.6E–4) < >	1.9664E–2(1.5E–3) > >	3.8183E–2(3.9E–3) > >	1.6868E–2(1.1E–3)	1.1092E–2(2.7E–4)
WFG5	6.7165E–2(1.7E–4) < >	6.8558E–2(6.3E–4) ≈ >	7.7588E–2(1.8E–3) > >	6.8390E–2(5.3E–4)	6.6540E–2(6.9E–5)
WFG6	6.1760E–2(9.2E–3) < ≈	6.4056E–2(8.2E–3) ≈ >	8.6534E–2(1.2E–2) > >	6.7004E–2(8.3E–3)	5.9407E–2(7.8E–3)
WFG7	1.4264E–2(3.1E–4) < >	1.5242E–2(3.3E–4) < >	2.9628E–2(2.2E–3) > >	1.6195E–2(2.9E–4)	1.1408E–2(7.8E–5)
WFG8	8.0848E–2(6.1E–3) < >	8.5839E–2(5.6E–3) < >	1.0653E–1(8.1E–3) > >	8.9378E–2(4.7E–3)	7.7623E–2(4.9E–3)
WFG9	7.4934E–2(5.2E–2) ≈ >	5.6954E–2(4.5E–2) ≈ >	7.3108E–2(3.9E–2) ≈ >	5.7340E–2(4.8E–2)	3.7320E–2(4.4E–2)
VNT1	1.2731E–1(3.3E–3) < >	1.2240E–1(2.7E–3) < >	1.2544E–1(2.7E–3) < >	1.9823E–1(4.0E–2)	1.0868E–1(2.2E–2)
VNT2	1.2219E–2(3.2E–4) < <	1.2121E–2(2.8E–4) < <	1.2436E–2(2.5E–4) < <	2.4536E–2(4.4E–3)	1.3633E–2(7.0E–5)
VNT3	3.8870E–2(1.4E–3) < <	3.8536E–2(1.2E–3) < <	3.8244E–2(1.1E–3) < <	2.4137E–1(1.9E–1)	2.3838E–1(3.9E–2)
DTLZ1	2.2154E–2(2.0E–3) ≈ >	2.1149E–2(1.8E–3) ≈ >	2.2134E–2(2.4E–2) ≈ >	2.2708E–2(7.1E–3)	1.9170E–2(8.2E–5)
DTLZ2	5.3051E–2(1.0E–3) > <	5.1708E–2(7.7E–4) > <	5.1369E–2(8.3E–4) > <	5.0930E–2(3.7E–4)	7.2422E–2(6.9E–4)
DTLZ3	5.3253E–1(4.3E–1) < <	1.6158E+0(1.3E+0) ≈ <	1.4062E+0(1.1E+0) ≈ <	2.1496+0(1.3E+0)	2.6143E+0(8.9E–1)
DTLZ4	2.3409E–1(3.4E–1) > <	5.2207E–2(9.4E–4) > <	5.1944E–2(9.8E–4) > <	5.0989E–2(4.4E–4)	5.5827E–1(4.8E–1)
DTLZ5	4.1949E–3(2.4E–4) < <	4.4611E–3(3.2E–4) < <	4.5173E–3(3.4E–4) < <	1.1340E–2(1.7E–3)	5.1063E–3(2.0E–4)
DTLZ6	7.3724E–2(2.5E–2) < <	3.9000E–1(3.3E–2) < >	7.6720E–1(4.8E–2) > >	5.2645E–1(4.1E–2)	9.1480E–2(9.1E–3)
DTLZ7	2.7161E–1(2.5E–1) ≈ ≈	5.6166E–2(1.1E–3) < <	5.8486E–2(9.7E–4) < <	1.1630E–1(1.4E–2)	7.3461E–2(8.2E–4)
UF1	3.5263E–2(5.4E–3) ≈ ≈	1.6430E–3(1.6E–4) < <	9.3327E–3(2.1E–2) < <	3.3575E–2(3.4E–3)	3.3638E–2(9.8E–4)
UF2	2.2549E–2(2.8E–3) < <	6.5645E–3(1.4E–3) < <	1.1159E–2(2.7E–3) < <	4.3056E–2(2.2E–3)	3.9715E–2(1.1E–3)
UF3	4.2563E–2(2.3E–2) < <	9.5736E–3(1.0E–2) < <	1.3594E–2(1.5E–2) < <	7.8088E–2(2.2E–2)	1.0859E–1(8.0E–3)
UF4	5.0860E–2(2.6E–3) > ≈	6.6063E–2(5.7E–3) > >	6.6512E–2(5.9E–3) > >	4.9504E–2(2.2E–3)	5.0426E–2(1.1E–3)
UF5	5.1121E–1(1.4E–1) < <	4.0341E–1(1.4E–1) < <	4.7409E–1(1.5E–1) < <	1.1570E+0(1.2E–1)	1.1627E+0(3.9E–2)
UF6	2.8400E–1(1.1E–2) < <	4.2500E–1(1.4E–1) ≈ ≈	4.4913E–1(1.5E–1) ≈ ≈	3.9800E–1(1.9E–2)	3.8820E–1(9.9E–3)
UF7	1.5350E–2(1.1E–3) > >	1.2116E–2(5.0E–3) < <	1.4598E–2(3.8E–3) ≈ ≈	1.4883E–2(9.8E–4)	1.4730E–2(2.7E–4)
UF8	1.3722E–1(3.5E–2) ≈ ≈	5.6104E–2(1.1E–2) < <	8.9914E–2(3.3E–2) < <	1.2600E–1(3.5E–3)	1.2589E–1(5.1E–3)
UF9	1.1081E–1(4.1E–2) ≈ >	1.3153E–1(3.4E–2) > >	1.2588E–1(4.1E–2) > >	1.1108E–1(6.0E–3)	9.8454E–2(4.3E–3)
UF10	2.3864E+0(1.8E–1) < <	4.6496E–1(7.0E–2) < <	4.2631E–1(6.1E–2) < <	2.5358E+0(2.9E–1)	2.4508E+0(7.7E–2)
DTLZ2(4)	9.8155E–2(4.8E–3) > <	9.3172E–2(3.2E–3) > <	9.3679E–2(4.6E–3) > <	8.7617E–2(2.0E–4)	1.1543E–1(7.6E–4)
DTLZ2(6)	2.3931E–1(2.7E–3) < <	2.3766E–1(1.9E–3) < <	2.4064E–1(7.6E–4) < <	2.5721E–1(1.6E–4)	2.9009E–1(1.4E–3)
DTLZ2(10)	4.8722E–1(4.8E–3) < <	4.6966E–1(7.4E–3) < <	4.7015E–1(2.7E–3) < <	4.9596E–1(8.0E–4)	5.4540E–1(1.6E–3)
DTLZ5(2,10)	2.1183E–3(3.2E–5) < <	2.1620E–3(7.2E–5) < <	2.1492E–3(6.8E–5) < <	1.2713E+0(6.3E–1)	4.8257E–1(2.9E–1)
DTLZ5(3,10)	3.7996E–2(3.1E–3) < <	3.7136E–2(1.1E–3) < <	3.8918E–2(2.1E–3) < <	5.6242E–2(4.4E–3)	4.9480E–2(2.3E–3)



(a) $k = 2$ (b) $k = 3$ (c) $k = 4$ (d) $k = 6$ (e) $k = 10$

Fig. 18. The solution sets obtained by BCE-MOEA/D+TCH with different k values in the niche radius setting on DTLZ2.

TABLE VII
IGD RESULTS (MEAN AND SD) OF BCE-MOEA/D+TCH WITH DIFFERENT k VALUES ON THE WFG PROBLEMS. THE BEST AND SECOND BEST MEANS FOR EACH PROBLEM ARE SHOWN WITH DARK AND LIGHT GRAY BACKGROUNDS, RESPECTIVELY

Problem	$k = 2$	$k = 3$	$k = 4$	$k = 6$
WFG1	9.3814E–1(6.1E–2)	9.6279E–1(5.4E–2)	9.8176E–1(6.0E–2)	9.8774E–1(5.5E–2)
WFG2	2.1396E–2(6.5E–3)	2.0238E–2(4.1E–3)	2.0754E–2(2.5E–3)	2.3346E–2(5.6E–3)
WFG3	2.1886E–2(3.0E–3)	2.3228E–2(3.3E–3)	2.4681E–2(6.9E–3)	2.3794E–2(3.6E–3)
WFG4	1.8452E–2(1.4E–3)	1.9664E–2(1.5E–3)	2.0035E–2(1.8E–3)	2.0269E–2(1.7E–3)
WFG5	6.8565E–2(5.1E–4)	6.8558E–2(6.3E–4)	6.8577E–2(5.1E–4)	6.8871E–2(8.2E–4)
WFG6	6.4045E–2(9.1E–3)	6.4056E–2(8.2E–3)	6.4176E–2(9.7E–3)	6.3763E–2(7.0E–3)
WFG7	1.5289E–2(4.8E–4)	1.5242E–2(3.3E–4)	1.5779E–2(4.0E–4)	1.5528E–2(4.1E–4)
WFG8	8.5569E–2(4.9E–3)	8.5839E–2(5.6E–3)	8.7812E–2(6.8E–3)	8.6347E–2(4.6E–3)
WFG9	6.0298E–2(4.7E–2)	5.6954E–2(4.5E–2)	6.6903E–2(4.8E–2)	5.8558E–2(4.4E–2)

Then, how does the BCE algorithm perform in terms of separate convergence or diversity? In fact, BCE is designed to use a non-Pareto algorithm (as a drive) to lead the PC evolution forward and at the same time use the PC evolution to compensate for the possible diversity loss during the search process of this non-Pareto algorithm. Thus, the BCE algorithm usually performs worse than the embedded non-Pareto algorithm in terms of convergence, but better in terms of diversity. However, if the NPC used in the embedded algorithm struggles to steer the evolution forwards, Pareto dominance would drive the evolution instead. In this case, the BCE algorithm has better convergence than the embedded non-Pareto algorithm. Fig. 7 in Section IV is precisely such a case.

VII. Conclusions

This paper has presented a bi-criterion evolution (BCE) framework of Pareto criterion and non-Pareto criterion to deal with MOPs. In BCE, the two criteria work collaboratively, attempting to use their strengths to facilitate each other's evolution. In general, the NPC evolution drives the PC evolution forward while the PC evolution compensates for the possible diversity loss of the NPC evolution. In the proposed framework, the two populations communicate constantly, with their information being fully shared and compared in a generational manner. Any new individual produced in one population will be tested and applied in the other. The information comparison of the two populations reflects the current status of the NPC evolution, thus making the search more focused on some undeveloped (or not well-developed) but promising regions.

Systematic experiments have been carried out by investigating three representative non-Pareto EMO algorithms on seven categories of 42 test problems. The results have revealed the effectiveness of the BCE approach in providing a good balance between convergence and diversity. The three BCE algorithms work well, whether on problems where NPC could struggle, such as MOPs with a highly irregular or a discontinuous Pareto front, or on problems where the PC is likely to fail, such as MOPs with a complex Pareto set or a high-dimensional objective space.

Moreover, the performance verification of the embedded non-Pareto algorithms indicates that both the PC and NPC evolutions benefit from the information share and exchange under the BCE framework. In addition, two key operations of BCE, individual exploration and population maintenance, have been investigated and analyzed. The variation of the number of explored individuals during the evolutionary process has shown the adaptiveness of individual exploration, depending on the performance of the embedded algorithm. As to population maintenance, despite clear differences having been observed from the results in comparison with Pareto-based algorithms, the Pareto dominance and density-based maintenance strategy could have an impact on the performance of the BCE algorithm. Finally, a comparison with NSGA-III and SMS-EMOA has verified the competitiveness of the three BCE algorithms as independent algorithms to deal with MOPs.

The bi-criterion evolution of Pareto and non-Pareto criteria is a general framework in EMO. It can be especially of practical value in the area, given its applicability for any non-Pareto algorithm, no requirement of parameter tuning in the implementation, and the reliability on various problems with distinct characteristics.

Finally, note that the study in this paper focuses on the design of the BCE framework and the implementation of selection operations, while the variation operation is not fixed and it uses the same search operators from the embedded non-Pareto algorithm. In the subsequent work, we will attempt to introduce other search operators into BCE. This includes integrating existing operators (such as those from MO-CMA-ES [29], MTS [65], MTS2 [11] and SBS [46]) or designing new operators specially for the individual exploration in the PC evolution.

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