The local structure theorem

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THE LOCAL STRUCTURE THEOREM: THE WREATH PRODUCT CASE

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Dedicated to the memory of Kay Magaard

Abstract. Groups with a large p-subgroup, p a prime, include almost all of the groups of Lie type in characteristic p and so the study of such groups adds to our understanding of the finite simple groups. In this article we study a special class of such groups which appear as wreath product cases of the Local Structure Theorem [MSS2].

1. Introduction

Throughout this article p is a prime and G is a finite group. We say that $L \leq G$ has *characteristic* p if

$$C_G(O_p(L)) \leq O_p(L).$$

For T a non-trivial p-subgroup of G, the subgroup $N_G(T)$ is called a *p-local* subgroup of G. By definition G has *local characteristic* p if all p-local subgroups of G have characteristic p and G has *parabolic characteristic* p if all p-local subgroups containing a Sylow p-subgroup of G have characteristic p.

A group K is called a *$\mathcal{K}$-group* if all its composition factors are from the known finite simple groups. So, if K is a simple $\mathcal{K}$-group, then K is a cyclic group of prime order, an alternating group, a simple group of Lie type or one of the 26 sporadic simple groups. A group G is a *$\mathcal{K}_p$-group*, provided all subgroups of all p-local subgroups of G are $\mathcal{K}$-groups. This paper is part of a programme to investigate the structure of certain $\mathcal{K}_p$-groups. See [MSS1, MSS2] for an overview of the project.

Of fundamental importance to the development of the programme are large subgroups of G: a p-subgroup Q of G is *large* if

(i) $C_G(Q) \leq Q$; and
(ii) $N_G(U) \leq N_G(Q)$ for all $1 \neq U \leq C_G(Q)$.

For example, if G is a simple group of Lie type defined in characteristic p, $S \in \mathrm{Syl}_p(G)$ and $Q = O_p(C_G(Z(S)))$, then Q is a large subgroup of G unless there is some degeneracy in the Chevalley commutator relations which define G. This means that Q is a large subgroup of G unless G is one of $\mathrm{Sp}_{2n}(2^k)$, $n \geq 2$, $\mathrm{F}_4(2^k)$ or $\mathrm{G}_2(3^k)$.

If Q is a large subgroup of G, then it is easy to see that $O_p(N_G(Q))$ is also a large p-subgroup of G. Thus we also assume that

(iii) $Q = O_p(N_G(Q))$.

One of the consequences of G having a large p-subgroup is that G has parabolic characteristic p. In fact any p-local subgroup of G containing Q is of characteristic p [MSS2, Lemma 1.5.5 (e)]. Further, if $Q \leq S \in \mathrm{Syl}_p(G)$, then Q is weakly closed in S with respect to G (Q is the unique G-conjugate of Q in S) [MSS2, Lemma 1.5.2 (e)]. A significant part of the programme described in [MSS1] aims to determine the groups which possess a large p-subgroup. This endeavour extends and generalizes earlier work of Timmesfeld and others in the original proof of the classification theorem where groups with a so-called large extraspecial 2-subgroup were investigated. The state of play at the moment is that the Local Structure Theorem has been completed and published [MSS2]. To describe this result we need some further notation.

For a finite group L, Y_L denotes the unique maximal elementary abelian normal p-subgroup of L with $O_p(L/C_L(Y_L)) = 1$. Such a subgroup exists [MSS1, Lemma 2.0.1(a)]. From now on assume that G is a finite $\mathcal{K}_p$-group, S a Sylow p-subgroup of G and Q a large p-subgroup of G with $Q \leq S$ and $Q = O_p(N_G(Q))$. We define

$$\mathcal{L}_G(S) = \{L \leq G \mid S \leq L, O_p(L) \neq 1, C_G(O_p(L)) \leq O_p(L)\}.$$

Under the assumption that S is contained in at least two maximal p-local subgroups, for $L \in \mathcal{L}_G(S)$ with $L \not\leq N_G(Q)$, the Local Structure Theorem provides information about $L/C_L(Y_L)$ and its action on Y_L. Given the Local Structure Theorem there are two cases to treat in order to fully understand groups with a large p-subgroup. Either there exists $L \in \mathcal{L}_G(S)$ with $Y_L \not\leq Q$ or, for all $L \in \mathcal{L}_G(S)$, $Y_L \leq Q$. Research in the first case has just started and, for this situation, this paper addresses the wreath product scenario in the Local Structure Theorem [MSS2, Theorem A (3)]. This case is separated from the rest because of the special structure of L and Y_L. This structure allows us to use arguments measuring the size of certain subgroups to reduce to three exceptional configurations and has a distinct flavour from the remaining cases. For instance, the groups which are examples in the wreath product case typically have Q of class 3 whereas in the more typical cases it has class at most 2. The configurations in the Local Structure Theorem which are not in the wreath product case and have $Y_L \not\leq Q$ will be examined in a separate publication as there are methods which apply uniformly to cover many possibilities at once. Contributions to the $Y_L \leq Q$ for all $L \in \mathcal{L}_G(S)$ are the subject of [PPS].

For $L \in \mathcal{L}_G(S)$ with Q not normal in L we set

$$L^\circ = \langle Q^L \rangle, \overline{L} = L/C_L(Y_L) \text{ and } V_L = [Y_L, L^\circ]$$

and use this notation throughout the paper. Set $q = p^a$. We recall from [MSS2, Remark A.25] the definition of a *natural wreath* $\mathrm{SL}_2(q)$*-module* for the group X with respect to $\mathcal{K}$: suppose that X is a group, V is a faithful X-module and $\mathcal{K}$ is a non-empty X-invariant set of subgroups of X. Then V is a *natural* $\mathrm{SL}_2(q)$*-wreath product module* for X with respect to $\mathcal{K}$ if and only if

$$V = \bigoplus_{K \in \mathcal{K}} [V, K] \text{ and } \langle \mathcal{K} \rangle = \bigtimes_{K \in \mathcal{K}} K,$$

and, for each $K \in \mathcal{K}$, $K \cong \mathrm{SL}_2(q)$ and $[V, K]$ is the natural $\mathrm{SL}_2(q)$-module for K.

We now describe the wreath product case in [MSS2, Theorem A (3)]. For $L \in \mathcal{L}_G(S)$ with $L \not\leq N_G(Q)$, L is in the *wreath product case* provided

- there exists a unique $\overline{L}$-invariant set $\mathcal{K}$ of subgroups of $\overline{L}$ such that V_L is a natural $\mathrm{SL}_2(q)$-wreath product module for $\overline{L}$ with respect to $\mathcal{K}$.
- $\overline{L^\circ} = O^p(\langle \mathcal{K} \rangle)\overline{Q}$ and Q acts transitively on $\mathcal{K}$ by conjugation.
- $Y_L = V_L$ or $p = 2$, $|Y_L : V_L| = 2$, $\overline{L^\circ} \cong \mathrm{SL}_2(4)$ or $\Gamma\mathrm{SL}_2(4)$ and $V_L \not\leq Q$.

We say that $\overline{L}$ is *properly wreathed* if $|\mathcal{K}| > 1$.

There are overlaps between the wreath product case and some other divisions in the Local Structure Theorem.

If $\overline{L^\circ} \cong \mathrm{SL}_2(q)$ with $V_L = Y_L$, then this situation can be inserted in the linear case of [MSS2, Theorem A (1)] by including $n = 2$ is that case. Suppose that $|\mathcal{K}| = 2$ and $K \cong \mathrm{SL}_2(2)$. If $\overline{Q}$ is a fours group, then, as $\overline{Q}$ conjugates $\overline{K_1}$ to $\overline{K_2}$,

$$\overline{L^\circ} \cong \Omega_4^+(2) \cong \mathrm{SL}_2(2) \times \mathrm{SL}_2(2)$$

and Y_L is the tensor product module. This is an example in the tensor product case of [MSS2, Theorem A (6)]. We declare L to be in the *unambiguous wreath product case* if these two *ambiguous* configurations do not occur. The ambiguous cases will be handled in a more general setting in a forthcoming paper mentioned earlier.

Main Theorem. *Suppose that p is a prime, G is a finite group, S a Sylow p-subgroup of G and $Q \leq S$ is a large p-subgroup of G with $Q = O_p(N_G(Q))$. If there exists $L \in \mathcal{L}_G(S)$ with L in the unambiguous wreath product case and $V_L \not\leq Q$, then $G \cong \mathrm{Mat}(22)$, $\mathrm{Aut}(\mathrm{Mat}(22))$, $\mathrm{Sym}(8)$, $\mathrm{Sym}(9)$ or $\mathrm{Alt}(10)$.*

The proof of this theorem splits into four parts. First, in Section 3, we show that in the properly wreathed case we must have $q = |\mathcal{K}| = 2$ and, as L is unambiguous, $\overline{S} = \overline{Q} \cong \mathrm{Dih}(8)$ and $\overline{L^\circ} \cong \mathrm{O}_4^+(2)$. If $|\mathcal{K}| = 1$, we show that $\overline{L^\circ} \cong \Gamma\mathrm{SL}_2(4)$ or $\mathrm{SL}_2(4)$ and V_L is the natural module with $|Y_L : V_L| \leq 2$, where, if $\overline{L^\circ} \cong \mathrm{SL}_2(4)$, $|Y_L : V_L| = 2$ holds. In the following three sections, we determine the groups corresponding to these three cases. Finally the Main Theorem follows by combining Propositions 3.5, 4.1, 5.1 and 6.2.

In [PPS] the authors proved that the unambiguous wreath product case does not lead to examples if for all $L \in \mathcal{L}_G(S)$ we have $Y_L \leq Q$, with the additional assumption that G is of local characteristic p. In this paper we do not make the assumption that G is of local characteristic p.

In the Local Structure Theorem there is also a possibility that $L \in \mathcal{L}_G(S)$ is of weak wreath type. Any such group is contained in one, which is of unambiguous wreath type. A corollary of our theorem is

Corollary. *Suppose that p is a prime, G is a finite group, S a Sylow p-subgroup of G and $Q \leq S$ is a large p-subgroup of G with $Q = O_p(N_G(Q))$. If $L \in \mathcal{L}_G(S)$ is of weak wreath product type, then either G is as in the Main Theorem or $V_L \leq Q$.*

In addition to the notation already introduced, we will use the following

Notation. *For p a prime, G a group with a large p-subgroup $Q = O_p(N_G(Q))$ and $L \in \mathcal{L}_G(S)$, we set $Q_L = O_p(L)$ and assume that $V_L \not\leq Q$. Define $D = \langle V_L^{N_G(Q)} \rangle (L \cap N_G(Q)) \in \mathcal{L}_G(S)$. Furthermore, set*

$$W = \langle (V_L \cap Q)^D \rangle,$$

$$U_L = \langle (W \cap Q_L)^L \rangle$$

and

$$Z = C_{V_L}(Q).$$

Notice that for $L_0 = N_L(S \cap C_L(Y_L))$, we have $L = C_L(Y_L)L_0$ and $C_L(Y_L) \leq D$. Further

$$Y_{L_0} = Y_L = \Omega_1(Z(O_p(L_0)))$$

by [MSS2, Lemma 1.2.4 (i)]. Since $C_L(Y_L)$ normalizes Q,

$$L^\circ = \langle Q^L \rangle = \langle Q^{C_L(Y_L)L_0} \rangle = \langle Q^{L_0} \rangle = L_0^\circ.$$

Therefore, if L is in the unambiguous wreath product case, then so is L_0. Hence we also assume that $L = L_0$ and so

$$Y_L = \Omega_1(Z(Q_L)).$$

2. Preliminaries

In this section we present some lemmas which will be used in the forthcoming sections.

Lemma 2.1. *Suppose that X is a group, $E = O_2(X)$ is elementary abelian of order 16 and $X/E \cong \mathrm{Alt}(6)$ induces the non-trivial irreducible part of the 6-point permutation module on E. Then X splits over E.*

Proof. Choose $R \leq X$ such that $R/E \cong \mathrm{Sym}(4)$ and $Z(R) = 1$. Let $T \in \mathrm{Syl}_3(R)$. As T acts fixed-point freely on $O_2(R)$, $N_R(T) \cong \mathrm{Sym}(3)$ and so there are involutions in X/E. Hence, as X/E has one conjugacy class of involutions, there are involutions in $O_2(R) \setminus E$. Therefore $O_2(R)/Z(O_2(R))$ is elementary abelian of order 16. Now we consider $O_2(R)$. The fixed-point free action of T on $O_2(R)/Z(O_2(R))$ implies there is partition of this group into five T-invariant subgroups of order 4. As T acts fixed-point freely on $O_2(R)$ the preimages of all these fours groups are abelian. As there are involutions in $O_2(R) \setminus E$, there is a T-invariant fours group $F^* \leq O_2(R)/Z(O_2(R))$ with $F^* \neq E/Z(O_2(R))$ and such that the preimage F of F^* is elementary abelian of order 16. Now the action of X on E shows that for any involution $i \in R \setminus E$ all involutions in the coset Ei are conjugate to i by an element of E. Hence all involutions in $O_2(R) \setminus E$ are in F. This shows that F is invariant under $N_R(T)$.

Again there is a partition of F into five groups of order four invariant under T. Let t be an involution in $N_R(T)$. Then $|C_F(t)| = 4$, where $|C_{E \cap F}(t)| = 2$. Hence there is some fours group $F_1 \leq F$, $F_1 \neq E \cap F$ and $C_{F_1}(t) \neq 1$. This shows that F_1 is normalized by t. Then $F_1\langle t \rangle \cong \mathrm{Dih}(8)$ is a complement to E. Using a result of Gaschütz [GLS2, Theorem 9.26] , X splits over E. $\square$

The next lemma is well-known.

Lemma 2.2. *Suppose that $X \cong \mathrm{Sym}(5)$, F_1 and F_2 are fours groups of X with $F_1 \le \mathrm{Alt}(5)$ and V is a non-trivial irreducible $\mathrm{GF}(2)X$-module. Then*

(i) *V is either the non-trivial irreducible part of the permutation module, which is the same as the natural $\mathrm{O}_4^-(2)$-module, or V is the natural $\Gamma\mathrm{L}_2(4)$-module.*

(ii) *F_1 acts quadratically on V if and only if V is the natural $\Gamma\mathrm{L}_2(4)$-module.*

(iii) *F_2 acts quadratically on V if and only if V is the natural $\mathrm{O}_4^-(2)$-module.*

Lemma 2.3. *Suppose that p is a prime, X is a group of characteristic p and U is a normal p-subgroup of X. Let R be a normal subgroup of X with $R \le C_X(U/[U, O_p(X)])$. If $[O_p(X), O^p(R)] \le U$, then $R \le O_p(X)$.*

Proof. It suffices to prove that $O^p(R) = 1$. Suppose that $n \ge 1$ is such that $[U, O^p(R)] \le [U, O_p(X); n]$. Then

$$[O_p(X), O^p(R)] = [O_p(X), O^p(R), O^p(R)] \le [U, O^p(R)] \le [U, O_p(X); n]$$

and so

$$[O_p(X), O^p(R), U] \le [[U, O_p(X); n], O_p(X)] = [U, O_p(X); n+1].$$

We also have

$$[U, O^p(R), O_p(X)] \le [[U, O_p(X); n], O_p(X)] = [U, O_p(X); n+1]$$

and thus the Three Subgroups Lemma implies

$$[U, O_p(X), O^p(R)] \le [U, O_p(X); n+1].$$

This yields

$$[U, O^p(R)] = [U, O^p(R), O^p(R)] \le [U, O_p(X), O^p(R)] \le [U, O_p(X); n+1].$$

Since $O_p(X)$ is nilpotent, we deduce $[U, O^p(R)] = 1$. Hence

$$[O_p(X), O^p(R)] = [O_p(X), O^p(R), O^p(R)] \le [U, O^p(R)] = 1.$$

As X has characteristic p, $O^p(R) = 1$ and so $R \le O_p(X)$ as claimed. □

Lemma 2.4. *Assume that X is a group, Y is a normal subgroup of X and $xC_X(Y) \in Z(X/C_X(Y))$. If $[Y, x] \le Z(Y)$, then $Y/C_Y(x) \cong [Y, x]$ as X-groups.*

Proof. Define

$$\begin{aligned} \theta : Y &\to [Y, x] \\ y &\mapsto [y, x]. \end{aligned}$$

Then θ is independent of the choice of the coset representative in $xC_X(Y)$.

For $y, z \in Y$,

$$(yz)\theta = [yz, x] = [y, x]^z [z, x] = [y, x][z, x] = (y)\theta(z)\theta,$$

and, for $y \in Y$ and $\ell \in X$, as $[x, \ell] \in C_R(Y)$, $x^\ell = xc$ for some $c \in C_X(Y)$, and so

$$(y\theta)^\ell = [y, x]^\ell = [y^\ell, x^\ell] = [y^\ell, xc] = [y^\ell, c][y^\ell, x]^c = [y^\ell, x] = (y^\ell)\theta.$$

Thus θ is an X-invariant homomorphism from Y to $[Y, x]$. As $\ker\theta = C_Y(x)$, we have $Y/C_Y(x) \cong [Y, x]$ as X-groups. □

Lemma 2.5. *Assume that p is a prime, X is a group, Y is an abelian normal p-subgroup of X and R is a normal p-subgroup of X which contains Y. Suppose that $Y = [Y, O^p(X)]$, $[R, O^p(X)] \le C_R(Y)$ and R acts quadratically or trivially on Y. Suppose that no non-central X-chief factor of $Y/C_Y(R)$ is isomorphic to an X-chief factor of $[Y, R]$. Then $Y \le Z(R)$.*

Proof. Assume that $R > C_R(Y)$. Using $[R, O^p(X)] \le C_R(Y)$, we may select $x \in R \backslash C_R(Y)$ such that $xC_X(Y) \in Z(X/C_X(Y))^{\#}$. As Y is abelian, $[Y, x] \le Z(Y)$ and so Lemma 2.4 applies to give $Y/C_Y(x) \cong [Y, x]$ as X-groups. As R acts quadratically on Y,

$$C_Y(x) \ge C_Y(R) \ge [Y, R] \ge [Y, x]$$

and so the hypothesis on non-central X-chief factors now gives $Y/C_Y(x)$ and $[Y, x]$ only have central X-chief factors. In particular, $Y = [Y, O^p(X)] \le C_Y(x)$ and this contradicts the initial choice of $x \in R \setminus C_R(Y)$. Hence $Y \le Z(R)$. □

Lemma 2.6. *Suppose that p is a prime, X is a group, $V \le U$ are normal p-subgroups of X, and Q is a large p-subgroup of X which is not normal in X. Assume that V is a non-trivial irreducible* $\mathrm{GF}(p)X$*-module and U/V is centralized by $O^p(X)$. Then*

(i) *U is elementary abelian; and*
(ii) *if $U \not\le \Omega_1(Z(O_p(X)))$, then $O_p(X)/C_{O_p(X)}(U)$ contains a non-central chief factor isomorphic to V as a* $\mathrm{GF}(p)X$*-module.*

Proof. Set $Z_X = \Omega_1(Z(O_p(X)))$. We have $[U, O^p(X)] \le V \le Z_X$ as V is irreducible. As $O^p(X)$ does not centralize $U/\Phi(U)$ by Burnside's Lemma [GLS2, Proposition 11.1] and V is a non-trivial irreducible X-module, $V \not\le \Phi(U)$ and $\Phi(U)$ is centralized by $O^p(X)$. Therefore $\Phi(U) \cap Z_X$ is centralized by $O^p(X)$ and is normalized by Q. Since Q is large and $O^p(X) \not\le N_X(Q)$, we deduce $\Phi(U) \cap Z_X = 1$. Thus $\Phi(U) = 1$ and so U is elementary abelian. Hence (i) holds.

Set $Y = O_p(X)$ and assume that $U \not\le Z_X$. Select $x \in U \setminus Z_X$ such that $[X, x] \le U \cap Z_X \le Z(Y)$. Then $xC_X(Y) \in Z(X/C_X(Y))$. Thus Lemma 2.4 implies $Y/C_Y(x) \cong [Y, x] \le U \cap Z_X$ and this isomorphism is as X-groups. Since $[Y, x]$ is normalized by Q, $[Y, x] \ne 1$ and Q is large, $O^p(X)$ does not centralize $[Y, x]$. Thus $[Y, x] \ge V$ as $[U, O^p(X)] \le V$. This proves (ii). □

Lemma 2.7. *Assume that p is a prime, X is a group, U is an elementary abelian normal subgroup of X, $U = [U, O^p(X)]$ and $O_p(X)$ acts quadratically and non-trivially on U. Set $R = O_p(X)$, $W = R/C_R(U)$, and $Z = [U, R]$. Then W, U/Z and Z are X/R-modules and W is isomorphic to an X/R-submodule of* $\mathrm{Hom}(U/Z, Z)$*. In particular, if Z is centralized by X, then the set of X-chief factors of W can be identified with a subset of the* $\mathrm{GF}(p)$*-duals of the X-chief factors of U/Z.*

Proof. Since R acts quadratically on U, W is elementary abelian. Furthermore, R centralizes W, U/Z and Z. Hence all of these groups can be regarded as $\mathrm{GF}(p)X/R$-modules. For $w \in R$, define

$$\begin{array}{rcl} \theta : R & \to & \mathrm{Hom}(U/Z, Z) \\ w & \mapsto & \begin{array}{rcl} \theta_w : U/Z & \to & Z \\ uZ & \mapsto & [u, w] \end{array} \end{array}.$$

The calculation in the proof of Lemma 2.4 shows that the commutator $[u, w]$ defines a homomorphism from U to Z and, as w centralizes Z, θ_w is a well-defined homomorphism from U/Z to Z. Thus θ is a well-defined map. Consider $w_1, w_2 \in R$, $uZ \in U/Z$ and $\ell \in X$. Then

$$(uZ)\theta_{w_1 w_2} = [u, w_1 w_2] = [u, w_2]^{w_1}[u, w_1] = [u, w_1][u, w_2] = (uZ)\theta_{w_1}(uZ)\theta_{w_2}$$

which means $\theta_{w_1 w_2} = \theta_{w_1}\theta_{w_2}$ and so θ is a group homomorphism. We show that θ is an X-module homomorphism. So let $\ell \in X$, $uZ \in U/Z$ and $w \in R$. Then $(w^\ell)\theta = \theta_{w^\ell}$ and

$$(uZ)\theta_{w^\ell} = [u, w^\ell] = [u^{\ell^{-1}}, w]^\ell = (u)(\theta_w \cdot \ell).$$

Since $\ker \theta = C_R(U)$, this completes the proof of the main claim.

If Z is centralized by X, then

$$\mathrm{Hom}(U/Z, Z) \cong (U/Z)^* \otimes Z = \bigoplus_{i=1}^{n} (U/Z)^*$$

where n is such that $|Z| = p^n$. This completes the proof of the lemma. □

Lemma 2.8. *Suppose that V is a p-group and X is a group which acts faithfully on V with $O_p(X) = 1$. Assume $A \leq X$ is an elementary abelian p-subgroup of order at least p^2 which has the property $C_V(A) = C_V(a)$ for all $a \in A^\#$. If L is a non-trivial subgroup of X and $L = [L, A]$, then A acts faithfully on L.*

In particular, A centralizes every p'-subgroup which it normalizes, $[A, F(X)] = 1$, $E(X) \neq 1$ and, if L is a component of X which is normalized but not centralized by A, then A acts faithfully on L.

Proof. Suppose that $L = [L, A]$ is a non-trivial subgroup of X. Assume that there is $b \in A^\#$ with $[L, b] = 1$. Then L normalizes $C_V(b)$ and so, as $C_V(b) = C_V(A)$, $L = [L, A]$ centralizes $C_V(b)$. Since $L = [L, A]$, $L = O^p(L)$ and the Thompson $A \times B$-Lemma implies $[L, V] = 1$, a contradiction. Hence A acts faithfully on L.

Let F be a p'-subgroup of X which is normalized by A. Then $F = \langle C_F(a) \mid a \in A^\# \rangle$. If A does not centralizes F, then there exists $a \in A^\#$ such that $1 \neq [C_F(a), A] = [C_F(a), A, A]$. Hence, taking $L = [C_F(a), A]$, we have $L = [L, A]$ and $a \in C_A(L)$, a contradiction. Hence $[F, A] = 1$. Now A centralizes $F(X)$ and therefore $E(X) \neq 1$.

If L is a component of X which is normalized by A, then either $[L, A] = L$ or $[L, A] = 1$. If $[L, A] \neq 1$, then we have A acts faithfully on L. □

Lemma 2.9. *Let X be a group, N a normal subgroup of G and $T \in \mathrm{Syl}_p(X)$. Assume that $X = NT$, $C_T(N) = 1$, $q = p^a$ and*

$$N = N_1 \times N_2 \cdots \times N_s,$$

where $N_i \cong \mathrm{SL}_2(q)$ for $1 \leq i \leq s$. Then the p-rank of G is sa.

Proof. Assume first that $q = 2$. Then T acts faithfully on $O_3(N)$. As the 2-rank of $\mathrm{GL}_s(3)$ is s, we are done. Similarly, if $q = 3$, then T acts faithfully on $O_2(N)/Z(N)$, which is elementary abelian of order 2^{2s} we are done as $\mathrm{GL}_{2s}(2)$ has 3-rank s.

Thus we may assume that $q > 3$. In particular, the subgroups N_i are quasisimple and T permutes the set $\{N_i \mid 1 \le i \le s\}$.

Assume that p is odd. Let A be an elementary abelian subgroup in T of maximal rank and assume that $A \not\le N$. Then by Thompson replacement [GLS2, Theorem 25.2] we may assume that A acts quadratically on $T \cap N$. This shows that A has to normalize each N_i. As non-trivial field automorphisms are not quadratic on $T \cap N_i$, we get that A centralizes $T \cap N$ and so $A \le T \cap N$, the assertion.

Assume that $q = 2^a$ with $a \ge 2$. Let $B = N_N(T \cap N)$. We have that T normalizes B and $T/(T \cap N)$ acts faithfully on $B/(T \cap N)$. Now the Thompson dihedral Lemma [GLS2, Lemma 24.1] says that for any elementary abelian subgroup A of T we have a B-conjugate A^g such that $U = \langle A, A^g\rangle(T \cap N)/(T \cap N)$ is a direct product of r dihedral groups where $2^r = |A/(A \cap N)| \le 2^s$ and $A(T \cap N)/(T \cap N)$ is a Sylow 2-subgroup of U. Set $T_1 = [O_{2'}(U), T \cap N]$. As U is generated by two conjugates of A we see that $|T_1| = |C_{T_1}(A/A \cap N)|^2$. This now shows that $|A| \le |T \cap N|$, the assertion again. This proves the lemma. □

In the next two lemmas we use the notation presented in the introduction though we do not assume that L is unambiguous.

Lemma 2.10. *Suppose that $L \in \mathcal{L}_G(S)$, $L \not\le N_G(Q)$ and $V_L = [Y_L, L^\circ]$. Then*

(i) $C_{Y_L}(L^\circ) = 1$.
(ii) $\Omega_1(Z(S)) \le V_L$.
(iii) *If V_L is an irreducible L-module, $V_L \not\le Q$ and $\Omega_1(Z(Q_L)) < Q_L$, then $V_L \le Q_L' \le \Phi(Q_L)$.*

Proof. As $C_{Y_L}(L^\circ) \le C_G(Q)$ is normalized by L, (i) is a consequence of Q being large.

By [MSS2, Lemma 1.24 (g)], $\Omega_1(Z(S)) \le Y_L$ now Gaschütz Theorem [GLS2, Theorem 9.26] and (i) give (ii).

Assume that N is a non-trivial normal p-subgroup of L. Then $\Omega_1(Z(S)) \cap N \ne 1$. Since V_L is irreducible as a L-module, (ii) gives $V_L \le N$. In particular, as $V_L \not\le Q$, $N \not\le Q$.

Suppose that Q_L is abelian. Then, as $Q = O_p(N_G(Q))$ and $[Q, Q_L, Q_L] \le Q_L' = 1$, Q_L is quadratic on Q, and hence Q_LQ/Q is elementary abelian and so $\Phi(Q_L) \le Q$. By the remark earlier taking $N = \Phi(Q_L)$ we obtain $\Phi(Q_L) = 1$, contrary to $\Omega_1(Z(Q_L)) < Q_L$. Hence Q_L is non-abelian. Thus $Q_L' \ne 1$ and so, as V_L is irreducible, $V_L \le Q_L' \le \Phi(Q_L)$. This proves (iii). □

Lemma 2.11. *Suppose that $L \in \mathcal{L}_G(S)$, $L \not\le N_G(Q)$ and $V_L = [Y_L, L^\circ]$. Assume that $Y_L = \Omega_1(Z(Q_L))$, $m \in L$ and $O^p(L)Q_L \le KQ_L$, where $K = \langle W, W^m\rangle$. Then $O^p(L) \le K$ and the following hold*

(i) $[O^p(L), Q_L] \le [W, Q_L][W^m, Q_L] \le (W \cap Q_L)(W^m \cap Q_L) = U_L$.
(ii) *If $[W, W] \le V_L$, then W acts quadratically on the non-central chief factors of Q_L/V_L.*

Assume, in addition, that V_L is irreducible as a K-module, $[V_L, W, W] \neq 1$, and $[W, W] \leq V_L$. Then

(iii) *$W \cap W^m \cap Q_L \leq Y_L$;*
(iv) *U_L/Y_L is elementary abelian or trivial; and*
(v) *either $Q_L = Y_L$ or $U_L' \geq V_L$.*

Proof. Since W and W^m are normalized by Q_L, $K = \langle W, W^m \rangle$ is normalized by $Q_L K$ and so $O^p(L) \leq K$. Since W, W^m, $[Q_L, W]$ and $[Q_L, W^m]$ are normalized by Q_L, we have

$$[Q_L, O^p(L)] \leq [Q_L, \langle W, W^m \rangle] = [Q_L, W][Q_L, W^m] \leq (W \cap Q_L)(W^m \cap Q_L).$$

In particular, $A = (W \cap Q_L)(W^m \cap Q_L)$ is normalized by $O^p(L)$. Since $(W \cap Q_L)^L = (W \cap Q_L)^{SO^p(L)} = (W \cap Q_L)^{O^p(L)}$, we have $A = U_L$. Thus (i) holds.

By the additional hypothesis,

$$[Q_L, W, W] \leq [W, W] \leq V_L$$

and so W acts quadratically on all the non-central L-chief factors in Q_L/V_L, which is (ii).

Notice that part (ii), V_L irreducible as a K-module and $[V_L, W, W] \neq 1$ together imply that the non-central K-chief factors in Q_L/V_L are not isomorphic to V_L.

Set $I = W \cap W^m \cap Q_L$. Then $I \leq W \cap W^m$ and so

$$[I, W] \leq [W, W] \leq V_L$$

and

$$[I, W^m] \leq [W^m, W^m] \leq V_L^m = V_L.$$

Hence IV_L/V_L is centralized by $\langle W, W^m \rangle = K$. As W acts quadratically on all the non-central chief factors of K in Q_L/V_L by (ii) and by assumption, W does not act quadratically on V_L, Lemma 2.6 implies that $I \leq \Omega_1(Z(Q_L)) = Y_L$. This proves (iii).

Since W is generated by elements of order p, $W/[W, W]$ is elementary abelian and therefore, as $[W, W] \leq V_L$, WV_L/V_L is also elementary abelian. Since $W \cap Q_L$ and $Q_L \cap W^m$ normalize each other parts (i) and (iii) give (iv).

If $V_L \not\leq U_L'$ and $Q_L \neq Y_L$, then, as U_L/Y_L is elementary abelian by (iv), Lemma 2.10 (ii) implies U_L is elementary abelian. Select E with $Q_L \geq E > V_L$ of minimal order such that $E = [E, O^p(L)]$ and E/V_L has a non-central K-chief factor. Then

$$E \leq [Q_L, O^p(L)] \leq [Q_L, W][Q_L, W^m] \leq U_L \leq C_L(E).$$

Furthermore, $V_L[E, Q_L] < E$ and so $[[E, Q_L], O^p(L)] \leq V_L$. Therefore Lemma 2.6 implies that $[E, Q_L] \leq Y_L$ and so Q_L acts quadratically on E. Hence Lemma 2.5 implies that $E \leq Y_L$, a contradiction. Hence U_L' is non-trivial and it follows that $V_L \leq U_L'$. □

3. The reduction

We use the notation presented in the introduction. For the rest of this article we have $L \in \mathcal{L}_G(S)$ with Q not normal in L and L is in the unambiguous wreath product case. This means that $Y_L = V_L$ unless we are in the special case that $\overline{L^\circ} \cong \mathrm{SL}_2(4)$ or $\Gamma\mathrm{SL}_2(4)$, $|Y_L : V_L| = 2$ and

$$V_L \not\leq Q.$$

We start with a general result which just requires $V_L \not\leq Q$.

Lemma 3.1. *The following hold.*

(i) $\langle V_L^D \rangle$ *is not a* p*-group;*
(ii) $[Q, \langle V_L^D \rangle] \leq W$*; and*
(iii) $W \not\leq C_G(V_L)$.

Proof. Let $\tilde{C} = N_G(Q)$ and $K = \langle V_L^{\tilde{C}} \rangle$. As $D = KN_L(Q)$ and $N_L(Q)$ acts on V_L we have $\langle V_L^D \rangle = \langle V_L^K \rangle$ is subnormal in H. If $\langle V_L^D \rangle$ is a p-group, we obtain $V_L \leq O_p(N_G(Q)) = Q$ which is a contradiction. This proves (i).

We have $[Q, V_L] \leq Q \cap V_L \leq W$. As W and Q are normalized by D, (ii) holds.

Assume $W \leq C_G(V_L)$. Then $[W, V_L] = 1$ and so $[W, \langle V_L^D \rangle] = 1$. Hence $X = O^p(\langle V_L^D \rangle)$ centralizes Q by (ii). Since $C_G(Q) \leq Q$, we have $X \leq Q$. Thus $X = 1$ and $\langle V_L^D \rangle$ is a p-group, which contradicts (i). Hence $W \not\leq C_G(V_L)$. □

We adopt the following notation. Let $B \geq C_L(V_L)$ be such that $\overline{B} = \langle \mathcal{K} \rangle$ and let $S_0 = S \cap B$. We write $B = K_1 \dots K_s$ where $K_i \geq C_L(V_L)$, $\overline{K_i} \in \mathcal{K}$, $\overline{K_i} \cong \mathrm{SL}_2(q)$ and, for $1 \leq i \leq s$, put

$$S_i = S \cap K_i$$

$$V_L^i = [V_L, K_i],$$

$$Z_i = C_{V_L^i}(S_i) = C_{V_L^i}(S_0)$$

and

$$Z_0 = Z_1 \dots Z_s = C_{V_L}(S_0).$$

We begin by showing that $\overline{W}$ is not contained in the base group $\overline{B}$.

Lemma 3.2. *Suppose that* $\overline{L}$ *is either properly wreathed, or* $q = p^a$ *(where* p *divides* a*) and some element of* $\overline{L^\circ}$ *induces a non-trivial field automorphism on* $O^p(\overline{L^\circ}) \cong \mathrm{SL}_2(q)$*. Then* W *is not contained in* S_0*. In particular, if* $\overline{L}$ *is properly wreathed with* $q = s = 2$*, then* $\overline{Q}$ *is not cyclic of order* 4.

Proof. Set $F = \bigcap_{g \in D} C_Q(V_L)^g$.

Suppose that W is contained in S_0. As $\overline{Q}$ normalizes $\overline{W}$ and acts transitively on $\mathcal{K}$ when $\overline{L}$ is properly wreathed and, as V_L is the natural $\mathrm{SL}_2(q)$-module when $s = 1$, and field automorphisms are present, the structure of V_L yields

$$[V_L, S_0] = [V_L, W] = C_{V_L}(W) = Z_0.$$

Suppose that $g \in D$. Then using Lemma 3.1(ii) and $(V_L)^g = V_{L^g}$ yields

(3.2.1) $[Z_0, [V_{L^g}, Q]] \leq [Z_0, W] = 1.$

We also remark that as $W \leq Q$, $Z_0 \leq [V_L, Q] \leq W = W^g \leq S_0^g$ and $Z_0 \leq Z(W)$. In particular, as S_0^g normalizes every element of $\mathcal{K}^g$, so does Z_0. Therefore, for $1 \leq i \leq s$, Z_0 also normalizes each K_i^g and so also $[Y_L^g, K_i^g] = (V_L^i)^g$.

If $s = 1$ and we have field automorphisms in $\overline{L^\circ}$, then $[V_L, Q] > Z_0$ and so (3.2.1) provides $Z_0 \leq C_Q([V_{L^g}, Q]) = C_Q(V_{L^g})$. Thus

$$[V_L, W] = Z_0 \leq F$$

in this case.

We will show that the same holds in the properly wreathed case. Because Q acts transitively on $\mathcal{K}^g$,

$$V_{L^g} = V_{L^g}^1[V_{L^g}, Q] = V_{L^g}^2[V_{L^g}, Q].$$

As $[Z_0, [V_{L^g}, Q]] = 1$ by (3.2.1),

$$\begin{aligned}[V_{L^g}, Z_0] &= [V_{L^g}^1[V_{L^g}, Q], Z_0] \cap [V_{L^g}^2[V_{L^g}, Q], Z_0] \\ &= [V_{L^g}^1, Z_0] \cap [V_{L^g}^2, Z_0] \leq V_{L^g}^1 \cap V_{L^g}^2 = 1.\end{aligned}$$

Hence $Z_0 \leq C_Q(V_{L^g})$ and this implies that

$$[V_L, W] = Z_0 \leq F$$

in the properly wreathed case too. Therefore,

$$\begin{aligned}[Q, V_L] &\leq W \\ [W, V_L] &= Z_0 \leq F \cap W \\ [F \cap W, V_L] &= 1.\end{aligned}$$

Hence V_L stabilizes the normal series $Q \geq W \geq W \cap F \geq 1$ in D and so $V_L \leq O_p(D)$. But then $\langle V_L^D \rangle$ is a p-group contrary to Lemma 3.1 (i). We conclude that $W \not\leq S_0$ as claimed.

If $q = s = 2$ and $\overline{Q}$ is cyclic of order four, then, as $\overline{W}$ is generated by involutions, $\overline{W} = \overline{Q} \cap \overline{S_0}$, a contradiction. Thus $\overline{Q}$ is not cyclic of order 4 in this case. □

We now reduce the properly wreathed case to one specific configuration which will be handled in detail in Section 4.

Proposition 3.3. *Assume that $\overline{L}$ is properly wreathed and unambiguous. Then $|\mathcal{K}| = 2$, $q = 2$, and $\overline{W}$ permutes $\mathcal{K}$ transitively by conjugation. Furthermore, $\overline{Q} = \overline{S} \cong \mathrm{Dih}(8)$, $\overline{L^\circ} \cong \mathrm{O}_4^+(2)$ and $Y_L = V_L$ is the natural $\mathrm{O}_4^+(2)$-module.*

Proof. Since Q permutes $\mathcal{K}$ transitively by conjugation and S_0 normalizes Q, we have

(3.3.1)

(i) $\overline{Q \cap S_0}$ contains $[\overline{Q}, \overline{S_0}]$;
(ii) $|\overline{S_0} : \overline{Q \cap S_0}| \leq |\overline{S_0} : [\overline{Q}, \overline{S_0}]| \leq q$; and
(iii) $\overline{[Q, S_0]} C_{\overline{L}}(\overline{K_i})/C_{\overline{L}}(\overline{K_i}) \in \mathrm{Syl}_p(\overline{K_i} C_{\overline{L}}(\overline{K_i})/C_{\overline{L}}(\overline{K_i}))$. ■

As $W = \langle V_{L^g} \cap Q \mid g \in D \rangle$, Lemma 3.2 implies there exists $g \in D$ such that $V_{L^g} \cap Q \not\leq S_0$. We fix this g.

(3.3.2) We have $\overline{V_{L^g} \cap Q} \cap \overline{S_0} \neq 1$.

Suppose that $\overline{V_{L^g} \cap Q} \cap \overline{S_0} = 1$. Then, as $\overline{Q \cap S_0}$ and $\overline{V_{L^g} \cap Q}$ normalize each other, $\overline{V_{L^g} \cap Q}$ centralizes $\overline{Q \cap S_0}$. If $\overline{V_{L^g} \cap Q}$ normalizes some $\overline{K_i} \in \mathcal{K}$, then, as $\overline{Q}$ acts transitively on $\mathcal{K}$ and normalizes $\overline{V_{L^g} \cap Q}$, $\overline{V_{L^g} \cap Q}$ normalizes every member of $\mathcal{K}$. As $\overline{V_{L^g} \cap Q}$ centralizes $\overline{[Q, S_0]}$, (3.3.1) (iii) implies that

$$\overline{V_{L^g} \cap Q} \leq \overline{[Q, S_0]} C_{\overline{L}}(\overline{K_i}).$$

Since Q acts transitively on $\mathcal{K}$, this is true for each $\overline{K_i} \in \mathcal{K}$. Thus

$$\overline{V_{L^g} \cap Q} \leq \bigcap_{i=1}^{s} \overline{[Q, S_0]} C_{\overline{L}}(\overline{K_i}) = \bigcap_{i=1}^{s} \overline{S_i} C_{\overline{L}}(\overline{K_i}) = \overline{S_0},$$

which contradicts the choice of $g \in D$.

Hence $\overline{V_{L^g} \cap Q}$ does not normalize any member of $\mathcal{K}$. As $\overline{B}$ is a direct product we calculate that $C_{\overline{S}_0}(\overline{V_{L^g} \cap Q})$ has index at least q^{p-1} in $\overline{S_0}$. However (3.3.1) (ii) states that $\overline{Q \cap S_0}$ has index at most q in $\overline{S_0}$ and, as this subgroup is centralized by $\overline{V_{L^g} \cap Q}$, we deduce that

$$p = 2.$$

Furthermore, as $\overline{V_{L^g} \cap Q}$ does not normalize any member of $\mathcal{K}$, if $s > 2$, we have $C_{\overline{S}_0}(\overline{V_{L^g} \cap Q})$ has index at least q^2 in $\overline{S_0}$, and so we must have

$$s = 2.$$

Since $\overline{V_{L^g} \cap Q}$ centralizes $[\overline{S_0}, \overline{Q}]$ by (3.3.1)(iii), no element in $\overline{V_{L^g} \cap Q}$ can act as a non-trivial field automorphism on $\overline{K_1}$ and so we infer from $\overline{V_{L^g} \cap Q} \cap \overline{S_0} = 1$, that $|\overline{V_{L^g} \cap Q}| = 2$. In particular, $|C_{V_L}(V_{L^g} \cap Q)| = q^2$ as $V_{L^g} \cap Q$ exchanges V_L^1 and V_L^2.

We know that $|V_{L^g}| = q^4$. As $|[V_{L^g}, Q]| \geq q^3$, we have

$$|V_{L^g} : V_{L^g} \cap Q| \leq q,$$

and we have just determined that

$$|V_{L^g} \cap Q : V_{L^g} \cap Q \cap C_G(V_L)| = |\overline{V_{L^g} \cap Q}| = 2.$$

Hence $V_{L^g} \cap Q \cap C_G(V_L)$ has order at least 2^{3a-1}, where $q = 2^a$.

Assume that $a \neq 1$. Then, as $V_{L^g}^1$ has order q^2,

$$V_{L^g} \cap Q \cap C_G(V_L) \cap V_{L^g}^1 \neq 1.$$

It follows that $V_L \cap Q$ normalizes both K_1^g and K_2^g. As $(V_L \cap Q) C_{L^g}(V_{L^g})/C_{L^g}(V_{L^g})$ is normalized by Q and Q permutes $\{K_1^g, K_2^g\}$ transitively, $(V_L \cap Q) C_{L^g}(V_{L^g})/C_{L^g}(V_{L^g})$ does not centralize $K_i^g/C_{L^g}(V_{L^g})$ for $i = 1, 2$. Thus $|C_{V_{L^g}^i}(V_L \cap Q)| \leq q$ for $i = 1, 2$. But then

$$2^{3a-1} \leq |V_{L^g} \cap Q \cap C_G(V_L)| \leq |C_{V_{L^g}}(V_L \cap Q)| \leq 2^{2a},$$

which contradicts $a \neq 1$. We conclude that $q = s = 2$ and $|\overline{V_{L^g} \cap Q}| = 2$. Furthermore, $\overline{V_{L^g} \cap Q}$ is centralized by $\overline{Q}$ and so $\overline{Q}$ is elementary abelian of order 4. It follows that $\overline{L^\circ} \cong \Omega_4^+(2)$ and V_L is the natural module. Hence L

is ambiguous and we conclude that $\overline{V_{L^g} \cap Q} \cap \overline{S_0} \neq 1$. ■

(3.3.3) We have $|C_{V_L}(V_{L^g} \cap Q)| \leq q^{s/p}$.

We know $\overline{V_{L^g} \cap Q} \not\leq \overline{S_0}$ and $\overline{V_{L^g} \cap Q} \cap \overline{S_0} \neq 1$ by (3.3.2). As $\overline{V_{L^g} \cap Q}$ is normalized by $\overline{Q}$, $\overline{V_{L^g} \cap Q} \cap \overline{S_0} \neq 1$ implies that

$$C_{V_L}(\overline{V_{L^g} \cap Q}) = C_{Z_0}(\overline{V_{L^g} \cap Q}).$$

If some element $d \in V_{L^g} \cap Q$ induces a non-trivial field automorphism on $\overline{K_i}$ for some $\overline{K_i} \in \mathcal{K}$, then $C_{V_L^i}(V_{L^g} \cap Q) \leq C_{Z_i}(d)$ has order $q^{1/p}$ and the result follows by transitivity of $\overline{Q}$ on $\mathcal{K}$. On the other hand, if $d \in V_{L^g} \cap Q$ has an orbit of length p on $\mathcal{K}$, then $C_{\langle (V_L^1)^{\langle d \rangle} \rangle}(V_{L^g} \cap Q) \leq C_{\langle Z_1^{\langle d \rangle} \rangle}(d)$ which has order q. Using the transitivity of Q on $\mathcal{K}$, we deduce $|C_{V_L}(V_{L^g} \cap Q)| \leq q^{s/p}$. This proves the result. ■

As Q acts transitively on the $\{V_i \mid 1 \leq i \leq s\}$, we have $V_L = [V_L, Q]V_1$. By (3.3.2) $\overline{Q} \cap \overline{S_0} \neq 1$ and so $|[V_1, Q]| \geq q$. In particular

$$|V_L : [V_L, Q]| \leq q.$$

Since $V_L \cap Q \cap C_{L^g}(V_{L^g}) \leq C_{V_L}(V_{L^g} \cap Q)$, (3.3.3) and $|V_L| = q^{2s}$ together give

$$|(V_L \cap Q)C_{L^g}(V_{L^g})/C_{L^g}(V_{L^g})| \geq q^{2s-1-s/p}.$$

On the other hand, by Lemma 2.9 the p-rank of $\overline{L}$ is as where $q = p^a$. Hence

$$s \geq 2s - 1 - s/p$$

and so

$$s = p = 2.$$

In particular, Lemma 2.9 implies

(3.3.4) $|(V_L \cap Q)C_{L^g}(V_{L^g})/C_{L^g}(V_{L^g})| = q^2 = 2^{2a}$.

Assume that $q > 2$. Since S^g/S_0^g has 2-rank 2 and $V_L \cap Q$ is elementary abelian, $(V_L \cap Q \cap S_0^g)C_{L^g}(V_{L^g})/C_{L^g}(V_{L^g})$ has rank at least $2a - 2 \neq 1$. Since $V_L \cap Q \cap S_0^g$ is normalized by Q and Q permutes $\{K_1^g, K_2^g\}$ transitively, $V_L \cap Q \cap S_0^g$ contains an element which projects non-trivially on to both $S_1^g C_{L^g}(V_{L^g})/C_{L^g}(V_{L^g})$ and $S_2^g C_{L^g}(V_{L^g})/C_{L^g}(V_{L^g})$. Thus $V_L \geq [V_L \cap Q, [V_{L^g}, Q]] \geq Z_0^g$. But then, using (3.3.3) yields the contradiction

$$q^2 = |Z_0^g| \leq |C_{V_L}(V_{L^g} \cap Q)| \leq q.$$

Thus $q = s = 2$. It follows from Lemma 3.2 that W is transitive on $\mathcal{K}$ and $\overline{Q} \cong \mathrm{Dih}(8)$ or $\overline{Q}$ is elementary abelian of order 4. The second possibility gives $\overline{L^\circ} \cong \Omega_4^+(2)$, which is ambiguous. This proves Proposition 3.3. □

Next we deal with the case $s = 1$.

Proposition 3.4. *Suppose that $O^p(\overline{L^\circ}) \cong \mathrm{SL}_2(q)$ where $q = p^a = r^p$, $V_L = Y_L$ is the natural $O^p(\overline{L^\circ})$-module and that some element of $\overline{L^\circ}$ induces a non-trivial field automorphism on $O^p(\overline{L^\circ})$. Then $p = 2 = r$.*

Proof. We may assume that $r^p > 4$. By Lemma 3.2 we have that $W \not\leq S_0$ and, as W is generated by elements of order p, we have that $|S_0W : S_0| = p$. As Q is normal in S, $1 \neq \overline{Q} \cap \overline{S}_0$, so $Z_0 \leq Q \cap Y_L$. Furthermore, as $\overline{Q}$ contains elements which act as field automorphisms on $O^p(\overline{L^\circ})$,

$$|V_L \cap Q : Z_0| \geq |[V_L, Q] : Z_0| \geq r^{p-1} > p,$$

by assumption. Thus no element in $S \setminus Q_L$ centralizes a subgroup of index p in $V_L \cap Q$.

Set $W_1 = \langle Z_0^D \rangle$. As Z_0 centralizes $W \cap S_0$, every element of Z_0 centralizes a subgroup of index at most p in W. As W_1 is generated by conjugates of Z_0, and these conjugates all contain elements which centralize a subgroup of index at most p in W, W_1 is generated by elements which centralize a subgroup of index at most p in $V_L \cap Q$. As no element in $S \setminus Q_L$ has this property, we conclude that $W_1 \leq Q_L$. Hence $[V_L, W_1] = 1$. In particular $[V_L \cap Q, W_1] = 1$ and so also $[W, Z_0] = [W, W_1] = 1$. This shows $W \leq S_0$ and contradicts Lemma 3.2. $\square$

We collect the results of this section in the following proposition:

Proposition 3.5. *Suppose that $L \in \mathcal{L}_G(S)$, $L \not\leq N_G(Q)$, $V_L \not\leq Q$ and L is in the unambiguous wreath product case. Then one of the following holds:*

(i) $\overline{L^\circ} \cong \mathrm{O}_4^+(2)$, $\overline{Q} = \overline{S} \cong \mathrm{Dih}(8)$ *and* $Y_L = V_L$ *is the natural module.*
(ii) $\overline{L^\circ} \cong \Gamma\mathrm{SL}_2(4)$*,* V_L *is the natural* $\mathrm{SL}_2(4)$*-module and* $|Y_L : V_L| \leq 2$.
(iii) $\overline{L^\circ} \cong \mathrm{SL}_2(4)$*,* V_L *is the natural module and* $|Y_L : V_L| = 2$.

Proof. If $|\mathcal{K}| > 1$, then (i) holds by Proposition 3.3, so we may assume that $|\mathcal{K}| = 1$. As L is unambiguous, either $Y_L \neq V_L$ or $\overline{L^\circ} \not\cong \mathrm{SL}_2(q)$. If $Y_L \neq V_L$, then by definition of the wreath product case, (ii) or (iii) holds. So we may assume $Y_L = V_L$ and $\overline{L^\circ} \not\cong \mathrm{SL}_2(q)$. Now (ii) holds by Proposition 3.4. $\square$

4. $\overline{L^\circ} \cong \mathrm{O}_4^+(2)$

In this section we analyse the configuration from Proposition 3.5(i). We prove

Proposition 4.1. *Suppose that $L \in \mathcal{L}_G(S)$, $L \not\leq N_G(Q)$ and L in the unambiguous wreath product case. If $Y_L \not\leq Q$ and $\overline{L^\circ} \cong \mathrm{O}_4^+(2)$, then $G \cong$* $\mathrm{Sym}(8)$*,* $\mathrm{Sym}(9)$ *or* $\mathrm{Alt}(10)$.

Proof. By Proposition 3.5 we have $\overline{Q} \cong \mathrm{Dih}(8)$. Since Y_L is the natural $\mathrm{O}_4^+(2)$-module for $L/C_L(Y_L)$ and V_L is also the wreath product module for $L/C_L(Y_L)$ with respect to $\{\overline{K_1}, \overline{K_2}\}$, we have the following well known facts.

(4.1.1)

(i) $|[Y_L, Q]| = 2^3$, $|[Y_L, Q, Q]| = 2^2$ and $C_{Y_L}(Q) = [Y_L, Q, Q, Q]$ has order 2.
(ii) $[Y_L, S_0] = C_{Y_L}(S_0)$ has order 2^2;
(iii) $|[Y_L, Q']| = 2^2$;
(iv) $C_L([Y_L, Q]) \leq C_L(Y_L)$.

Our first aim is to prove

(4.1.2) $\overline{W}$ is elementary abelian of order 2^2, $[Y_L, W] = [Y_L, Q] = Y_L \cap Q$ and $[Y_L, W, W] = C_{Y_L}(W) = C_{Y_L}(Q) = Z$.

Applying Lemma 3.1, we consider $x \in D$ such that $Y_{L^x} \cap Q \not\leq C_L(Y_L)$. Then $Y_{L^x} \cap Q$ is normalized by Q and so

$$\overline{Y_{L^x} \cap Q} \text{ contains a 2-central involution in } \overline{Q}.$$

In particular, (4.1.1)(iii) gives

$$|[Y_L, Y_{L^x} \cap Q]| \geq 2^2.$$

As Y_L is elementary abelian, $\overline{Y_{L^x} \cap Q}$ is elementary abelian.

Suppose that $[Y_L, Y_{L^x} \cap Q, Y_{L^x} \cap Q] = 1$. Then

$$[Y_L, Y_{L^x} \cap Q] \leq C_{S^x}([Y_{L^x}, Q]) = Q_{L^x}$$

by (4.1.1) (iv). Hence $[Y_L, Y_{L^x} \cap Q, Y_{L^x}] = 1$. Then as $|[Y_L, Y_{L^x} \cap Q]| = 2^2$ and $|Y_L \cap Q| = 2^3$, we conclude that $(Y_L \cap Q)C_{L^x}(Y_{L^x})/C_{L^x}(Y_{L^x})$ has order 2. Thus $[Y_{L^x}, Y_L \cap Q, Y_L \cap Q] = 1$. Now the argument just presented implies that $|\overline{Y_{L^x} \cap Q}| = 2$ and so, as Q normalizes $Y_{L^x} \cap Q$, $\overline{Y_{L^x} \cap Q} = Z(\overline{Q})$. In particular, as $[Y_L, S_0, S_0] = 1$, we have proved that

$$\text{if } \overline{Y_{L^x} \cap Q} \leq \overline{S_0}, \text{ then } \overline{Y_{L^x} \cap Q} = Z(\overline{Q}).$$

For a moment let $\overline{Q_1}$ be the fours subgroups of $\overline{Q}$ not equal to $\overline{S_0}$. Then as $\Phi(Y_{L^x} \cap Q) = 1$ the displayed line implies that $\overline{W} \leq \overline{Q_1}$ and Lemma 3.2 and $\overline{Q}' \leq \overline{Y_{L^x} \cap Q}$ imply $\overline{W} = \overline{Q_1}$. The remaining statements in (4.1.2) now follow from the action of L on Y_L. ■

We have that $Z(Q)$ centralizes $[Y_L, Q]$ and so $Z(Q) \leq S \cap C_L(Y_L) = Q_L$. Hence using (4.1.2) we obtain

$$\begin{aligned}
[W, W] &= [\langle [Y_L, Q]^D \rangle, W] = \langle [[Y_L, Q], W]^D \rangle \\
&= \langle Z^D \rangle = Z[Z, \langle V_L^{N_G(Q)} \rangle] \leq Z[Z(Q), \langle V_L^{N_G(Q)} \rangle] \\
&= Z\langle [Z(Q), V_L]^{N_G(Q)} \rangle = Z.
\end{aligned}$$

(4.1.3) We have $Q_L = Y_L$.

Suppose that $Q_L > Y_L$. Let $m \in L$ be such that $\overline{K} \cong \mathrm{SL}_2(2) \times \mathrm{SL}_2(2)$, where $K = \langle W, W^m \rangle$. Recall that by the choice of L in the Notation at the end of the introduction, we have $Y_L = \Omega_1(Z(Q_L))$ and by Proposition 3.5 and (4.1.2), K acts irreducibly on $Y_L = V_L$. Hence we may apply Lemma 2.11 (iii), (iv) and (v) which combined yield U_L/Y_L is elementary abelian and

$$U_L' = Y_L.$$

Since $[Q_L, W, W] \leq [W, W] = Z \leq Y_L$, we have W acts quadratically on every chief factor of L in Q_L/Y_L. In particular, no non-central L-chief factor of Q_L/Y_L is isomorphic to Y_L.

Let E be the preimage of $C_{U_L/Y_L}(K)$. Then E is normal in L and application of Lemma 2.6 implies that $E = Y_L$. Let $X \in \mathrm{Syl}_3(K)$. By Lemma 2.11(i), $[K, C_L(Y_L)] \leq U_L$, so XU_L is normal in L. As L is solvable, $C_L(Y_L) = C_X(Y_L)Q_L$ and either $C_X(Y_L) = 1$ or $X \cong 3_+^{1+2}$. The latter case is impossible as W is quadratic on U_L/Y_L. Hence $U_L = [U_L, O^2(L)]$ and U_L/Y_L contains no central L-chief factors. We know that every L-chief factor in

U_L/Y_L is a wreath product module for $\mathrm{SL}_2(2) \wr 2$ with $\overline{W}$ acting quadratically. In particular, for every non-central chief factor F of L in U_L/Y_L we have $[F, \overline{W}] = [F, Z(\overline{Q})]$. Set $W_1 = [W, D]$. Then

$$\overline{W_1} \geq [\overline{W}, \overline{Q}] = Z(\overline{Q}).$$

Hence $[F, W] = [F, W_1]$ for every non-central chief factor F of L in U_L/Y_L. Set $\widetilde{L} = L/Y_L$ and let $z \in Q$ with $Z(\overline{Q}) = \langle \overline{z} \rangle$. As $C_F(Z(\overline{Q})) = [F, Z(\overline{Q})]$ for each F, we have $C_{\widetilde{U_L}}(z) = [\widetilde{U_L}, z]$; then as W acts quadratically on $\widetilde{U_L}$, we have $[W, \widetilde{U_L}] = C_{\widetilde{U_L}}(W)$. Thus $[U_L, W]Y_L = [U_L, W_1]Y_L$. In particular,

$$[W/W_1, U_L] = [U_L, W]W_1/W_1 = (Y_L \cap Q)[U_L, W_1]W_1/W_1 = (Y_L \cap Q)W_1/W_1$$

and so U_L acts quadratically on W/W_1. Therefore $U_L C_D(W/W_1)/C_D(W/W_1)$ is elementary abelian. Hence

$$Y_L = U_L' \leq C_D(W/W_1).$$

Set $R = \langle Y_L^D \rangle$. Then, as $Y_L \not\leq O_2(D)$ by Lemma 3.1 (i), $Y_L \cap O_2(D) = Y_L \cap Q \leq W$ and so R centralizes $O_2(D)/W$ and W/W_1. Lemma 2.3 yields $Y_L \leq O_2(D)$ and this contradicts Lemma 3.1 (i). We have shown $Q_L = Y_L$. ■

(4.1.4) $|S| = 2^7$ and $N_G(Q)/Q \cong \mathrm{Sym}(3)$.

Since $Q_L = Y_L = V_L$ and $\overline{Q} \cong \mathrm{Dih}(8)$, $|S| = 2^7$ and $|Q| = 2^6$. Then $N_G(Q) = SX$, where X is a Hall $2'$-subgroup of $N_G(Q)$ and QX is normal in $N_G(Q)$. Furthermore W is extraspecial of order 2^5. As $W/Z = J(Q/Z)$, we have W is normal in $N_G(Q)$. Hence X acts faithfully on W and embeds in $\mathrm{O}_4^+(2)$. As $[\overline{W}, \overline{Q}] = Z(\overline{Q})$, S/W is faithful on W/Z, so $N_G(Q)/W$ embeds into $\mathrm{O}_4^+(2)$. Because $\mathrm{O}_4^+(2) \cong \mathrm{Sym}(3) \wr 2$, and $O_2(N_G(Q)/W) \neq 1$, we get the claim. ■

Taking $T \in \mathrm{Syl}_3(L)$, we have $N_L(T)$ is a complement to Q_L and so $L = Q_L N_L(T)$ is a split extension of Q_L by $\mathrm{O}_4^+(2)$. In particular, the isomorphism type of S is uniquely determined. As $\mathrm{Sym}(8)$ has a subgroup isomorphic to L and $\mathrm{Sym}(8)$ has odd index in $\mathrm{Alt}(10)$, we have S is isomorphic to a Sylow 2-subgroup of $\mathrm{Alt}(10)$.

Let $z \in C_{Y_L}(Q)^\#$, then as Y_L is a $+$-type space for L, there is a fours group A of Y_L which has all non-trivial elements L-conjugate to z. Since $C_G(z)$ has characteristic 2, $C_{O(G)}(z) = 1$ and so by coprime action

$$O(G) = \langle C_{O(G)}(a) \mid a \in A^\# \rangle = 1.$$

Assume that G has no subgroup of index two. Then S is isomorphic to a Sylow 2-subgroup of $\mathrm{Alt}(10)$. Therefore [Mas, Theorem 3.15] implies that $F^*(G) \cong \mathrm{Alt}(10)$, $\mathrm{Alt}(11)$, $\mathrm{PSL}_4(r)$, $r \equiv 3 \pmod 4$, or $\mathrm{PSU}_4(r)$, $r \equiv 1 \pmod 4$. Notice that $Z(Q) = C_{Y_L}(Q) = \langle z \rangle$ and so $C_G(z) = N_G(Q)$ has characteristic 2. In $\mathrm{Alt}(11)$, z corresponds to $(12)(34)(56)(78)$ and so $C_G(z) \leq (\mathrm{Alt}(8) \times Z_3) : 2$, which implies that $C_G(z)$ is not of characteristic 2. In the linear and unitary groups $C_G(z)$ has a normal subgroup isomorphic to $\mathrm{SL}_2(r) \circ \mathrm{SL}_2(r)$, and this contradicts (4.1.4). Hence $G \cong \mathrm{Alt}(10)$.

Assume now that G has a subgroup of index two. As $V_L \le G'$ we also have $W \le G'$. Therefore $(G' \cap L)/Y_L \cong \Omega_4^+(2)$ and so G' has Sylow 2-subgroups isomorphic to those of $\mathrm{Alt}(8)$. Applying [GH, Corollary A*] we have $F^*(G) \cong \mathrm{Alt}(8)$, $\mathrm{Alt}(9)$ or $\mathrm{PSp}_4(3)$. Again in $G' \cong \mathrm{PSp}_4(3)$, we have that G' contains a subgroup of shape $\mathrm{SL}_2(3)\circ\mathrm{SL}_2(3)$. This contradicts (4.1.4) and proves the proposition. □

5. $\overline{L^\circ} \cong \Gamma\mathrm{SL}_2(4)$

In this section we attend to the case from Proposition 3.5(ii). Hence we have $p = 2$, $\overline{L^\circ} \cong \Gamma\mathrm{SL}_2(4)$, V_L is the natural $\mathrm{SL}_2(4)$-module and either $Y_L = V_L$ or $|Y_L/V_L| = 2$. Notice that as $L \not\le N_G(Q)$ and L centralizes Y_L/V_L, if $Y_L > V_L$, Y_L does not split over V_L and $C_{Y_L}(Q) = C_{V_L}(Q)$ has order 2. Furthermore, $C_S([Y_L, Q]) = Q_L$.

Our aim is to prove

Proposition 5.1. *Suppose $L \in \mathcal{L}_G(S)$ and $L \not\le N_G(Q)$ with $\overline{L}$ in the unambiguous wreath product case. If $Y_L \not\le Q$ and $\overline{L^\circ} \cong \Gamma\mathrm{SL}_2(4)$, then $G \cong \mathrm{Mat}(22)$ or $\mathrm{Aut}(\mathrm{Mat}(22))$.*

Notice that as $Q_L \in \mathrm{Syl}_2(C_L(Y_L))$, $C_L(Y_L)/Q_L$ is centralized by L°, and so $C_{L^\circ}(Y_L) = Q_L \cap L^\circ$ as the Schur multiplier of $\mathrm{SL}_2(4)$ has order 2. We also have $|\overline{Q}| \ge 4$ and $|Z(Q) \cap V_L| = 2$.

Lemma 5.2. *For $N = N_G(Q_L)$ we have $(Z(Q) \cap V_L)^N \cap Y_L \subseteq V_L$. In particular, N normalizes V_L.*

Proof. If $V_L = Y_L$, there is nothing to prove. Assume that $|Y_L : V_L| = 2$. Choose $g \in N$, put $U = (Z(Q) \cap V_L)^g$ and assume that $U \not\le V_L$. Recall that $Y_L = \Omega_1(Z(Q_L))$ and so $U \le Y_L$ and Y_L is normalized by N. Then $C_L(U)C_N(Y_L)/C_N(Y_L) \cong 5 : 4$ or $2 \times \mathrm{Sym}(3)$. As $C_N(U^{g^{-1}})$ normalizes $Q \cap Y_L$, $C_N(U^{g^{-1}})$ is not irreducible on $Y_L/U^{g^{-1}}$. This excludes the possibility $C_L(U)C_N(Y_L)/C_N(Y_L) \cong 5 : 4$ which is irreducible on Y_L/U. Hence we see that $Z(Q) \cap V_L$ has exactly $15 + 10 = 25$ conjugates under N, but 25 does not divide the order of $\mathrm{SL}_5(2) = \mathrm{Aut}(Y_L)$. This contradiction proves the lemma. □

Lemma 5.3. *We have $Q_L = Y_L$ and either*

(i) *$|S| = 2^7$, $L/Q_L \cong \Gamma\mathrm{SL}_2(4)$, $N_G(Q)/Q \cong \mathrm{SL}_2(2)$, there exists a subgroup $E \le S$ of order 2^4 which is normalized by $N_G(Q)$ such that $N_G(E)/E \cong \mathrm{Alt}(6)$ and $N_L(E)$ has index 5 in L. Furthermore all the involutions in $\langle N_G(E), L\rangle$ are conjugate.*

(ii) *G has a subgroup of index 2 which satisfies the conditions in* (i).

Proof. We have $\overline{S} \cong \mathrm{Dih}(8)$ and $\overline{Q} \not\le \overline{S}_0$ as $\overline{L^\circ} \cong \Gamma\mathrm{SL}_2(4)$. Lemma 3.2 implies that $\overline{W} \not\le \overline{S_0}$. By assumption, we either have $Y_L = V_L$ or $|Y_L : V_L| = 2$. In particular, $2^4 \le |Y_L| \le 2^5$. Since $\overline{Q}$ is normal in $\overline{S}$ and contains $\overline{W}$ we know

(5.3.1) Either $\overline{Q}$ is elementary abelian of order 4 or $\overline{Q} = \overline{S}$

As V_L is a natural $\mathrm{SL}_2(4)$-module and $L \not\le N_G(Q)$, we have $C_{Y_L}(Q) = C_{Y_L}(S)$ has order 2 and $[Y_L, Q] = [V_L, Q]$ has order 8. Furthermore, as W

is normal in S and is not contained in S_0, we have $[Y_L, Q, W] = Z$ where $Z = C_{V_L}(S)$ has order 2. Thus, arguing exactly as before (4.1.3) and in the proof of (4.1.2) we obtain

(5.3.2) $|\overline{W}| = 4$, $[W, W] = Z$ and $[Q_L, W, W] \leq Y_L$.

(5.3.3) Assume that $Q_L > Y_L$. Then $[Q_L, O^2(L)] \not\leq Y_L$.

Suppose that $[Q_L, O^2(L)] \leq Y_L$. Then $V_L \not\leq \Phi(Q_L)$ by Burnside's Lemma [GLS2, Proposition 11.1], which contradicts Lemma 2.10(iii). This proves the claim ■

(5.3.4) If $V_L < Y_L$, then $\overline{Q} = \overline{S}$.

If $\overline{Q}$ has order 4, then $\overline{Q} = \overline{W}$ by (5.3.2), so $\overline{Q}$ normalizes a Sylow 3-subgroup $\overline{T}$ of $\overline{L}$ and so Q normalizes $C_{Y_L}(T)$ which has order 2 and complements V_L. Hence $C_{Y_L}(T) \leq Z(Q)$, so $T \leq N_G(Q)$ and therefore $L = \langle T, S\rangle \leq N_G(Q)$, a contradiction. Thus $\overline{Q} = \overline{S}$ has order 8. ■

(5.3.5) We have $Q_L = Y_L$.

Suppose false. By (5.3.2) W acts quadratically on Q_L/Y_L and $|\overline{W}| = 4$. Also $\overline{W} \not\leq \overline{S_0}$, so Lemma 2.2 implies that the non-central L-chief factors in Q_L/Y_L are orthogonal modules for $\overline{L} \cong \mathrm{O}_4^-(2)$. In particular, as L-modules, the non-central L-chief factors of Q_L/Y_L are not isomorphic to V_L.

Choose $E \leq Q_L$ normal in L and minimal so that E/Y_L contains a non-central L-chief factor and let F be the preimage of $C_{E/Y_L}(O^2(L))$. Then $[F, O^2(L)] \leq Y_L$ and Lemma 2.6 applies to yield $F \leq Y_L$. In particular, $[E, E] \leq Y_L$.

We claim $E' \leq V_L$. This is obviously the case if $V_L = Y_L$. So suppose that $|Y_L : V_L| = 2$. If $E' \not\leq V_L$. Then the minimal choice of E and $E'V_L = Y_L$ implies that E/V_L is extraspecial of order 2^5. Notice that $[E, W] \leq W$ and W/Z is elementary abelian as $[W, W] = Z$ by (5.3.2). Hence, as $[E, W]Y_L/V_L$ has order 2^3, we infer that E/V_L has $+$-type contrary to $\overline{L} \cong \Gamma\mathrm{SL}_2(4)$. Hence E/V_L is elementary abelian. If $[Q_L, E] = V_L$, then E/V_L has order 2^4 by Lemma 2.2 and so $Q_L/C_{Q_L}(E)$ embeds into

$$\mathrm{Hom}_L(E/V_L, V_L) \cong (E/V_L)^* \otimes V_L \cong (E/V_L) \otimes V_L$$

by Lemma 2.7. Since $Q_L/C_{Q_L}(E)$ involves only trivial and orthogonal modules this contradicts [Pr, Lemma 2.2].

Thus $[E, Q_L] = Y_L > V_L$.

By (5.3.4)

$$\overline{Q} = \overline{S} \text{ has order } 8.$$

In summary we now know $|\overline{W}| = 4$ and $\overline{[W, Q]} = \overline{[W, S]} = Z(\overline{S})$.

We calculate using Z is normal in D by (5.3.2) that

$$[W, Q, Q] = \langle [V_L, Q, Q, Q]^D\rangle = \langle Z^D\rangle = Z.$$

Therefore

$$[E, [W, Q], Q] \leq E \cap [[W, Q], Q] \leq Z \leq Y_L.$$

As $|[Z(\overline{S}), E/Y_L]| = 4$ and $\overline{Q} = \overline{S}$, this implies that $|C_{E/Y_L}(\overline{S})| = 4$. As E/Y_L is the orthogonal $\mathrm{O}_4^-(2)$-module for L, this is impossible. We have proved the claim. ■

(5.3.6) Suppose that $Y_L = V_L$. Then L is a maximal 2-local subgroup of G, $N_G(Q)/Q \cong \mathrm{SL}_2(2)$, there exists a subgroup $E \le S$ of order 2^4 which is normalized by $N_G(Q)$ such that $N_G(E)/E \cong \mathrm{Alt}(6)$ and $N_L(E)$ has index 5 in L.

By (5.3.5) we have $|S| = 2^7$, and $|\overline{W}| = 2^2$. Also $|[W, Y_L]| = 8$ and $Y_L \not\le Q$, so $Q \cap Y_L = [W, Y_L] = W \cap Y_L$, Therefore $|W| = 2^5$. Set $C = C_Q(W)$. Then C centralizes $[Y_L, Q]$ which has order 2^3 and so $C \le C_L([Y_L, Q]) = Y_L$. Thus $C \le C_{Y_L}(W)$ which has order 2. Then, by (5.3.2), $W' = Z = C$ and, as W is generated by involutions, we have W is extraspecial. Since $[Y_L, Q] \le W$, W has $+$-type.

Observe $W/Z = J(Q/Z)$, so W is normal in $N_G(Q)$ and $N_G(Q)/Z$ embeds into $\mathrm{Aut}(W) \cong 2^4{:}\mathrm{O}_4^+(2)$.

Assume that Y_LQ/Q normalizes a subgroup T of $O_3(N_G(Q))/Q$ which has fixed points on W/Z. Then $W = [W,T]C_W(T)$ and $[W,T] \cong C_W(T) \cong \mathrm{Q}_8$ and these subgroups are normalized by Y_L. But then

$$[W, Y_L] = [C_W(T), Y_L][W, T, Y_L].$$

Since $[W, Y_L]$ is elementary abelian and $\Omega_1(P) = Z(P)$ for $P \cong \mathrm{Q}_8$, we conclude that

$$[C_W(T), Y_L] = [W, T, Y_L] = Z$$

and then $[W, Y_L]$ has order 2 which is nonsense as Y_L is the natural module. Therefore Y_L normalizes no such subgroup.

Let $F = O_{2,3}(N_G(Q))$. Assume that $|F/Q| = 9$. Then the previous argument implies that $C_{F/Q}(Y_L) \ne 1$. Let T_1 be the preimage of this subgroup. Then $[Y_L, Q]$ is normalized by T_1. Hence $Y_L = C_{Y_LQ}([Y_L, Q])$ is normalized by T_1. Using the fact that Q is weakly closed in any 2-group which contains it, for $w \in Y_L^{\#}$, we let Q_w be the unique conjugate of Q in $O_2(C_G(w))$. Then T_1 permutes the elements of Y_L and so T_1 normalizes $L^\circ = \langle Q_w \mid w \in Y_L^{\#}\rangle$. Since $L = L^\circ Y_L$, we have that T_1 normalizes L. On the other hand, WY_L is normalized by T_1 and, as T_1 acts fixed-point freely on W/Z, T_1 acts transitively on $WY_L/Y_L \cong W/[Y_L, Q] \cong 2^2$ and this is impossible as $W \cap O^2(L)$ is a maximal subgroup of W and is normalized by T_1.

Hence $|F/Q| = 3$, $N_G(Q) = FS$ and $N_G(Q)/Q \cong \mathrm{SL}_2(2)$. In particular, $|Q| = 2^6$, $S = Y_LQ$, and $FY_L/W \cong 2 \times \mathrm{SL}_2(2)$. It follows that

$$[W, Q] \text{ is elementary abelian of order 8.}$$

et $E = C_S([W, Q])$. As W is normal in $N_G(Q)$, so is E. As $|S| = 2^7$ and $|\mathrm{GL}_3(2)|_2 = 2^3$, we have $|E| \ge 2^4$. Since F acts fixed-point freely on W/Z (being normalized by Y_L), we have $E \le Q$ and then E is normal in $N_G(Q)$. Since $E \cap W = [W, Q]$, we find $|E| = 2^4$. Let $S \le L_1 \le L$ be such that $L_1/Q_L \cong \mathrm{Sym}(4)$ has index 5 in L. Notice that $O_2(L_1) = S_0$. Then $E \le C_L([Y_L, Q, Q]) = Y_LS_0$. Also $Y_L \le S_0$, so $S_0 = Y_LS_0$. Therefore $E \le S_0$. Now EY_L/Y_L acts as a Sylow 2-subgroup of $\mathrm{SL}_2(4)$ on the natural module. In particular for any involution $e \in E \setminus Y_L$ we have that $C_{Y_L}(e) =$

$E \cap Y_L$. This implies that all involutions in EY_L are contained in $Y_L \cup E$ and therefore E and Y_L are the only elementary abelian subgroups of S_0 of order 2^4. In particular, L_1 normalizes E. Now $N_G(E) \geq \langle L_1, N_G(Q)\rangle \in \mathcal{L}_G(S)$. Notice that L_1 has orbits of lengths 3, and 12 on E and that $N_G(Q)$ does not preserve these orbits. Hence $N_G(E)$ acts transitively on $E^{\#}$. As $N_G(Q) = C_G(Z)$, we now have that $|N_G(E)| = 15|N_G(Q)| = 2^7 \cdot 3^2 \cdot 5$. We have that $X = N_G(E)/E$ is isomorphic to a subgroup of $\mathrm{GL}_4(2) \cong \mathrm{Alt}(8)$ of order $2^3 \cdot 3^2 \cdot 5$. We consider the action of X on a set of size 8. As Alt(8) has no subgroups of order 45, X is not transitive. Hence X is isomorphic to a subgroup of Alt(7), Sym(6) or $X \cong (\mathrm{Alt}(5) \times 3){:}2$. Suppose that $X \cong (\mathrm{Alt}(5) \times 3){:}2$. As $N_G(Q)/Q \cong \mathrm{Sym}(4)$, we see that $EQ/E \leq \mathrm{Alt}(5)$. Since E is the natural $\mathrm{SL}_2(4)$-module, we get that $|Z(Q)| = 4$. But, by (5.3.2), $|Z(Q)| = 2$. Hence we have one of the first two possibilities and then obviously $X = N_G(E)/E \cong \mathrm{Alt}(6)$.

We just have to show that L is a maximal 2-local subgroup. Let M be a 2-local subgroup with $L \leq M$. As $Q \leq M$, we have that M is of characteristic 2. Then $Y_L = Y_M$ and $C_G(Y_L) = Y_L$. As $|N_G(Q) : S| = 3$ and Y_L is not normal in $N_G(Q)$, we have $N_M(Q) = S = N_L(Q)$. As L acts transitively on $Y_L^{\#}$, we conclude $M = N_M(Q)L = N_L(Q)L = L$. ■

(5.3.7) If $Y_L = V_L$, then G has just one conjugacy class of involutions.

By (5.3.6) $N_G(E)/E \cong \mathrm{Alt}(6)$. As $Y_L \not\leq E$, there is an involution $y \in Y_L \setminus E$. Now y inverts an element of order 5 in $N_G(E)$ and so $|[E, y]| = |C_E(y)| = 4$. This shows that all involutions in Ey are conjugate. As all involutions in S/E are conjugate in Alt(6) and all the involutions in Y_L are L-conjugate, this proves the claim. ■

We have now proved that (i) holds when $Y_L = V_L$.

(5.3.8) Suppose that $Y_L > V_L$. Then G has a subgroup of index 2.

We have that $|S| = 2^8$. By (5.3.4) and (5.3.5), $S = QY_L$. We are going to show that $J(S) = Y_L$. For this let $A \leq S$ be elementary abelian of maximal order and assume that $A \neq Y_L$. Then $|AY_L/Y_L| \leq 4$. As there are no transvections on V_L, we get $|AY_L/Y_L| = 4$ and we may assume that A acts quadratically on Y_L by [GLS2, Theorem 25.2]. As $W \not\leq S_0$ by Lemma 3.2 and $|\overline{W}| = 4$ by (5.3.2), W does not act quadratically on Y_L, $AY_L/Y_L \leq S_0/Y_L$ and $S_0 = AY_L$. Now $A \cap Y_L$ has order 8 and so $|C_{Y_L}(S_0)| = 8$. But $(L^\circ)'$ is generated by two conjugates of S_0, which gives $C_{Y_L}(L^\circ) \neq 1$ a contradiction to Lemma 2.10(i). Thus $Y_L = J(S)$ is the Thompson subgroup of S. In particular, $N_G(Y_L)$ controls G-fusion of elements in Y_L. As $S \in \mathrm{Syl}_2(G)$ and $C_S(Y_L) = Q_L$, $Q_L \in \mathrm{Syl}_2(C_G(Y_L))$ and we have $N_G(Y_L) = C_G(Y_L)N_{N_G(Y_L)}(Q_L)$. By Lemma 5.2

$$V_L \text{ is normal in } N_G(Y_L).$$

Suppose that $O^2(L) \geq Y_L$. Then $O^2(L)/V_L \cong \mathrm{SL}_2(5)$ has quaternion Sylow 2-subgroups and $|L : O^2(L)| = 2$. On the other hand, there exists $g \in N_G(Q) \setminus N_G(Y_L)$ with $WY_L \geq (Y_L^g \cap Q)Y_L \neq Y_L$ and $(Y_L^g \cap Q)V_L/V_L$ is elementary abelian, which is a contradiction. Therefore $O^2(L)/V_L \cong \mathrm{SL}_2(4)$

and, as W does not act quadratically on Y_L, we see that $|W : W \cap O^2(L)| = 2$ and thus $O^2(L)W/V_L \cong \Gamma\mathrm{SL}_2(4)$. Hence L has a subgroup $L_0 = O^2(L)W$ of index 2 with $Y_L \cap L_0 = V_L$.

Let $T \in \mathrm{Syl}_2(L_0)$ and $w \in Y_L \setminus T$. Suppose that for some $x \in G$, $w^x \in T$ and $|C_S(w^x)| \geq |C_S(w)|$. As L° has orbits of length 6 and 10 on $Y_L \setminus V_L$, we may assume $|C_S(w^x)| \geq |S|/2$. But then as V_L is the natural module, it does not admit transvections and so $w^x \in V_L$. As $N_G(Y_L) = N_G(V_L)$ and $N_G(Y_L)$ controls fusion in Y_L, this is not possible. Hence the supposed condition cannot hold. Application of [GLS2, Proposition 15.15], shows that G has a subgroup of index 2. This proves (5.3.8). ■

Let G_0 be a subgroup of G of index 2, and set $Q_0 = Q \cap G_0$. We have $V_L \leq L^\circ \leq G_0$. Hence $W = \langle [V_L, Q]^D \rangle \leq G_0$. In particular, $W \leq Q_0$ and so $Z(Q_0) = Z$ and Q_0 is large in G_0. Set $L_0 = O^2(L)Q_0 = O^2(L)W$. Then $L_0^\circ/V_L \cong \Gamma\mathrm{SL}_2(4)$ and $Y_{L_0} = V_{L_0} = V_L \not\leq Q_0$. Thus (G_0, L_0) satisfies the hypotheses of (i). This proves (ii) holds if $V_L \neq Y_L$. □

Proof of Proposition 5.1*:* By Lemma 5.3 we just have to examine the structure in Lemma 5.3(i), so we may assume that Lemma 5.3(i) holds.

By Lemma 2.1

$$N_G(E) \text{ splits over } E.$$

As $N_G(Q) \leq N_G(E)$, for a 2-central involution z we have that $C_G(z)$ is a split extension of E by $\mathrm{Sym}(4)$. As $O(C_G(z)) = 1$ coprime action yields $O(G) = \langle C_{O(G)}(e) \mid e \in E^\# \rangle = 1$. In particular $F(G) = 1$ and $E(G) \neq 1$. Suppose that J^* is a non-trivial subnormal subgroup of G normalized by $\langle L, N_G(E) \rangle$. Then $S \cap J^* \neq 1$. Since $1 \neq J^* \cap N_G(E)$ is normal in $N_G(E)$ and $1 \neq J^* \cap L$ is normal in L, it follows that $J^* \cap N_G(E) \geq J^* \cap S \geq EY_L$. Hence $J^* \geq \langle Y_L^{N_G(E)} \rangle = N_G(E) \geq S$ and $J^* \geq \langle S^L \rangle = L$. Therefore there is a unique non-trivial subnormal subgroup of G of minimal order normalized by $\langle L, N_G(E) \rangle$. It follows that $\langle L, N_G(E) \rangle$ is contained in a component J of G. Since $O(G) = 1$ and $S \leq J$, $J = E(G)$. As J has just one conjugacy class of involutions by Lemma 5.3(i) and, for $z \in E^\#$, $C_G(z) \leq N_G(E)$, it follows that $G = J$ is simple. Using G has just one conjugacy class of involutions and applying [J, Theorem] yields $G \cong \mathrm{Mat}(22)$. This proves the proposition when Lemma 5.3(i) holds. If Lemma 5.3(ii) holds, then $G \cong \mathrm{Aut}(\mathrm{Mat}(22))$. □

6. $\overline{L^\circ} \cong \mathrm{SL}_2(4)$

In this section we investigate the configuration in Proposition 3.5(iii). Thus $\overline{L^\circ} \cong \mathrm{SL}_2(4)$, $|Y_L : V_L| = 2$ and V_L is the natural $\mathrm{SL}_2(4)$-module.

As $Q \leq L^\circ$, $C_{V_L}(S_0) = C_{V_L}(Q) \leq Z(Q)$, so Q is normal in $N_{L^\circ}(C_{V_L}(S_0))$ and hence $\overline{Q} = \overline{S_0}$ is a Sylow 2-subgroup of $\overline{L^\circ}$. In particular $Z(Q) \cap Y_L = Z(Q) \cap V_L$ is of order 4.

Lemma 6.1. *The subgroup Q is elementary abelian. In particular, $Q \cap Y_L = Q \cap V_L = C_{Y_L}(Q) = Z$, $|Y_L Q/Q| = 2^3$ and $|V_L Q/Q| = 2^2$.*

Proof. We know that $[Q, V_L] = C_{V_L}(Q) = Q \cap V_L$ and, as $\overline{Q}$ is elementary abelian, $\Phi(Q) \leq Q_L$. If $\Phi(Q) \neq 1$, then, since $Z(S) \cap \Phi(Q) \neq 1$, we deduce

$\Phi(Q) \cap V_L \neq 1$. As $N_L(QQ_L)$ normalizes Q and is irreducible on $[V_L, Q]$, $[V_L, Q] \leq \Phi(Q)$. But then V_L centralizes $Q/\Phi(Q)$, so $V_L \leq O_p(N_G(Q)) = Q$, a contradiction. This shows Q is elementary abelian and then also $Y_L \cap Q = V_L \cap Q = C_{Y_L}(Q)$. □

Proposition 6.2. *Suppose $L \in \mathcal{L}_G(S)$ and $L \not\leq N_G(Q)$ with $\overline{L}$ in the unambiguous wreath product case. If $Y_L \not\leq Q$, $\overline{L^\circ} \cong \mathrm{SL}_2(4)$ and $|Y_L : V_L| = 2$, then G is* Aut(Mat(22)).

Proof. We start by observing that the action of L on Y_L gives

(6.2.1)
(i) $|V_LQ/Q| = |Q : C_Q(V_L)| = 2^2$;
(ii) for all $v \in V_L \setminus Q$, $C_Q(v) = C_Q(V_L)$; and
(iii) for all $w \in Q \setminus Q_L$, $[w, V_L] = [Q, V_L]$.

Let $B = N_L(QQ_L)$. Then B contains an element β of order 3 which acts fixed-point freely on V_L and irreducibly on $[V_L, Q] = C_{Y_L}(Q)$.

Using (6.2.1) (ii) and Lemma 2.8 yields $[V_L, F(N_G(Q)/Q)] = 1$. Let $K \geq Q$ be the preimage of

$$[E(N_G(Q)/Q), V_LQ/Q].$$

Then K is non-trivial, normalized by B and Lemma 2.8 implies V_LQ/Q acts faithfully on K/Q.

The three involutions of QQ_L/Q_L each centralize a subgroup of Y_L of order 2^3 and by Lemma 2.10(i), there are three elements of Y_LQ/Q which act on Q as GF(2)-transvections, they generate Y_LQ/Q and are permuted transitively by B/Q. As B normalizes K and as V_LQ/Q acts faithfully on K/Q, at least one and hence all of the transvections in Y_LQ/Q act faithfully on K/Q.

If $C_Q(K) \neq 1$, then $C_{C_Q(K)}(S) \neq 1$. As $\Omega_1(Z(S)) = C_{V_L}(S)$ by Lemma 2.10 (ii), and $C_Q(K)$ is normalized by B, we have $[Q, V_L] \leq C_Q(K)$. But then $K = \langle V_L^K \rangle Q$ centralizes $Q/C_Q(K)$ contrary to $C_K(Q) = Q$. Hence $C_Q(K) = 1$.

Let V be a non-trivial minimal KY_L-invariant subgroup of Q. Then KY_L acts irreducibly on V. Moreover, as Y_L does not centralize V, $V \not\leq Q_L$ and, as V_L is the natural $\overline{L^\circ}$-module we have $[Y_L, V] = [Y_L, Q] = Y_L \cap Q \leq V$. It follows that K centralizes Q/V and so K/Q acts faithfully on $V = [Q, K]$ which is normalized by B. Hence $C_{Y_L}(V) = Y_L \cap V = Y_L \cap Q$ and Y_LQ/Q acts faithfully on V. Recall that Y_LQ/Q is generated by elements which operate as transvections on Q and hence on V. Therefore [McL, Theorem] applies to give $KY_L/Q \cong \mathrm{SL}_m(2)$ with $m \geq 3$, $\mathrm{Sp}_{2m}(2)$ with $m \geq 2$, $\mathrm{O}^{\pm}_{2m}(2)$ with $m \geq 2$, or $\mathrm{Sym}(m)$ with $m \geq 7$. Furthermore, $V = [Q, K]$ is the natural module in each case.

Since $C_{Y_LQ/Q}(S/Q)$ contains a transvection and has order 2^2, $KY_L/Q \not\cong \mathrm{SL}_m(2)$ with $m \geq 3$ or $\mathrm{O}^{\pm}_{2m}(2)$ with $m \geq 2$. Suppose that $KY_L/Q \cong \mathrm{Sym}(m)$ with $m \geq 7$. Then, as Y_LQ/Q is generated by three transvections, we see that Y_LQ/Q is generated by three commuting transpositions in KY_L/Q. Let t be the product of these transpositions. Then, as $m \geq 7$, $|[V, t]| = 2^3$. However, $|[V, Y_L]| = 2^2$, and so we have a contradiction. We have demonstrated

(6.2.2) $KY_L/Q \cong \mathrm{Sp}_{2m}(2)$, $m \geq 2$ and $[Q,K] = [Q,KY_L]$ is the natural module.

Since $[Q,K]$ is the natural KY_L/Q-module and $[V_L,Q] \leq [Q,K]$ has order 2^2, we have $[[V_L,Q],S] \neq 1$. In particular, $QQ_L/Q_L < S/Q_L \cong \mathrm{Dih}(8)$ and $SQ/Q \cap K/Q$ acts non-trivially on $[Q,V_L]$.

Consider $Q^* = O_2(KS)$. Since Q^* centralizes $[Q,K]$, Q^* centralizes $[V_L,Q]$ and so $Q^*Q_L = QQ_L$. Hence $\Phi(Q^*) \leq Q_L$. If $\Phi(Q^*) \neq 1$, then

$$[Q,K] = \langle \Omega_1(Z(S))^K \rangle \leq \Phi(Q^*)$$

and so also $[Q^*,K] = [Q^*,K,K] \leq [Q,K] \leq \Phi(Q^*)$ which is impossible. Hence Q^* is elementary abelian and it follows that $Q \leq Q^* = C_{Q^*}(Q) \leq Q$. Since KS acts on $[Q,K]$ and $KY_L/Q \cong \mathrm{Sp}_{2m}(2)$, we now deduce $S \leq KY_L$ from the structure of $\mathrm{Out}(K/Q)$. Hence $B = \langle S^B \rangle \leq KY_L$ as B normalizes KV_L. It follows that B/Q is the minimal parabolic subgroup P of K/Q irreducible on $[Y_L,V]$ and with $O^2(P)$ centralizing $[Y_L,V]^\perp/[Y_L,V] = C_{Y_L}(V)/[Y_L,V]$. Therefore there is $\beta \in K$ of order three such that $\langle\beta\rangle$ is transitive on the transvections in Y_LQ/Q and normalizes Q_LQ/Q which has index 2 in S/Q. In particular, from the structure of the natural $\mathrm{Sp}_{2m}(2)$-module β centralizes

$$C_V(Y_L)/[V,Y_L] = (V \cap Q_L)/(V \cap Y_L) = (V \cap Q_L)Y_L/Y_L \leq [Q_L,V]Y_L/Y_L.$$

As V is abelian, V acts quadratically on Q_L/V_L. By Lemma 2.2, Q_L/V_L involves only natural $\mathrm{SL}_2(4)$-modules and trivial modules as L-chief factors. We know β acts fixed-point freely on the natural module and so, as β centralizes $[Q_L,V]Y_L/Y_L$, all the L-chief factors of Q_L/V_L are centralized by L. In particular, V_L is the unique non-central L-chief factor in Q and so $Y_L \cap \Phi(Q_L) = 1$. As $\Omega_1(Z(S)) \leq V_L$ by Lemma 2.10 (ii), $\Phi(Q_L) = 1$, so $Q_L = \Omega_1(Z(Q_L)) = Y_L$, which together with $S/Q_L \cong \mathrm{Dih}(8)$ implies

(6.2.3) $Y_L = Q_L$ has order 2^5 and $|S| = 2^8$.

Together (6.2.2) and (6.2.3) give

(6.2.4) $|Q| = 2^4$ and $N_G(Q)/Q \cong \mathrm{Sym}(6)$.

We next show that G has a subgroup of index two. In $N_G(Q)$ we have a subgroup U of index 2 of shape $2^4.\mathrm{Alt}(6)$. Furthermore $Y_L \not\leq U$ and $V_L \leq U$. Since $[v,Q] = C_Q(v)$ for $v \in V_L \setminus Q$ and U/Q has one conjugacy class of involutions, all the involutions in $U \setminus Q$ are U-conjugate. Since L acts transitively on V_L and U is transitive on $Q^\#$, we have that all the involutions in U are G-conjugate. As Q is large, we have $C_G(z) \leq N_G(Q)$ for $z \in Q^\#$. Hence all the involutions in U have centralizer which is a $\{2,3\}$-group. There is an involution t in $Y_L \setminus V_L$, which is not in U and centralized by an element of order 5 in L. Hence t is not conjugate to any involution of U. Application of [GLS2, Proposition 15.15] gives a subgroup G_1 of index two in G. We have $N_{G_1}(Q)/Q \cong \mathrm{Alt}(6)$. By Lemma 2.1 this extension splits and we have that the centralizer of a 2-central involution $z \in G_1$ is a split extension of an elementary abelian group of order 16 by $\mathrm{Sym}(4)$. In particular $O(C_G(z)) = 1$ and so by coprime action $O(G) = \langle C_{O(G)}(e) \mid e \in Q^\# \rangle = 1$. As $Y_L \not\leq Q$, there is an involution $y \in N_{G_1}(Q) \backslash Q$. Since all involutions in Qy and in $N_{G_1}(Q)/Q$ are conjugate, G_1 has just one conjugacy class of involutions. In particular

$F^*(G_1)$ is simple. Application of [J, Theorem] gives that $F^*(G_1) \cong \mathrm{Mat}(22)$ and so $G \cong \mathrm{Aut}(\mathrm{Mat}(22))$. $\square$

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