

## Cycle-Complete Ramsey Numbers

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# Cycle-complete Ramsey numbers

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May 1, 2019

## Abstract

The Ramsey number  $r(C_\ell, K_n)$  is the smallest natural number  $N$  such that every red/blue edge-colouring of a clique of order  $N$  contains a red cycle of length  $\ell$  or a blue clique of order  $n$ . In 1978, Erdős, Faudree, Rousseau and Schelp conjectured that  $r(C_\ell, K_n) = (\ell - 1)(n - 1) + 1$  for  $\ell \geq n \geq 3$  provided  $(\ell, n) \neq (3, 3)$ .

We prove that, for some absolute constant  $C \geq 1$ , we have  $r(C_\ell, K_n) = (\ell - 1)(n - 1) + 1$  provided  $\ell \geq C \frac{\log n}{\log \log n}$ . Up to the value of  $C$  this is tight since we also show that, for any  $\varepsilon > 0$  and  $n > n_0(\varepsilon)$ , we have  $r(C_\ell, K_n) \gg (\ell - 1)(n - 1) + 1$  for all  $3 \leq \ell \leq (1 - \varepsilon) \frac{\log n}{\log \log n}$ .

This proves the conjecture of Erdős, Faudree, Rousseau and Schelp for large  $\ell$ , a stronger form of the conjecture due to Nikiforov, and answers (up to multiplicative constants) two further questions of Erdős, Faudree, Rousseau and Schelp.

## 1 Introduction

Graph Ramsey numbers are a central topic of research in Combinatorics. Given two graphs  $G$  and  $H$ , the Ramsey number  $r(G, H)$  is the smallest natural number  $N$  such that every red/blue colouring of the edges of the complete graph  $K_N$  on  $N$  vertices contains a red copy of  $G$  or a blue copy of  $H$ . The existence of  $r(G, H)$  follows from Ramsey's theorem [42], but determining or accurately estimating these parameters presents many challenging problems.

The classical Ramsey numbers are the graph Ramsey numbers  $r(G, H)$  where  $G$  and  $H$  are cliques. Erdős and Szekeres [23] showed  $r(K_n, K_n) \leq 2^{(1+o(1))2n}$ , and later Erdős [20] showed  $r(K_n, K_n) \geq 2^{(1+o(1))n/2}$ , in one of the first instances of the probabilistic method. Both bounds changed very little over the past 70 years, despite progress by Thomason [51] and Conlon [17] on the upper bound, and by Spencer [48] on the lower bound. Another intensively studied Ramsey number is  $r(K_3, K_n)$ ; it was a long-standing open problem to determine its order of magnitude, which is now known to be  $\Theta\left(\frac{n^2}{\log n}\right)$ , due to theorems of Ajtai, Komlós and Szemerédi [4] and Kim [31]. Recent analyses of the triangle-free process independently by Bohman and Keevash [5] and by Fiz Pontiveros, Griffiths and Morris [25], together with an improved upper bound due to Shearer [46], have now determined  $r(K_3, K_n)$  to within a multiplicative factor of  $4 + o(1)$ .

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At the other end of the spectrum, sparse graphs tend to have small Ramsey numbers. In this context, Chvátal, Rödl, Szemerédi and Trotter [16] proved that if  $G$  and  $H$  have bounded maximum degree then  $r(G, H) = O(v(G) + v(H))$ , where  $v(G)$  denotes the number of vertices of the graph  $G$ . A similar bound was obtained by Chen and Schelp [13] under the assumption of bounded arrangeability. After intense effort [2, 26, 27, 33, 34, 35], a longstanding conjecture of Burr and Erdős [10] that such bounds hold only assuming bounded degeneracy was recently confirmed by Lee [36].

In this paper, we will focus on the cycle-complete Ramsey numbers  $r(C_\ell, K_n)$ . For any connected graph  $H$ , Chvátal and Harary [15] observed that  $r(H, K_n) \geq (v(H) - 1)(n - 1) + 1$ . This is shown by the red/blue edge-coloured clique of order  $(v(H) - 1)(n - 1)$ , in which the red edges consist of  $n - 1$  disjoint cliques of order  $v(H) - 1$  and all the remaining edges are blue. Burr and Erdős [11] asked when equality holds in the Chvátal–Harary bound (the ‘Ramsey goodness’ question, see e.g. [1]). When  $H = C_\ell$ , for  $\ell \geq n^2 - 2$  Bondy and Erdős [6] showed the equality

$$r(C_\ell, K_n) = (\ell - 1)(n - 1) + 1. \quad (1)$$

Erdős, Faudree, Rousseau and Schelp [21] noted that  $r(C_3, K_n) = r(K_3, K_n)$  grows much faster than a linear function of  $n$  (as discussed above), and posed the problem of determining the critical  $\ell$  at which the change in behaviour of  $r(C_\ell, K_n)$  occurs. They conjectured (see also [14, Chapter 2]) that (1) holds for  $\ell \geq n \geq 3$  provided  $(\ell, n) \neq (3, 3)$ .

There is a large literature on  $r(C_\ell, K_n)$ . An improved lower bound on  $r(C_\ell, K_n)$  for small  $\ell$  was given by Spencer [47]. Caro, Li, Rousseau and Zhang [12] improved the upper bound on  $r(C_\ell, K_n)$  of Erdős et al. [21] for small even  $\ell$ ; Sudakov [49] gave a similar improvement for small odd  $\ell$ . Several authors [24, 43, 52, 8, 44] confirmed the Erdős–Faudree–Rousseau–Schelp conjecture for small values of  $n$ . Schiermeyer [45] improved the result of Bondy and Erdős by showing that (1) holds for  $\ell \geq n^2 - 2n > 3$ . Nikiforov [40] substantially extended this range, proving that (1) holds for  $\ell \geq 4n + 2$ . Moreover, he conjectured (Conjecture 2.14 in [40]) that in fact (1) already holds at a much lower threshold, namely that for all  $\varepsilon > 0$  there is  $n_0$  such that  $r(C_\ell, K_n) = (\ell - 1)(n - 1) + 1$  provided  $\ell \geq n^\varepsilon$  and  $n \geq n_0$ .

Our main result proves both the Erdős–Faudree–Rousseau–Schelp conjecture for large  $\ell$  and Nikiforov’s conjecture. In fact, we prove (1) for a much wider range of parameters.

**Theorem 1.1.** *There is  $C \geq 1$  so that  $r(C_\ell, K_n) = (\ell - 1)(n - 1) + 1$  for  $n \geq 3$  and  $\ell \geq C \frac{\log n}{\log \log n}$ .*

*Remarks:* All logarithms in this paper are to base 2. Note that  $r(C_\ell, K_1) = 1$  and  $r(C_\ell, K_2) = \ell$  for all  $\ell \geq 3$ ; we include the condition  $n \geq 3$  only to avoid division by 0 in the lower bound on  $\ell$ .

The bound in Theorem 1.1 is best possible up to the value of  $C$ , as shown by our next result.

**Theorem 1.2.** *Given  $\varepsilon > 0$  there is  $n_0(\varepsilon)$  so that  $r(C_\ell, K_n) > n \log n \gg (\ell - 1)(n - 1) + 1$  for all  $n \geq n_0(\varepsilon)$  and  $3 \leq \ell \leq (1 - \varepsilon) \frac{\log n}{\log \log n}$ .*

In combination, Theorems 1.1 and 1.2 answer (up to the constant  $C$ ) two further questions of Erdős et al. [21] regarding  $r(C_\ell, K_n)$ , namely (i) the location of the critical value of  $\ell$  for the transition in behaviour of  $r(C_\ell, K_n)$ , and (ii) the choice of  $\ell$  that minimises  $r(C_\ell, K_n)$ . The answer to both questions is  $\ell = \Theta\left(\frac{\log n}{\log \log n}\right)$ .

An overview of the proof of Theorem 1.1 and the organisation of the paper is as follows. We suppose for a contradiction that there is some  $C_\ell$ -free graph  $G$  with  $v(G) = N = (\ell - 1)(n - 1) + 1$  and independence number  $\alpha(G) \leq n - 1$ . By induction we can also assume  $G$  has minimum degree  $\delta(G) \geq \ell - 1$ . The main task of the paper is to prove the stability result (Lemma 5.1)

that  $G$  is close in structure to the lower bound construction described above, i.e.  $G$  can be mostly partitioned into approximate cliques of size about  $\ell$  (and also less than  $\ell$ , as there is no  $C_\ell$ ). Then in Section 6, following various arguments to clean up the approximate structure, we will see that it is incompatible with our assumptions, and so obtain a contradiction that proves the theorem.

In the next section, after the short proof of Theorem 1.2, we gather various tools needed for the proof of the stability result. Over the following three sections we prove the existence of approximate decompositions of  $G$  into pieces whose properties are gradually strengthened: in Section 3 the pieces are quite dense, in Section 4 they are ‘hubs’ (highly connected in a certain sense), and in Section 5 they are ‘almost cliques’, as required for the stability result.

## 2 Preliminaries

We start in the next subsection with some notation, then we prove Theorem 1.2. In the third subsection we collect various well-known results that we use in our proofs. The final subsection of this section describes two applications of Breadth First Search.

### 2.1 Notation

We summarise some (mostly) standard graph theory notation (see e.g. [7]) used in this paper. Let  $G$  be a finite graph. We write  $v(G) := |V(G)|$  for the number of vertices and  $e(G) := |E(G)|$  for the number of edges. Given a vertex  $v \in V(G)$ , the neighbourhood of  $v$  in  $G$  is  $N_G(v) := \{y \in V(G) : xy \in E(G)\}$ . The degree of  $v$  is  $d_G(v) := |N_G(v)|$ . The minimum degree is  $\delta(G) := \min\{d(v) : v \in V(G)\}$ , the maximum degree is  $\Delta(G) := \max\{d(v) : v \in V(G)\}$ , and the average degree is  $d(G) := 2e(G)/v(G)$ . Given  $A \subset V(G)$ , the induced graph  $G[A]$  has vertex set  $A$  and edge set  $\{e \in E(G) : e \subset A\}$ . Given disjoint sets  $A, B \subset V(G)$ , we let  $G[A, B]$  denote the bipartite graph with parts  $A$  and  $B$  and edge set  $\{e \in E(G) : |e \cap A| = |e \cap B| = 1\}$ . A path  $P = x_0x_1 \dots x_\ell$  of length  $\ell$  consists of  $\ell + 1$  distinct vertices  $x_0, \dots, x_\ell$ , where  $x_i x_{i+1}$  is an edge for  $i \in \{0, \dots, \ell - 1\}$ . We call  $x_0$  and  $x_k$  the end vertices of  $P$  and say that  $P$  is an  $x_0 x_k$ -path. We say  $P$  is internally disjoint from a set  $X$  if  $X$  contains none of the interior vertices  $\{x_1, \dots, x_{\ell-1}\}$  of  $P$ . A cycle of length  $\ell$ , or  $\ell$ -cycle, is a graph obtained from a path  $P = x_0x_1 \dots x_{\ell-1}$  of length  $\ell - 1$  by adding the edge  $x_{\ell-1}x_0$ . Edges of cycles will often be listed modulo  $\ell$ , so that  $x_{\ell-1}x_\ell$  represents  $x_{\ell-1}x_0$ . We say  $I \subset V(G)$  is independent if  $G[I]$  has no edges. The independence number  $\alpha(G)$  is the size of a largest independent set in  $G$ . Given integers  $m \leq n$  we let  $[m, n] := \{m, m + 1, \dots, n\}$ . To simplify the presentation, we may omit floor and ceiling signs when they are not crucial.

### 2.2 The lower bound

The lower bound construction comes from the following standard application of the probabilistic method.

*Proof of Theorem 1.2.* Let  $\varepsilon \in (0, 1)$ ,  $n > n_0(\varepsilon)$  and  $N = 2n \log n$ . It suffices to prove that there is a graph  $G$  on at least  $N/2$  vertices with  $\alpha(G) < n$  which does not contain a cycle  $C_\ell$  with  $\ell \leq \ell_0 := (1 - \varepsilon) \frac{\log n}{\log \log n}$ . We consider a random graph  $G_1 \sim G(N, p)$ , where  $p := \frac{3 \log \log n}{n-1}$ . The expected number of independent sets of order  $n$  in  $G_1$  is

$$\binom{N}{n} (1-p)^{\binom{n}{2}} \leq \left( \frac{eN}{n} e^{-p(n-1)/2} \right)^n = (2e(\log n) e^{-3 \log \log n/2})^n \ll 1.$$

On the other hand, the expected number of cycles of length at most  $\ell_0$  is  $\sum_{i \in [3, \ell_0]} (Np)^i \leq 2(Np)^{\ell_0} \leq 2(7 \log n \log \log n)^{(1-\varepsilon) \log n / \log \log n} \ll N$ . By Markov's inequality applied to both of these expectations, with positive probability  $G_1$  satisfies  $\alpha(G_1) < n$  and has  $\leq N/2$  cycles of length at most  $\ell_0$ . Fixing a choice of such  $G_1$  and deleting a vertex from each cycle of length at most  $\ell_0$  leaves a graph  $G$  with the required properties.  $\square$

## 2.3 Tools

In this subsection we collect several well-known results. The first is very simple, but we include a short proof for the convenience of the reader.

**Proposition 2.1.** *For any graph  $G$ ,*

- (i) *if  $G$  has no subgraph of minimum degree at least  $k$  then  $e(G) \leq \binom{k}{2} + (v(G) - k)(k - 1)$ ;*
- (ii)  *$G$  contains a subgraph  $G_1$  with  $\delta(G_1) \geq d(G)/2$ ;*
- (iii)  *$G$  contains a bipartite subgraph  $G_2$  with  $d(G_2) \geq d(G)/2$ .*

*Proof.* To see (i), note that as any subgraph of  $G$  contains a vertex with degree at most  $k - 1$ , we may iteratively delete such vertices until we obtain a subgraph on  $k$  vertices. The bound follows by counting edges. Similarly, for (ii), if there were no such  $G_1$  we could reduce  $G$  to an empty graph by deleting vertices of degree less than  $d(G)/2$ , but then  $e(G) < v(G)d(G)/2$  would be a contradiction. Lastly, for (iii), note that, for a uniformly random partition  $A, B$  of  $V(G)$ , the induced bipartite subgraph  $G_2 = G[A, B]$  has  $\mathbb{E}d(G_2) = d(G)/2$ .  $\square$

Next we state several classical results from extremal graph theory.

**Theorem 2.2** (Turán [50]). *Any graph  $G$  satisfies  $\alpha(G) \geq \frac{v(G)}{d(G)+1}$ .*

**Theorem 2.3** (Dirac [18]). *Any graph  $G$  with  $d(G) \geq v(G)/2$  contains a Hamilton cycle.*

**Theorem 2.4** (Bondy [9]). *Any graph  $G$  with  $d(G) \geq v(G)/2$  is either a complete bipartite graph or is pancyclic, i.e. contains cycles of all lengths in  $[3, v(G)]$ .*

**Theorem 2.5** (Erdős and Gallai [22]). *Any graph  $G$  with  $d(G) > k - 1$  has a path of length  $k$ .*

We conclude by stating a version of Dependent Random Choice (see [28, Lemma 7.2]).

**Theorem 2.6.** *Given  $\varepsilon > 0$  there is  $\delta > 0$  so that the following holds for  $N \geq N_0(\varepsilon)$  and any  $N$ -vertex graph  $G$  with at least  $N^{2-\delta}$  edges. There are non-empty disjoint sets  $U_1, U_2 \subset V(G)$  such that, for  $i = 1, 2$ , every  $a, a' \in U_i$  satisfies  $|N_G(a, U_{3-i}) \cap N_G(a', U_{3-i})| \geq N^{1-\varepsilon}$ .*

## 2.4 Breadth First Search

Here we give two applications of Breadth First Search, namely finding short cycles, and a nice decomposition of a substantial part of any graph.

We start by describing the well-known construction of a breadth first search tree  $T$  in a graph  $G$  rooted at some vertex  $x \in V(G)$ . At each step  $i \geq 0$ , we construct a tree  $T_i$  with layers  $V_0, \dots, V_i$  which are disjoint subsets of  $V(G)$ . Initially,  $T_0$  is a tree with one vertex, namely  $V(T_0) = V_0 = \{x\}$ . Given  $T_{i-1}$  for some  $i > 0$ , we let  $V_i := N_G(V_{i-1}) \setminus V(T_{i-1})$ . If  $V_i = \emptyset$  we terminate with  $T = T_{i-1}$ , otherwise we obtain  $T_i$  from  $T_{i-1}$  by adding an arbitrary edge of  $G$  from each vertex in  $V_i$  to some vertex in  $V_{i-1}$ . It will be useful to consider the first layer which does not cause the tree to grow significantly, in the sense of the following simple proposition.

**Proposition 2.7.** *Let  $\gamma > 1$  and let  $G$  be an  $N$ -vertex graph. Let  $T$  be a breadth first search tree in  $G$  rooted at  $x \in V(G)$  with layers  $V_0, \dots, V_r$ . Suppose  $m \in \mathbb{N} \cup \{0\}$  is minimal such that  $|\cup_{i=0}^{m+1} V_i| \leq \gamma |\cup_{i=0}^m V_i|$ . Then  $m \leq \frac{\log N}{\log \gamma} = \log_\gamma(N)$ .*

*Proof.* By definition of  $m$  we have  $N \geq |\cup_{i=0}^m V_i| > \gamma |\cup_{i=0}^{m-1} V_i| > \dots > \gamma^m |V_0| = \gamma^m$ .  $\square$

Our first application is to finding short cycles within an approximate range.

**Lemma 2.8.** *Let  $G$  be an  $N$ -vertex graph with  $d(G) \geq d = 16\gamma d_1$ , where  $\gamma > 1$  and  $d_1 \geq 2$ . Then  $G$  contains an  $\ell$ -cycle for some  $\ell \in [d_1, d_1 + 2 \log_\gamma(N)]$ .*

*Proof.* By Lemma 2.1 (ii) and (iii) there is a bipartite subgraph  $G'$  of  $G$  with  $\delta(G') \geq d(G)/4$ . Let  $T$  be a breadth first search tree in  $G'$  rooted at some  $x \in V(G')$  with layers  $V_0, \dots, V_r$ . Let  $m \leq \log_\gamma(N)$  be as in Proposition 2.7. As  $G'$  is bipartite,  $G'[V_i]$  is an empty graph for all  $i \in [r]$ , so

$$\sum_{i \in [0, m]} e(G'[V_i, V_{i+1}]) = e(G'[\cup_{i \in [0, m+1]} V_i]) \geq \frac{\delta(G')}{2} \sum_{i \in [0, m]} |V_i| \geq \sum_{i \in [0, m]} \frac{d(G)}{16\gamma} (|V_i| + |V_{i+1}|),$$

using  $\sum_{i \in [0, m]} (|V_i| + |V_{i+1}|) \leq (1 + \gamma) \sum_{i \in [0, m]} |V_i| \leq 2\gamma \sum_{i \in [0, m]} |V_i|$ . Thus  $d(G'[V_i, V_{i+1}]) \geq d(G)/16\gamma \geq d_1$  for some  $i \in [m]$ . By Theorem 2.5, there is a path of length  $d_1$  in  $G[V_i, V_{i+1}]$ . By possibly removing vertices we can obtain an  $xy$ -path in  $G[V_i, V_{i+1}]$  of length between  $d_1 - 2$  and  $d_1$  with  $x, y \in V_i$ . Combining this with the unique  $xy$ -path in  $T$  of length at most  $2m \leq 2 \log_\gamma(N)$  gives a cycle of length in  $[d_1, d_1 + 2 \log_\gamma(N)]$ , as required.  $\square$

Our second application is to construct the following partial decomposition of a graph  $G$ , consisting of a family of disjoint sets  $X_i \subset V(G)$ , which are mutually non-adjacent in  $G$ , with each  $X_i$  entirely at a fixed distance in some tree  $T_i$  from the root  $x_i$ .

**Lemma 2.9.** *Let  $\gamma > 1$  and let  $G$  be an  $N$ -vertex graph. Then there are triples  $\{(x_i, X_i, T_i)\}_{i \in [t]}$ , where each  $T_i$  is a subtree of  $G$  rooted at  $x_i$  and  $X_i \subset V(T_i)$ , such that:*

- (i) *there is  $d_i \in [0, \log_\gamma(N)]$  such that for all  $x'_i \in X_i$  the unique  $x_i x'_i$ -path in  $T_i$  has length  $d_i$ ;*
- (ii)  *$\{X_i\}_{i \in [t]}$  are disjoint and satisfy  $|\cup_{i \in [t]} X_i| \geq N/2\gamma$ ;*
- (iii) *there are no edges of  $G$  between  $X_i$  and  $X_j$  for distinct  $i, j \in [t]$ .*

*Proof.* We prove the statement by induction on  $v(G)$ , noting that it is trivial if  $v(G) = 1$ .

Let  $T$  be a breadth first search tree in  $G$  rooted at some  $x \in V(G)$  with layers  $V_0, \dots, V_r$ . Let  $m \leq \log_\gamma(N)$  be as in Proposition 2.7. Let  $\{X_i\}_{i \in [s]}$  be  $\{V_{2i}\}_{2i \in [m]}$  or  $\{V_{2i+1}\}_{2i+1 \in [m]}$  according to which set  $\cup_{2i \in [m]} V_{2i}$  or  $\cup_{2i+1 \in [m]} V_{2i+1}$  is larger. Setting  $X := \cup_{i \in [s]} X_i$ , we note that  $|X| + |N_G(X)| = |X \cup N_G(X)| \leq |\cup_{i \in [m+1]} V_i| \leq \gamma |\cup_{i \in [m]} V_i| \leq 2\gamma |X|$ .

For each  $i \in [s]$ , set  $x_i = x$ ,  $T_i = T$  and  $d_i = j$ , where  $X_i = V_j$ , so that (i) holds by the definition of  $V_j$ . As  $\{X_i\}_{i \in [s]}$  are non-consecutive layers of a breadth first search tree, they are disjoint and there are no edges between  $X_i$  and  $X_j$  for distinct  $i, j \in [s]$ .

Now let  $W = V(G) \setminus (X \cup N_G(X))$  and apply induction on  $G[W]$  to obtain  $\{(x_i, X_i, T_i)\}_{i \in [s+1, t]}$ . We claim that  $\{(x_i, X_i, T_i)\}_{i \in [t]}$  satisfy the statement of the lemma. Indeed, (i) holds by construction. For (ii), disjointness is clear, and we have  $\sum_{i \in [s]} |X_i| = |X| \geq |X \cup N_G(X)|/2\gamma$  and  $\sum_{i \in [s+1, t]} |X_i| \geq |W|/2\gamma$  by induction. Finally, (iii) holds by construction and as each  $X_j$  with  $j \in [s+1, t]$  is contained in  $W$ , which is disjoint from  $X \cup N_G(X)$ .  $\square$

### 3 Quite dense subgraphs

In this section we take our first steps towards the stability result described above, by showing that any supposed counterexample to Theorem 1.1 can be partitioned almost entirely into vertex-disjoint subgraphs, each of which is quite large (has  $\ell^{1-o(1)}$  vertices) and is quite dense (has  $\ell^{2-o(1)}$  edges). We start by showing that a graph of large minimum degree has a long path or a dense subgraph.

**Lemma 3.1.** *Fix  $D \in \mathbb{N}$ . Then any graph  $G$  either*

- (i) *contains paths of length at least  $D$  starting at any given vertex, or*
- (ii) *has a subgraph  $H$  with  $v(H) \leq D$  and  $e(H) \geq \binom{\delta(G)+1}{2}$ .*

*Proof.* Suppose that (i) fails, i.e. there is  $x_0 \in V(G)$  such that any path starting at  $x_0$  has length less than  $D$ . We must show that (ii) holds. We construct a path  $P$  starting at  $x_0$  as follows. At step  $i \geq 0$ , having chosen a path  $P_{i-1} = x_0 \dots x_{i-1}$ , we select  $x_i \in N_G(x_{i-1}) \setminus \{x_0, \dots, x_{i-1}\}$  that maximises  $|N_G(x_i) \cap \{x_0, \dots, x_{i-1}\}|$ . If no such  $x_i$  exists we terminate with  $P = P_{i-1}$ . Let  $P = x_0 x_1 \dots x_\ell$  be the final path, where by choice of  $x_0$  we have  $\ell < D$ . By the termination rule, we have  $N_G(x_\ell) \subset V(P)$ . Let  $N_G(x_\ell) = \{x_{i_1}, \dots, x_{i_s}\}$ , where  $s \geq \delta(G)$ , ordered so that  $i_1 < \dots < i_s$ . As  $x_\ell$  is adjacent to  $x_{i_j}$  for each  $j \in [s]$ , the rule for choosing  $x_{i_j}$  guarantees  $|N_G(x_{i_j+1}) \cap \{x_0, \dots, x_{i_j}\}| \geq |N_G(x_\ell) \cap \{x_0, \dots, x_{i_j}\}| = j$  for each  $j \in [s]$ . Then  $H = G[V(P)]$  satisfies  $v(H) \leq D$  and  $e(H) \geq \sum_{j=1}^s j \geq \binom{\delta(G)+1}{2}$ .  $\square$

*Remark:* An unpublished result of the second author in [38] used a variant of Lemma 3.1 to prove that subgraphs of the cube graph with average degree  $d$  contain paths and cycles of length at least  $2^{\Omega(\sqrt{d})}$ . This result was later improved to  $2^{\Omega(d)}$  in [37] via a different approach.

We combine the previous lemma with two applications of the breadth first search decomposition of the previous section to show that any  $C_\ell$ -free graph with small independence number contains a small dense subgraph.

**Lemma 3.2.** *Let  $N, \ell \in \mathbb{N}$ ,  $\gamma > 1$ , where  $\ell \geq 3 \log_\gamma(N)$ , and  $d \geq 8\gamma^2$ . Suppose  $G$  is a  $C_\ell$ -free graph on  $N$  vertices with  $\alpha(G) \leq N/d$ . Then  $G$  has a subgraph  $H$  with  $v(H) \leq \ell$  and  $e(H) \geq d^2/2^9\gamma^4$ .*

*Proof.* Let  $\{(x_i, X_i, T_i)\}_{i \in [t]}$  be obtained by applying Lemma 2.9 to  $G$ . Let  $X = \bigcup_{i \in [t]} X_i$ , and note that  $|X| \geq N/2\gamma$ . Let  $\{(y_i, Y_i, T'_i)\}_{i \in [s]}$  be obtained by applying Lemma 2.9 again, this time to  $G[X]$ . Let  $Y = \bigcup_{i \in [s]} Y_i$ , and note that  $|Y| \geq |X|/2\gamma \geq N/4\gamma^2 \geq d\alpha(G)/4\gamma^2$ . By Theorem 2.2 (Turán's Theorem),  $d(G[Y]) \geq d/4\gamma^2 - 1 \geq d/8\gamma^2$ , as  $d \geq 8\gamma^2$ . Then Proposition 2.1 (ii) applied to  $G[Y]$  gives some  $G' = G[Y']$  with  $Y' \subset Y$  such that  $\delta(G') \geq d/16\gamma^2$ . By Lemma 3.1, to complete the proof of the lemma, it suffices to show that  $G'$  does not contain a path of length  $\ell$ .

For contradiction, suppose  $P = z_0 z_1 \dots z_\ell$  is a path in  $G'$ . As  $z_0 \in Y$  there is a triple  $(y_j, Y_j, T'_j)$  with  $z_0 \in Y_j$ . As  $T'_j$  is a tree, and so a connected subgraph of  $G[X]$ , by Lemma 2.9 (iii) there is a triple  $(x_i, X_i, T_i)$  with  $V(T'_j) \subset X_i$ , and by (i) there is  $d_i \in [0, \log_\gamma(N)]$  so that every vertex in  $X_i$  is at distance  $d_i$  from  $x_i$  in  $T_i$ . In particular, the  $x_i y_j$ -path and  $x_i z_0$ -path in  $T_i$  only intersect  $X_i$  in  $y_j$  and  $z_0$ . We let  $P_1$  be the  $y_j z_0$ -path in  $T_i$ . Then  $P_1$  has length  $\ell_1 \leq 2 \log_\gamma(N)$  and intersects  $X_i$  only in  $y_j$  and  $z_0$ .

We now use the triple  $(y_j, Y_j, T'_j)$ . As  $P$  is a connected subgraph of  $G[Y]$ , by Lemma 2.9 (iii) we have  $V(P) \subset Y_j$ , and by (i) there is  $d'_j \in [0, \log_\gamma(N)]$  so that every vertex of  $Y_j$  is at distance  $d'_j$  from  $y_j$  in  $T'_j$ . Let  $\ell_2 = \ell - \ell_1 - d'_j$  and consider the subpath  $P_2 = z_0 z_1 \dots z_{\ell_2}$  of  $P$ . Let  $P_3$  be the  $y_j z_{\ell_2}$ -path in  $T'_j$ . Then  $P_3$  has length  $d'_j$  and intersects  $Y_j$  only in  $z_{\ell_2}$ . As  $V(P_3) \subset V(T'_j) \subset X_i$ , we can combine  $P_1, P_2$  and  $P_3$  to form a cycle of length  $\ell$ . This contradiction completes the proof.  $\square$

*Remark:* We note that larger values  $\gamma$  above allow us to argue for small values of  $\ell$  in Lemma 3.2, and later in Theorem 1.1. In particular, for our applications it is important that  $\gamma$  tends to infinity, and we will take  $\gamma = (\log N)^c$  for some constant  $c > 0$ . The trade-off here however is that the subgraph  $H$  obtained from Lemma 3.2 becomes sparser.

By iterating the previous lemma it is a short step to obtain the following approximate decomposition of the vertex set of  $G$ .

**Corollary 3.3.** *Given  $\varepsilon > 0$  there is  $C \geq 1$  so that the following holds for all  $\ell, n \in \mathbb{N}$  with  $n \geq 3$  and  $\ell \geq C \frac{\log n}{\log \log n}$ . Suppose  $G$  is a  $C_\ell$ -free graph on  $N = (\ell - 1)(n - 1) + 1$  vertices with  $\alpha(G) \leq n - 1$ . Then there is a partition  $V(G) = W \cup \bigcup_{i \in [L]} V_i$  so that  $|V_i| < \ell$  and  $e(G[V_i]) > \ell^{2-\varepsilon}$  for all  $i \in [L]$ , and  $|W| \leq \varepsilon N$ .*

The decomposition in Corollary 3.3 is fact too coarse to be used in the proof of Theorem 1.1, so we will omit its proof. However, in the next section we provide a closely related stronger decomposition (see Corollary 4.3).

## 4 Hubs

Continuing our progress towards the stability result, we next prove that there is a decomposition of  $G$  as in Corollary 3.3 in which each set  $V_i$  forms a ‘hub’, which has the property that any small set of vertices can be joined together via disjoint paths of essentially any desired lengths. The precise definition is as follows.

**Definition 4.1.** *Let  $G$  be a graph and  $A, B \subset V(G)$  be disjoint sets. Given distinct  $x, y \in A \cup B$ , we call  $\ell \in \mathbb{N}$  a bipartite length for  $\{x, y\}$  in  $G[A, B]$  if (a)  $\ell$  is even and  $\{x, y\} \subset A$  or  $\{x, y\} \subset B$ , or (b)  $\ell$  is odd and  $|\{x, y\} \cap A| = |\{x, y\} \cap B| = 1$ .*

*For  $\varepsilon \in (0, 1)$  and  $u \in \mathbb{N}$ , we call a triple  $(A, B, D)$  an  $(u, \varepsilon)$ -hub in a graph  $G$  if  $|A| = |B| = u$ ,  $|D| \leq \varepsilon u$ , and for any distinct  $s_1, \dots, s_m, t_1, \dots, t_m$  in  $A \cup B$  with  $m \leq u^{1-\varepsilon}$  we have the following connection property: for any  $\ell_1, \dots, \ell_m \geq 2$  with  $\sum_{i \in [m]} (\ell_i + 1) \leq 2(1 - \varepsilon)u$ , where each  $\ell_i$  is a bipartite length for  $\{s_i, t_i\}$  in  $G[A, B]$ , there are vertex-disjoint paths  $P_1, \dots, P_m$  in  $G[A \cup B \cup D]$ , where each  $P_i$  is an  $s_i t_i$ -path of length  $\ell_i$ .*

The main lemma of this section shows that quite dense graphs contain large hubs.

**Lemma 4.2.** *Given  $\varepsilon \in (0, 1)$  there is  $\delta > 0$  so that for  $N \geq N_0(\varepsilon)$  and any integer  $u \in [N^\varepsilon, N^{1-\varepsilon}]$ , every  $N$ -vertex graph  $G$  with  $d(G) \geq N^{1-\delta}$  contains a  $(u, \varepsilon)$ -hub.*

*Proof.* We assume throughout the proof that  $\delta$  is sufficiently small and  $N$  is sufficiently large. By Proposition 2.1 (iii) we may assume  $G$  is bipartite. Let  $\delta_1 = \varepsilon^2/10$ . By Theorem 2.6, applied with  $\delta_1$  in place of  $\varepsilon$ , there are disjoint  $U_1, U_2 \subset V(G)$  so that  $|N_G(a, U_{3-i}) \cap N_G(a', U_{3-i})| \geq N^{1-\delta_1}$  for every  $a, a' \in U_i$  with  $i \in [2]$ . As  $G$  is bipartite,  $U_1$  and  $U_2$  must lie on opposite sides of the bipartition. We construct an alternating cycle  $C$  of length  $2u$  in  $G[U_1, U_2]$  by fixing distinct vertices  $a_1, \dots, a_u \in U_1$  and greedily selecting a common neighbour in  $U_2$  of each consecutive pair  $\{a_i, a_{i+1}\}$  (including  $\{a_u, a_1\}$ ) so that all selected vertices are distinct. This is possible as  $u \leq N^{1-\varepsilon} \ll N^{1-\delta_1}$ . We let  $A = V(C) \cap U_1 = \{a_1, \dots, a_u\}$  and  $B = V(C) \cap U_2$ .

We let  $D$  be a random subset of  $(U_1 \cup U_2) \setminus (A \cup B)$  where each element is included independently with probability  $p = \varepsilon u/2N$ . By Markov’s inequality,  $|D| \leq 2pN \leq \varepsilon u$  with probability at least  $1/2$ . Furthermore, for each pair  $a, a' \in A$ , we have

$$\mathbb{E}(|N_G(a) \cap N_G(a') \cap D|) \geq p(|N_G(a, U_2) \cap N_G(a', U_2)| - |U_2 \cap C|) \geq \varepsilon u/4N^{\delta_1} \geq 2u^{1-\varepsilon/2},$$



and similarly for each pair in  $B$ . By Chernoff's inequality (see [3, Appendix A]), with positive probability  $D$  satisfies  $|D| \leq \varepsilon u$  and  $|N_G(c) \cap N_G(c') \cap D| \geq u^{1-\varepsilon/2}$  for all  $\{c, c'\} \subset A$  or  $\{c, c'\} \subset B$ . We fix any set  $D$  with these properties.

It remains to show that  $(A, B, D)$  is a  $(u, \varepsilon)$ -hub. Suppose  $S = \{s_1, \dots, s_m\}$  and  $T = \{t_1, \dots, t_m\}$  are disjoint subsets of  $A \cup B$  with  $m \leq u^{1-\varepsilon}$ . Let  $\ell_1, \dots, \ell_m \geq 2$  with  $\sum_{i \in [m]} (\ell_i + 1) \leq 2(1 - \varepsilon)u$ , where each  $\ell_i$  is a bipartite length for  $\{s_i, t_i\}$  in  $G[A, B]$ . We want to find vertex-disjoint paths  $P_1, \dots, P_m$  in  $G[A \cup B \cup D]$ , where each  $P_i$  is an  $s_i t_i$ -path of length  $\ell_i$ .

First we claim that there is a path  $R$  with  $V(R) \cap (S \cup T) = \emptyset$ ,  $|V(R) \cap D| \leq 2m$  and  $|(A \cup B) \setminus V(R)| \leq 4m$ . To see this, we consider  $C \setminus (S \cup T)$ , which is the vertex-disjoint union of some paths  $R_1, \dots, R_k$ , where  $k \leq 2m$ . By deleting at most two vertices from each such path  $R_i$ , we can assume that each starts and ends in  $A$ . We form  $R$  by 'stitching' these paths together greedily, using distinct vertices from  $D \cap U_2$  to link successive paths  $R_i$  and  $R_{i+1}$  for all  $i \in [k-1]$ . This is possible by the common neighbourhood property, as  $2m \ll u^{1-\varepsilon/2}$ , so the claim follows.

Now we will construct the paths  $P_1, \dots, P_m$  by chopping  $R$  into suitable subpaths and connecting these to the endpoint sets  $S$  and  $T$ . To construct  $P_1$ , we consider separately the cases  $\ell_1 = 2$ ,  $\ell_1 = 3$  and  $\ell_1 \geq 4$ . If  $\ell_1 = 2$  we let  $P_1 = s_1 u_1 t_1$  for any common neighbour  $u_1 \in D$  of  $s_1$  and  $t_1$  disjoint from all previous choices. If  $\ell_1 = 3$  we let  $P_1 = s_1 u_1 v_1 t_1$  where  $u_1 \in D$  is a neighbour of  $s_1$  and  $v_1 \in D$  is a common neighbour of  $t_1$  and  $u_1$ , with  $\{u_1, v_1\}$  disjoint from all previous choices. Lastly, if  $\ell_1 \geq 4$  we consider a subpath  $R_1$  starting at one end of  $R$  with length  $\ell_1 - 3$ . As  $\ell_1$  is a bipartite length for  $\{s_1, t_1\}$ , it is possible to delete a vertex from one end of  $R_1$  to obtain a subpath  $R'_1$  of length  $\ell_1 - 4$  which starts on the same side of the partition as  $s_1$  and ends on the same side as  $t_1$ . Writing  $x_1$  and  $y_1$  for the ends of  $R'_1$ , we form the  $s_1 t_1$ -path  $P_1$  of length  $\ell_1$  from  $R'_1$  by adding paths  $s_1 u_1 x_1$  and  $t_1 v_1 y_1$  where  $u_1 \in D$  is a common neighbour of  $s_1$  and  $x_1$ , and  $v_1 \in D$  is a common neighbour of  $t_1$  and  $y_1$ , with  $\{u_1, v_1\}$  disjoint from all previous choices. To continue, we modify  $R$  by removing  $R_1$ , then repeat the process to find  $P_2$ , and so on.

It remains to show that the above process succeeds, i.e. that we do not ever exhaust  $R$  or any common neighbourhoods in  $D$ . To see this, note that initially  $|R| \geq |\bigcup_{i \in [k]} R_i| \geq 2u - |S \cup T| - 2m \geq \sum_{i \in [m]} \ell_i$ . As we remove at most  $\ell_i$  vertices from  $R$  to build each path  $P_i$ , we never run out of vertices in  $R$ . Also, we used at most 2 vertices from  $D$  to build each  $P_i$ , and so at most  $4m \leq u^{1-\varepsilon/2}/2$  from  $D$  in total. As  $|N_G(a) \cap N_G(a') \cap D| \geq u^{1-\varepsilon/2}$  for all  $\{a, a'\} \subset A$  or  $\{a, a'\} \subset B$ , we never run out of common neighbours in  $D$ .  $\square$

We conclude this section by showing that any supposed counterexample to Theorem 1.1 can be partitioned almost entirely into quite large hubs.

**Corollary 4.3.** *Given  $\varepsilon > 0$  there is  $C \geq 1$  so that the following holds for all  $\ell, n \in \mathbb{N}$  with  $n \geq 3$  and  $\ell \geq C \frac{\log n}{\log \log n}$ . Suppose  $G$  is a  $C_\ell$ -free graph on  $N = (\ell - 1)(n - 1) + 1$  vertices with  $\alpha(G) \leq n - 1$ . Then there is a partition  $V(G) = W \cup \bigcup_{i \in [L]} (A_i \cup B_i \cup D_i)$  so that  $|W| \leq \varepsilon N$  and each  $(A_i, B_i, D_i)$  is a  $(u, \varepsilon)$ -hub with  $u := \ell^{1-\varepsilon}$ .*

*Proof.* Let  $\delta > 0$  be such that Lemma 4.2 applies with  $\varepsilon/2$  in place of  $\varepsilon$ . Let  $\beta = \delta/7$  and  $C \geq 4/\beta$  be sufficiently large. It suffices to show that any  $W \subset V(G)$  with  $|W| > \varepsilon N$  contains a  $(u, \varepsilon)$ -hub, as then iteratively removing such hubs proves the lemma.

To see this, we claim that we can apply Lemma 3.2 to  $G[W]$  with  $\gamma = \ell^\beta$  and  $d = \ell^{1-\beta}$ . Indeed, for  $C$  large we have  $d \geq 8\gamma^2$  and  $\alpha(G) \leq n - 1 \leq |W|/d$ , and also  $\ell \geq \frac{3 \log(N)}{\log(\ell^\beta)} \geq 3 \log_\gamma(|W|)$ , as  $\ell \geq (4/\beta) \frac{\log n}{\log \log n}$ . Thus Lemma 3.2 gives a subgraph  $H$  of  $G[W]$  with  $v(H) \leq \ell$  and  $e(G[U]) \geq d^2/2^9 \gamma^4 = 2^{-9} \ell^{2-6\beta} \geq \ell^{2-\delta}$ . Now Lemma 4.2 gives a  $(u, \varepsilon)$ -hub in  $G[W]$ .  $\square$

## 5 Stability

In this section we upgrade the decomposition provided by Corollary 4.3 to obtain our main stability result, namely that any supposed counterexample to Theorem 1.1 can be partitioned almost entirely into quite large approximate cliques, and furthermore there are no edges between parts. The precise statement is as follows.

**Lemma 5.1.** *Given  $\eta > 0$  there is  $C \geq 1$  so that the following holds for all  $\ell, n \in \mathbb{N}$  with  $n \geq 3$  and  $\ell \geq C \frac{\log n}{\log \log n}$ . Suppose  $G$  is a  $C_\ell$ -free graph on  $N = (\ell - 1)(n - 1) + 1$  vertices with  $\alpha(G) \leq n - 1$ . Then there are disjoint sets  $V_1, \dots, V_s \subset V(G)$  such that:*

- (i)  $|V_i| \in [(\ell - 1)\eta, \ell - 1]$  for all  $i \in [s]$ ;
- (ii)  $|\bigcup_{i \in [s]} V_i| \geq (1 - \eta)N$ ;
- (iii)  $G[V_i]$  has minimum degree at least  $(1 - \eta)\ell$  for all  $i \in [s]$ ;
- (iv) There are no edges of  $G$  between  $V_i$  and  $V_j$  for all distinct  $i, j \in [s]$ .

Throughout the section we will assume the following setup.

Let  $G$  be as in Lemma 5.1, with  $\varepsilon < \varepsilon_0(\eta)$  sufficiently small and  $C$  sufficiently large so that Corollary 4.3 gives a partition  $V(G) = W \cup \bigcup_{i \in [L]} (A_i \cup B_i \cup D_i)$  with  $|W| \leq \varepsilon N$ , (2) where each  $(A_i, B_i, D_i)$  is a  $(u, \varepsilon)$ -hub with  $u = \ell^{1-\varepsilon}$ .

The proof proceeds in several stages, gradually refining the structure provided from the hubs to that in Lemma 5.1. In the next subsection we show how to find cycles of specified lengths in a system of hubs and ‘handles’ (suitable paths connecting the hubs). There is a potential parity obstacle due to the bipartite structure of hubs, but we can eliminate this obstacle using the bound on  $\alpha(G)$ ; this is achieved in the second subsection. In the third subsection we study the interaction between hubs: roughly speaking, we consider an auxiliary graph  $H_3$ , where  $V(H_3)$  consists of most of the hubs and we join two hubs if they are connected by a large matching. We show that  $H_3$  cannot have large components, and then in the final subsection we show that these components identify the approximate cliques needed to prove Lemma 5.1.

### 5.1 Cycles from hubs and handles

In this subsection we show how to find cycles from a suitable system of hubs and connecting paths. Our first lemma concerns the following condition under which we can drop the parity restriction on lengths of paths within a hub. We say that a  $(u, \varepsilon)$ -hub  $(A, B, D)$  is *parity broken* if  $G[A]$  contains a matching of size  $2u^{1-\varepsilon}$ .

**Lemma 5.2.** *Suppose  $(A, B, D)$  is a parity broken  $(u, \varepsilon)$ -hub in  $G$ . Let  $s_1, \dots, s_m, t_1, \dots, t_m \in A \cup B$  be distinct and  $\ell_1, \dots, \ell_m \geq 2$  with  $\sum_{i \in [m]} (\ell_i + 1) \leq 2(1 - \varepsilon)u$ . Suppose also that, for each  $i \in [m]$ , if  $\ell_i$  is not a bipartite length for  $\{s_i, t_i\}$  in  $G[A, B]$  then  $\ell_i \geq 7$ . Then there are vertex-disjoint paths  $P_1, \dots, P_m$  in  $G[A \cup B \cup D]$ , where each  $P_i$  is an  $s_i t_i$ -path of length  $\ell_i$ .*

*Proof.* As  $(A, B, D)$  is parity broken and  $2u^{1-\varepsilon} - 2m \geq m$ , there is a matching  $\mathcal{M} = \{x_i y_i : i \in [m]\}$  in  $G[A]$  which is vertex-disjoint from  $\{s_1, \dots, s_m, t_1, \dots, t_m\}$ . We will apply the connection property of  $(A, B, D)$  to a collection of pairs  $(s_{i,k}, t_{i,k})$  where there are one or two pairs for each original pair  $(s_i, t_i)$ . If  $\ell_i$  is a bipartite length for  $\{s_i, t_i\}$  then we take one pair  $(s_{i,1}, t_{i,1}) = (s_i, t_i)$  with the same length  $\ell_{i,1} = \ell_i$ . Otherwise, we take two pairs  $(s_{i,1}, t_{i,1}) = (s_i, x_i)$  and  $(s_{i,2}, t_{i,2}) = (y_i, t_i)$  with lengths  $\ell_{i,1}, \ell_{i,2} \geq 2$  chosen such that both  $\ell_{i,k}$  are bipartite lengths for  $\{s_{i,k}, t_{i,k}\}$  in  $G[A, B]$  with  $\ell_{i,1} + \ell_{i,2} + 1 = \ell_i$ . By the connection property of  $(A, B, D)$  we find vertex-disjoint  $s_{i,k} t_{i,k}$ -paths of lengths  $\ell_{i,k}$ , which combine with edges from  $\mathcal{M}$  to produce the required paths  $P_1, \dots, P_m$ .  $\square$

Let  $\mathcal{H}$  be a set of vertex-disjoint  $(u, \varepsilon)$ -hubs and  $\mathcal{P} = \{P_1, \dots, P_k\}$  be a set of vertex-disjoint paths in a graph  $G$ . Suppose  $P_i$  is an  $b_i a_{i+1}$ -path for  $i \in [k]$ , writing  $a_{k+1} := a_1$ . We call  $\mathcal{P}$  a *handle system* for  $\mathcal{H}$  if

- (i) each  $P_i$  is internally disjoint from  $\bigcup\{V(H) : H \in \mathcal{H}\}$ ,
- (ii) for each  $i \in [k]$  there is  $H_i \in \mathcal{H}$  with  $\{a_i, b_i\} \subset V(H_i)$ ,
- (iii) each  $H \in \mathcal{H}$  contains at most  $u^{1-\varepsilon}/2$  of  $\{a_1, \dots, a_k\}$ .

Note that we often apply the above definition with some paths  $P_i$  consisting only of the edge  $b_i a_{i+1}$  (in which case condition (i) is vacuous). The next lemma shows how handle systems provide cycles of specified lengths.

**Lemma 5.3.** *Let  $\mathcal{H}$  be a set of vertex-disjoint  $(u, \varepsilon)$ -hubs in  $G$  and  $\mathcal{P} = \{P_1, \dots, P_k\}$  be a handle system for  $\mathcal{H}$ , where each  $P_i$  is an  $b_i a_{i+1}$ -path of length  $\ell_i$ . Let  $\ell_{\text{sum}} = \sum_{i \in [k]} \ell_i$ . Then:*

- (i) *If  $\{a_i, b_i\} \subset A_i$  for all  $i \in [k]$  then  $G$  contains an  $\ell$ -cycle for any  $\ell \in [2k + \ell_{\text{sum}}, 2(1 - \varepsilon)u|\mathcal{H}| + \ell_{\text{sum}} - 2k]$  of the same parity as  $\ell_{\text{sum}}$ .*
- (ii) *If some  $\{a_j, b_j\}$  with  $j \in [k]$  is contained in a parity broken hub of  $\mathcal{H}$  then  $G$  contains an  $\ell$ -cycle for any  $\ell \in [7k + \ell_{\text{sum}}, 2(1 - \varepsilon)u|\mathcal{H}| + \ell_{\text{sum}} - 2k]$ .*

*Proof.* We write  $\ell - \ell_{\text{sum}} = \sum_{i \in [k]} \ell'_i$ , where  $\ell_j \geq 7$  (for (ii)), each  $\ell'_i \geq 2$  with  $i \neq j$  is a bipartite length for its hub, and for each  $H \in \mathcal{H}$  we have  $\sum\{\ell'_i + 1 : \{a_i, b_i\} \subset V(H)\} \leq 2(1 - \varepsilon)u$ . By the connection property of hubs, and Lemma 5.2 for the parity broken hub, we can find vertex-disjoint  $a_i b_i$ -paths of length  $\ell'_i$  for each  $i \in [k]$ , which combine with  $\mathcal{P}$  to produce an  $\ell$ -cycle.  $\square$

## 5.2 Breaking parity

In this subsection we will prove that almost all hubs of  $G$  are parity broken. This will use the bound on the independence number of  $G$ , via the following proposition.

**Proposition 5.4.** *Let  $m, d, s \in \mathbb{N}$  with  $m \geq 3d$ . Suppose  $G$  is a graph with  $V(G) = \bigcup_{i \in [s]} I_i$ , where  $I_1, \dots, I_s$  are disjoint independent sets of order  $m$ . Suppose also that  $\alpha(G) < v(G)/12d$ . Then there is  $\{i_0, \dots, i_d\} \subset [s]$  and a matching of size  $d$  with one edge in each  $G[I_{i_{j-1}}, I_{i_j}]$  for  $j \in [d]$ .*

*Proof.* Consider a maximal matching  $\mathcal{M}'$  in  $G$  with the property that  $\mathcal{M}'$  contains at most one edge of  $G[I_i, I_j]$  for all distinct  $i, j \in [s]$ . We use  $\mathcal{M}'$  to define a graph  $H$  with  $V(H) = [s]$ , where  $ij \in E(H)$  if and only if  $\mathcal{M}'$  contains an edge from  $G[I_i, I_j]$ . To prove the proposition, it suffices to show that  $H$  contains a path of length  $d$ . By Theorem 2.5, it suffices to prove  $d(H) > d - 1$ .

For contradiction, suppose  $d(H) \leq d - 1$ . Let  $S \subset V(H)$  with  $|S| = s/2$  be such that  $d_H(i) \leq d_H(j)$  for all  $i \in S, j \notin S$ . Then  $d_H(i) \leq 2(d - 1)$  for all  $i \in S$ . By Theorem 2.2 (Turán's Theorem), there is an independent set  $S' \subset S$  in  $H$  with  $|S'| \geq |S|/(2d - 1) \geq s/4d$ . For each  $i \in S'$ , let  $J_i$  be obtained from  $I_i$  by deleting all vertices contained in an edge of  $\mathcal{M}'$ . By the definition of  $\mathcal{M}'$  and  $S$ , we have  $|J_i| \geq |I_i| - 2d \geq m/3$ . Since  $\mathcal{M}'$  is maximal, there are no edges between  $J_i$  and  $J_j$  for any distinct  $i, j$ , so  $\bigcup_{i \in S'} J_i$  is independent. We deduce  $\alpha(G) \geq |S'|(m/3) \geq ms/12d = v(G)/12d$ . This contradiction completes the proof.  $\square$

We can now show that almost all  $(u, \varepsilon)$ -hubs of  $G$  are parity broken.

**Lemma 5.5.** *Assume setup (2). Then at least  $(1 - \varepsilon)L$  hubs are parity broken.*

*Proof.* First we note that if  $u \geq 4n$  then every hub  $(A, B, D)$  must be parity broken. Indeed, as  $\alpha(G) < n$ , any maximal matching in  $A$  has size at least  $u/3 > 2u^{1-\varepsilon}$ . Thus we may assume  $n \geq u/4 = \ell^{1-\varepsilon}/4$ .

For contradiction, suppose the hubs  $\{(A_i, B_i, D_i)\}_{i \in [s]}$  are not parity broken, where  $s = \varepsilon L \geq \varepsilon N/4u$ . We will obtain a contradiction by using Lemma 5.3 to find an  $\ell$ -cycle. Specifically, it suffices to show that there is a set of hubs  $\mathcal{H} = \{H_1, \dots, H_k\}$  for some  $k \geq \ell/u$ , and a handle system  $\mathcal{P} = \{P_1, \dots, P_k\}$  for  $\mathcal{H}$ , where each  $P_i$  has length  $\ell_i$ , starts in  $H_i$  and ends in  $H_{i+1}$ , and  $\ell_{\text{sum}} = \sum_{i \in [k]} \ell_i \leq \ell/4$  has the same parity as  $\ell$ .

To achieve this, we look for a cycle of suitable length in the auxiliary graph  $H$  with  $V(H) = [s]$ , where  $ij \in E(H)$  if and only if there is an edge between  $A_i$  and  $A_j$ . We apply Lemma 2.9 to  $H$  with  $\gamma = \ell^{1-2\varepsilon}$  to obtain triples  $\{(x_i, X_i, T_i)\}_{i \in [t]}$  so that for each  $i \in [t]$  there is  $d_i \in [0, \log_\gamma(N)]$  such that each vertex of  $X_i$  is at distance  $d_i$  from  $x_i$  in  $T_i$ . We let  $X = \bigcup_{i \in [t]} X_i$  and note that  $|X| \geq s/2\gamma \geq \varepsilon N/8u\gamma \geq \varepsilon n\ell^{-1+3\varepsilon}/16$ .

We will construct a cycle by applying Proposition 5.4 to find a long path in  $H[X]$ . Consider a maximal matching in each  $G[A_i]$  and let  $I_i \subset A_i$  denote the vertices not covered by the matching. Then each  $I_i$  is independent and  $|I_i| \geq u/2$  as  $(A_i, B_i, D_i)$  is not parity broken. Deleting some vertices if necessary we can assume  $|I_i| = u/2$  for all  $i \in [s]$ . Fix  $d \in \mathbb{N}$  of the same parity as  $\ell$  with  $d = \ell^\varepsilon + 2 \pm 1$ . Then  $u/2 > 3d$ , and for large  $\ell$  we have

$$\left| \bigcup_{i \in X} I_i \right| / 12d \geq (\varepsilon n \ell^{-1+3\varepsilon} / 16) \cdot (\ell^{1-\varepsilon} / 24d) > n > \alpha(G).$$

Thus Proposition 5.4 applies to  $G[\bigcup_{i \in X} I_i]$ , giving some  $\{i_0, \dots, i_d\} \subset X$  and a matching  $M$  of size  $d$  with one edge in each  $G[I_{i_{j-1}}, I_{i_j}]$  for  $j \in [d]$ .

Note that  $P = i_0 \dots i_d$  is a path in  $H$ , so Lemma 2.9 (iii) implies that it is contained in some  $X_i$ . By the distance property of  $X_i$ , the unique  $i_0 i_d$ -path  $Q$  in  $T_i$  is internally disjoint from  $P$  and has length  $\ell(Q)$  which is even with  $\ell(Q) \leq 2d_i \leq 2\log_\gamma(N)$ . Let  $S$  be a set of edges obtained by choosing one edge in  $G[A_x, A_y]$  for each edge  $xy$  of  $Q$  (which exists by definition of  $H$ ). Then  $M \cup S$  consists of a set of vertex-disjoint paths, which we denote  $P_1, \dots, P_k$ , with lengths  $\ell_1, \dots, \ell_k$ , where  $k \geq d - 1 > \ell/u$  (as  $M$  is a matching) and  $\ell_{\text{sum}} = \sum_{i \in [k]} \ell_i = \ell(Q) + d \leq \ell/4$  has the same parity as  $\ell$ . Furthermore,  $\{P_1, \dots, P_k\}$  is a handle system for a set of hubs  $\{H_1, \dots, H_k\}$  such that each  $P_i$  starts in  $H_i$  and ends in  $H_{i+1}$ . Now Lemma 5.3 (i) gives an  $\ell$ -cycle, which is the required contradiction.  $\square$

*Remark:* Henceforth, we will assume all hubs of  $G$  are parity broken:

Let  $G$  be as in Lemma 5.1, with  $\varepsilon < \varepsilon_0(\eta)$  sufficiently small and  $C$  sufficiently large so that there is a partition  $V(G) = W \cup \bigcup_{i \in [L]} (A_i \cup B_i \cup D_i)$  with  $|W| \leq \varepsilon N$ , where each  $(A_i, B_i, D_i)$  is a parity broken  $(u, \varepsilon)$ -hub with  $u = \ell^{1-\varepsilon}$ . (3)

This can be guaranteed by taking  $\varepsilon$  slightly smaller in Corollary 4.3 and moving into  $W$  any hubs that are not parity broken.

### 5.3 Interaction between hubs

We will now organise most of the hubs into ‘components’, so that there is no large matching between two hubs in different components. To do so, we write  $U_i = A_i \cup B_i$  for each  $i \in [L]$  and consider a maximum matching  $\mathcal{M}$  in  $G[\bigcup_i U_i]$  such that (a) every  $uv \in \mathcal{M}$  goes between distinct hubs, and (b) between any two distinct hubs there is at most one edge of  $\mathcal{M}$ . We define an auxiliary graph

$H_1$  on  $[L]$  where  $ij \in E(H_1)$  iff there is an edge of  $\mathcal{M}$  between  $U_i$  and  $U_j$ . We start by bounding the average degree of  $H_1$ .

**Lemma 5.6.** *Assuming (3),  $H_1$  has average degree at most  $\ell^{1-3\varepsilon}$ .*

*Proof.* For contradiction, suppose  $d(H_1) \geq \ell^{1-3\varepsilon}$ . We apply Lemma 2.8 to  $H_1$  with  $\gamma = \ell^{1-5\varepsilon}$  and  $d_1 = \ell^\varepsilon$ , noting that  $d(H_1) \geq 16\gamma d_1$ , to find an  $\ell_1$ -cycle for some  $\ell_1 \in [d_1, d_1 + 2\log_\gamma(N)] \subset [d_1, d_1 + \ell/10]$ , using  $\ell \geq C \log n / \log \log n$ . Its edges correspond to a submatching  $\mathcal{M}'$  of  $\mathcal{M}$  of size  $\ell$ , which forms a handle system for a set of  $\ell$  hubs. As  $8\ell_1 \leq \ell \leq u\ell_1$  and each hub is parity broken, Lemma 5.3 (ii) gives an  $\ell$ -cycle, which is a contradiction.  $\square$

By Lemma 5.6, at most  $\varepsilon L$  vertices of  $H_1$  have degree greater than  $\varepsilon^{-1}\ell^{1-3\varepsilon}$  in  $H_1$ . Let  $H_2$  be obtained from  $H_1$  by deleting these high degree vertices, so that  $v(H_2) \geq (1 - \varepsilon)L$ . We will now restrict attention to the subgraph  $H_3$  of  $H_2$  where  $ij \in E(H_3)$  iff  $G[U_i, U_j]$  has a matching of size  $2\ell^\varepsilon$ . We show that  $H_3$  does not have large components.

**Lemma 5.7.** *Assuming (3), all connected components of  $H_3$  have fewer than  $(1 + 2\varepsilon)\ell^\varepsilon/2$  vertices.*

*Proof.* For contradiction, suppose  $H_3$  contains a tree  $T$  with  $(1 + 2\varepsilon)\ell^\varepsilon/2$  vertices. By definition of  $H_3$ , we can greedily choose a matching  $\mathcal{P} = \{P_1, \dots, P_k\}$  that contains two edges of  $G[U_i, U_j]$  for each  $ij \in E(T)$ . We can regard  $\mathcal{P}$  as a handle system for the hubs  $\mathcal{H} = \{(A_i, B_i, D_i) : i \in V(T)\}$ . To see this, we note that condition (i) is vacuous, and (iii) holds as  $|\mathcal{P}| = 2e(T) < \ell^\varepsilon < u^{1-\varepsilon}/2$ . To achieve (ii), we order the edges of  $\mathcal{P}$  cyclically according to a closed walk in  $T$  that uses every edge exactly twice (which is well-known to exist, e.g. by embedding  $T$  in the plane and walking around its outside). As  $8|\mathcal{P}| \leq \ell \leq 2(1 - \varepsilon)u|\mathcal{P}| - |\mathcal{P}|$  and all hubs are parity broken, Lemma 5.3 (ii) gives an  $\ell$ -cycle, which is a contradiction.  $\square$

## 5.4 Proof of stability

We now combine the results of this section to prove our stability result.

*Proof of Lemma 5.1.* Let the matching  $\mathcal{M}$  and graphs  $H_1$ ,  $H_2$  and  $H_3$  be as in the previous subsection. Fix a maximal matching  $M_{ij}$  in  $G[U_i, U_j]$  for each  $ij \in E(H_2) \setminus E(H_3)$ ; by definition of  $H_3$  each  $|M_{ij}| < 2\ell^\varepsilon$ . For each  $i \in V(H_2) = V(H_3)$ , let  $U'_i = U_i \setminus \left( V(\mathcal{M}) \cup \bigcup_{ij} V(M_{ij}) \right)$ ; by definition of  $H_2$  each  $|U'_i| \geq |U_i| - \varepsilon^{-1}\ell^{1-3\varepsilon} \cdot 2\ell^\varepsilon \geq (1 - \varepsilon)2u$  for large  $\ell$ . Let  $U' = \bigcup \{U'_i : i \in V(H_3)\}$  and  $G' = G[U']$ . We have  $|U'| \geq |V(H_3)| \cdot (1 - \varepsilon)2u \geq (1 - \varepsilon)^2 2uL \geq (1 - 3\varepsilon)N$ , so by Theorem 2.2 (Turán's Theorem)  $d(G') \geq (1 - 4\varepsilon)\ell$ .

Note that all edges of  $G'$  lie within some hub or join two hubs in the same connected component of  $H_3$ . By Lemma 5.7 the number of vertices in any component of  $G'$  is at most  $(1 + 2\varepsilon)(\ell^\varepsilon/2) \cdot (2\ell^{1-\varepsilon}) = (1 + 2\varepsilon)\ell$ . Let  $B$  be obtained from  $U'$  by deleting  $V(C)$  for any component  $C$  of  $G'$  with  $d(C) \leq (1 - \varepsilon^{1/2})\ell$ . Then  $|U'| - (1 - 4\varepsilon)\ell \leq 2e(G') \leq |B|(1 + \varepsilon)\ell + (|U'| - |B|)(1 - \varepsilon^{1/2})\ell$ , which gives  $|B|(\varepsilon^{1/2} + \varepsilon) \geq |U'|(\varepsilon^{1/2} - 4\varepsilon)\ell$ , and so  $|B| \geq (1 - 6\varepsilon^{1/2})|U'| \geq (1 - 7\varepsilon^{1/2})N$ .

We conclude by taking subgraphs of high minimum degree in each component of  $G'[B]$ . Letting  $k = (1 - \eta/2)\ell$ , each such component  $C$  has  $e(C) = d(C)v(C)/2 \geq (1 - \varepsilon^{1/2})\ell v(C)/2 \geq \binom{k}{2} + (v(C) - k)(1 - \eta/2)\ell$ , as  $(1 + \varepsilon)\ell \geq v(C) \geq d(C) \geq (1 - \varepsilon^{1/2})\ell$  and  $\varepsilon \ll \eta$ . Proposition 2.1 (i) gives a subgraph  $C'$  of  $C$  with  $\delta(C') \geq k \geq (1 - \eta/2)\ell \geq (1 - 3\eta/4)v(C)$ . We let  $V_1, \dots, V_s$  be the vertex-sets of these subgraphs  $C'$  for all components  $C$  of  $G'[B]$ . Then each  $|V_i| \geq \delta(G[V_i]) \geq (1 - \eta)\ell$  and  $\sum_{i=1}^s |V_i| \geq (1 - 3\eta/4)|B| \geq (1 - \eta)N$ . Lastly, suppose for contradiction that some  $|V_i| \geq \ell$ . We may delete  $|V_i| - \ell \leq 3\varepsilon\ell$  vertices from  $V_i$  and apply Theorem 2.3 (Dirac's Theorem) to find an  $\ell$ -cycle in  $G$ . This contradiction shows that all  $|V_i| \leq \ell - 1$ .  $\square$

## 6 The upper bound

In this section we will prove our main result, Theorem 1.1, which establishes the upper bound on cycle-complete Ramsey numbers; the proof will be given in the last subsection. Most of this section will be occupied with cleaning up the approximate structure of a supposed counterexample, as provided by the stability result in the last section, until it becomes clear that its properties are contradictory, so it cannot exist.

Throughout the section we fix a graph  $G$  and ‘approximate cliques’  $V_1, \dots, V_s$  satisfying the hypotheses and conclusions of Lemma 5.1. In the first subsection we give conditions under which the approximate cliques can absorb additional vertices from the remainder  $R := V(G) \setminus \bigcup_{i=1}^s V_i$ , while maintaining pancyclicity and also the property that any pair of vertices can be connected by paths with a large range of possible lengths. In the second subsection we clean up  $R$  by absorbing some of its vertices into the approximate cliques. In the third subsection we show that the remaining part of  $R$  can be separated from most of the approximate cliques, in the sense they have each have a large subset with no neighbours in  $R$ . In the fourth subsection we show that one of the approximate cliques has a vertex that can absorb its neighbours. This final property quickly leads to a contradiction, which will complete the proof.

### 6.1 Absorbable paths

In this subsection we consider the following set-up which is very similar to the handle systems used for hubs. Given a set of paths  $\mathcal{P} = \{P_1, \dots, P_m\}$  in a graph  $H$  and a set  $V \subset V(H)$ , we say  $\mathcal{P}$  is absorbable into  $V$  if it consists of paths that are vertex-disjoint and disjoint from  $V$ , and there are distinct vertices  $\{a_1, \dots, a_m, b_1, \dots, b_m\} \subset V$  such that  $a_i$  is adjacent to one end of  $P_i$  and  $b_i$  is adjacent to the other end of  $P_i$ ; we say that  $P_i$  attaches to  $a_i$  and  $b_i$ . The following lemma will be used to absorb paths into approximate cliques.

**Lemma 6.1.** *Let  $H$  be a graph with a partition  $V(H) = U \cup V$ , where  $\delta(H[V]) \geq 0.9|V|$  and  $|U| \leq 0.1|V|$ . Suppose that  $\mathcal{P}$  is a set of paths of length at most 2 which is absorbable into  $V$  and has  $\bigcup_{P \in \mathcal{P}} V(P) = U$ . Then*

- (i)  $H$  contains an  $xy$ -path of length  $\ell$  for any distinct  $x, y$  in  $V(H)$  and  $\ell \in [6, 2v(H)/3]$ ,
- (ii)  $H$  is pancyclic.

*Proof.* For (i), we suppose first that both  $x$  and  $y$  are in  $V$ , and show that there is an  $xy$ -path of length  $\ell$  for any  $\ell \in [2, 2v(H)/3]$ . To see this, we use  $\delta(H[V]) \geq 0.9|V|$  to greedily choose an  $xy'$ -path  $P$  of length  $\ell - 2$  in  $H[V]$  that avoids  $y$ . As  $|N_{H[V]}(y) \cap N_{H[V]}(y')| - |V(P)| \geq 0.8|V| - (2/3)1.1|V| > 0$  we can choose a common neighbour of  $y$  and  $y'$  in  $V \setminus V(P)$ , and so obtain the required  $xy$ -path of length  $\ell$ . Next we suppose that  $x$  is in  $V$  and  $y$  is in  $U$ . Then  $y$  lies on a path  $P \in \mathcal{P}$ . Let  $a$  and  $b$  be the attachments of  $P$ , where without loss of generality  $a \neq x$ . The subpath of  $P$  from  $y$  to  $a$  has length  $\ell' \leq 3$ . Adding a path of length  $\ell - \ell'$  from  $a$  to  $x$  gives the required  $xy$ -path of length  $\ell$ . Finally, suppose  $x$  and  $y$  are both in  $U$ . Then we can find  $a$  and  $b$  in  $V$  so that there is an  $xa$ -path and  $yb$ -path that are vertex-disjoint and both of length at most 2. Adding an  $ab$ -path of the appropriate length completes the proof of (i).

For (ii), we first note that by Theorem 2.4 (Bondy’s Theorem)  $H[V]$  is pancyclic. It remains to show there is an  $\ell$ -cycle whenever  $|V| < \ell \leq |V(H)|$ . Let  $S$  be the set of attachments of  $\mathcal{P}$ , and fix any  $V' \subset V \setminus S$  with  $|V'| = \ell - |S| - |U|$ . As  $|U| + |S| \leq 3|U| \leq 0.3|V|$ , we have  $|V'| \geq 0.7|V|$ . Let  $H'$  be the graph obtained from  $H[V']$  by adding a new vertex  $v_P$  for each  $P \in \mathcal{P}$ , which is joined to all common neighbours in  $V'$  of the attachments of  $P$ . Note that  $v(H') = |V'| + |\mathcal{P}|$

and  $\delta(H') \geq |V'| - 0.2|V| \geq |V|/2 > v(H')/2$ , and so by Theorem 2.3 (Dirac's Theorem)  $H'$  has a Hamilton cycle. Replacing each  $v_P$  by  $P$  and the edges to its attachments produces a cycle of length  $\ell$  in  $H$ , as required.  $\square$

## 6.2 Cleaning up the remainder

Here we clean up the remainder  $R = V(G) \setminus \bigcup_{i=1}^s V_i$  by absorbing some of its vertices into the approximate cliques, according to the following algorithm. For each  $i \in [s]$  we keep track of two sets during the algorithm: (a) a set  $W_i = V_i \cup R_i$ , where  $R_i \subset R$  has been absorbed by  $V_i$ , and (b) a subset  $A_i$  of  $V_i$ , which is available for further attachments in the sense of the previous subsection. We start with  $W_i = A_i = V_i$  for each  $i \in [s]$ . In a given round:

- Consider any path  $P$  of length at most 2 in  $G[R]$  that attaches to some distinct vertices  $a, b$  in  $A_i$  for some  $i \in [s]$ . If there is no such  $P$  then stop. Otherwise, move  $V(P)$  from  $R$  to  $R_i$ , delete  $a$  and  $b$  from  $A_i$ , and proceed to the next round.

We claim that the algorithm terminates with  $|W_i| \leq \ell - 1$  for all  $i \in [s]$ . Indeed, otherwise in some round some  $|W_i| \in [\ell, \ell + 2]$ , as  $W_i$  increments by at most 3 vertices in each round. Then  $W_i$  has a partition  $W_i = V_i \cup R_i$ , where  $|V_i| \geq \delta(G[V_i]) \geq (1 - \eta)|V_i| \geq 0.9|V_i|$  and  $|R_i| \leq \eta\ell + 2 \leq 0.1|V_i|$ . By construction,  $R_i$  is the union of paths  $\mathcal{P}_i$  absorbable into  $V_i$ , so Lemma 6.1 (ii) gives an  $\ell$ -cycle in  $G[W_i]$ . This contradiction proves the claim. We deduce  $|R_i| = |W_i| - |V_i| < \eta\ell$ . Furthermore, each  $A_i$  decreased by two vertices for each path added to  $R_i$ , so  $|A_i| \geq |V_i| - 2|R_i| \geq (1 - 3\eta)\ell$ .

## 6.3 Separating the remainder

Now we show that the cleaned up remainder  $R := V(G) \setminus \bigcup_{i=1}^s W_i$  can be separated from most of the approximate cliques, in the following sense. For  $i \in [s]$  let  $A'_i$  be the set of  $v \in A_i$  such that  $v$  has a neighbour in  $R$ . We partition  $[s]$  as  $S \cup T$ , where  $T = \{i \in [s] : |A'_i| < \ell^{2/3}\}$ .

**Lemma 6.2.**  $|T| \geq s/2$ .

*Proof.* We start by constructing a partition  $R = U_1 \cup \dots \cup U_r$ , where each  $G[U_j]$  has diameter at most 2 and  $r \leq 2N\ell^{-1/2}$ . To see that this is possible, we repeatedly remove stars from  $R$  of order  $\ell^{1/2}$  until none remain. We can remove at most  $\eta N/\ell^{1/2}$  such stars. The remaining set  $R'$  must have  $d(G[R']) < \ell^{1/2} - 1$ . By Theorem 2.2 (Turán's Theorem)  $|R'|/\ell^{1/2} \leq \alpha(G[R']) \leq \alpha(G) < n$ , so  $|R'| < n\ell^{1/2} < \frac{3}{2}N\ell^{-1/2}$ . We let the parts  $U_1, \dots, U_r$  consist of all removed stars and singleton parts for each vertex of  $R'$ . Then  $r \leq 2N\ell^{-1/2}$ , as required.

Now suppose for contradiction that  $|T| < s/2$ , so  $|S| > s/2$ . For each  $v \in \bigcup_{i=1}^s A'_i$  we fix any  $u_v \in N(v) \cap R$  (such a vertex exists by definition of  $A'_i$ ). We consider an auxiliary bipartite graph  $H$  with parts  $A = \{W_i\}_{i \in [s]}$  and  $B = \{U_j\}_{j \in [r]}$ , where we add an edge from  $W_i$  to  $U_j$  for each  $v \in A'_i$  with  $u_v \in U_j$ . To see that this gives a (simple) graph we use the termination condition of the algorithm in the previous subsection: there cannot be distinct  $v_1, v_2 \in A'_i$  with neighbours  $u_1, u_2 \in U_j$ , as  $U_j$  has diameter at most 2, so we would have a  $u_1 u_2$ -path of length at most 2 attaching to  $A_i$ .

We will obtain a contradiction by finding a short cycle in  $H$  and using it to construct an  $\ell$ -cycle in  $G$ . We have  $v(H) = s + r \leq 2N\ell^{-1} + 2N\ell^{-1/2} \leq 4N\ell^{-1/2}$  and  $e(H) = \sum_{i \in [s]} |A'_i| \geq |S|\ell^{2/3} > \frac{1}{2} \frac{N}{2\ell} \ell^{2/3} \geq \ell^{1/6} v(H)/16$ , so  $d(H) > \ell^{1/6}/8$ . As  $\ell \geq C \frac{\log n}{\log \log n}$ , we can apply Lemma 2.8 with  $\gamma = d_1 = \ell^{1/14}$  to find a cycle in  $H$  with length in  $[\ell^{1/14}, \ell^{1/14} + 2 \log_{\ell^{1/14}}(v(H))] \subset [4, \ell/8]$ .

As  $H$  is bipartite, we can write this cycle as  $W_{i_1}U_{i_1}W_{i_2}\cdots W_{i_L}U_{i_L}W_{i_1}$ , for some  $2 \leq L \leq \ell/16$ . Each  $U_{i_j}$  has diameter at most 2, so by construction of  $H$  there is a path  $Q_j$  of length at most 4, starting with the edge  $b_ju_{b_j}$  for some  $b_j \in W_{i_j}$  and ending with the edge  $u_{a_{j+1}}a_{j+1}$  for some  $a_{j+1} \in W_{i_{j+1}}$ . Furthermore,  $a_j, b_j \in W_{i_j}$  are distinct, as  $u_{a_j} \neq u_{b_j}$ . We fix  $\ell_j \in [2, \ell/2]$  for each  $j \in [L]$  with  $\sum_{j \in [L]} \ell_j = \ell - \sum_{j \in [L]} e(Q_j)$ . and apply Lemma 6.1 (i) to choose  $a_j b_j$ -paths  $P_j$  in  $W_j$  of length  $\ell_j$ . Combining these with the paths  $Q_j$  produces an  $\ell$ -cycle, which is a contradiction.  $\square$

## 6.4 Absorbing neighbours

Now we will show that one of the approximate cliques has a vertex that can absorb its neighbours. To do so, we now analyse the edges crossing between the approximate cliques. For each  $i \in T$  let  $B_i = A_i \setminus A'_i$  denote the set of  $v \in A_i$  with no neighbour in  $R$ . By definition of  $T$ , for each  $i \in T$ , we have  $|B_i| \geq |A_i| - \ell^{2/3} \geq 2|V_i|/3$ . For each  $i \in T$  we consider a maximum matching  $\mathcal{M}_i$  in the induced bipartite subgraph  $G[B_i, V(G) \setminus W_i]$ . We will show that these matchings cannot all be large.

**Lemma 6.3.** *There is  $i^* \in T$  with  $|\mathcal{M}_{i^*}| \leq \ell^{1/3}$ .*

Before giving the proof, we show how this lemma allows us to find a vertex that can absorb its neighbours. Recall that  $W_{i^*} = V_{i^*} \cup R_{i^*}$  and  $R_{i^*}$  is a union of vertex-disjoint paths  $\mathcal{P}_{i^*}$  that is absorbable into  $V_{i^*}$ . Set  $\overline{W_{i^*}} = V(G) \setminus W_{i^*}$ .

**Lemma 6.4.** *There is  $v \in B_{i^*}$  such that for any neighbours  $y_1, \dots, y_k$  of  $v$  in  $\overline{W_{i^*}}$  with  $k \leq \ell - |W_{i^*}|$ , letting  $\mathcal{P}'_{i^*}$  be obtained from  $\mathcal{P}_{i^*}$  by adding each  $y_i$  as a path of length 0, we have  $\mathcal{P}'_{i^*}$  absorbable into  $V_{i^*}$ .*

*Proof.* We apply the following algorithm to construct a set  $D_{i^*} \subset B_{i^*}$  such that every vertex in  $B_{i^*}$  has the stated property. We start with  $D_{i^*} = B_{i^*}$  and  $X = \emptyset$ . First note that any vertex  $x \in \overline{W_{i^*}}$  which has a neighbour in  $D_{i^*}$  must be from  $\bigcup_{j \neq i^*} W_j$ . While there is  $x \in \bigcup_{j \neq i^*} W_j$  with  $1 \leq d_G(x, D_{i^*}) \leq 2\ell^{1/3}$  we add  $x$  to  $X$  and delete  $N_G(x) \cap D_{i^*}$  from  $D_{i^*}$ . This process terminates with a set  $D_{i^*}$  such that  $d_G(x, D_{i^*}) = 0$  or  $d_G(x, D_{i^*}) > 2\ell^{1/3}$  for all  $x \in (\bigcup_{j \neq i^*} W_j) \setminus X$ . Each  $x \in X$  has a private neighbour in  $B_{i^*}$ , so by choice of  $\mathcal{M}_{i^*}$  we have  $|X| \leq |\mathcal{M}_{i^*}| \leq \ell^{1/3}$ , and so  $|D_{i^*}| \geq |B_{i^*}| - (2\ell^{1/3})|X| \geq \ell/2 > 0$ .

Consider any  $v \in D_{i^*}$  and neighbours  $y_1, \dots, y_k$  of  $v$  in  $\overline{W_{i^*}}$  with  $k \leq \ell - |W_{i^*}|$ . Each  $y_i$  is not in  $X$  (otherwise we would have deleted  $v$  from  $D_{i^*}$ ) so has at least  $2\ell^{1/3}$  neighbours in  $D_{i^*}$ . This implies  $k \leq |\mathcal{M}_{i^*}| \leq \ell^{1/3}$ , or otherwise we could greedily construct a matching of size  $|\mathcal{M}_{i^*}| + 1$  between  $\{y_1, \dots, y_k\}$  and  $B_{i^*}$ , which is contrary to the choice of  $\mathcal{M}_{i^*}$ . We can therefore greedily choose two attachments for each  $y_i$  in  $D_{i^*}$ , which are distinct from each other, and distinct from the attachments of  $\mathcal{P}_{i^*}$  as  $D_{i^*} \subset B_{i^*} \subset A_{i^*}$ . Thus  $\mathcal{P}'_{i^*}$  is absorbable into  $V_{i^*}$ .  $\square$

We conclude this subsection by returning to the proof of Lemma 6.3.

*Proof of Lemma 6.3.* For contradiction, suppose  $|\mathcal{M}_i| > \ell^{1/3}$  for all  $i \in T$ . Note that every edge in  $\mathcal{M}_i$  has one end in  $B_i$  and the other end in  $\bigcup_{j \neq i} W_j$  (it is not in  $R$  by definition of  $B_i$ ). Consider a uniformly random partition  $[s] = S_1 \cup S_2$ . Say that  $bc \in \mathcal{M}_i$  with  $b \in B_i$  and  $c \in W_j$  is *good* if  $i \in S_1$  and  $j \in S_2$ . Each edge is good with probability  $1/4$ , so we can fix a partition so that the number of good edges is at least  $\frac{1}{4} \sum_{i \in T} |\mathcal{M}_i| > |T|\ell^{1/3}/4 \geq s\ell^{1/3}/8$ .

Consider the auxiliary bipartite graph  $H$  with parts  $A = \{W_i\}_{i \in S_1}$  and  $B = \{W_j\}_{j \in S_2}$ , where we add an edge from  $W_i \in A$  to  $W_j \in B$  for each good edge  $bc \in \mathcal{M}_i$  with  $b \in B_i$  and  $c \in W_j$ . We claim that  $H$  is a (simple) graph. To see this, suppose on the contrary we have  $b_1c_1$  and  $b_2c_2$  in



$\mathcal{M}_i$  with  $\{c_1, c_2\} \subset W_j$ . By Lemma 6.1 (i) there is a  $b_1b_2$ -path  $P_1$  in  $G[W_i]$  of length  $\lfloor \ell/2 \rfloor - 1$  and a  $c_1c_2$ -path  $P_2$  in  $G[W_j]$  of length  $\lfloor \ell/2 \rfloor - 1$ . Combining the paths  $P_1$  and  $P_2$  with the edges  $b_1c_1$  and  $b_2c_2$  gives a  $\ell$ -cycle. This contradiction proves the claim.

We deduce  $e(H) \geq s\ell^{1/3}/8 = v(H)\ell^{1/3}/8$ , so  $d(H) \geq \ell^{1/3}/4$ . We use this to obtain the required contradiction by finding a short cycle in  $H$ , and so an  $\ell$ -cycle in  $G$ . This part of the proof is very similar to that of Lemma 6.2. Lemma 2.8 provides an even cycle  $W_{i_1}W_{j_1} \cdots W_{i_L}W_{j_L}W_{i_1}$ , for some  $2 \leq L \leq \ell/16$ , where each  $i_\alpha \in S_1$  and  $j_\alpha \in S_2$ . By definition of  $H$ , for each  $\alpha \in [L]$  there are edges  $a_\alpha x_\alpha$  and  $b_\alpha y_\alpha$  in  $\mathcal{M}_{i_\alpha}$  with  $\{a_\alpha, b_\alpha\} \subset W_{i_\alpha}$ ,  $x_\alpha \in W_{j_{\alpha-1}}$  and  $y_\alpha \in W_{j_\alpha}$ . By Lemma 6.1 (i) there is a path  $Q_\alpha$  of length at most 4 from  $b_{\alpha-1}$  to  $a_\alpha$  through  $W_{j_{\alpha-1}}$  via  $y_{\alpha-1}$  and  $x_\alpha$  (whether or not these coincide). We fix  $\ell_\alpha \in [2, \ell/2]$  for each  $\alpha \in [L]$  with  $\sum_{\alpha \in [L]} \ell_\alpha = \ell - \sum_{\alpha \in [L]} e(Q_\alpha)$ . and apply Lemma 6.1 (i) to choose  $a_\alpha b_\alpha$ -paths  $P_\alpha$  in  $W_{i_\alpha}$  of length  $\ell_\alpha$ . Combining these with the paths  $Q_\alpha$  produces an  $\ell$ -cycle, and so the required contradiction.  $\square$

## 6.5 Proof of Theorem 1.1

We now complete the proof of our main theorem.

*Proof of Theorem 1.1.* We fix  $\ell \in \mathbb{N}$  and prove the following statement (\*) by induction on  $n \geq 1$  such that if  $n \geq 3$  we have  $\ell \geq C \frac{\log n}{\log \log n}$  (for some large absolute constant  $C$ ):

(\*) there is no  $C_\ell$ -free graph  $G$  with  $v(G) = N = (\ell - 1)(n - 1) + 1$  and  $\alpha(G) \leq n - 1$ .

The case  $n = 1$  holds as every graph  $G$  with  $v(G) \geq 1$  has an independent set of order 1. The case  $n = 2$  holds as every graph  $G$  with  $v(G) \geq \ell$  contains an independent set of order 2 or a clique of order  $\ell$ .

Now we give the induction step for  $n \geq 3$ . For contradiction, suppose we have a  $C_\ell$ -free graph  $G$  with  $v(G) = N = (\ell - 1)(n - 1) + 1$  and  $\alpha(G) \leq n - 1$ .

If there is any vertex  $v$  of degree less than  $\ell - 1$  we delete  $N(v) \cup \{v\}$  from  $G$  and apply induction. The remaining subgraph  $G_1$  satisfies  $v(G_1) \geq (\ell - 1)(n - 2) + 1 = r(C_\ell, K_{n-1})$  by induction, so it contains a cycle  $C$  of length  $\ell$  or an independent set  $I$  of order  $n - 1$ . Then  $G$  contains an  $\ell$ -cycle  $C$  or  $\{v\} \cup I$  forms an independent set of order  $n$ . Thus we may assume  $\delta(G) \geq \ell - 1$ .

We let  $V_1, \dots, V_s$  be the approximate cliques provided by the stability result (Lemma 5.1), let  $W_i = V_i \cup R_i$  for  $i \in [s]$  be the enlarged approximate cliques obtained in the previous section by absorbing part of the remainder, and let  $v \in B_{i^*}$  be given by Lemma 6.4. As  $v$  has at least  $\ell - 1$  neighbours, we can choose neighbours  $y_1, \dots, y_k$  of  $v$  in  $\overline{W_{i^*}}$  with  $k = \ell - |W_{i^*}|$ . As the path system  $\mathcal{P}'_{i^*}$  in Lemma 6.4 is absorbable, Lemma 6.1 gives a cycle of length  $|V_i| + |R_i| + k = \ell$  in  $G$ . This gives a contradiction and completes the proof of the theorem.  $\square$

## 7 Concluding remarks

Our results answer the questions of Erdős et al. [21] up to a constant factor, which we did not compute explicitly, although with more work it seems that a reasonable value (less than 20, say) can be obtained. It would be interesting to obtain an asymptotic formula for the  $\ell$  minimising  $r(C_\ell, K_n)$ . The constructions for the lower bound on  $r(C_\ell, K_n)$  avoid a range of cycles. For large  $\ell$ , this range consists of all cycles of length at least  $\ell$ , and for small  $\ell$ , it consists of all cycles of length at most  $\ell$ . This suggests that the finer nature of the threshold may be connected to the problem of improving the Moore bound (see [39]) on the number of edges in a graph of given order and diameter.

The problem of obtaining good estimates on  $r(C_\ell, K_n)$  for small  $\ell > 3$  remains widely open. The most significant gap in the current state of knowledge is the case  $\ell = 4$ , for which the known bounds (see [12, 47]) are  $c(n/\log n)^{3/2} \leq r(C_4, K_n) \leq C(n/\log n)^2$  for some constants  $c$  and  $C$ .

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