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THE LOCAL STRUCTURE THEOREM, THE NON-CHARACTERISTIC 2 CASE

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ABSTRACT. Let p be a prime, G a finite $\mathcal{K}_p$-group, S a Sylow p-subgroup of G and Q be a large subgroup of G in S. The aim of the Local Structure Theorem [11] is to provide structural information about subgroups L with $S \le L$, $O_p(L) \ne 1$ and $L \not\le N_G(Q)$. There is, however, one configuration where no structural information about L can be given using the methods in [11]. In this paper we show that for $p = 2$ this hypothetical configuration cannot occur. We anticipate that our theorem will be used in the programme to revise the classification of the finite simple groups.

1. Introduction

The proof of the classification of the finite simple groups took different directions depending upon the structure of normalizers of non-trivial 2-subgroups. Such subgroups are called 2*-local subgroups*. If M is such a 2-local subgroup, then there are two possibilities $C_M(O_2(M)) \le O_2(M)$ or $C_M(O_2(M)) \not\le O_2(M)$. In the former case, we say that M has *characteristic* 2. If all the 2-local subgroups of a finite group G have characteristic 2, then we say that G is of *local characteristic* 2. The classification divides into the investigation of groups which are of local characteristic 2 and those which are not. In the latter case the objective is to show that there is a 2-local subgroup which has a fairly simple structure (a subnormal $\mathrm{SL}_2(q)$, standard subgroups). One of the main obstructions for proving the existence of such a 2-local subgroup is the existence of non-trivial normal subgroups of odd order in 2-local subgroups. A new approach due to M. Aschbacher (for an overview see [1, Chapter 2]) using fusion systems avoids this problem. The first steps of this programme can be found in a preprint [2].

For groups of local characteristic 2, the problem is the complexity of the structure of the 2-local subgroups. In the original classification, to avoid this complexity problem the strategy was to move to p-local subgroups for suitable odd primes p, which then eventually have a fairly restricted structure similar to standard subgroups.

In the years following the classification, methods for working with 2-local subgroups of groups of local characteristic 2 have been refined and developed. These new methods inspired a novel approach to the classification of groups of local characteristic 2 initiated by U. Meierfrankenfeld, B. Stellmacher and G. Stroth (see [10] for an overview), the MSS-programme for short, which stays in the 2-local world and intends to pin down the structure of G. The Local Structure Theorem [11] provides information about important subgroups and quotients of certain 2-local subgroups and further work is in progress. A tempting possibility is that there is a bridge between Aschbacher's programme and the MSS-programme which means they can be merged to give a new proof of the classification of the finite simple groups. One of the purposes of this paper is to build part of such a bridge.

We now explain how these two programmes can possibly be joined. For this we have to say a little bit more about Aschbacher's approach. As an example, let us assume that we have a 2-local subgroup $M \cong 2 \times \mathrm{Alt}(5)$. Then our target simple group is the sporadic simple group J_1, but the groups $\mathrm{SL}_2(16){:}2$ and $\mathrm{Alt}(5) \wr 2$ also have such a 2-local subgroup, of course they are not simple. However to detect this fact takes a lot of work. To avoid this problem Aschbacher assumes that M contains an elementary abelian 2-subgroup of G of maximal order. With this extra condition J_1 is the unique solution (assuming $O_2(G) = O(G) = 1$). The problem is that an approach to the classification based on Aschbacher's new work no longer has the tidy division into two cases: local characteristic 2 or not local characteristic 2. To take the discussion further, we introduce the notion of *parabolic characteristic* 2. This means that we require M has characteristic 2 only for those 2-local subgroups M of odd index in G. If we could classify the groups of parabolic characteristic 2, then this would be a counterpart to Aschbacher's work and together they would provide an alternative proof of the classification. At the moment providing such classification seems to be out of reach. However, it is also more than is required. Fix $S \in \mathrm{Syl}_2(G)$ and recall the *Baumann subgroup* of S is defined to be $B(S) = C_S(\Omega_1(Z(J(S))))$. For a saturated fusion system $\mathcal{F}$ on a 2-group T, Aschbacher considers components of $C_{\mathcal{F}}(t)$ where t is an involution with $m_2(T) = m_2(C_T(t))$ (see [2, page 5]). So, if Aschbacher's programme is successful, then we can assume that $C_G(t)$ has characteristic 2 for all involutions in $\Omega_1(Z(J(S)))$. Hence, if we could determine the groups in which every 2-local subgroup containing $B(S)$ has characteristic 2, then we could meld the two programme and produces an alternative proof of the classification. Such groups are called groups of *Baumann characteristic* 2.

So far the investigation in MSS focuses on groups which possess a large subgroup Q (the exact definition will be given later on). A consequence of the existence of such a group is that G has parabolic characteristic 2. The Local Structure Theorem in [11] gives information about the structure of groups of parabolic characteristic 2, which have a large subgroup Q. In fact this has been done for arbitrary primes p. For a p-local subgroup M of characteristic p, there is a unique non-trivial normal elementary abelian p-subgroup Y_M maximal subject to $O_p(M/C_M(Y_M)) = 1$. The Local Structure Theorem gives information about Y_M and the action of $M/C_M(Y_M)$ on Y_M provided Q is not normal in M and M contains a Sylow p-subgroup of G. To take the investigation further there are two cases to be investigated. Either $Y_M \not\leq Q$ for some such M or $Y_M \leq Q$ for all such M. In both instances define

$$H = \langle K \mid O_p(K) \neq 1, S \leq K \rangle.$$

In the first case, the H-Structure Theorem (work in preparation) builds on the Local Structure Theorem and determines the group H. Using this, for $p = 2$, $F^*(G)$ can be identified. If p is odd, then either $F^*(G)$ is determined, or $F^*(H)$ is demonstrated to be a group of Lie type in characteristic p and rank at least three, or H is a weak BN-pair. Up to this point the MSS-programme fits well with Aschbacher's point of view. In the second case, $Y_M \leq Q$ for all M, and again we intend to determine the group H. For this, the first question is: which of the p-local subgroups from the Local Structure Theorem can show up? This has been partly answered in [9, 12] but only under the assumption that G has local characteristic p and this assumption is not compatible with Aschbacher's approach. Hence we must replace it by a more applicable premise. The starting point for [9, 12] is [11, Corollary B] which lists the cases from the Local Structure Theorem which may appear when $Y_M \leq Q$ and G is of local characteristic p. Using this information [9, 12] basically exclude what is called the wreath product case in the Local Structure Theorem. From now on assume that $p = 2$ as this is the relevant prime for Aschbacher's approach. The first question is: what happens if we remove the assumption of local characteristic 2 in [11, Corollary B]? The answer is that two further configurations for the group M appear. One is that $M/C_M(Y_M)$ induces the natural $\mathrm{O}^{\pm}_{2n}(2)$-module on $[Y_M, M]$. This possibility has been handled by Chr. Pröseler [15] in his PhD thesis. The second is that [11, Theorem A (10-1)] holds and this is the situation handled in this paper. We will show that no groups satisfy this hypothesis. Hence we may investigate the proofs of [9, 12] starting with the same set of possible 2-local subgroups as provided by [11, Corollary B]. However the proofs in [9, 12] also exploit that G has local characteristic 2 but only for 2-local subgroups K which contain the Baumann subgroup $B(S)$. Hence the local characteristic 2 assumption can be replaced with Baumann characteristic 2 and we then obtain the same conclusion as in [9, 12]. Therefore, provided we can prove the analogue of the H-structure theorem in the case that $Y_M \le Q$ for all M when G has Baumann characteristic 2, we will have a companion to Aschbacher's approach.

To explain further the context of the results in this article, we give a simplified overview of the Local Structure Theorem for the particular case when $p = 2$ and then outline the contribution of the research in this paper. We work in an environment compatible with being a counter example to the classification. Thus we call G a $\mathcal{K}_2$-group if and only if every simple section of every 2-local subgroup of G is in the set of known simple groups $\mathcal{K}$ where $\mathcal{K}$ consists of the groups of prime order, the alternating groups, the simple groups of Lie type and the sporadic simple groups.

A subgroup Q of S is called *large* if

- $Q = O_2(N_G(Q))$;
- $C_G(Q) \le Q$; and
- for any $1 \ne A \le Z(Q)$ we have that Q is normal in $N_G(A)$.

For $K \le H \le G$ we define

$$\mathcal{L}_H(K) = \{K \le L \le H \mid O_2(L) \ne 1, C_H(O_2(L)) \le O_2(L)\}.$$

As we asserted earlier, the existence of the large subgroup Q implies that G has parabolic characteristic 2 and so, in this case, $\mathcal{L}_G(S)$ contains all of the 2-local subgroups of G which contain S. Define $\mathfrak{M}_G(S)$ to be the subset of $\mathcal{L}_G(S)$ which contains the subgroups $M \in \mathcal{L}_G(S)$ such that, setting $M^\dagger = MC_G(Y_M)$,

- $\mathcal{L}_G(M) = \mathcal{L}_{M^\dagger}(M)$ and $Y_M = Y_{M^\dagger}$; and
- $C_M(Y_M)/O_2(M) \le \Phi(M/O_2(M))$.

For $L \in \mathcal{L}_G(S)$ with $L \not\le N_G(Q)$, define

$$L^\circ = \langle Q^L \rangle.$$

With these assumptions and notation the Local Structure Theorem states:

Suppose that G is a $\mathcal{K}_2$-group, $S \in \mathrm{Syl}_2(G)$ and S is contained in at least two maximal 2-local subgroups of G. Assume that Q is a large subgroup of G contained in S and $M \in \mathfrak{M}_G(S)$ with $M \not\le N_G(Q)$. Then the structure of $M^\circ/C_{M^\circ}(Y_M)$ and its action on Y_M are described **unless** [11, Theorem A (10–1)] **holds**.

So far so good, but what is this mysterious clause (10–1) in [11, Theorem A]? In this case, all that is proved is that Y_M is tall and asymmetric in G, but importantly Y_M is not characteristic p-tall in G. We will now explain in detail what this means, as our intention is to remove this restriction to the Local Structure Theorem for the situation when $p = 2$ so that the we can provide the bridge to the Aschbacher project.

Let $M \in \mathfrak{M}_G(S)$ and $T \in \mathrm{Syl}_2(C_G(Y_M))$. The subgroup Y_M is

- *tall*, if there exists K with $T \le K \le G$ such that $O_2(K) \ne 1$ and $Y_M \not\le O_2(K)$, and
- *asymmetric* in G, if whenever $g \in G$ and $[Y_M, Y_M^g] \le Y_M \cap Y_M^g$, then $[Y_M, Y_M^g] = 1$.

Further Y_M is *characteristic* 2*-tall* provided

- there is some K with $T \le K \le G$ such that $C_K(O_2(K)) \le O_2(K)$ and $Y_M \not\le O_2(K)$.

We can now state the theorem we shall prove in this paper

Theorem. *Let G be a finite $\mathcal{K}_2$-group and $S \in \mathrm{Syl}_2(G)$. Suppose that S is contained in at least two maximal* 2*-local subgroups and that Q is a large subgroup of G in S. Assume that there exists $M \in \mathfrak{M}_G(S)$ such that Y_M is asymmetric and tall. Then Y_M is characteristic* 2*-tall.*

This theorem shows that for $p = 2$ [11, Theorem A (10-1)] does not arise for $M \in \mathfrak{M}_G(S)$ and so also implies that Theorem A (10-1) of the Local Structure Theorem does not occur for arbitrary $L \in \mathcal{L}_G(S)$ with $L \not\le N_G(Q)$ as [11, Theorem A (10-1)] states that there exists $M \in \mathfrak{M}_G(S)$ with $Y_L = Y_M$ and $L^\circ = M^\circ$.

For the proof of the theorem of this paper we assume that there exists $M \in \mathfrak{M}_G(S)$ with Y_M asymmetric and tall, but not characteristic 2-tall. This means that for $T \in \mathrm{Syl}_2(C_M(Y_M))$ if K is a subgroup of G containing T with $O_2(K) \ne 1$ and $Y_M \not\le O_2(K)$, then $C_G(O_2(K)) \not\le O_2(K)$. Thus there are involutions $y \in Y_M$ such that $C_G(O_2(C_G(y))) \not\le O_2(C_G(y))$. We study these centralizers and would like to show that $E(C_G(y)) \ne 1$. That is, the centralizers have components. The obstruction to this is the existence of normal subgroups of odd order. The key for removing this obstacle is that $C_G(y)$ contains a 2-central element z and, as $z \in Q$, using Q large, yields $C_G(O_2(C_G(z))) \le O_2(C_G(z))$. This implies that z inverts any normal subgroup of odd order in $C_G(y)$ and so such subgroups are abelian. In addition, we prove that $|Y_M| \ge 2^3$ and we know $Y_M \le C_G(y)$. So signalizer functor methods can be employed to obtain $E(C_G(y)) \neq 1$ (see Lemmas 2.3 and 5.2). The arguments used to prove this do not transfer to odd primes as in this case we only find that $O_{p'}(C_G(y))$ is nilpotent and this prevents us demonstrating the balance condition required for use of the signalizer functor theorem.

From among all the components involved in the centralizers of elements in Y_M we select one, K say, with first $|K/Z(K)|$ maximal and then $|K|$ maximal. Then from all the elements of $Y_M^{\#}$ that contain components we select those y that have $|E_y|$-maximal where E_y is the subgroup of $E(C_G(y))$ generated by components J with $J/Z(J) \cong K/Z(K)$ and $|J| = |K|$. The set of such elements is denoted by $\mathcal{Y}^*$ and the members of $\mathcal{Y}^*$ are the focus of attention. With these choices, for $y \in \mathcal{Y}^*$, Lemma 5.10 shows that, if $C_S(y) \in \mathrm{Syl}_2(C_G(y))$ and $|C_S(y)|$ is maximal, then $C_{O_2(M)}(E_y)$ is a trivial intersection subgroup of M. Roughly speaking, the contradictions which lead to the proof of the Theorem come about by finding that either M normalizes a non-trivial subgroup of $Z(Q)$ which, as Q is large, contradicts $M \not\leq N_G(Q)$, or that M is the unique maximal 2-local subgroup containing S which contradicts the fact that S is contained in at least two such subgroups. These two observations are encoded in Lemmas 4.5 and 4.11.

We give a little more detail, select $y \in \mathcal{Y}^*$, fix a component $K \leq E_y$ and set $L_K = C_K(z)$ where z is an involution in $Z(S)$. Then L_K has characteristic 2, and the examination of the various possibilities for K take markedly different routes dependent upon whether or not L_K is a 2-group. If L_K is not a 2-group, it is often possible to show that $K = E(C_G(y))$. Furthermore, Lemma 5.23 asserts that $O^2(L_K)$ cannot act irreducibly on $O_2(L)/Z(O_2(L_K))$(the root of this observation lies in Lemma 4.5). This fact eliminates many candidates for $K/Z(K)$. The detailed arguments are presented in Sections 7, 8, 9 and 10 where, for the more difficult cases, the 2-local structure of K plays a central role in the proof. The data needed for this is provided in Section 3.

By the end of Section 10, we are left with two possibilities. Either $K/Z(K) \cong \mathrm{PSL}_3(4)$ or $K \cong \mathrm{Sp}_4(2^a)$. Interestingly in this situation we are unable to bound the number of components involved in E_y. We quickly prove that $Z(K)$ is elementary abelian and that $z \in \Omega_1(Z(S)) \leq Y_M$ does not project to a root element when $K \cong \mathrm{Sp}_4(2^a)$. In Lemma 11.3 we show that the Thompson subgroup of $O_2(M)$ is equal to $(S \cap E_y)J(C_S(E_y))$ and this provides our way into the study of these cases. We eventually show that M either normalizes E_y, or there is a further subgroup $K_{r+1} \cong K$ which commutes with E_y such that M normalizes E_yK_{r+1}. Our objective is to prove that every elementary abelian normal subgroup of S is contained in Y_M, once this is done the contradiction is provided by Lemma 4.11.

2. Preliminary group theoretical results

In this section we collect some group theoretical facts that we require. In this work we assume that all groups are finite. Recall that for a prime p, a group X has *characteristic* p provided $C_X(O_p(X)) \leq O_p(X)$ or, equivalently, if $F^*(X) = O_p(X)$. Our first lemma which is a consequence of coprime action and the Thompson $A \times B$-Lemma [5, Lemma 11.7] is well-known and plays a critical role in our proof of the Theorem.

Lemma 2.1. *Suppose that p is a prime, X is a group, B is a p-subgroup of X and C is a normal subgroup of B. If $N_X(C)$ has characteristic p, then $N_X(B)$ and $C_X(B)$ have characteristic p.*

Proof. Set $A = O_{p'}(N_X(B))E(N_X(B))$. Then A centralizes B and so A also centralizes $C \leq B$. Therefore $AB \leq N_X(C)$ and AB normalizes $P = O_p(N_X(C))$. We have $C_P(B)$ normalizes A and, as $[B, A] = 1$, $C_P(B)$ is normalized by A. Hence

$$[C_P(B), A] = [C_P(B), A, A] \leq [O_p(A), A] \leq [Z(E(N_X(B))), A] = 1.$$

As A is generated by p'-elements, the Thompson $A \times B$-Lemma implies that A centralizes P and hence $A = 1$ as $N_X(C)$ has characteristic p. Therefore $F^*(N_X(B)) = O_p(N_X(B))$ and so $N_X(B)$ has characteristic p. Since $F^*(C_X(B)) \leq F^*(N_X(B))$, we also have $C_X(B)$ has characteristic p. □

As an example of how we might use Lemma 2.1 consider the case X has characteristic p. Then we may take $C = 1$, and obtain $N_X(B)$ has characteristic p.

Lemma 2.2. *Let X be a group of characteristic p and Y be subnormal in X. Then Y is a group of characteristic p.*

Proof. If Y is subnormal in X, then $F^*(Y) \leq F^*(X)$. Hence $F^*(Y)$ is a p-group. □

The next lemma will be used to show that certain involutions have components in their centralizers.

Lemma 2.3. *Suppose that X is a group and Y is an elementary abelian 2-subgroup of X of order at least 8. Assume that $E(C_X(x)) = 1$ for all $x \in Y^\#$ and that there exists $z \in Y^\#$ such that $F^*(C_X(z)) = O_2(C_X(z))$. Then $\langle O(C_X(y)) \mid y \in Y^\# \rangle$ has odd order and is normalized by $N_X(Y)$.*

Proof. Suppose that $a, b \in Y^{\#}$ are such that $F^*(C_X(a)) = O_2(C_X(a))$ and $O(C_X(b)) \neq 1$. Then $C_{C_X(a)}(b) = C_{C_X(b)}(a)$ has characteristic 2 by Lemma 2.1. In particular, $C_{O(C_X(b))}(a) = 1$ and so a inverts $O(C_X(b))$. This means that $O(C_X(b))$ is abelian. Since there exists $z \in Y^{\#}$ such that $F^*(C_X(z)) = O_2(C_X(z))$, we have $O(C_X(b))$ is abelian and is inverted by z for all $b \in Y^{\#}$.

Suppose that $a, b \in Y^{\#}$ are arbitrary. We claim that

$$O(C_X(b)) \cap C_X(a) \leq O(C_X(a)).$$

If $F^*(C_X(b)) = O_2(C_X(b))$, then $O(C_X(b)) = 1$ and there is nothing to prove. Suppose that $F^*(C_X(a)) = O_2(C_X(a))$, then we have already argued that $O(C_X(b)) \cap C_X(a) = 1$ and so the claimed containment also holds in this case. Suppose that $O(C_X(b)) \neq 1 \neq O(C_X(a))$. Set $U = O(C_X(b)) \cap C_X(a)$. Then $\langle b \rangle \times U$ normalizes $O_2(C_X(a))$ and $[C_{O_2(C_X(a))}(b), U] \leq O_2(C_X(a)) \cap O(C_X(b)) = 1$. Thus again the Thompson $A \times B$-Lemma implies that $[U, O_2(C_X(a))] = 1$. Now consider $UO(C_X(a))$. This group is normalized by z and, as z inverts U and inverts $O(C_X(a))$, we have z inverts $UO(C_X(a))$. But then $UO(C_X(a))$ is abelian. Consequently U centralizes $F^*(C_X(a)) = O(C_X(a))O_2(C_X(a))$ and so $U \leq O(C_X(a))$ as claimed.

As $|Y| \geq 8$ by hypothesis, the Soluble Signalizer Functor Theorem [5, Theorem 21.3] implies that the completeness subgroup

$$\Sigma = \langle O(C_X(b)) \mid b \in Y_M^{\#} \rangle$$

has odd order. Finally we note that $N_X(Y)$ normalizes Σ as it permutes the generating subgroups by conjugation. This completes the proof of the lemma. □

Recall from [5, Definition 4.5] that a 2-*component* of a group X is a subnormal perfect subgroup F of X such that $F/O(F)$ is quasisimple. The subgroup $L_{2'}(X)$ is defined to be the subgroup of X generated by all the 2-components of X and is called the 2-*layer* of X. The subgroup X^∞ of X is the last member in the derived series of X.

Lemma 2.4. *We have*

$$C_{L_{2'}(X)}(O(L_{2'}(X))) = E(X)Z(O(L_{2'}(X)))O_2(L_{2'}(X)).$$

Proof. Plainly $C_{L_{2'}(X)}(O(L_{2'}(X))) \geq E(X)Z(O(L_{2'}(X)))O_2(L_{2'}(X))$. We may as well suppose that $X = L_{2'}(X)$. Set $\overline{X} = X/O(X)$ and $C = C_X(O(X))$. Then $\overline{X} = E(\overline{X})$ by [5, Proposition 4.7 (iii)]. Therefore $\overline{C}$ is a product of components of $\overline{X}$ together with $\overline{O_2(X)}$. Assume that $K \leq C$ is such that $\overline{K}$ is a component in $\overline{C}$. Then $KO(X)$ is normal in X and K^∞ is a 2-component of X. If K^∞ is not a component of X, then $O(K^\infty) \not\leq Z(K^\infty)$. As $K \leq C$, this is impossible. Hence $K \leq E(X)$. Thus $\overline{C} = \overline{E(X)O_2(X)}$ and consequently $C \leq E(X)O_2(X)O(X)$. Using the Dedekind Modular Law we obtain

$$\begin{aligned} C &= C \cap E(X)O_2(X)O(X) = E(X)O_2(X)(C \cap O(X)) \\ &= E(X)O_2(X)Z(O(X)), \end{aligned}$$

as claimed. □

Lemma 2.5. *Suppose that K is a component of the group X and $T \in \mathrm{Syl}_2(X)$. If Y is an abelian normal subgroup of T and Y does not normalize K, then $K/Z(K)$ has abelian Sylow 2-subgroups.*

Proof. See [14, Lemma 2.28]. □

Lemma 2.6. *Assume that $R \leq G$ be a 2-group which normalizes the subgroup $P \leq N_G(R)$. Set $V = [O_2(P), O^2(P)]$ and assume that the non-central $O^2(P)$-chief factors in $V/\Phi(V)$ are pairwise non-isomorphic. Then $[V, R] \leq \Phi(V)$.*

Proof. Set $\overline{V} = V/\Phi(V)$. As R normalizes P, R operates on $\overline{V}$ and, as R is normalized by P, coprime action yields

$$[R, O^2(P)] = [R, O^2(P), O^2(P)] \leq [O_2(P), O^2(P)] = V \leq C_{O_2(P)}(\overline{V}).$$

Assume $R > C_R(\overline{V})$ and select $x \in R \backslash C_R(\overline{V})$ such that $x^2 \in C_R(\overline{V})$. Then $\overline{V} > C_{\overline{V}}(x)$ and $[\overline{V}, x] \neq 1$ are $O^2(P)$-invariant as $[x, O^2(P)]$ centralizes $\overline{V}$. Additionally, $\overline{V}/C_{\overline{V}}(x) \cong [\overline{V}, x]$ as $O^2(P)$-modules. In addition, as $x^2 \in C_R(V)$, we have $C_{\overline{V}}(x) \geq [\overline{V}, x]$. Thus, the condition on the non-central $O^2(P)$-chief factors in $\overline{V}$ implies that $\overline{V}/C_{\overline{V}}(x)$ is centralized by $O^2(P)$. But then $V = [V, O^2(P)] < V$, a contradiction. Hence $R = C_R(\overline{V})$ as claimed. □

We recall that the *Thompson subgroup* $J(X)$ of a group X, is the subgroup of X generated by the elementary abelian subgroups of X of maximal rank. One of the main tools in the proof of the theorem of this paper requires that we locate $J(O_2(M))$ in certain subgroups of centralizers of elements in Y_M. The next two results are important when such subgroups have more than one component.

Proposition 2.7. *Suppose that X is a group, $O(X) = 1$, K is a component of X and S is a Sylow 2-subgroup of X. Assume that K satisfies the following two properties.*

(i) *For $\bar{x} \in K/Z(K)$ an involution, there is a preimage x such that*
   (a) *x is an involution; and*

(b) *any involution in* $\mathrm{Aut}(K)$, *which centralizes* $\bar{x}$ *also centralizes* x.

(ii) *If* $K/Z(K)$ *has dihedral or semidihedral Sylow* 2*-subgroups, then* $\mathrm{Aut}(K/Z(K))$ *does not contain a fours-group disjoint from* $\mathrm{Inn}(K/Z(K))$.

Then $J(S)$ *normalizes* K. *In particular, if* $K \in \mathcal{K}$ *is simple, then* $J(S)$ *normalizes* K.

Proof. This is [5, Proposition 8.5] and the remark thereafter. □

Lemma 2.8. *Suppose* $G = SE$ *where* $S \in \mathrm{Syl}_2(G)$ *and* E *is a direct product of simple components* $K \in \mathcal{K}$ *of* G. *Assume that each component* K *of* E *satisfies*

$$\text{(1)} \qquad \text{if } T \in \mathrm{Syl}_2(\mathrm{Aut}(K)), \text{ then } J(T) = J(T \cap \mathrm{Inn}(K)).$$

Then

$$J(S) = J(C_S(E)) \times J(S \cap E)$$

and

$$J(S \cap E) = \prod_{K \text{ a component of } G} J(S \cap K).$$

Proof. Assume that A is an elementary abelian 2-subgroup of S of maximal rank. By Proposition 2.7 every component K of G is normalized by A. Furthermore $AC_G(K)/C_G(K)$ is an elementary abelian 2-subgroup of $\mathrm{Aut}(K)$. Assume that $AC_G(K)/C_G(K)$ is not a maximal rank elementary abelian 2-subgroup of $N_G(K)/C_G(K)$. Then, by (1), there exists an elementary abelian p-subgroup $B \le K$ such that

$$|B| = |BC_G(K)/C_G(K)| > |AC_G(K)/C_G(K)| = |A : A \cap C_G(K)|.$$

Set $B_0 = B(A \cap C_G(K))$. Then B_0 is elementary abelian and $|B_0| = |B||A \cap C_G(K)| > |A|$, contrary to the choice of A. Hence $AC_G(K)/C_G(K)$ is a maximal rank elementary abelian 2-subgroup of $N_G(K)/C_G(K)$ and therefore $A \le KC_G(K)$ and $A = (A \cap K)(A \cap C_G(K))$ with $A \cap K$ a maximal rank elementary abelian 2-subgroup of K. Assume that $E = K_1 \cdots K_\ell$. Then, for $1 \le j \le \ell$, we have shown that

$$A = C_A(K_j)(A \cap K_j)$$

where $A \cap K_j$ is a maximal rank elementary abelian 2-subgroup of K_j and $C_A(K_j)$ is a maximal rank elementary abelian 2-subgroup of

$C_S(K_j)$. Notice that by the Modular Law

$$\begin{aligned} A &= C_A(K_1)(A\cap K_1)\cap C_A(K_2)(A\cap K_2) \\ &= (C_A(K_1)(A\cap K_1)\cap C_A(K_2))(A\cap K_2) \\ &= (C_A(K_1)\cap C_A(K_2))(A\cap K_1)(A\cap K_2) \\ &= C_A(K_1K_2)(A\cap K_1)(A\cap K_2) \end{aligned}$$

and continuing in this way yields $A = C_A(E)(A\cap K_1)\dots(A\cap K_\ell)$. This proves the claim. □

Remark 2.9. If K is a simple group, then the statement in Proposition 2.7 can be proved for all primes p provided $O_{p'}(X) = 1$, where we do not need $K \in \mathcal{K}$ for $p > 2$. The statement of Lemma 2.8 also holds for all primes.

3. Properties of $\mathcal{K}$-groups

We require detailed information about the 2-local structure of certain of the groups of Lie type defined in characteristic 2. What we require can mostly be found in [13], but we present the statements here for the convenience of the reader. We start with groups defined over a field of characteristic 2. In the next lemma we use the notation V_n to denote a natural module for a classical group defined in dimension n but considered as a GF(2)-module. Thus, if X is a classical group defined over $\mathrm{GF}(2^e)$, then $|V_n| = 2^{ne}$.

Lemma 3.1. *Let X be a simple group of Lie type defined in characteristic 2 and R be a long root subgroup of X. Set $Q = O_2(N_X(R))$ and $L = O^{2'}(N_X(R)/Q)$. Then for specified X, the following table displays the Levi section $L/Z(L)$, the 2-rank of Q/R and, for the classical groups X, describes the action of L on Q/R.*

X	$L/Z(L)$	$m_2(Q/R)$	Q/R
$\mathrm{PSL}_m(2^e), m \geq 5$	$\mathrm{PSL}_{m-2}(2^e)$	$2(m-2)e$	$V_{m-2} \oplus V^*_{m-2}$
$\mathrm{PSU}_m(2^e), m \geq 5$	$\mathrm{PSU}_{m-2}(2^e)$	$(m-2)2e$	V_{m-2}
$\mathrm{P}\Omega^{\pm}_{2m}(2^e), m \geq 4$	$\mathrm{PSL}_2(2^e) \times \mathrm{P}\Omega^{\pm}_{2(m-2)}(2^e)$	$4(m-2)e$	$V_2 \otimes V_{2m-4}$
$\mathrm{P}\Omega^{\pm}_6(2^e)$	$\mathrm{PSL}_2(2^e)$	$4e$	$V_2 \oplus V_2$
$\mathrm{E}_6(2^e)$	$\mathrm{PSL}_6(2^e)$	$20e$	
${}^2\mathrm{E}_6(2^e)$	$\mathrm{PSU}_6(2^e)$	$20e$	
$\mathrm{E}_7(2^e)$	$\mathrm{P}\Omega^+_{12}(2^e)$	$32e$	
$\mathrm{E}_8(2^e)$	$\mathrm{E}_7(2^e)$	$56e$	
$\mathrm{G}_2(2^e), e \geq 3$	$\mathrm{PSL}_2(2^e)$	$4e$	
${}^3\mathrm{D}_4(2^e)$	$\mathrm{PSL}_2(2^{3e})$	$8e$	

Furthermore, other than for $X \cong \mathrm{PSL}_m(2^e)$ and $\mathrm{P}\Omega^{\pm}_6(2^e)$, Q/R is an irreducible L-module and, for the exceptional groups, it is defined over $\mathrm{GF}(2^e)$. If $X \cong \mathrm{P}\Omega^-_6(2^e)$, then $C_X(R)$ acts irreducibly on Q/R.

Proof. This is [13, Lemmas D.1 and D.10]. □

Lemma 3.2. *Let $X \cong \mathrm{PSL}_4(2^a)$ with $a > 1$, R be a root subgroup of X, $L = C_X(R)$ and $Q = O_2(L)$. Then the non-central chief factors of Q/R are not isomorphic as L-modules.*

Proof. This is checked by direct calculation. Let λ be a primitive element in $\mathrm{GF}(2^a)$ and put $\delta = \mathrm{diag}(\lambda, \lambda^{-2}, 1, \lambda)$. Then δ is non-central in X and centralizes $Z(S)$, where S is taken to be the subgroup of lower unitriangular matrices. Let

$$E_1 = \left\{ \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ \beta & 0 & 1 & 0 \\ \gamma & 0 & 0 & 1 \end{smallmatrix} \right) \mid \alpha, \beta, \gamma \in \mathrm{GF}(2^a) \right\}$$

and

$$E_2 = \left\{ \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma & \beta & \alpha & 1 \end{smallmatrix} \right) \mid \alpha, \beta, \gamma \in \mathrm{GF}(2^a) \right\}.$$

Then $Q = E_1E_2$ and we calculate that conjugation of E_1 by δ scales β by λ and conjugation of E_2 by δ leaves β unchanged. It follows that the $\langle \delta \rangle$-invariant subgroups of $Z_2(S)$ are in $Z_2(S) \cap E_1$ or $Z_2(S) \cap E_2$. From this we deduce that E_1 and E_2 are the only normal subgroups of L contained in Q which have order 2^{3a}. This proves the result. □

Lemma 3.3. *Suppose that $K \cong \mathrm{SL}_2(2^{e+1})$ or ${}^2\mathrm{B}_2(2^{2e+1})$ with $e \geq 1$. Let $T \in \mathrm{Syl}_2(\mathrm{Aut}(K))$. Then either $K \cong \mathrm{SL}_2(2^2)$ or $J(T) = J(T \cap \mathrm{Inn}(K)) = \Omega_1(T \cap \mathrm{Inn}(K))$.*

Proof. We identify K with $\mathrm{Inn}(K)$. If K is a Suzuki group then $T \leq K$ by [6, Theorem 2.5.12] and $\Omega_1(T \cap K) = Z(T \cap K)$ and we are done. So suppose that $K \cong \mathrm{SL}_2(2^{e+1})$. Let $x \in T \setminus K$ be an involution. Then x acts as a field automorphism on K and $e + 1$ is even. Thus $C_K(x) \cong \mathrm{SL}_2(2^{(e+1)/2})$ by [6, Theorem 4.9.1].

Assume that $A \leq T$ has maximal rank. Then $|A| \geq 2^{e+1}$ and $T \cap K$ is elementary abelian of order 2^{e+1}. Assume $A \not\leq T$. Then

$$1 + (e + 1)/2 \geq e + 1$$

as K has 2-rank $e + 1$. Hence either $K \cong \mathrm{SL}_2(4)$ or $J(T) \leq K$. In the latter case, we have $J(T) = J(T \cap K) = \Omega_1(T \cap K) = T \cap K$. □

We need the following well-known result about representations of $\mathrm{SL}_2(2^e)$.

Lemma 3.4. *Let V be a non-split extension of a trivial module by the natural module for $X = \mathrm{SL}_2(2^e)$. Let S be a Sylow 2-subgroup of X and A be a fours-group in S. Then $[V, A] = [V, S]$.*

Proof. By a result of Gaschütz [8, Satz I.17.4], we may assume that $C_V(X) \le [V,S]$. Hence, if $[V,A] \ne [V,S]$, as $[V/C_V(X), S] = [V/C_V(X), A]$, there is a hyperplane in $C_V(X)$ which contains $[V,A] \cap C_V(X)$. Thus we may assume that $|C_V(X)| = 2$. Choose $\nu \in X$, of order $q+1$ and $\nu^a = \nu^{-1}$ for some $a \in A$. We have that $|[V,\nu]| = q^2$ has index 2 in V and $V = [V,\nu] + C_V(X)$. Therefore $[V,a] \le [V,\nu]$. Let $A = \langle a, b\rangle$. We have that $[V,\nu] + [V,b]$ is invariant under $\langle A, \nu\rangle = X$. Hence $[V,\nu] + [V,b] = V$ and so $[V,A] > [V,a]$, which implies $C_V(X) \le [V,A]$ and then $[V,A] = [V,S]$. □

Lemma 3.5. *Suppose that $X \cong \mathrm{Sp}_{2n}(q)$ with $q = 2^e$ and $n \ge 3$, and let R_1 be a long root subgroup and R_2 be a short root subgroup of X. For $i = 1, 2$, set $Q_i = O_2(N_X(R_i))$ and*

$$L_i = O^{2'}(N_X(R_i)/Q_i).$$

Then

(i) *$L_1 \cong \mathrm{Sp}_{2n-2}(q)$, Q_1 is elementary abelian and Q_1/R_1 is a natural $\mathrm{Sp}_{2n-2}(q)$-module; and*

(ii) *$L_2 \cong \mathrm{Sp}_{2n-4}(q) \times \mathrm{SL}_2(q)$, $\Phi(Q_2) = Q_2' = R_2$, $Z(Q_2)/R_2$ is a natural $\mathrm{SL}_2(q)$-module and $Q_2/Z(Q_2)$ is the tensor product of natural modules of the direct factors of L_2. In addition, if $q > 2$, then $Z(Q_2)$ does not split over R_2 as an L_2-module.*

Proof. See [13, Lemma D.5]. □

Lemma 3.6. *Suppose that $X \cong \mathrm{Sp}_{2n}(q)$, $q = 2^e$, with $n \ge 3$. Let V be the natural symplectic module, P be the stabilizer of a maximal isotropic subspace of V and $S \in \mathrm{Syl}_2(P)$. Then $J(S) = O_2(P)$ is elementary abelian.*

Proof. See [13, Lemma D.6]. □

Lemma 3.7. *Suppose that $X \cong \mathrm{PSp}_4(q)$, $q = 2^e > 2$, let T be a Sylow 2-subgroup of $\mathrm{Aut}(X)$ and set $S = T \cap X$. Then X has exactly two parabolic subgroups P_1, P_2 which contain S. For $i = 1, 2$, $E_i = O_2(P_i)$ is elementary abelian of order q^3 and $P_i/E_i \cong \mathrm{GL}_2(q)$. We have that E_i is an indecomposable module for P_i and $Z(O^{2'}(P_i)) = R_i$ is a root group. Furthermore $Z(S) = R_1R_2 = S'$, $J(T) = J(S) = S = E_1E_2$ and any involution in S is contained in $E_1 \cup E_2$.*

Proof. See [13, Lemmas D.3 and D.4]. □

Lemma 3.8. *Suppose that $X \cong \mathrm{PSp}_4(q)$, $q = 2^e > 2$, and $S \in \mathrm{Syl}_2(X)$. If D is a non-abelian normal subgroup of S, then either $Z(S) \le D$ or $C_S(D) = Z(S)$ and $|DE_1/E_1| = |DE_2/E_2| = 2$.*

Proof. We use the notation from Lemma 3.7. Assume that $Z(S) \not\leq D$. Then $|DE_1/E_1| = |DE_2/E_2| = 2$ for otherwise $Z(S) = [E_1, D]$ by Lemma 3.4. Since D is non-abelian, $|DZ(S)/Z(S)| \geq 4$. Hence there exists $t_i \in (E_i \cap D) \setminus Z(S)$ for $i = 1, 2$. As $C_{E_{3-i}}(t_i) = Z(S)$ by the last line of Lemma 3.7, we have $C_S(t_i) = E_i$. Therefore $C_S(D) \leq E_1 \cap E_2 = Z(S)$. □

Lemma 3.9. *Suppose that X is quasisimple and $X/Z(X) \cong \mathrm{PSL}_3(4)$ and $S \in \mathrm{Syl}_2(X)$. If $Z(X)$ has an element of order 4, then $Z(S) \leq Z(X)$.*

Proof. See [7, Chapter 10, Lemma 2.3 (a)]. □

Lemma 3.10. *Suppose that X is a group with $F^*(X) \cong \mathrm{PSL}_3(2^e)$, $e \geq 1$. Let $T \in \mathrm{Syl}_2(X)$ and $S = T \cap F^*(X)$. Then*

- (i) *$F^*(X)$ possesses exactly two parabolic subgroups P_1, P_2 which contain S. For $i = 1, 2$, $E_i = O_2(P_i)$ is elementary abelian of order 2^{2e}, $O^{2'}(P_i/E_i) \cong \mathrm{SL}_2(2^e)$ and E_i is a natural module for $O^{2'}(P_i)$. Furthermore $S = E_1E_2$ and any involution in S is contained in $E_1 \cup E_2$.*
- (ii) *every elementary abelian normal subgroup of T is contained in S;*
- (iii) *$J(T) = J(S) = E_1E_2$.*

Proof. See [13, Lemmas D.2 and D.4]. □

Lemma 3.11. *Suppose that X is a group of Lie type in characteristic 2. If σ is an automorphism of X of order 2 which centralizes a Sylow 2-subgroup of X, then either σ is inner or $X \cong \mathrm{PSp}_4(2)'$.*

Proof. This follows from [3, Chapter 19] when $X \not\cong {}^2\mathrm{F}_4(q)$. For $X \cong {}^2\mathrm{F}_4(q)$ we can use [6, Theorem 9.1] for $q > 2$ and for $q = 2$ the result follows from [6, Theorems 2.5.12, 2.5.15 and 3.3.2]. □

Lemma 3.12. *Suppose that X is a group with $F^*(X) \cong \mathrm{PSL}_3(2^e)$, $e \geq 1$. Let $T \in \mathrm{Syl}_2(X)$ and $S = T \cap F^*(X)$. Then $C_T(S) = Z(S)$.*

Proof. Set $Y = C_T(S)$. Then Y is normalized by $B = N_{F^*(X)}(S)$. Let C be a Cartan subgroup of B, then $Y = C_Y(C)[Y, C]$ and $[Y, C] = Z(S)$. In particular, if $C_Y(C) \neq 1$, then $C_Y(C)$ contains an involution. This contradicts Lemma 3.11. □

Lemma 3.13. *Suppose that X is quasisimple with $X/Z(X) \cong \mathrm{PSL}_3(4)$ and $Z(X)$ elementary abelian. Let $T \in \mathrm{Syl}_2(\mathrm{Aut}(X))$, $S = T \cap X \in \mathrm{Syl}_2(X)$ and $\overline{X} = X/Z(X)$.*

(i) $\overline{S}$ *has exactly two elementary abelian subgroups* $\overline{E_1}$ *and* $\overline{E_2}$ *of order* 16*. Every involution of* $\overline{S}$ *is in* $\overline{E_1} \cup \overline{E_2}$*,* $\overline{S} = \overline{E_1 E_2} = J(\overline{T}) = \overline{J(T)}$ *and* $C_{\overline{S}}(x) = \overline{E_i}$ *for all* $x \in \overline{E_i} \setminus Z(\overline{S})$*. For* $i = 1, 2$*, let* E_i *be the preimage of* $\overline{E_i}$*. Then* E_i *is elementary abelian.*

(ii) $[S, E_1] = [S, E_2] = S' = Z(S) = E_1 \cap E_2 \geq Z(X)$*.*

(iii) *If* D *is a non-abelian normal subgroup of* S*, then* $[\overline{S}, \overline{D}] = Z(\overline{S}) = \overline{Z(S)} = C_{\overline{S}}(\overline{D})$*.*

(iv) *Every normal elementary abelian subgroup of* T *is contained in* S*.*

Proof. For part (i) and (iv) see Lemma 3.10 and [7, Chapter 10, Lemma 2.1 (h) and 2.2].

We now prove (ii). Since E_1 is elementary abelian, we may regard it as a GF(2)H-module for $H = N_X(E_1)$. As E_1 centralizes $\overline{E_1}$, Lemma 3.10 implies that $N_X(E_1)$ induces the natural $\mathrm{SL}_2(4)$-module on $\overline{E_1}$. We claim $[E_1, S] = Z(S) \geq Z(X)$. Certainly we have

$$[E_1, S] = [E_1, E_1 E_2] = [E_1, E_2] \leq E_1 \cap E_2 \leq Z(S)$$

and $[E_1, S]Z(X) = E_1 \cap E_2 = Z(S)$. To prove that $Z(X) \leq [E_1, S]$, we may suppose that $|Z(X)| = 2$. Suppose that $[E_1, S] < E_1 \cap E_2$. Then $[E_1, S] \cap Z(X) = 1$. For $x \in N_X(E_1) \setminus N_X(S)$, we have $N_X(E_1) = \langle S, S^x \rangle$ and so $[E_1, S][E_1, S^x]$ as order 2^4 and is normalized by $N_X(E_1)$. Since E_2 is elementary abelian, we obtain $S/[E_1, S][E_1, S^x]$ is elementary abelian. Hence $N_X(E_1)/[E_1, S][E_1, S^x]$ splits as $2 \times \mathrm{SL}_2(4)$. It follows that S splits over $Z(X)$ and we have a contradiction via Gaschütz's Theorem [8, (I.17.4)]. Hence (ii) holds.

For (iii), suppose that D is a non-abelian normal subgroup of S. Then $D \not\leq E_1$ and so $[\overline{E_1}, \overline{D}] = Z(\overline{S}) = \overline{Z(S)}$ as $\overline{E_1}$ is a natural $N_X(E_1)/E_1$-module. We now determine $C_{\overline{S}}(\overline{D})$. We have that $\overline{D}$ has order at least 16 and $\overline{D}$ contains $Z(\overline{S})$. If $\overline{D} \cap \overline{E_1} > Z(\overline{S})$, then $C_{\overline{S}}(\overline{D}) \leq C_{\overline{E_1}}(\overline{D}) = Z(\overline{S})$, the assertion. So assume $\overline{D} \cap \overline{E_1} = Z(\overline{S})$. Then $S = DE_1$ and $|\overline{D}| = 16$. Thus we can apply Lemma 3.4 to see that $D \geq [E_1, D] = [E_1, S] = Z(S)$. In particular $\overline{Z(S)} = Z(\overline{S}) = Z(\overline{D}) = C_{\overline{S}}(\overline{D})$, as claimed. □

Lemma 3.14. *Let* X *be quasisimple with* $X/Z(X) \cong \mathrm{PSL}_3(4)$ *and* $Z(X)$ *elementary abelian. Then* X *satisfies assumption (i) of Proposition* 2.7*.*

Proof. By [6, Corollary 5.1.4] we can lift $\mathrm{Aut}(X)$ to a group of automorphisms of the universal covering group of $X/Z(X)$ and then restrict it to a group X_1 such that $|Z(X_1)|$ is elementary abelian of order 4 and

$X_1/Z(X_1) \cong X/Z(X)$. Hence it is enough to prove the assertion when $|Z(X)| = 4$.

We follow the notation in Lemma 3.13. Set $P = N_X(E_1)$. Since E_1 is elementary abelian by Lemma 3.13 (ii) and $X/Z(X)$ has just one conjugacy class of involutions, there are no elements of X of order 4 with square in $Z(X)$. This is the condition (i)(a) of Proposition 2.7.

Assume that $\overline{x}$ is an involution in $X/Z(X)$ and let Y be the preimage of $\langle \overline{x} \rangle$. Then Y is elementary abelian of order 8. We will show that there is some $x \in Y \setminus Z(X)$ which is centralized by any automorphism of X centralizes $\overline{x}$. From [6, Table 6.3.1] we know $\mathrm{Out}(X/Z(X)) \cong \mathrm{Sym}(3) \times 2$ and acts on $Z(X)$ with an element of order three non-trivial. Since $\mathrm{Inn}(X)$ acts transitively on the involutions in $X/Z(X)$, $N_{\mathrm{Aut}(X)}(Y)\mathrm{Inn}(X) = \mathrm{Aut}(X)$. As $C_{\mathrm{Inn}(X)}(Y) = \overline{T}$, and $|Y| = 2^3$, the subgroup structure of $\mathrm{SL}_3(2)$ yields $N_{\mathrm{Aut}(X)}(Y)/C_{\mathrm{Aut}(X)}(Y) \cong \mathrm{Sym}(3)$. Let $\rho \in N_{\mathrm{Aut}(X)}(Y)$ have order three. Then $Y = [Y, \rho] \times \langle x \rangle$ and $\langle x \rangle$ is centralized by $N_{\mathrm{Aut}(X)}(Y)$. Thus x is a preimage of $\overline{x}$, which is centralized by any automorphism which normalizes Y. This element satisfies the assumption (i)(b) of Proposition 2.7. □

Lemma 3.15. *Suppose that $X \cong \mathrm{F}_4(q)$ with $q = 2^e$ and let R_1 be a long root subgroup and R_2 be a short root subgroup of X. For $i = 1, 2$, set $Q_i = O_2(N_X(R_i))$ and $L_i = O^{2'}(N_X(R_i)/Q_i)$. Then, for $i = 1, 2$, we have $L_i \cong \mathrm{Sp}_6(q)$ and $\Phi(Q_i) = R_i$. Furthermore, as L_i-modules, $Z(Q_i)/R_i$ is a natural module of dimension 6, $Q_i/Z(Q_i)$ is a spin module of dimension 8 and the modules $Z(Q_i)$ and Q_i/R_i are indecomposable.*

Proof. See [13, Lemma D.7]. □

Lemma 3.16. *Suppose that $X \cong \mathrm{F}_4(q)$ with $q = 2^e$, $S \in \mathrm{Syl}_2(X)$ and $\Omega_1(Z(S)) = R_1R_2$ with R_1 a long root subgroup of X and R_2 a short root subgroup of X. We use the notation introduced in Lemma 3.15 and additionally set $I_{12} = C_X(R_1R_2)$, $Q_{12} = O_2(I_{12})$ and $L_{12} = I_{12}/Q_{12}$. For $i = 1, 2$, define*

$$V_i = [Z(Q_i), Q_{12}]R_1R_2,$$

put $V_{12} = V_1V_2$ and $W_{12} = Z(Q_1)Z(Q_2)$.

Then the following hold:

(i) *$L_{12} \cong \mathrm{Sp}_4(q)$ and $Q_{12} = Q_1Q_2$.*
(ii) *V_{12} and W_{12} are normal in I_{12} and*

$$1 < R_1R_2 < V_{12} < W_{12} < Q_{12}.$$

In addition, we have $Z(Q_1) \cap Z(Q_2) = R_1R_2$, $Q_1 \cap Q_2 = V_{12}$ *is elementary abelian and, setting* $\overline{V_{12}} = V_{12}/R_1R_2$,

$$\overline{V_{12}} = \overline{V_1} \oplus \overline{V_2},$$

where $\overline{V_1}$ *and* $\overline{V_2}$ *are irreducible* L_{12}*-modules of* $\mathrm{GF}(q)$*-dimension* 4 *which are not isomorphic as* $\mathrm{GF}(2)L_{12}$*-modules. Furthermore, if* $q > 2$, $W'_{12} = R_1R_2$ *whereas, if* $q = 2$, $W'_{12} = \langle r_1r_2\rangle$ *where* $r_i \in R_i^{\#}$.

(iii) $[V_{12}, W_{12}] = 1$ *and* W_{12}/V_{12} *has order* q^2 *and is centralized by* L_{12}.

(iv) *We have*

$$Q_{12}/W_{12} \cong Q_1W_{12}/W_{12} \oplus Q_2W_{12}/W_{12},$$

Q_1W_{12}/W_{12} *and* Q_2W_{12}/W_{12} *are irreducible, non-isomorphic* L_{12}*-modules of* $\mathrm{GF}(q)$*-dimension* 4*. Furthermore, as* L_{12}*-modules, for* $i = 1, 2$,

$$Q_iW_{12}/W_{12} \cong V_{3-i}/R_1R_2.$$

(v) *We have*

$$Q_{12}/V_{12} = Q_1/V_{12} \oplus Q_2/V_{12}$$

is a direct sum of two indecomposable L_{12}*-modules of* $\mathrm{GF}(q)$*-dimension* 5*.*

(vi) *The group* $\mathrm{Aut}(Q_{12})$ *has a subgroup of index* 2 *which normalizes all of* R_1, R_2, Q_1, Q_2, $Z(Q_1)$, $Z(Q_2)$, V_{12} *and* W_{12}.

Proof. See [13, Lemma D.8]. □

Lemma 3.17. *Suppose that* $X \cong {}^2\mathrm{F}_4(q)$ *with* $q = 2^{2e+1}$, $S \in \mathrm{Syl}_2(X)$, R *is a long root subgroup in* $Z(S)$, $P = C_X(R)$ *and* $Q = O_2(P)$*. Then*

(i) $P/Q \cong {}^2\mathrm{B}_2(q)$.

(ii) $R = Z(Q)$, $Z_2(Q)$ *is elementary abelian and* $Z_2(Q)/R$ *is an irreducible* 4*-dimensional module for* P/Q.

(iii) $C_Q(Z_2(Q))$ *is non-abelian of order* q^6, $\Phi(C_Q(Z_2(Q))) = R$ *and* $Q/C_Q(Z_2(Q))$ *is the natural* P/Q*-module.*

(iv) *If* $q > 2$*, then* $Q/Z_2(Q)$ *is an indecomposable module.*

(v) *If* $q = 2$*, then* $F^*(X) = {}^2\mathrm{F}_4(2)'$ *has index* 2 *in* X*. We have that* $R = Z(O_2(P \cap F^*(X)))$, $Z_2(Q) = Z_2(Q \cap F^*(X))$ *and* $|(Q \cap F^*(X))/Z_2(Q)| = 16$*. Furthermore,* $(Q \cap F^*(X))/Z_2(Q)$ *and* $Z_2(Q)/R$ *admit* $P \cap F^*(X)$ *irreducibly.*

(vi) *Let* $P_1 = N_X(Z_2(S))$*. Then* P_1 *is a maximal parabolic subgroup of* X, $P_1 \neq P$, P_1 *normalizes* $Z_3(S)$ *which has order*

q^3 and P_1 induces $\mathrm{GL}_2(q)$ on $Z(O_2(P_1)) = Z_2(S)$. Furthermore for $q > 2$, we have $W = \langle (Z_4(S) \cap Z_2(Q))^{P_1} \rangle$ is elementary abelian and $W/Z_3(S)$ is the natural $\mathrm{GL}_2(q)$-module. Further $C_S(Z_3(S))/W$ is an irreducible 4-dimensional module for $\mathrm{SL}_2(q)$ and $O_2(P_1)/C_S(Z_3(S))$ is the natural $\mathrm{SL}_2(q)$-module.

Proof. For the structure of P see [6, Example 3.2.5, page 101] or [4, 12.9]. For part (vi) we refer to [4, 12.9]. □

Lemma 3.18. *Suppose that $X \cong \mathrm{G}_2(4)$, S is a Sylow 2-subgroup of X and R is a long root subgroup contained in $Z(S)$. Set $P = N_X(R)$, $Q = O_2(P)$ and $L = O^{2'}(P)$. Then $Z(Q) = R = Q'$, $L/Q \cong \mathrm{SL}_2(4) \cong \mathrm{Alt}(5)$, P acts irreducibly on Q/R while L induces a direct sum of two natural $\mathrm{Alt}(5)$-modules on Q/R. Furthermore, if $R < E \leq Q$ is normalized by L, then E is not abelian.*

Proof. See [13, Lemma D.10]. □

Lemma 3.19. *Suppose that X is quasisimple and $X/Z(X) \cong {}^2\mathrm{B}_2(8)$. Let S be a Sylow 2-subgroup of X. If $Z(X) \neq 1$, then $Z(S) = Z(X)$.*

Proof. We may assume that $|Z(X)| = 2$. There is an element ν of order 7 normalizing S such that $[Z(S/Z(X)), \nu] = Z(S/Z(X))$. Let Y be the preimage of $Z(S/Z(X))$. Then $|[Y, \nu]| = 8$. Assume $Z(S) > Z(X)$, then $Z(S) = Y$ as ν normalizes $Z(S)$. Now $[Y, \nu]$ is normal in S and so $S/[Y, \nu]$ is of order 16. Since $C_{S/[Y,\nu]}(\nu) = Y/[Y, \nu]$ and $S/[Y, \nu]$ is not extraspecial, $S/[Y, \nu]$ is elementary abelian. Thus $S = [S, \nu] \times Z(X)$ and Gaschütz's Theorem [8, (I.17.4)] provides a contradiction. Hence $Z(S) = Z(X)$. □

In the next lemma we adopt the notation introduced in [6, Table 4.5.1] for inner-diagonal and graph automorphisms of order 2 of groups of Lie type defined over fields of odd characteristic.

Lemma 3.20. *Suppose that p is an odd prime and K is quasisimple with $K/Z(K)$ a group of Lie type defined in characteristic p. Let $\alpha \in \mathrm{Aut}(K)$ be an automorphism of order 2. If $E(C_K(\alpha)) = 1$, then $C_K(\alpha)$ is soluble and one of the following holds where the bold face notation indicates an automorphism which centralizes a Sylow 2-subgroup of K.*

(i) *$K/Z(K) \cong \mathrm{PSL}_2(p^e)$ and either α is an inner-diagonal automorphism or $p^e = 9$ and α is a field automorphism;*
(ii) *$K \cong \mathrm{PSL}_3(3)$ and $\alpha \in \{\mathbf{t_1}, \gamma_1\}$;*
(iii) *$K \cong \mathrm{PSU}_3(3)$ and $\alpha \in \{\mathbf{t_1}, \gamma_1\}$;*
(iv) *$K/Z(K) \cong \mathrm{PSL}_4(3)$ and $\alpha \in \{\mathbf{t_2}, \gamma_2\}$;*
(v) *$K/Z(K) \cong \mathrm{PSU}_4(3)$ and $\alpha \in \{\mathbf{t_2}, \gamma_2\}$;*

(vi) $K/Z(K) \cong \mathrm{PSp}_4(3)$ *and* $\alpha \in \{\mathbf{t_1}, t_2, t_2'\}$*;*

(vii) $K/Z(K) \cong \mathrm{P}\Omega_7(3)$ *and* $\alpha = \mathbf{t_2}$*;*

(viii) $K/Z(K) \cong \mathrm{P}\Omega_8^+(3)$ *and* $\alpha = \mathbf{t_2}$*;*

(ix) $K/Z(K) \cong \mathrm{G}_2(3)$ *and* $\alpha = \mathbf{t_1}$*; or*

(x) $K \cong {}^2\mathrm{G}_2(3)'$ *and* $\alpha = \mathbf{t_1}$.

In particular, in all but case (i), $C_K(\alpha)$ *is a* $\{2,3\}$*-group and* $F^*(C_K(\alpha))$ *is a* 2*-group.*

Proof. If $K/Z(K) \cong \mathrm{PSL}_2(p^e)$, then we read the statement from [6, Table 4.5.1 and Proposition 4.9.1 (a) and (e)]. Suppose that $K/Z(K) \not\cong \mathrm{PSL}_2(p^e)$. Then [6, Table 4.5.1 and Proposition 4.9.1 (a) and (e)] yields $p^e = 3$ and that $C_K(\alpha)$ can only involve Lie components of type $\mathrm{A}_1(3)$ and $\mathrm{D}_2(3)$. This observation then leads to the groups listed. □

4. Elementary properties of the configuration

For the convenience of the reader we repeat the most important notions that we presented in the introduction. The group G is a $\mathcal{K}_2$-group, S is a Sylow 2-subgroup of G and Q is a large subgroup of S. This means $C_G(Q) \le Q$, $Q = O_p(N_G(Q))$ and $Q \trianglelefteq N_G(A)$ for all $1 \ne A \le Z(Q)$. We define

$$\mathcal{L}_G(S) = \{S \le L \le G \mid O_2(L) \ne 1, C_G(O_2(L)) \le O_2(L)\}$$

and for $L \in \mathcal{L}_G(S)$ with $L \not\le N_G(Q)$, set

$$L^\circ = \langle Q^L \rangle.$$

Denote by Y_L the largest normal elementary abelian subgroup of L such that $O_2(L/C_L(Y_L)) = 1$ and set $C_L = C_L(Y_L)$.

Let $\mathfrak{M}_G(S)$ be the subset of those $M \in \mathcal{L}_G(S)$, for which C_M is 2-closed, $C_M/O_2(M) \le \Phi(M/O_2(M))$ and $M^\dagger = MC_G(Y_M)$ is the only maximal element in $\mathcal{L}_G(S)$ with $M \le M^\dagger$. In particular, $Y_M = Y_{M^\dagger}$ by [11, Lemma 1.24 (h)].

Suppose that $M \in \mathfrak{M}_G(S)$ and $T \in \mathrm{Syl}_2(C_G(Y_M))$. Then Y_M is

- *tall*, if there exists K with $T \le K \le G$ such that $O_2(K) \ne 1$ and $Y_M \not\le O_2(K)$,
- *characteristic* 2*-tall* provided there is some K with $T \le K \le G$ such that $C_K(O_2(K)) \le O_2(K)$ and $Y_M \not\le O_2(K)$, and
- *asymmetric* in G, if whenever $g \in G$ and $[Y_M, Y_M^g] \le Y_M \cap Y_M^g$, then $[Y_M, Y_M^g] = 1$.

We intend to prove the theorem of this paper by contradiction. Specifically, we work under the following hypothesis.

Hypothesis 4.1. *The group G is a $\mathcal{K}_2$-group, $S \in \mathrm{Syl}_2(G)$ is contained in at least two maximal 2-local subgroups and $Q \leq S$ is a large subgroup of G. Furthermore, there exists $M \in \mathfrak{M}_G(S)$ such that $M \not\leq N_G(Q)$ and Y_M is asymmetric and tall but not characteristic 2-tall.*

In this section we collect the rudimentary facts about the configuration of Hypothesis 4.1.

Lemma 4.2. *Suppose that $Q \leq K \leq G$ and $O_2(K) \neq 1$. Then*

(i) *$C_G(O_2(K))$ is a 2-group;*
(ii) *K has characteristic 2; and*
(iii) *If, in addition, $O_2(C_M) \leq K$, then $Y_M \leq O_2(K)$.*

In particular, $Y_M \leq Q$.

Proof. Parts (i) and (ii) are [11, Lemma 1.55 (a)]. Taking $T = S \cap C_G(Y_M) = O_2(M)$, part (iii) follows from the fact that Y_M is not characteristic 2-tall. The final statement follows from (iii) by taking $K = N_G(Q)$. □

Lemma 4.3. *If $1 \neq R \leq O_2(M)$ is normalized by M, then*

$$M \leq N_G(R) \leq M^\dagger.$$

Proof. We have that $M \leq N_G(R)$. By assumption $M^\dagger$ is the unique maximal element in $\mathcal{L}_G(S)$, which contains M. As $N_G(R) \in \mathcal{L}_G(S)$ by Lemma 4.2(ii), $N_G(R) \leq M^\dagger$. □

Recall that if X is a group, $A \leq B \leq X$, then A is *weakly closed* in B with respect to X provided whenever $x \in X$ and $A^x \leq B$, then $A^x = A$.

Lemma 4.4. *The following hold:*

(i) *$O_2(M) \in \mathrm{Syl}_2(C_G(Y_M))$;*
(ii) *Q is weakly closed in S with respect to G;*
(iii) *$O_2(M)$ is weakly closed in S with respect to G;*
(iv) *$Y_M = \Omega_1(Z(O_2(M))) \geq \Omega_1(Z(S))$;*
(v) *$O_2(\langle \mathcal{L}_G(S)\rangle) = 1$;*
(vi) *if $N \geq O_2(M)$ with $O_2(N) \neq 1$ and N has characteristic 2, then $Y_M \leq O_2(N)$;*
(vii) *if U is a 2-group which is normalized by $O_2(M)$, then $U \leq M^\dagger$, in particular, U normalizes Y_M;*
(viii) *$M^\circ = \langle Q^{M^\circ}\rangle$ and $[C_G(Y_M), M^\circ] \leq O_2(M^\circ)$.*

Proof. The first four parts come from [11, Lemma 2.2(b), (e), (f) and (e) and Lemma 2.6(b)].

Part (v) follows from the fact that $\mathcal{L}_G(S)$ contains at least two maximal members and part (vi) is a consequence of Y_M being non-characteristic 2-tall and part (i).

For (vii) consider $O_2(M)U$, which is a 2-group. By (iii) we have that $O_2(M)$ is normal in $UO_2(M)$ and so $U \leq N_G(O_2(M)) \leq M^\dagger$. Thus $Y_{M^\dagger}$ is normalized by U, since $Y_M = Y_{M^\dagger}$ by definition, U normalizes Y_M.

Part (viii) follows from [11, Lemmas 1.46 (c) and 1.52 (c)]. □

We can now formulate the fact that Y_M is not characteristic 2-tall, in terms of $O_2(M)$: if $K \geq O_2(M)$ with $O_2(K) \neq 1$ and $Y_M \not\leq O_2(K)$, then $F^*(K) \neq O_2(K)$.

Lemma 4.5. *If X is a non-trivial 2-group normalized by Q, then X does not centralize $O^2(M^\circ)$.*

Proof. Suppose that $O^2(M^\circ) \leq C_G(X)$. As Q normalizes X, we have $Z(Q) \cap X \neq 1$ and so $O^2(M^\circ) \leq N_G(Z(Q) \cap X)$. As Q is large, $O^2(M^\circ) \leq N_G(Q)$. Therefore,

$$Q = \langle Q^{O^2(M^\circ)} \rangle = \langle Q^{QO^2(M^\circ)} \rangle = \langle Q^{M^\circ} \rangle = M^\circ$$

by Lemma 4.4 (viii). Thus $M \leq N_G(Q)$, which is a contradiction. □

The next lemma plays a very important role in the proof of our theorem.

Lemma 4.6. *There exists $y \in Y_M^\#$ such that $F^*(C_G(y)) \neq O_2(C_G(y))$. That is $C_G(y)$ does not have characteristic 2. In particular, $M^\dagger$ does not act transitively on $Y_M^\#$.*

Proof. The first statement is [11, Theorem F (page 131)]. The remaining part follows as $\Omega_1(Z(S)) \cap Y_M \neq 1$ and the centralizer of this group has characteristic 2 by Lemma 4.2 (ii). □

Lemma 4.7. *Suppose that $y \in Y_M^\#$ and $1 \neq R_1 \leq R \leq C_G(y)$ are 2-groups.*

- (i) *If $Q \leq N_G(R_1)$ and $R \leq N_G(Q)$, then $N_G(R)$, $C_G(R)$, $N_{C_G(y)}(R)$ and $C_{C_G(y)}(R)$ have characteristic 2.*
- (ii) *If $Z(Q) \cap R \neq 1$, then $N_G(R)$, $C_G(R)$, $N_{C_G(y)}(R)$ and $C_{C_G(y)}(R)$ have characteristic 2.*
- (iii) *If R is a non-trivial subgroup of $O_2(M)$ which is normalized by M, then $N_{C_G(y)}(R)$ and $C_{C_G(y)}(R)$ have characteristic 2.*
- (iv) *$N_{C_G(y)}(Y_M)$ has characteristic 2.*
- (v) *If $z \in Z(Q)^\#$ and J is a subnormal subgroup of $C_G(y)$, then $C_J(z)$ has characteristic 2.*

(vi) *If* $z \in Z(Q)^{\#}$ *and* K *is a component of* $C_G(y)$, *then* $C_K(z)$ *has characteristic* 2.

Proof. Suppose the hypotheses of (i) hold. Then, as R_1 is normalized by Q, we have $R_1 \cap Z(Q) \neq 1$. As $R_1 \leq R$ also $R \cap Z(Q) \neq 1$. Set $K = N_G(R \cap Z(Q))$. Then $Q \leq K$ and K has characteristic 2. As R normalizes Q, it also normalizes $R \cap Z(Q)$ and so $R \leq K$. Furthermore $C_G(R) = C_K(R)$. Now application of Lemma 2.1 (with $C = R \cap Z(Q)$, $X = G$ and $B = R$) yields $N_G(R)$ and $C_G(R)$ have characteristic 2.

Since $y \in N_G(R)$, $N_{C_G(y)}(R) = C_{N_G(R)}(y)$ and $C_{C_G(y)}(R) = C_{C_G(R)}(y)$ have characteristic 2 by Lemma 2.1 (with $C = 1$, $B = \langle y \rangle$ and $X = N_G(R)$, $X = C_G(R)$, respectively). This proves (i).

For (ii) take $R_1 = R \cap Z(Q)$, for (iii) take $R_1 = R$, and then apply (i).

Part (iv) is a special case of (iii).

For (v), we take $R = R_1 = \langle z \rangle$ and use (i) to get $C_{C_G(y)}(z)$ has characteristic 2. By Lemma 4.2, $Y_M \leq Q$ and so $[y, z] = 1$. As this property passes to subnormal subgroups by Lemma 2.2, we have $C_J(z)$ has characteristic 2. Part (vi) follows from (v). □

The next lemma is often used to help conclude that $|Y_M|$ small.

Lemma 4.8. *Suppose that* $J \leq G$ *is normalized by* $O_2(M)$ *and* J *has characteristic* 2. *Then* $Y_M \trianglelefteq O_2(JY_M)$ *and* $\langle Y_M^J \rangle$ *is elementary abelian.*

Proof. We have $O_2(M)J$ has characteristic 2 and $O_2(M) \in \mathrm{Syl}_2(C_G(Y_M))$ by Lemma 4.4(i). Since Y_M is not characteristic 2-tall, $Y_M \leq O_2(O_2(M)J)$. Hence

$$Y_M \leq O_2(O_2(M)J) \cap Y_M J \leq O_2(JY_M).$$

By Lemma 4.4(vii), Y_M is normal in $O_2(JY_M)$ and, as Y_M is asymmetric, we also have $\langle Y_M^J \rangle$ is elementary abelian. □

Define

$$U_Q = \langle Y_M^{N_G(Q)} \rangle.$$

Lemma 4.9. *The following hold:*

(i) $Y_M \leq U_Q \leq Q \cap O_2(M)$ *and* U_Q *is elementary abelian;*
(ii) $1 \neq \Omega_1(Z(Q)) \cap Y_M < Y_M$; *and*
(iii) $Y_M \not\leq [U_Q, Q] < U_Q$.

Proof. Since $N_G(Q) \in \mathcal{L}_G(S)$, $Y_M \leq Q$ by Lemma 4.2(iii) and U_Q is elementary abelian by Lemma 4.8. Thus $U_Q \leq C_Q(Y_M) = Q \cap C_S(Y_M) = Q \cap O_2(M)$ by Lemma 4.4 (i). This is (i).

If $Y_M \leq \Omega_1(Z(Q))$, then $M \leq M^\dagger \leq N_G(Y_M) \leq N_G(Q)$ as Q is large. This is against the choice of M and so (ii) holds.

As Q acts on U_Q, we have $[Q, U_Q] < U_Q$. As $[Q, U_Q]$ is normal in $N_G(Q)$, we get $Y_M \not\leq [Q, U_Q]$, which is (iii). □

Lemma 4.10. *Assume $N_G(Q) \leq M^\dagger$. Then there is at least one $L \in \mathcal{L}_G(S)$ such that $Y_L \not\leq Y_M$.*

Proof. Assume that for all $L \in \mathcal{L}_G(S)$, $Y_L \leq Y_M$. Then $O_2(M) \leq C_M \leq C_L$. As $O_2(M) \leq S \cap C_L$ and $O_2(M)$ is weakly closed in S by Lemma 4.4, we have that $N_L(S \cap C_L) \leq N_L(O_2(M)) \leq M^\dagger$ by Lemma 4.3. As $C_L \leq N_G(Q) \leq M^\dagger$ by assumption, we get $L = N_L(S \cap C_L)C_L \leq M^\dagger$. Hence $\langle \mathcal{L}_G(S) \rangle \leq M^\dagger$ and this contradicts Lemma 4.4 (v). Thus there exists $L \in \mathcal{L}_G(S)$ with $Y_L \not\leq Y_M$. □

We use the previous lemma as follows:

Lemma 4.11. *There exists an elementary abelian normal subgroup of S contained in $O_2(M)$ which strictly contains Y_M. In particular, $Y_M \neq \Omega_1(O_2(M))$, $O_2(M)$ is not abelian and $Y_M \neq J(O_2(M))$.*

Proof. Suppose that Y_M is a maximal elementary abelian subgroup of $O_2(M)$, which is normal in S. By Lemma 4.9 (i), $Y_M \leq U_Q \leq O_2(M)$ and U_Q is elementary abelian. Thus $U_Q = Y_M$ and so $N_G(Q) \leq M^\dagger$. Let $L \in \mathcal{L}_G(S)$, then by Lemma 4.2(iii) $Y_M \leq O_2(L)$ and so $[Y_L, Y_M] = 1$. Hence $Y_L \leq C_S(Y_M) = O_2(M)$ by Lemma 4.4(i). As Y_L is normal in S, we have $Y_L \leq Y_M$ by assumption. Now Lemma 4.10 yields a contradiction. This proves the first claim. Furthermore $Y_M < \Omega_1(O_2(M))$.

As $Y_M \neq \Omega_1(O_2(M))$, $Y_M \neq J(O_2(M))$. That $O_2(M)$ is not abelian, follows as $Y_M = \Omega_1(Z(O_2(M)))$ by Lemma 4.4 (iv). □

We finish this section with a look at what happens when Y_M has small order.

Lemma 4.12. *We have $|Y_M| \geq 16$.*

Proof. Assume false. Since $1 \neq \Omega_1(Z(Q)) \cap Y_M \leq Y_M$, Lemma 4.9(ii) implies $|Y_M| = 4$ or 8. By Lemma 4.6 $M^\dagger/C_M$ cannot act transitively on $Y_M^\#$.

If $|Y_M| = 4$, then $|M^\dagger/C_M| = 2$, but by the definition of $Y_M = Y_{M^\dagger}$ we have that $O_2(M^\dagger/C_M) = 1$, a contradiction.

Thus $|Y_M| = 8$. Then $M^\dagger/C_M$ is a subgroup of $\mathrm{SL}_3(2)$ and, as $M^\dagger$ does not act transitively on $Y_M^\#$, $M^\dagger/C_M$ is a $\{2,3\}$-group. In particular, $M^\dagger/C_M$ is soluble. As $O_2(M^\dagger/C_M) = 1$, we have that $M^\dagger/C_M \cong \mathrm{Sym}(3)$ or is cyclic of order 3. In both cases there is some $w \in Y_M^\#$, with $M^\dagger \leq C_G(w)$. In particular $[w, Q] = 1$ and so, as Q is large, $M^\dagger \leq C_G(w) \leq N_G(Q)$, a contradiction. □

5. The components of $C_G(y)$

By Lemma 4.6, there is some $y \in Y_M^{\#}$ such that $F^*(C_G(y)) \neq O_2(C_G(y))$. In this section we show that we can carefully select y such that $C_G(y)$ has a structure which can be used to reach a contradiction in the sections which will follow.

Lemma 5.1. *Assume that $z \in Z(Q)$ and $y \in C_S(z)$ are involutions. Then $C_{C_G(y)}(z)$ has characteristic 2 and z inverts $O(C_G(y))$. Furthermore, if K is a component of $C_G(y)$ which is normalized by z, then $K = [K, z]$ and, if z induces an inner automorphism on K, then $Z(K)$ is a 2-group.*

Proof. By Lemma 4.2(ii), $C_G(z)$ has characteristic 2 and therefore so does $C_{C_G(y)}(z)$ by Lemma 2.1. In particular $C_{O(C_G(y))}(z) = 1$ and so z inverts $O(C_G(y))$.

Suppose that K is a component of $C_G(y)$ which is normalized by z. If z centralizes K, then K is a component of $C_{C_G(y)}(z)$, a contradiction. Hence z acts non-trivially on K and so $K = [K, z]$. Finally, if z induces as an inner automorphism of K, then $K\langle z\rangle = KC_{K\langle z\rangle}(K) \leq C_G(Z(K))$. As z inverts $O(C_G(y))$, we infer that $Z(K)$ is a 2-group. □

The next lemma is of fundamental importance.

Lemma 5.2. *There exists $y \in Y_M^{\#}$ such that $E(C_G(y)) \neq 1$.*

Proof. There exists $z \in C_{Y_M}(S)^{\#} \subseteq C_{Y_M}(Q)^{\#}$ and, for such z, $C_G(z)$ has characteristic 2 by Lemma 4.2 (ii). Furthermore, $|Y_M| \geq 16$ by Lemma 4.12.

Assume that for all $y \in Y_M^{\#}$, $E(C_G(y)) = 1$. Then Lemmas 2.3 and 4.6 imply that

$$\Sigma = \langle O(C_G(b)) \mid b \in Y_M^{\#}\rangle \neq 1$$

has odd order.

Because $M^\dagger$ permutes the elements of $Y_M^{\#}$, Σ is normalized by $M^\dagger$. Since $C_{M^\dagger}(\Sigma)$ is normal in $M^\dagger$ and $F^*(M^\dagger)$ is a 2-group and is a maximal 2-local subgroup of G implies that $C_{M^\dagger}(\Sigma) = 1$. In addition, as $O_2(C_G(z)) \leq S \leq M$,

$$[C_\Sigma(z), O_2(C_G(z))] \leq \Sigma \cap O_2(C_G(z)) = 1$$

and so z inverts Σ. Hence $[z, M^\dagger] \leq C_{M^\dagger}(\Sigma) = 1$ and so $z \in Z(M^\dagger)$. But then $M^\dagger \leq C_G(z) \leq N_G(Q)$, a contradiction. □

From now on we focus our interest on the following subset of elements of Y_M:

$$\mathcal{Y} = \{y \in Y_M^{\#} \mid E(C_G(y)) \neq 1\},$$

which by Lemma 5.2 is non-empty. We also put

$$\mathcal{Y}_S = \{y \in \mathcal{Y} \mid C_S(y) \in \mathrm{Syl}_2(C_G(y))\}.$$

From among all the components that appear in $C_G(y)$ for $y \in \mathcal{Y}$ select C such that first $|C/Z(C)|$ is maximal and second that $|C|$ is maximal. Then for $y \in \mathcal{Y}$ set

$$E_y = \langle J \mid J \text{ is a component of } C_G(y), J/Z(J) \cong C/Z(C) \text{ and } |J| = |C|\rangle.$$

Let

$$\mathcal{Y}^* = \left\{ y \in \mathcal{Y} \;\middle|\; \begin{array}{ll} (a) & \text{the number of components in } E_y \text{ is maximal} \\ (b) & |E_y| \text{ maximal} \end{array} \right\},$$

and

$$\mathcal{Y}_S^* = \mathcal{Y}^* \cap \mathcal{Y}_S.$$

Lemma 5.3. *For $y \in \mathcal{Y}$, there exists $g \in M$ such that $y^g \in \mathcal{Y}_S$.*

Proof. As $O_2(M) \le C_G(Y_M) \le C_G(y)$, we may choose $R \in \mathrm{Syl}_2(C_G(y))$ such that $O_2(M) \le R$. Then $R \le M^\dagger$ by Lemma 4.4 (vii). Since $S \in \mathrm{Syl}_2(M^\dagger)$, there exists $h \in M^\dagger$ such that $R^h \le S$. Hence $R^h = C_S(y^h) \in \mathrm{Syl}_2(C_G(y^h))$. As $M^\dagger = MC_{M^\dagger}(Y_M)$ we have $h = gh_1$ with $g \in M$ and $h_1 \in C_{M^\dagger}(Y_M)$. Now $y^h = y^g$. This proves the claim. □

By Lemma 5.3 every member of $\mathcal{Y}^*$ is conjugate to an element of $\mathcal{Y}_S^*$, thus $\mathcal{Y}_S^* \neq \emptyset$.

Lemma 5.4. *Suppose that $y \in \mathcal{Y}_S$ and K is a component of $E(C_G(y))$. If $w \in C_{C_S(y)}(K)$ is an involution, then $K \le E(C_G(w))$. In particular, if $w \in C_{Y_M}(K)^\#$, then $w \in \mathcal{Y}$.*

Proof. Set $X = C_G(w)$. Then

$$K \le E(C_X(y)) \le L_{2'}(C_X(y)) \le L_{2'}(X)$$

by $L_{2'}$-balance [5, Theorem 5.17]. By Lemma 5.1, z inverts $O(L_{2'}(X))$ for $z \in \Omega_1(Z(S))^\#$. Since z centralizes $C_S(y) \in \mathrm{Syl}_2(C_G(y))$, z normalizes K and so $[K, z] = K$ by Lemma 5.1. Since z inverts $O(L_{2'}(X))$, we also have $K = [K, z] \le C_X(O(L_{2'}(X)))$. Thus

$$K \le C_{L_{2'}(X)}(O(L_{2'}(X))) = E(X)Z(O(L_{2'}(X)))O_2(L_{2'}(X)).$$

by Lemma 2.4. We conclude that $K \le E(X)$, as claimed. □

Lemma 5.5. *Suppose that $y \in \mathcal{Y}_S$, $w \in \mathcal{Y}$ and K is a component in E_y which is centralized by w. Then either K is a component of E_w or $K \le J_1J_2$ where J_1 and $J_2 = J_1^y$ are components of E_w, $J_1/Z(J_1) \cong K/Z(K)$ and $|J_1| = |K|$. In particular, $K \le E_w$.*

Proof. By Lemma 5.4, $K \leq E(C_G(w))$. Let $J = \langle K^{E(C_G(w))} \rangle$. Then J is a product of components of $C_G(w)$. By [5, Theorem 5.24 (ii)], $\langle y \rangle$ acts transitively on the components of $C_G(w)$ in J. If J is a component of $E(C_G(w))$, then the maximal selection of K implies that $K = J$ and so $K \leq E_w$. So suppose that $J = J_1^y J_1$. Then $K \cap J_1$ is centralized by y and so $K \cap J_1 \leq J_1^y \cap J_1$. Thus

$$K/(K \cap J_1) \cong KJ_1/J_1 \leq J_1^y J_1/J_1 \cong J_1^y/(J_1^y \cap J_1).$$

In particular, the maximal choice of $|K/Z(K)|$ implies that $K/Z(K) \cong J_1/Z(J_1)$. Moreover, we calculate

$$|K||J_1^y \cap J_1| \leq |J_1^y||K \cap J_1| \leq |J_1^y||J_1^y \cap J_1|$$

and so from the maximal choices of $|K|$ we deduce that $|K| = |J_1^y| = |J_1|$. Thus, by definition, $K \leq J \leq E_w$, and this completes the proof. □

Lemma 5.6. *Suppose that $y \in \mathcal{Y}_S^*$ and $w \in C_{Y_M}(E_y)^\#$. Then $E_y = E_w$. In particular, $w \in \mathcal{Y}^*$.*

Proof. By Lemma 5.4, $w \in \mathcal{Y}$ and then, by Lemma 5.5, $E_y \leq E_w$. The maximal choice of $|E_y|$ shows $E_y = E_w$. In particular $w \in \mathcal{Y}^*$. □

For $y \in \mathcal{Y}_S^*$, define

$$S_y = C_S(y) \cap E_y \in \mathrm{Syl}_2(E_y); \text{ and}$$
$$T_y = C_{C_S(y)}(E_y).$$

Observe that Lemma 5.6 implies that $(Y_M \cap T_y)^\# \subseteq \mathcal{Y}^*$.

Lemma 5.7. *If $y \in \mathcal{Y}_S$ and $F \leq E(C_G(y))$ is a component of $C_G(y)$, then $C_{C_S(y)}(F) \cap Z(Q) = 1$. In particular, $Z(Q) \cap T_y = 1$.*

Proof. This follows by Lemma 5.1. □

Lemma 5.8. *Suppose that $y \in \mathcal{Y}_S^*$ is chosen with $|C_S(y)|$ maximal. Then $C_S(y) \in \mathrm{Syl}_2(N_G(E_y))$. In particular, $C_S(y) = N_S(E_y)$ and $T_y = C_S(E_y)$*

Proof. Plainly $C_S(y) \leq N_G(E_y)$. Assume that $R \in \mathrm{Syl}_2(N_G(E_y))$ with $R > C_S(y)$ and pick $t \in N_R(C_S(y)) \setminus C_S(y)$. As t normalizes $C_S(y) \geq O_2(M)$, Lemma 4.4 (iii) and (iv) imply that t normalizes Y_M. Hence $\langle t \rangle C_S(y)$ normalizes $Y_M \cap C_{C_S(y)}(E_y) \geq \langle y \rangle$. Thus there exists $w \in (Y_M \cap C_{C_S(y)}(E_y))^\#$ which is centralized by $\langle t \rangle C_S(y)$. Lemma 5.6 implies $w \in \mathcal{Y}^*$ and then the maximal choice of $|C_S(y)|$ together with Lemma 5.3 provide a contradiction. Therefore $C_S(y) \in \mathrm{Syl}_2(N_G(E_y))$ and this proves the main claim. It follows at once that $C_S(y) = N_S(E_y)$ and $C_S(E_y) = T_y$. □

Lemma 5.9. *Let $y \in \mathcal{Y}_S^*$ with $|C_S(y)|$ maximal. Then $N_S(T_y) = C_S(y)$.*

Proof. Assume the statement is false and choose $t \in N_S(C_S(y)) \setminus C_S(y)$ with $T_y^t = T_y$. Then t normalizes $U = Z(C_S(y)) \cap T_y \cap Y_M$. As $y \in U$, $U \neq 1$. Hence there is $1 \neq w \in U$ such that $w^t = w$. Since $w \in T_y$, $E_y = E_w$ by Lemma 5.6. But then, by Lemma 5.8, $t \in N_S(E_w) = N_S(E_y) = C_S(y)$, a contradiction. □

Suppose that W is a group. A subgroup H of W is called a *trivial intersection subgroup* in W provided that H is not normal in W and, for all $g \in W \setminus N_W(H)$, we have $H \cap H^g = 1$. The following lemma will play an important role in the proof of our theorem.

Lemma 5.10. *Suppose that $y \in \mathcal{Y}_S^*$ is chosen with $|C_S(y)|$ maximal. Then T_y is a trivial intersection subgroup in S and $T_y \cap O_2(M)$ is a trivial intersection subgroup in $N_G(O_2(M))$.*

Proof. By Lemma 5.7, $Z(S) \cap T_y = 1$. Hence T_y is not normal in S and also $T_y \cap O_2(M)$ is not normal in $N_G(O_2(M)) \geq S$. Suppose that $g \in N_G(O_2(M))$ and assume $T_y \cap T_y^g \neq 1$. Since $O_2(M) = O_2(M)^g$, $O_2(M)$ normalizes $T_y \cap T_y^g$. Therefore Lemma 4.4 (iv) implies $Y_M \cap T_y \cap T_y^g \neq 1$. Pick $w \in (Y_M \cap T_y \cap T_y^g)^\#$. Then, by Lemma 5.6,

$$E_y = E_w.$$

As $y^g \in \mathcal{Y}_{S^g}^*$, and $w \in T_y^g$, we also obtain by Lemma 5.6

$$E_{y^g} = E_w$$

and therefore

$$E_y^g = E_{y^g} = E_w = E_y.$$

Hence, as $g \in N_G(O_2(M))$, using Lemma 5.8 for the first and last equality yields

$$T_y^g \cap O_2(M) = C_{O_2(M)}(E_y)^g = C_{O_2(M)}(E_{y^g}) = C_{O_2(M)}(E_y) = T_y \cap O_2(M).$$

This proves the $T_y \cap O_2(M)$ is a trivial intersection subgroup in $N_G(O_2(M))$. If, in fact, $g \in S \leq N_G(O_2(M))$, then, again using Lemma 5.8, we have

$$T_y^g = C_S(E_y)^g = C_S(E_{y^g}) = C_S(E_y) = T_y$$

which shows that T_y is a trivial intersection subgroup in S. □

Lemma 5.11. *Suppose that $y \in \mathcal{Y}_S^*$ is chosen with $|C_S(y)|$ maximal. Assume that $X \leq Y_M$ is normalized by $C_S(y)Q$. Then*

$$|X \cap T_y|^2 \leq |X| \leq |XT_y/T_y|^2.$$

In particular, these bounds hold for $X = [Q, y]$ and $X = Y_M$.

Proof. As, by Lemma 4.2(ii), $y \notin Z(Q)$, we can choose $t \in N_Q(C_S(y)) \setminus C_S(y)$ with $t^2 \in C_S(y)$. If T_y is normalized by t, then $T_y \cap Y_M \geq \langle y \rangle$ is normalized by $C_S(y)\langle t \rangle$ and so by Lemma 5.6 there exists $w \in \mathcal{Y}^*$ with $|C_S(w)| \geq 2|C_S(y)|$, a contradiction. Hence $t \notin N_S(T_y)$ and so

$$(T_y \cap X) \cap (T_y \cap X)^t \leq T_y \cap T_y^t = 1$$

by Lemma 5.10. Thus $|X \cap T_y|^2 \leq |X|$. As

$$|X \cap T_y| = |(X \cap T_y)^t| = |(X \cap T_y)^t(X \cap T_y)/(X \cap T_y)| \leq |XT_y/T_y|,$$

we also obtain

$$|X| = |XT_y/T_y||(X \cap T_y)| \leq |XT_y/T_y|^2.$$

Since Y_M and $[Q, y] \leq Y_M$ are both normalized by $QC_S(y)$, the displayed bounds apply to these subgroups. □

Lemma 5.12. *Assume that $y \in \mathcal{Y}_S^*$ and K is a component of E_y. Suppose that $N_G(S_yT_y)$ has characteristic 2. Then $Y_M \leq O_2(N_G(S_yT_y))$. In particular, if $N_{E_y}(S_y) > S_yZ(E_y)$, then Y_M normalizes K.*

Proof. We have that $O_2(M)$ normalizes T_yS_y. Hence the first assertion follows from Lemma 4.8.

Let K be a component of E_y and $X = S_y \cap K$. Then by hypothesis $N_K(X) > XZ(K)$. Let $w \in N_K(X) \setminus XZ(K)$ have odd order, then $w \in N_G(T_yS_y)$ and so, for $t \in Y_M$, $[w, t] \in O_2(N_G(S_yT_y))$. However, if $K \neq K^t$, then w and w^t commutes and so $[w, t] = w^{-1}w^t$ has odd order. We conclude that Y_M normalizes K. □

Definition 5.13. *Assume that W is a normal subgroup of a group X. Then W has the* Sylow centralizer property in X *provided that for $T \in \mathrm{Syl}_2(X)$ and $R = W \cap T \in \mathrm{Syl}_2(W)$,*

$$C_{TC_X(W)/C_X(W)}(RC_X(W)/C_X(W)) \leq WC_X(W)/C_X(W).$$

Lemma 5.14. *Assume that $y \in \mathcal{Y}_S^*$ and that every component K of E_y has the Sylow centralizer property in $N_{C_G(y)}(K)$. Then $\Omega_1(Z(S)) \leq S_yT_y \in \mathrm{Syl}_2(E_yT_y)$ and $N_G(S_yT_y)$ has characteristic 2.*

Proof. Since $\Omega_1(Z(S))$ normalizes every component in E_y and they each satisfy the Sylow centralizer property in $C_G(y)$, we have $\Omega_1(Z(S)) \leq S_yT_y$. The result now follows from Lemma 2.1. □

Lemma 5.15. *Suppose that $y \in \mathcal{Y}_S^*$ with $|C_S(y)|$ is maximal and $E_y = K$ is quasisimple and satisfies the Sylow centralizer property in $C_G(y)$. Assume that $C_G(x)$ has characteristic 2 for all $x \in \Omega_1(Z(S_y)) \setminus Z(K)$. Then T_y is isomorphic to a subgroup of $Z(S_y)/(S_y \cap Z(K))$.*

Proof. Lemma 5.7 implies that $S > N_S(T_y)$. Let $g \in N_S(N_S(T_y)) \setminus N_S(T_y)$ with $g^2 \in N_S(T_y)$. Then $T_yT_y^g \le N_S(T_y) = N_S(T_y^g)$ and, as $T_y \ne T_y^g$, Lemma 5.10 implies

$$[T_y, T_y^g] \le T_y \cap T_y^g = 1.$$

In particular, as $y \in T_y$, $T_y^g \le C_S(y)$ and so normalizes $E_y = K$ and thus also S_y. Assume that $T_y^g \cap S_y \ne 1$. Then, as S_y normalizes T_y^g, $T_y^g \cap \Omega_1(Z(S_y)) \ne 1$. By Lemma 5.4 the centralizer of every involution in T_y is not of characteristic 2. The hypothesis on elements of $\Omega_1(Z(S_y))$ implies that $T_y^g \cap \Omega_1(Z(S_y)) \le Z(K) \cap S_y \le C_{C_S(y)}(K) = T_y$. As $T_y \cap T_y^g = 1$, we have a contradiction. Hence $T_y^g \cap S_y = 1$. As T_y^g is normalized by $N_S(T_y) \ge S_y$, we have $[T_y^g, S_y] \le T_y^g \cap S_y = 1$. Thus the Sylow centralizer property in $C_G(y)$ yields

$$T_y^g \le T_y Z(S_y).$$

As $T_y \cap T_y^g = 1$, we conclude that T_y is abelian and isomorphic to a subgroup of $Z(S_y)/(S_y \cap Z(K)) = Z(S_y)/(S_y \cap T_y)$. □

Next, for $y \in \mathcal{Y}$, we study the action of $O_2(M)$ and Y_M on the components of $C_G(y)$.

Lemma 5.16. *Assume that $y \in \mathcal{Y}$ and K is a component of $E(C_G(y))$. If Y_M does not normalize K, then $K/Z(K)$ has elementary abelian Sylow 2-subgroups.*

Proof. We may assume that $y \in \mathcal{Y}_S$. Then Y_M is an abelian normal subgroup of $C_S(y) \in \mathrm{Syl}_2(C_G(y))$ which does not normalize K. Hence Lemma 2.5 provides the result. □

Lemma 5.17. *Suppose that $y \in \mathcal{Y}$ and K is a component of $C_G(y)$. If $Z(Q) \cap K \ne 1$, then $F^*(C_G(y)) = KO_2(C_G(y))$ and $O(C_G(y)) = 1$.*

Proof. Since $Z(Q) \cap K \ne 1$, we can select $z \in (\Omega_1(Z(Q)) \cap K)^\#$. As $z \in K$, z centralizes $O(C_G(y))$ as well as any component J of $C_G(y)$ with $J \ne K$. Applying Lemma 5.1 proves the claim. □

Lemma 5.18. *Suppose that $y \in \mathcal{Y}_S$ and K is a component of $E(C_G(y))$ which is normalized by Y_M. Assume that $S \cap Z(K) = 1$. Then either $S \cap K \le O_2(M)$ or $O_2(M)$ normalizes K.*

Proof. If $S \cap K$ centralizes Y_M, then $S \cap K \le C_S(Y_M) = O_2(M)$. Suppose that Y_M does not centralize $S \cap K$. Then $1 \ne [Y_M, S \cap K] \le S \cap K$ and $[Y_M, S \cap K]$ is centralized by $O_2(M)$. Thus, for $m \in O_2(M)$, $[Y_M, S \cap K] \le K \cap K^m$. If $K \ne K^m$ this yields $[Y_M, S \cap K] \le S \cap Z(K) = 1$, a contradiction. Thus K is normalized by $O_2(M)$. □

Lemma 5.19. *Suppose that $y \in \mathcal{Y}_S$ and $z \in \Omega_1(Z(S))^\#$. Assume that K is a component of $C_G(y)$ and $C_K(z)$ is not a 2-group. Then $C_Q(y)$ normalizes K. In particular, Y_M normalizes K.*

Proof. Assume that the lemma is false. As z inverts $O(C_G(y))$ by Lemma 5.1, z inverts $O(C_G(y)) \cap Z(K)$ and so, as $C_K(z)$ is not a 2-group, there is an odd prime r and $R \in \mathrm{Syl}_r(C_K(z))$ with $R \not\leq Z(K)$. Assume that $C_Q(y)$ does not normalize K. Then there exists $b \in C_Q(y)$ such that $K^b \neq K$. Because K is a component of $C_G(y)$, $[K, K^b] = 1$. Since $C_G(z) \leq N_G(Q)$ and $b \in Q$, we have

$$C_K(z)Q = (C_K(z)Q)^b = C_{K^b}(z)Q.$$

In particular, as $C_K(z)C_{K^b}(z) \leq C_K(z)Q$, $R \in \mathrm{Syl}_r(C_K(z)C_{K^b}(z))$ and so $RR^b = R \leq K \cap K^b \leq Z(K)$, a contradiction. Hence $C_Q(y)$ normalizes K. As $Y_M \leq C_Q(y)$, Y_M also normalizes K. □

Next we show that in many situations $E(C_G(y))$ is quasisimple.

Lemma 5.20. *Suppose that $y \in \mathcal{Y}_S$, $z \in \Omega_1(Z(S))^\#$ and K is a component of $C_G(y)$. Assume there is a non-trivial subgroup $J \leq C_K(z)$ such that*

(a) *$J = O^2(J)$ is normalized by $C_Q(y)$; and*
(b) *$[Q, y]$ is centralized by J,*

Then

(i) *Q normalizes J and $1 \neq Z(Q) \cap [Q, J] \leq K$; and*
(ii) *$F^*(C_G(y)) = KO_2(C_G(y))$.*

In particular, assumption (b) holds if, for all $W \leq Y_M$ with W normalized by J, we have $[W, J] = 1$.

Proof. By (a) and Lemma 5.19, $C_Q(y)$ normalizes K and, as $z \in Z(Q)$ and Q is large, $J = O^2(J) \leq N_G(Q)$ and $[Q, J] \neq 1$. Set $W = [Q, y]$. Then (b) implies $[W, J] = 1$ and, as $[J, y] = 1$, we have

$$[Q, J, y] = 1$$

by the Three Subgroups Lemma. In particular, as $J = O^2(J)$ and $C_Q(y)$ normalizes $C_K(z)$ by (a),

$$[Q, J] = [Q, J, J] \leq [C_Q(y), J] \leq J.$$

Because $[Q, J] \neq 1$ and $[Q, J]$ is normalized by Q, we have that

$$1 \neq Z(Q) \cap [Q, J] \leq Z(Q) \cap J \leq K.$$

Thus $Z(Q) \cap K \neq 1$ and so $F^*(C_G(y)) = KO_2(C_G(y))$ follows from Lemma 5.17. This proves (i) and (ii).

Now suppose for all $W \leq Y_M$ with W normalized by J, we have $[W, J] = 1$. Then as $[Q, y] \leq Y_M$ and is normalized by J, (b) holds. □

Lemma 5.21. *Suppose that $y \in \mathcal{Y}_S$ and $z \in \Omega_1(Z(S))^\#$. Let K be a component of $C_G(y)$ and set $L_K = C_K(z)$. Assume that L_K is not a 2-group. Then $C_G(Y_M)C_Q(y)$ normalizes K. Furthermore, if $Y_M \cap K \le C_K(O^2(L_K))$, then $F^*(C_G(y)) = KO_2(C_G(y))$.*

Proof. By Lemma 5.19, Y_M normalizes K. Assume that $Y_M \cap K \not\le Z(K)$. Then, for $m \in C_G(Y_M)$, $K^m \cap K \ge Y_M \cap K$. Hence $K^m = K$, and so $C_G(Y_M)$ normalizes K. Thus the main assertion holds in this case.

If $Y_M \cap K \le Z(K)$, then $Y_M \cap K \le C_K(O^2(L_K))$. Hence suppose that $Y_M \cap K \le C_K(O^2(L_K))$ and set $L_1 = O^2(L_K)$. If $W \le Y_M$ is normalized by L_1, then, as W normalizes K,

$$[W, L_1] \le W \cap K \le Y_M \cap K \le C_K(L_1).$$

Thus $[W, L_1] = [W, L_1, L_1] = 1$. Lemma 5.19 now provides the hypothesis for Lemma 5.20 which in turn yields $F^*(C_G(y)) = KO_2(C_G(y))$. In particular, $C_G(Y_M)$ normalizes K. This completes the proof. □

Lemma 5.22. *Suppose that $z \in \Omega_1(Z(S))^\#$, $y \in \mathcal{Y}_S$ and K is a component of $C_G(y)$. Assume that $J \le C_K(Y_M)$, $O_2(M)$ normalizes J and J is not a 2-group. Set $\widetilde{J} = JC_{C_S(y)}(K)/C_{C_S(y)}(K)$. Then the following hold*

(i) $M^\circ \le N_G(O^2(J))$.

(ii) *There exist distinct non-central $O^2(J)$-chief factors in*

$$O_2(O^2(J))/\Phi(O_2(O^2(J)))$$

which are isomorphic as $O^2(J)$-modules. In particular, $O_2(J)$ has at least two non-central $O^2(J)$-chief factors.

(iii) $F^*(C_G(y)) = KO_2(C_G(y))$ *and* $O(C_G(y)) = 1$.

(iv) $|Z(\widetilde{O^2(J)})| \ne 2$.

Proof. Lemma 4.4 states $[C_G(Y_M), M^\circ] \le O_2(M)$ and so $JO_2(M)$ is normalized M°. Hence, as J is normalized by $O_2(M)$, $O^2(JO_2(M)) = O^2(J)$ is normalized by M°. This is (i).

We have that $O^2(J) \ne 1$ by hypothesis. Further, by (i), Q normalizes $O^2(J)$. As $O^2(J)$ normalizes Q, we have $[Q, O^2(J)] \le Q \cap O^2(J)$. As $[Q, O^2(J)] \ne 1$, we have that $Q \cap O^2(J) \ne 1$ and so $O_2(O^2(J)) \ne 1$.

Assume that (ii) is false. Then the non-central $O^2(J)$-chief factors in $O_2(O^2(J))/\Phi(O_2(O^2(J)))$ are pairwise non-isomorphic. Since $Q \le M^\circ$ normalizes $O^2(J)$ and $O^2(J) \le N_G(Q)$, Lemma 2.6 shows that

$$[O_2(J), O^2(J)]/\Phi([O_2(J), O^2(J)])$$

is centralized by Q. Since M° operates on this factor, we conclude from Burnside's Lemma that $O_2(O^2(J))$ is centralized by $O^2(M^\circ)$. This contradicts Lemma 4.5 and completes the proof of (ii).

We have that $[Q, O^2(J)]$ is a non-trivial normal subgroup of Q contained in K. It follows that $Z(Q) \cap K \neq 1$. Part (iii) follows from Lemma 5.17.

Suppose that $|Z(\widetilde{O^2(J)})| = 2$. Observe that $Z(K)$ is a 2-group by (iii). Then, as $O^2(J)$ centralizes $Z(K)$,

$$|Z(O^2(J))Z(K)/Z(K)| = |Z(O^2(J)) : Z(O^2(J)) \cap Z(K)| \leq 2.$$

By (i), M° normalizes $Z(O^2(J))$. If $Z(K) \cap Z(O^2(J)) = 1$, then M° centralizes $Z(O^2(J))$ and this contradicts Lemma 4.5. So assume that $Z(K) \cap Z(O_2(J)) \neq 1$.

As $Z(K) \leq T_y$, Lemma 5.7 implies $Z(K) \cap Z(Q) = 1$. In particular $Z(K) \cap Z(O^2(J))$ is not normalized by Q and so there exists $x \in Q$ such that

$$(Z(K) \cap O^2(Z(J)))(Z(K) \cap Z(O^2(J)))^x = Z(O^2(J)).$$

Since T_y is a trivial intersection subgroup in S by Lemma 5.10, we conclude that $Z(O^2(J))$ has order 4. As $Z(O^2(J))$ is normalized by Q, we have that $Z(O^2(J))$ contains elements in $Z(K)^\#$ and elements in $Z(Q)^\#$. By Lemma 5.4, these elements are not conjugate in G, hence $O^2(M^\circ)$ centralizes $Z(O^2(J))$ and again we have a contradiction to Lemma 4.5. This proves (iv). $\square$

The next lemma will be used when K is a group of Lie type in characteristic 2 and also for some situations when K is a sporadic simple group. Recall that U_Q is defined by $U_Q = \langle Y_M^{N_G(Q)} \rangle$, U_Q is elementary abelian and $U_Q \leq C_Q(y)$ for all $y \in Y_M$ by Lemma 4.9.

Lemma 5.23. *Suppose that $z \in \Omega_1(Z(S))^\#$, $y \in \mathcal{Y}_S$ and K is a component of $C_G(y)$. Set $L_K = C_K(z)$ and $J_K = C_{O^2(L_K)}(Z(O_2(O^2(L_K))))$. If $O_2(O^2(L_K))$ is non-abelian, then $O^2(J_K)$ does not act irreducibly on $O_2(O^2(L_K))/Z(O_2(O^2(L_K)))$.*

Proof. Set $Z = Z(O_2(O^2(L_K)))$. Then J_K centralizes Z. By Lemma 4.7 (v), $F^*(L_K) = O_2(L_K)$. Suppose that

$$O^2(J_K) \text{ acts irreducibly on } O_2(O^2(L_K))/Z.$$

Since $O_2(O^2(L_K))$ is non-abelian, $O_2(O^2(L_K))/Z$ is not cyclic and so $O^2(J_K) \neq 1$. In particular, $O^2(L_K) \neq 1$ and so Lemma 5.21 yields

$$C_G(Y_M)C_Q(y) \text{ normalizes } K.$$

As $O^2(J_K) \leq L_K \leq C_G(z) \leq N_G(Q)$, $O^2(J_K)$ normalizes $U_Q = \langle Y_M^{N_G(Q)} \rangle$. Using $U_Q \leq C_Q(y)$ and $C_Q(y)$ normalizes K, yields U_Q normalizes K. Furthermore, U_Q normalizes $L_K = C_K(z)$ and therefore also $O^2(J_K)$ and so does $C_Q(y)$. It follows that

$$[U_Q, O^2(J_K)] \leq U_Q \cap O^2(J_K) \leq O_2(O^2(J_K)) \leq O_2(O^2(L_K)).$$

Since $[U_Q, O^2(J_K)]$ is normalized by $O^2(J_K)$, $O^2(J_K)$ acts irreducibly on $O_2(O^2(L_K))/Z$ and $O_2(O^2(L_K))$ is non-abelian, but U_Q is abelian, we get that $[U_Q, O^2(J_K)] \leq Z$. Therefore,

$$[U_Q, O^2(J_K), O^2(J_K)] \leq [Z, O^2(J_K)] = 1.$$

Hence U_Q is centralized by $O^2(J_K)$ and thus

$$O^2(J_K) \leq C_G(Y_M).$$

Since $C_G(Y_M)C_Q(y)$ normalizes K, $O_2(M)$ normalizes J_K. As $O_2(J_K)$ has exactly one non-central $O^2(J_K)$-chief factor, Lemma 5.22 (ii) provides the final contradiction. □

Lemma 5.24. *Suppose that $z \in \Omega_1(Z(S))^\#$, $y \in \mathcal{Y}_S$ and K is a component of $C_G(y)$. Set $L_K = C_K(z)$. Assume that L_K is not a 2-group, $O_2(O^2(L_K))$ is elementary abelian and contains exactly one non-central $O^2(L_K)$-chief factor. If Q normalizes $O^2(L_K)$, then $[O_2(L_K), O^2(L_K)] \leq Z(Q)$.*

Proof. Since Q normalizes $O^2(L_K)$ and L_K normalizes Q, $[Q, O^2(L_K)] \leq O_2(O^2(L_K))$. The result follows from Lemma 2.6. □

Lemma 5.25. *Assume that $z \in \Omega_1(Z(S))^\#$, $y \in \mathcal{Y}$ and K is a component of $C_G(y)$ which is normalized by Y_M. Then, setting $\widetilde{KN_{C_S(y)}(K)} = KN_{C_S(y)}(K)/C_{C_S(y)}(K)$, we have*

$$|\widetilde{Y_M}| \geq |\widetilde{\langle z^M \rangle}| > 2.$$

Proof. Let K be a component of $C_G(y)$ and assume that

$$|\widetilde{\langle z^M \rangle}| \leq 2.$$

Then, for all $m \in M$, $z^m \in Y_M^\#$ and Lemma 5.1 implies z^m acts non-trivially on K. Therefore

$$\widetilde{z} = \widetilde{z^m}$$

for all $m \in M$. Hence

$$[z, m] = z^{-1}z^m \in C_{C_S(y)}(K)$$

for all $m \in M$. Therefore $[\langle z \rangle, M] \leq C_{C_S(y)}(K)$. If $[\langle z \rangle, M] \neq 1$, then K is a component of $N_{C_G(y)}([\langle z \rangle, M])$ and this contradicts Lemma 4.7(iii).

Hence $M \leq C_G(z) \leq N_G(Q)$ and this is a contradiction to our basic assumption. $\square$

Lemma 5.26. *Suppose that $z \in \Omega_1(Z(S))^\#$ and $y \in \mathcal{Y}_S^*$. Let K be a component of $\widetilde{C_G(y)}$ and $L_K = C_K(z)$. Assume $L_K/Z(L_K)$ is not a 2-group and set $\widetilde{KN_{C_S(y)}(K)} = KN_{C_S(y)}(K)/C_{C_S(y)}(K)$.*

(i) *If $|\widetilde{Y_M}| \leq 4$, then $F^*(C_G(y)) = KO_2(C_G(y))$ and $O^2(L_K)$ is normalized by Q.*
(ii) *If $|\Omega_1(C_{\widetilde{KN_{C_S(y)}(K)}}(O^2(\widetilde{L_K}))) \cap \widetilde{Y_M}| = 2$, then $|\widetilde{Y_M}| \geq 8$.*

Proof. Suppose that $|\widetilde{Y_M}| \leq 4$. Assume that $W \leq Y_M$ is normalized by $O^2(L_K)$. If $|\widetilde{W}| = 2$, then $O^2(L_K)$ centralizes W and, if $|\widetilde{W}| = 4$, then, as $O^2(L_K)$ centralizes $\widetilde{\langle z\rangle}$, again $O^2(L_K)$ centralizes W. Lemma 5.19 implies Lemma 5.20(a) holds. Hence Lemma 5.20 yields $K = E(C_G(y))$ and $O^2(L_K)$ is normalized by Q. In particular (i) holds.

To prove (ii), assume that $|\widetilde{Y_M}| \leq 4$ and set $X = C_{Y_M}(O^2(L_K))$. Then, by (i), $K = E(C_G(y))$, X is normalized by Q and it is also normalized by $C_S(y)$. In particular, $C_{C_S(y)}(K) = T_y$. Since X is elementary abelian, $|XT_y/T_y| \leq 2$ holds by assumption. Using Lemma 5.11 yields $|X| \leq 4$. As $\widetilde{z} \in \widetilde{X}$ and $y \in C_{Y_M}(K) \leq X$, we deduce that $|X| = 4$ and $|C_{Y_M}(K)| = 2$. Hence $|Y_M| = |\widetilde{Y_M}||C_{Y_M}(K)| \leq 8$ and this contradicts Lemma 4.12. Therefore (ii) holds. $\square$

Lemma 5.27. *Suppose that $y \in Y_M^\#$ and K is a component of $C_G(y)$. Let P be a 2-local subgroup of K, and assume that both K and P are normalized by $O_2(M)$. Set $\widetilde{KN_{C_S(y)}(K)} = KN_{C_S(y)}(K)/C_{C_S(y)}(K)$. If P is of characteristic 2, then $\widetilde{Y_M} \leq O_2(\widetilde{PO_2(M)})$.*

Proof. Set $H = PO_2(M)C_{C_S(y)}(K) \leq C_G(y)$. Then $O^2(C_H(O_2(H))) \leq P$ and so $O^2(C_H(O_2(H))) \leq C_P(O_2(P)) \leq O_2(P)$, as P has characteristic 2. Hence H has characteristic 2 and so Lemma 4.8 gives $Y_M \leq O_2(H)$. Therefore $\widetilde{Y_M} \leq O_2(\widetilde{H})$. $\square$

6. The standard setup and consolidation of notation

Throughout the remainder of this paper Hypothesis 4.1 holds. We pick and fix $y \in \mathcal{Y}_S^*$ with $|C_S(y)|$ maximal. We continue the notation

$$S_y = C_S(y) \cap E_y \text{ and } T_y = C_{C_S(y)}(E_y)$$

where E_y is as defined just before Lemma 5.4. Recall that $C_S(y)$ is a Sylow 2-subgroup of $C_G(y)$ by the definition of $\mathcal{Y}_S$ and so S_y is a Sylow 2-subgroup of E_y. Furthermore by Lemma 5.8 we have that $C_S(E_y) =$

$C_{C_S(y)}(E_y)$. The subgroup K represents an arbitrary component of E_y. We denote by $\tilde{}$ the projection

$$\tilde{} : KN_{C_S(y)}(K) \to KN_{C_S(y)}(K)/C_{C_S(y)}(K).$$

Thus $\widetilde{K} = KC_{C_S(y)}(K)/C_{C_S(y)}(K) \cong K/Z(K)$. By Lemma 4.11, $O_2(M)' \neq 1$ and, by Lemma 4.4(iv), $Y_M = \Omega_1(Z(O_2(M)))$. Hence we will fix an involution

$$z \in \Omega_1(Z(S)) \cap Y_M \cap O_2(M)' \leq Z(Q).$$

Since z centralizes $C_S(y)$ and so S_y, z normalizes K. We know from Lemma 5.1 that $K = [K, z]$ and z inverts $O(C_G(y))$. We set

$$L_K = C_K(z).$$

Obviously, $L_K \leq C_G(z) \leq N_G(Q)$ and so $[Q, y] \leq Y_M$ is normalized by L_K. Furthermore, if L_K is not a 2-group, Lemma 5.21 implies that $C_G(Y_M)C_Q(y)$ normalizes K and, in particular, $O_2(M)$ normalizes L_K. We will often require the subgroup

$$U_Q = \langle Y_M^{N_G(Q)} \rangle$$

which is elementary abelian and contained in $Q \cap O_2(M)$ by Lemma 4.9.

The next five sections investigate the various possibilities for the isomorphism type of $K/Z(K)$.

7. Sporadic groups as components

The aim of this section is to show that $K/Z(K)$ cannot be a sporadic simple group or the Tits group ${}^2\mathrm{F}_4(2)'$. We begin with Ru and ${}^2\mathrm{F}_4(2)'$.

Lemma 7.1. *$K/Z(K) \not\cong {}^2\mathrm{F}_4(2)'$ or* Ru.

Proof. We first provide some structural detail about the groups $X^* \cong {}^2\mathrm{F}_4(2)'$, ${}^2\mathrm{F}_4(2)$ and Ru. Suppose that x is a 2-central involution in X^*. Then by Lemma 3.17 and [16, page 65] the centralizer $X = C_{X^*}(x)$ has the following normal subgroup structure:

$$1 \leq X_1 < X_2 \leq X_3 < X_4 = O_2(X) < X,$$

where $|X_1| = 2$, $X_1 = Z(X_4)$, X_2 is elementary abelian of order 32, $X_3 = C_X(X_2)$, $X_2 = \Omega_1(X_3)$ and $O^2(X)$ acts irreducibly on X_2/X_1 and on X_4/X_3 each of order 16. Furthermore,

$$X_3 \cong \begin{cases} X_2 & X^* \cong {}^2\mathrm{F}_4(2)' \\ 4 \times 2^4 & X^* \cong {}^2\mathrm{F}_4(2) \\ \mathrm{Q}_8 \times 2^4 & X^* \cong \mathrm{Ru}. \end{cases}$$

Finally, if $X^* \cong {}^2\mathrm{F}_4(2)$ or ${}^2\mathrm{F}_4(2)'$, then $X/X_4 \cong {}^2\mathrm{B}_2(2) \cong \mathrm{Frob}(20)$, while, if $X^* \cong \mathrm{Ru}$, then $X/X_4 \cong \mathrm{Sym}(5)$.

Recall the Sylow centralizer property from Definition 5.13. By [6, Table 5.3r], $\mathrm{Aut}(\mathrm{Ru}) = \mathrm{Ru}$ and so, when $K/Z(K) \cong \mathrm{Ru}$, the Sylow centralizer property holds for K in $N_{C_G(y)}(K)$. We read from [6, Theorem 2.5.12 and Theorem 2.5.15] that $\mathrm{Aut}({}^2\mathrm{F}_4(2)') = {}^2\mathrm{F}_4(2) = \mathrm{Aut}({}^2\mathrm{F}_4(2))$. As presented above for $X^* \cong {}^2\mathrm{F}_4(2)$, we have that $X_1 = Z(X_4) = Z(S)$. Thus the Sylow centralizer property also holds when $K/Z(K) \cong {}^2\mathrm{F}_4(2)'$.

Suppose $K/Z(K) \cong {}^2\mathrm{F}_4(2)'$ or Ru. Notice that either $Z(K) = 1$ or $K \cong 2{\cdot}\mathrm{Ru}$. As $L_K \geq S_y$, L_K projects mod $Z(K)$ onto X as described above (in the cases $X^* \cong \mathrm{Ru}$ and $X^* \cong {}^2\mathrm{F}_4(2)'$). For $1 \leq i \leq 4$, we define $B_i \leq L_K$ to be the preimage of the subgroup X_i. Since L_K is not a 2-group, Lemma 5.21 implies that K is normalized by $C_G(Y_M)C_Q(y)$. We have

$$\widetilde{KO_2(M)} \cong \mathrm{Ru}, {}^2\mathrm{F}_4(2)' \text{ or } {}^2\mathrm{F}_4(2)$$

and, as $C_{\widetilde{KN_{C_S(y)}(K)}}(\widetilde{O^2(L_K)}) = \widetilde{B_1}$ has order 2, Lemma 5.26 (ii) implies that

$$|\widetilde{Y_M}| \geq 8.$$

Suppose that $W \leq Y_M$ is normalized but not centralized by $O^2(L_K)$. Then $\widetilde{B_2} \leq \widetilde{W} \leq \widetilde{Y_M}$ and so $|\widetilde{Y_M}| \geq 2^5$. Hence, as $X_2 = \Omega_1(C_X(X_2))$,

$$\widetilde{Y_M} = \widetilde{B}_2.$$

Now we have

$$\Omega_1(C_{\widetilde{O_2(M)(S\cap K)}}(\widetilde{Y_M})) = \Omega_1(\widetilde{B}_3) = \widetilde{B}_2 = \widetilde{Y_M}.$$

It follows that $\Omega_1(O_2(M))C_{C_S(y)}(K) = Y_M C_{C_S(y)}(K)$, which means that $[\Omega_1(O_2(M)), O_2(M)] \leq C_{C_S(y)}(K)$. Since K does not centralize any element of $Z(Q)$ by Lemma 5.7, we have $\Omega_1(O_2(M)) = \Omega_1(Z(O_2(M))) = Y_M$ and this contradicts Lemma 4.11. Therefore

$O^2(L_K)$ centralizes every subgroup of Y_M which it normalizes.

By Lemma 5.20

$$F^*(C_G(y)) = KO_2(C_G(y)) \text{ and } Q \text{ normalizes } O^2(L_K).$$

Select $g \in N_Q(C_S(y)) \setminus C_S(y)$. We have $T_yT_y^g$ is centralized by $K \cap K^g$ and $K \cap K^g \geq O^2(L_K)$ as Q normalizes $O^2(L_K)$. Hence

$$\widetilde{T_y^g} \leq C_{\widetilde{KC_S(y)}}(O^2(\widetilde{L_K})) = \widetilde{B_1}$$

which has order 2. As $T_y^g \cap T_y = 1$ by Lemma 5.10 and Lemma 5.9, we conclude that $|T_y| = 2$.

Suppose that $K/Z(K) \cong \mathrm{Ru}$. Then by [6, Table 5.3r] there is a 2-local subgroup J of K containing S_y with

$$J/Z(K) \sim 2^{3+8}.\mathrm{SL}_3(2)$$

and J is normalized by $O_2(M)$. Hence Lemma 4.8 implies that $Y_M \le O_2(JY_M)$ and $\langle Y_M^J\rangle$ is an elementary abelian. Now the structure of J and the fact that $O_2(J/Z(K))$ is non-abelian implies that $|\widetilde{\langle Y_M^J\rangle}| = 2^3$. Hence $\widetilde{\langle Y_M^J\rangle} = \widetilde{Y_M}$ and $\widetilde{O_2(M)} \le O_2(\widetilde{J})$. Thus $O_2(M) \le O_2(O_2(M)J)$ and Lemma 4.4 implies J normalizes $O_2(M)$ and so also Y_M. Lemma 4.3 yields $J \le M^\dagger$ and J induces $\mathrm{SL}_3(2)$ on $Y_M/\langle y\rangle$. As $M^\dagger$ does not act transitively on $Y_M^\#$ and $O_2(M^\dagger/C_M) = 1$, the subgroup structure of $\mathrm{SL}_4(2)$ yields $M^\dagger = JC_M$. But then $C_{Y_M}(M^\dagger) = \langle y\rangle$, a contradiction as $y \notin Z(S)$. Hence $K/Z(K) \not\cong \mathrm{Ru}$.

Suppose that $K \cong {}^2\mathrm{F}_4(2)'$. As $T_y = \langle y\rangle$, $Y_MK = T_yK = \langle y\rangle \times K$ and so $Y_M \cap K$ has index 2 in Y_M. Since $F^*(C_G(y)) = KO_2(C_G(y)) = K\langle y\rangle$, we have $C_G(y) = KC_S(y)$. Because L_K normalizes Q, L_K normalizes $U_Q = \langle Y_M^{N_G(Q)}\rangle$ which is elementary abelian. Again we have $U_QK = \langle y\rangle \times K$ and so $U_Q \cap K$ is an elementary abelian subgroup of K normalized by L_K. We deduce that $K \cap U_Q = B_2 \ge Y_M \cap K$. Since $Y_M \cap K = C_{B_2}(O_2(M))$, $O_2(M)L_K/O_2(O_2(M)L_K) \cong \mathrm{Frob}(20)$ and $B_2/Z(O_2(L_K))$ is an irreducible 4-dimensional L_K-module, either $O_2(M) \le O_2(O_2(M)L_K)$ or $|Y_M \cap K| = 8$. In the former case $L_K \le M^\dagger$ as $O_2(M)$ is weakly closed in S. But then L_K normalizes Y_M and this contradicts $O^2(L_K)$ centralizing every subgroup of Y_M that it normalizes. Hence

$$|Y_M \cap K| = 8 \text{ and } |Y_M| = 16.$$

By Lemma 3.17 (iv), $|Z_2(S_y)| = 4$ and so $Z_2(S_y) \le Y_M$. In particular, $P_1 = N_K(Z_2(S_y))$ normalizes $C_{O_2(M)K}(Z_2(S_y)) \ge O_2(M)$ and so $O_2(M)$ and Y_M are normalized by P_1.

Since $O_2(M)' \ne 1$, we have $O_2(M)' \cap Y_M \ne 1$. As $O_2(M)' \le K$, we have $Y_M \cap O_2(M)'$ is either $Y_M \cap K$ or $Z_2(S_y)$. If $Y_M \cap O_2(M)' = Z_2(S_y)$, then MC_M/C_M embeds into the stabilizer of a 2-space in $\mathrm{SL}_4(2)$, which is isomorphic to $2^4.(\mathrm{Sym}(3)\times\mathrm{Sym}(3))$. Since $P_1/C_{P_1}(Y_M \cap K) \cong \mathrm{Sym}(4)$, this means that $O_2(MC_M/C_M) \ne 1$, a contradiction. Therefore M normalizes $Y_M \cap K = Y_M \cap O_2(M)'$ and, as $O_2(MC_M/C_M) = 1$ and MC_M/C_M is isomorphic to a subgroup $\mathrm{SL}_3(2)$, again using $P_1/C_{P_1}(Y_M \cap K) \cong \mathrm{Sym}(4)$ yields $MC_M/C_M \cong \mathrm{SL}_3(2)$. Now y^M has size 1, 7 or 8. In the first two cases y is centralized by a conjugate of S, a contradiction. In the latter case, y is centralized by an element of order 7 in M and this contradicts the fact that $|K|$ is coprime to 7. Hence $K \not\cong {}^2\mathrm{F}_4(2)'$. □

Proposition 7.2. *$K/Z(K)$ is not a sporadic simple group.*

Proof. We use the information from [6, Table 5.3] to see that K satisfies the Sylow centralizer property Definition 5.13 in $N_{C_G(y)}(K)$. Hence z induces a 2-central involution on K. By Lemma 4.7 (v), $F^*(L_K)$ is a 2-group and, in particular, L_K does not have a component. It follows that $K/Z(K)$ is not J_1, Co_3, McL, LyS, O'N or M(23). By Lemma 5.23, if $O_2(O^2(L_K/Z(K)))$ has derived group and Frattini subgroup of order 2, then $O^2(L_K)$ does not act irreducibly on $O_2(O^2(L_K))/Z(O_2(O^2(L_K)))$. Using [6, Table 5.3] shows that $K/Z(K) \not\cong \mathrm{Mat}(11)$, J_2, J_3, J_4, Co_1, Co_2, Suz, M(22), M(24)$'$, F_1, F_2, F_3, or F_5. Because of Lemma 7.1 the groups which remain to be considered are

$$K/Z(K) \cong \mathrm{Mat}(12), \mathrm{Mat}(22), \mathrm{Mat}(23), \mathrm{Mat}(24), \mathrm{HS}, \text{ and He}.$$

Using [6, Table 5.3] we observe that L_K is not a 2-group. In particular, $C_G(Y_M)C_Q(y)$ normalizes K by Lemma 5.21.

(7.2.1) Either $|\widetilde{Y_M}| \geq 8$ or $K/Z(K) \cong \mathrm{Mat}(22)$ and $\widetilde{Y_M} \not\leq \widetilde{K}$.

Using [6, Table 5.3] we see that $C_{\widetilde{KN_{C_S(y)}(K)}}(\widetilde{O^2(L_K)})$ has order 2 unless $K/Z(K) \cong \mathrm{Mat}(22)$ in which case it has order 4 and is not contained in $\widetilde{K}$. Hence Lemma 5.26 gives the result. ■

Suppose that $K/Z(K) \cong \mathrm{HS}$. Then $L_K/Z(K)$ has shape $(2^{1+4}_+ \circ 4).\mathrm{Sym}(5)$. As $L_K \leq N_G(Q)$ and $\langle Y_M^{L_K} \rangle \leq U_Q \cap O_2(L_K)Y_M$ is elementary abelian, we obtain from [6, Table 5.3m] that Y_M projects into $\Omega_1(Z(L_K/Z(K)))$. Thus $|Y_M/C_{Y_M}(K)| = 2$, contrary to (7.2.1)

Assume that $K/Z(K)$ is one of Mat(22), Mat(23), Mat(24) or He. Let $J \in \mathcal{L}_K(S \cap K)$ be normalized by $O_2(M)$. Then $Y_M \leq O_2(JY_M)$ and $\langle Y_M^J \rangle$ is elementary abelian by Lemma 4.8. Hence

(7.2.2) $Y_M \leq \bigcap_{\substack{J \in \mathcal{L}_K(S\cap K);\\ O_2(M) \leq N_G(J)}} O_2(JY_M)$.

Assume that $K \cong$ He or Mat(24). Then $S \cap K$ is isomorphic to a Sylow 2-subgroup of $\mathrm{SL}_5(2)$. Hence $S \cap K$ has exactly two elementary abelian subgroups E_1, E_2 of order 64 and they intersect in a group of order 2^4. Also note that $O_2(L_K)$ is the unique extraspecial subgroup of order 2^7 in $S \cap K$. For $i = 1, 2$, set $J_i = N_K(E_i)$. If J_1 and J_2 are conjugate in $KO_2(M)$, then $\widetilde{KO_2(M)} \cong \mathrm{Aut(He)}$ and $L_KO_2(M)$ acts irreducibly on $O_2(L_K)/Z(O_2(L_K))$. Hence $O_2(L_K) \leq \langle Y_M^{L_KO_2(M)} \rangle \leq U_Q$ which is a contradiction as U_Q is abelian. Therefore $O_2(M)$ normalizes

J_1 and J_2, and, as J_1 and J_2 have characteristic 2, we get by (7.2.2)

$$\widetilde{Y_M} \leq \widetilde{E_1} \cap \widetilde{E_2} \cap \widetilde{O_2(L_K)} = Z_2(\widetilde{S \cap K}).$$

Since $|\langle Z_2(S \cap K)\rangle| = 8$, (7.2.1) gives $\widetilde{Y_M} = Z_2(\widetilde{S \cap K})$. However, $\langle Z_2(\widetilde{S \cap K})^{\widetilde{L}_K}\rangle = \widetilde{O_2(L_K)}$ which is not abelian whereas $\langle \widetilde{Y_M}^{\widetilde{L}_K}\rangle \leq \widetilde{U}_Q$ which is abelian. As this is impossible, we conclude $K/Z(K) \ncong \mathrm{Mat}(24)$ or He.

Assume next that $K/Z(K) \cong \mathrm{Mat}(22)$ or $\mathrm{Mat}(23)$. Then from [6, Table 5.3c and 5.3d], $(S \cap K)/Z(K)$ has two elementary abelian subgroups $E_1/Z(K)$, $E_2/Z(K)$ of order 16 with normalizers in K that are of characteristic 2, where $N_K(E_1/Z(K)) \cong 2^4.\mathrm{Sym}(5)$ and $N_K(E_2/Z(K)) \cong 2^4.\mathrm{Alt}(6)$, $2^4.\mathrm{Alt}(7)$, respectively. Furthermore, they are normalized by $O_2(M)$. We have $O_2(\widetilde{N_{KO_2(M)}(E_2)}) \leq C_{\widetilde{KO_2(M)}}(\widetilde{E_2}) = \widetilde{E_2} \leq \widetilde{K}$ and thus by (7.2.2)

$$\widetilde{Y_M} \leq \widetilde{E_1} \cap \widetilde{E_2} \leq \widetilde{K}.$$

Since $(E_1 \cap E_2)/Z(K)$ has order 4, we have a contradiction to (7.2.1) in this case as well. Hence $K/Z(K) \ncong \mathrm{Mat}(22)$ or $\mathrm{Mat}(23)$.

Assume that $K/Z(K) \cong \mathrm{Mat}(12)$. In this case

$$L_K/Z(K) \sim 2_+^{1+4}.\mathrm{Sym}(3)$$

and by [6, Table 5.3 b, notes 2] an element τ of order 3 in L_K acts fixed point freely on $O_2(L_K/Z(K))/Z(O_2(L_K/Z(K)))$.

Set $U_1 = \langle Y_M^{L_K}\rangle \leq U_Q$. Then U_1 is elementary abelian. If some involution u of U_1 induces an outer automorphism of K, then so does some involution of $C_{U_1}(\tau)$; however, τ is in the K-conjugacy class $3A$ whereas the elements of order 3 in $C_K(u)$ are in the class $3B$ (see [6, Table 5.3 b, notes 3]). Therefore $\widetilde{U_1} \leq \widetilde{K}$. The action of τ now shows that $|\widetilde{U_1}| = 8$. Hence by (7.2.1)

$$\widetilde{Y_M} = \widetilde{U_1}.$$

We have

$$\widetilde{O_2(M)} \leq C_{\widetilde{KO_2(M)}}(\widetilde{U_1})$$

which has order at most 2^4, as $m_2(\mathrm{Aut}(\mathrm{Mat}(12))) \leq 4$ by [6, Table 5.6.1]. But then $\widetilde{O_2(M)}' = 1$ whereas we know it contains $\widetilde{z}$, a contradiction. Hence $K/Z(K) \ncong \mathrm{Mat}(12)$. □

8. Groups of Lie type in odd characteristic as components

The aim of this section is to show that if $K/Z(K)$ is a group of Lie type defined in odd characteristic, then $K \cong {}^2\mathrm{G}_2(3)' \cong \mathrm{SL}_2(8)$.

Lemma 8.1. *The following statements hold.*

(i) *$K/Z(K) \not\cong \mathrm{PSL}_2(p)$ with $p \geq 7$ a Fermat or Mersenne prime.*

(ii) *If $K/Z(K) \cong \mathrm{PSL}_2(9)$, then $|Z(K)|$ is odd and*
$$\widetilde{KN_{C_S(y)}}(K) \cong \mathrm{Sym}(6) \text{ or } \mathrm{Aut}(\mathrm{PSL}_2(9)).$$

(iii) *If $K/Z(K) \cong \mathrm{PSL}_2(5)$, then $Z(K) = 1$, $Y_M \leq KC_{C_S(y)}(K)$ and $\widetilde{Y_M} = \widetilde{S \cap K}$.*

Proof. Suppose that K is one of the groups itemised in the lemma with $\widetilde{KN_{C_S(y)}}(K) \not\cong \mathrm{Sym}(6)$ or $\mathrm{Aut}(\mathrm{PSL}_2(9))$. Thus, if $K \cong \mathrm{PSL}_2(p)$, then $\widetilde{KN_{C_S(y)}}(K) \cong \mathrm{PSL}_2(p)$ or $\mathrm{PGL}_2(p)$ and, if $K \cong \mathrm{PSL}_2(9)$, we have
$$\widetilde{KN_{C_S(y)}}(K) \cong X \in \{\mathrm{Alt}(6), \mathrm{PGL}_2(9), \mathrm{Mat}(10)\}.$$
Assume that z induces an outer automorphism on K. Then, as $\mathrm{Mat}(10)$ has semidihedral Sylow 2-subgroups, we have $K\langle z\rangle/Z(K) \cong \mathrm{PGL}_2(p)$ or $\mathrm{PGL}_2(9)$ and, in particular, the Sylow 2-subgroups of $K\langle z\rangle/Z(K)$ are dihedral groups of order at least 8. Since $zZ(K)$ centralizes $(S \cap K)Z(K)/Z(K)$, this is impossible. Hence z induces an inner automorphism on K. In particular, Lemma 5.1 yields $Z(K)$ is a 2-group.

Assume that $Z(K) \neq 1$. Then $S \cap K$ is a quaternion group. Since $K = [K, z]$, we have $z = ws$ for some $w \in C_{\langle z\rangle K}(K)$ and $s \in (S \cap K) \setminus Z(K)$. As $[S \cap K, z] = 1$, we have $[S \cap K, s] = 1$, a contradiction. Hence
$$Z(K) = 1.$$

We first prove parts (i) and (ii). By Lemma 5.16 as $S \cap K$ is not elementary abelian, Y_M normalizes all the components of E_y and by Lemma 5.25 we have
$$|\widetilde{Y_M}| > 2.$$
If $\widetilde{K} \not\cong \mathrm{PSL}_2(7)$ or $\mathrm{PSL}_2(9)$, then, as $\widetilde{Y_M}$ is normalized by $\widetilde{N_{C_S(y)}}(K)$ the structure of the Sylow 2-subgroup of $\widetilde{K}$ shows that the only normal elementary abelian 2-subgroup has order 2 and so $|\widetilde{Y_M}| \leq 2$, which is not the case.

Hence $\widetilde{K} \cong \mathrm{PSL}_2(7)$ or $\mathrm{PSL}_2(9)$, $\widetilde{S \cap K} \cong \mathrm{Dih}(8)$ and $|\widetilde{Y_M}| = 4$. Thus $[S \cap K, Y_M] = Z(S \cap K)$ and so $O_2(M)$ normalizes K. Since $\widetilde{O_2(M)}$ centralizes $\widetilde{Y_M}$, $\widetilde{O_2(M)} = \widetilde{Y_M}$. But as $z \in O_2(M)'$, we get $z \in C_S(K)$

which is a contradiction to Lemma 4.7. Thus (i) holds and to complete the proof of (ii) we just have to establish that, if $K\widetilde{N_{C_S(y)}}(K) \cong \mathrm{Sym}(6)$ or $\mathrm{Aut}(\mathrm{PSL}_2(9))$, then $|Z(K)|$ is odd. Since z centralizes $S\cap K$, we have that $K\langle z\rangle/C_{K\langle z\rangle}(K) \cong \mathrm{PSL}_2(9)$ or $\mathrm{Sym}(6)$. That $|Z(K)|$ is odd follows from these observations and [6, Proposition 5.2.8 (b)].

For the proof of (iii), we have already shown that $Z(K) = 1$ and so $K \cong \mathrm{PSL}_2(5)$. Thus $\widetilde{N_{C_G(y)}}(S \cap K) \cong \mathrm{Alt}(4)$ or $\mathrm{Sym}(4)$. Lemmas 5.12 and 5.14 imply that Y_M normalizes K and $Y_M \leq O_2(N_G(S_yT_y))$. Hence (iii) holds. It follows from Lemma 5.25 that $\widetilde{Y_M} = \widetilde{S \cap K}$. □

Lemma 8.2. *We have $K/Z(K) \not\cong \mathrm{PSL}_2(5)$.*

Proof. Assume $K/Z(K) \cong \mathrm{PSL}_2(5)$. By Lemma 8.1 we have that $K \cong \mathrm{PSL}_2(5)$. Furthermore $\widetilde{Y_M} = \widetilde{S \cap K}$ and this is true for all components K of E_y. In particular, $[S_y, Y_M] \leq C_K(E_y) \cap E_y = 1$ and so

$$S_y \leq C_S(Y_M) = O_2(M).$$

Set $F_y = E_yO_2(M)$. Then $O_2(M)$ is a Sylow 2-subgroup of F_y. As $O(F_y) = 1$ we have by Proposition 2.7 that $J(O_2(M))$ normalizes every component of E_y. Since $J(O_2(M))$ centralizes Y_M, for any fixed component K we have

$$[\widetilde{S \cap K}, J(\widetilde{O_2(M)})] = [\widetilde{Y_M}, J(\widetilde{O_2(M)})] = 1$$

and so $J(\widetilde{O_2(M)}) = \widetilde{Y_M}$. Therefore $\Phi(J(O_2(M))) \leq C_{C_S(y)}(K)$. Lemma 4.7 implies $J(O_2(M))$ is elementary abelian. Therefore

$$J(O_2(M)) = S_y \times J(O_2(M) \cap T_y)$$

and so $N_{E_y}(S_y) \leq N_G(J(O_2(M))) \leq M^\dagger$. Now the action of $N_{E_y}(S_y)$ on S_y yields $Y_M \cap E_y = S_y$ and $O_2(M) = C_{O_2(M)}(K) \times S_y$. In particular, $\widetilde{O_2(M)}$ is abelian. Then $z \in O_2(M)'$ is contained in $C_S(K)$, which contradicts Lemma 4.7. Hence $K/Z(K) \not\cong \mathrm{PSL}_2(5)$. □

Lemma 8.3. *We cannot have $K/Z(K) \cong \mathrm{PSL}_2(9)$.*

Proof. Assume $K/Z(K) \cong \mathrm{PSL}_2(9)$. By Lemma 5.16, K is normalized by Y_M and, by Lemma 8.1, $K\widetilde{N_{C_S(y)}}(K) \cong \mathrm{Sym}(6)$ or $\mathrm{Aut}(K)$ with

$$|Z(K)| \text{ is odd.}$$

Furthermore, by Lemma 5.25 we have $|\widetilde{Y_M}| \geq 4$.

Assume that $[\widetilde{S \cap K}, \widetilde{Y_M}] \neq 1$. Then $K \geq [S \cap K, Y_M] \neq 1$ and so $O_2(M)$ normalizes K by Lemma 5.18. Since $z \in O_2(M)'$ and $\widetilde{O_2(M)}' \leq \widetilde{K}$, we have $\widetilde{z} \in \widetilde{K}$. Now Lemma 5.25 implies that $\widetilde{O_2(M)}' \cap \widetilde{Y_M}$ has order 4. But then, as $\widetilde{O_2(M)}$ centralizes $\widetilde{Y_M}$, we have $\widetilde{O_2(M)}$ is abelian. As $z \in O_2(M)'$, we then get that $z \in C_S(K)$, contradicting Lemma 4.7. Hence

$$[\widetilde{S \cap K}, \widetilde{Y_M}] = 1.$$

As $|\widetilde{Y_M}| \geq 4$ by Lemma 5.25 and $[\widetilde{S \cap K}, \widetilde{Y_M}] = 1$, we have $|\widetilde{Y_M}| = 4$ and $\widetilde{Y_M}$ maps to the centre of a Sylow 2-subgroup of $\mathrm{Sym}(6)$. In particular, $S \cap KY_M$ is contained in $O_2(M)$. This applies to every component of E_y. Especially

$$S_y \leq O_2(M). \tag{1}$$

If z does not induce an inner automorphism on K, then $O^2(L_K) \cong \mathrm{Alt}(4)$. By Lemma 5.21 we have that $O_2(M)$ normalizes K, which contradicts $z \in O_2(M)'$. Thus z induces an inner automorphism and so by Lemma 5.1 $O(K) = 1$. Now by [5, Remark following Proposition 8.5] the assumptions of Proposition 2.7 are satisfied, which yields that $J(O_2(M))$ normalizes every component of E_y. Hence $J(\widetilde{O_2(M)}) \leq J(\widetilde{N_{C_S(y)}(K)}) \cong \mathrm{Dih}(8) \times 2$. Thus

$$|\Phi(J(\widetilde{O_2(M)}))| \leq 2. \tag{2}$$

Let A be a maximal elementary abelian subgroup of $O_2(M)$. Then A normalizes K and

$$m_2(A) = m_2(C_{AK}(K)) + m_2(AK/C_{AK}(K))$$

. Combining this with (1) we conclude that $J(AK) = A(S \cap K)$. In particular, $J(O_2(M))$ is not abelian.

As $\Phi(J(O_2(M))) \neq 1$, we may select $z_* \in C_{Y_M \cap \Phi(J(O_2(M)))}(S)^{\#}$, and obtain

$$\widetilde{\langle z_*{}^M \rangle} \leq \Phi(J(\widetilde{O_2(M)}))$$

contrary to (2) and Lemma 5.25. Hence $K/Z(K) \not\cong \mathrm{PSL}_2(9)$. □

Lemma 8.4. *We cannot have $K/Z(K) \cong \mathrm{PSL}_2(p^a)$ with p an odd prime.*

Proof. Suppose that $K/Z(K) \cong \mathrm{PSL}_2(p^a)$. By Lemmas 8.1, 8.2 and 8.3, p^a is not a Mersenne or Fermat prime and $p^a \neq 9$. If z induces an inner automorphism on K, then L_K has a normal 2-complement. Application of Lemma 4.7 (v) yields that L_K is a 2-group. Now [8, Hauptsatz 8.27] implies that p^a is a Fermat or Mersenne prime or $p^a = 9$, a contradiction.

Hence z induces an outer automorphism on K. If z induces an inner-diagonal automorphism, then $\langle z\rangle K/Z(K)$ has non-abelian dihedral Sylow 2-subgroups. Since z induces an outer automorphism which centralizes $S\cap K$ this is impossible.

Hence z is in the coset of the field automorphism (mod $\mathrm{PGL}_2(p^a)$) and hence is a field automorphism by [6, Proposition 4.9.1]. Thus, as $p^a\neq 9$, $F^*(L_KZ(K)/Z(K))\cong \mathrm{PSL}_2(p^{a/2})$ and this contradicts Lemma 4.7(i). Hence $K/Z(K)\not\cong \mathrm{PSL}_2(p^a)$. □

Proposition 8.5. *If $K/Z(K)$ is a group of Lie type in odd characteristic, then $K/Z(K)\cong {}^2\mathrm{G}_2(3)'\cong \mathrm{PSL}_2(8)$.*

Proof. By Lemma 8.4 we may assume that $K/Z(K)\not\cong \mathrm{PSL}_2(p^a)$ and we also suppose that $K/Z(K)\not\cong {}^2\mathrm{G}_2(3)'$. We know that $F^*(L_K)$ is a 2-group by Lemma 4.7 (v). Using Lemma 3.20 yields $K/Z(K)$ is one of the following groups.

$$\mathrm{PSL}_3(3),\mathrm{PSU}_3(3),\mathrm{PSL}_4(3),\mathrm{PSU}_4(3),\mathrm{PSp}_4(3),\mathrm{P}\Omega_7(3),\mathrm{P}\Omega_8^+(3),\mathrm{G}_2(3).$$

Furthermore, in each case the conjugacy class of $\widetilde{z}$ is uniquely determined and is contained in $\widetilde{K}$. Using [6, Table 4.5.1] with Lemma 3.20 we have

$$\widetilde{O_2(L_K)}=\begin{cases}\mathrm{Q}_8 & K/Z(K)\cong \mathrm{PSL}_3(3)\\ \mathrm{Q}_8\circ 4 & K/Z(K)\cong \mathrm{PSU}_3(3)\\ 2_+^{1+4} & K/Z(K)\cong \mathrm{PSL}_4(3),\mathrm{PSU}_4(3),\mathrm{PSp}_4(3),\mathrm{G}_2(3)\\ 2_+^{1+4}\times 2^2 & K/Z(K)\cong \mathrm{P}\Omega_7(3)\\ 2_+^{1+8} & K/Z(K)\cong \mathrm{P}\Omega_8^+(3).\end{cases}$$

Moreover, other than for $K/Z(K)\cong \mathrm{P}\Omega_7(3)$, $\widetilde{L_K}$ does not normalize any elementary abelian subgroup of $\widetilde{O_2(L_K)}$ of order greater that 2.

Suppose that $K\not\cong \mathrm{P}\Omega_7(3)$. Then

$$\widetilde{U_Q}\cap\widetilde{O_2(L_K)}=\Omega_1(Z(\widetilde{O_2(L_K)}))=\Omega_1(Z(\widetilde{L_K}))$$

which has order 2. Hence $O^2(L_K)$ centralizes U_Q and so also Y_M. Applying Lemma 5.22 (iv) provides a contradiction.

Therefore $\widetilde{K}\cong \mathrm{P}\Omega_7(3)$. Then $O^2(\widetilde{L_K})\cong \mathrm{Alt}(4)\times(\mathrm{SL}_2(3)\circ\mathrm{SL}_2(3))$. Set $J=O^2(C_{O^2(\widetilde{L_K})}(Z(O_2(O^2(\widetilde{L_K})))))$. Then $J\cong \mathrm{SL}_2(3)\circ\mathrm{SL}_2(3)$ and $O^2(J)=J$ centralizes every abelian subgroup of $O_2(O^2(\widetilde{L_K}))$ which it normalizes. In particular, J centralizes $U_Q\geq Y_M$. Thus Lemma 5.22 (iv) provides a contradiction. This completes the proof of the proposition. □

The group ${}^2\mathrm{G}_2(3)'$ will be handled as $\mathrm{PSL}_2(8)$ in Section 10.

9. Alternating groups as components

In this section we will show that $K/Z(K)$ is not an alternating group $\mathrm{Alt}(n)$, $n \geq 5$. The cases $n = 5, 6$ have been discussed in Lemma 8.2 and Lemma 8.3. Thus we may assume that $n \geq 7$. Therefore $K\widetilde{N_{C_S(y)}}(K)$ is isomorphic to either $\mathrm{Alt}(n)$ or $\mathrm{Sym}(n)$.

Lemma 9.1. *We have $C_G(Y_M)C_Q(y)$ normalizes K.*

Proof. We consider $X \cong \mathrm{Sym}(n)$. Then, as $n \geq 7$, every involution in X either centralizes an element of cycle shape 3 or 3^2. Hence L_K is not a 2-group. Lemma 5.21 gives the result. □

Because $O_2(M)$ normalizes K by Lemma 9.1 and $\widetilde{z} \in O_2(M)'$, $\widetilde{K\langle z\rangle} = \widetilde{K}$ is isomorphic to $\mathrm{Alt}(n)$. Under this isomorphism, we get $\widetilde{z}$ is even and we let $\mathrm{supp}(z)$ be the set of elements of $\{1, \ldots, n\}$ moved by the image of z. For a subgroup H of $\mathrm{Sym}(n)$, we use H^e to denote the subgroup of even elements of H. We set notation so that $|\mathrm{supp}(z)| = 2m$.

Lemma 9.2. *We have $n > 7$ and $Z(K) = 1$.*

Proof. If $n = 7$, then as $\widetilde{z}$ is even, we get $m = 2$ and then $O_3(C_K(z)) \neq 1$, which contradicts Lemma 4.7. Thus $n > 7$.

We have $K = [K, z]$ by Lemma 5.1 and so z induces a non-trivial automorphism of K of order 2 and z centralizes $S \cap K \in \mathrm{Syl}_2(K)$. Application of [6, Proposition 5.2.8 (b)] implies that $Z(K) = 1$. □

Lemma 9.3. *We have $n - 2m \leq 2$ and, if $2m = n - 2$, then $n \equiv 2 \pmod 4$. Furthermore either $O_2(L_K)/Z(K)$ is elementary abelian and involves exactly one non-trivial irreducible $O^2(L_K)$-module or $n \in \{8, 9, 10\}$ and $|\mathrm{supp}(z)| = 8$.*

Proof. By Lemma 9.2 $Z(K) = 1$, so $K \cong \mathrm{Alt}(n)$. If $2m \leq n - 4$, then $O^2(F^*(L_K))$ contains $\mathrm{Alt}(n - 2m)$ and this contradicts Lemma 4.7 (v), other than if $2m = n - 4$.

Suppose that $2m = n - 4$. We may assume that $z = (12)(34)\ldots(n-5, n-4)$. Then $L_K \cong (2 \wr \mathrm{Sym}(m) \times \mathrm{Sym}(4))^e$, which contains a Sylow 2-subgroup of K only if $n \equiv 4 \pmod 8$. By Lemma 9.1 we have that $O_2(M)$ normalizes K. We consider $H, J \in \mathcal{L}_K(S \cap K)$ with H stabilizing the partition $\{\{1, 2\}, \ldots \{n-1, n\}\}$ and J stabilizing $\{\{1, 2, 3, 4\}, \ldots, \{n-3, n-2, n-1, n\}\}$. Then H and J are normalized by $O_2(M)$. By Lemma 4.8, $Y_M \leq O_2(HO_2(M)) \cap O_2(JO_2(M))$. This shows that $\widetilde{Y_M}$ is contained in the subgroup

$$\langle (12)(34), \ldots, (n-3, n-2)(n-1, n)\rangle.$$

In particular any subgroup of Y_M, which is normalized by $O^2(L_K)$ is centralized by $O^2(L_K)$. By Lemma 5.20, Q normalizes $O^2(L_K)$. But then it also normalizes the fours-group

$$\langle (n-3,n-2)(n-1,n),(n-3,n-1)(n-2,n)\rangle,$$

as this subgroup is obviously characteristic in $O^2(L_K)$. This is trivial to observe if $2m > 8$. In the case $2m = 8$, it is $[Z(O_2(O^2(L_K))), O^2(L_K)]$ which is also characteristic. Therefore there exists $z_1 \in Z(Q)^\#$ with $|\mathrm{supp}(z_1)| = 4$, a contradiction to Lemma 4.7(v).

Therefore $|\mathrm{supp}(z)| \geq n-3$. If $|\mathrm{supp}(z)| = n-3$, then $O(L_K) \neq 1$, and we have a contradiction to Lemma 4.7. Hence $|\mathrm{supp}(z)| \geq n-2$. If $2m \neq n-2$, we have that $C_K(z) \cong (2 \wr \mathrm{Sym}(m))^e$. If $2m = n-2$, then $C_K(z) \cong (2 \times 2 \wr \mathrm{Sym}(m))^e$ which contains a Sylow 2-subgroup of K only if $n \equiv 2 \pmod 4$. As $n \geq 7$, we have $m \geq 3$. Now $\mathrm{Sym}(m)$ has a non-trivial normal 2-subgroup if and only if $m = 4$. Thus so long as $m \neq 4$, we have that $O_2(L_K)$ is elementary abelian and $L_K/O_2(L_K) \cong \mathrm{Sym}(m)$ induces the non-trivial irreducible part of the natural permutation module on the unique non-central chief factor in $O_2(L_K)$. Finally we note that we have $m = 4$ only when $n \in \{8,9,10\}$. □

We now deal with the three exceptional cases in Lemma 9.3.

Lemma 9.4. *We have $|\mathrm{supp}(z)| \neq 8$. In particular $n > 10$.*

Proof. Suppose that $|\mathrm{supp}(z)| = 8$. Then by Lemma 9.3, $n \in \{8,9,10\}$. By Lemma 9.2 $Z(K) = 1$. We may suppose that z corresponds to the permutation (12)(34)(56)(78). By Lemma 9.1, $O_2(M)$ normalizes K.

To start assume that $\widetilde{K} \cong \mathrm{Alt}(8)$ or $\mathrm{Alt}(9)$. Then there exist $J_1, J_2 \in \mathcal{L}_K(S \cap K)$ with $J_1 \cong J_2 \cong 2^3{:}\mathrm{SL}_3(2)$ and $J_1 \neq J_2$. Both these subgroups are normalized by $O_2(M)$ and hence

$$Y_M \leq O_2(Y_M J_1) \cap O_2(Y_M J_2) = C_{Y_M K}(K)(O_2(J_1) \cap O_2(J_2))$$

by Lemma 4.8. Since $|O_2(J_1) \cap O_2(J_2)| = 2$, this contradicts Lemma 5.25. Hence

$$\widetilde{KO_2(M)} \cong \mathrm{Sym}(8), \mathrm{Sym}(9), \mathrm{Alt}(10), \text{ or } \mathrm{Sym}(10).$$

Notice that in $\mathrm{Alt}(10)$, $(\mathrm{Sym}(8) \times \mathrm{Sym}(2))^e \cong \mathrm{Sym}(8)$.

We consider $J \in \mathcal{L}_K(S \cap K)$ stabilizing the partition

$$\{\{1,2,3,4\},\{5,6,7,8\},\Omega_0\}$$

where $|\Omega_0| \in \{0,1,2\}$. Then

$$J \cong \begin{cases} (\mathrm{Sym}(4) \wr 2)^e & n \in \{8,9\} \\ (\mathrm{Sym}(4) \wr 2 \times 2)^e & n = 10 \end{cases}.$$

Notice that J has characteristic 2 and is normalized by $O_2(M)$. Setting $J_1 = O_2(JO_2(M))$, we have $Y_M \leq J_1$ and $\langle Y_M^J \rangle$ is elementary abelian by Lemma 4.8. We calculate

$$\widetilde{J_1} = \begin{cases} \left\langle \substack{(12)(34),(13)(24),\\(56)(78),(57)(68)} \right\rangle & n = 8, 9 \text{ or } \widetilde{KO_2(M)} \cong \mathrm{Alt}(10) \\ \left\langle \substack{(12)(34),(13)(24),\\(56)(78),(57)(68),(9,10)} \right\rangle & \widetilde{KO_2(M)} \cong \mathrm{Sym}(10) \end{cases}$$

and

$$\widetilde{J_1} \cap \widetilde{O_2(L_K)} = \begin{cases} \left\langle \substack{(12)(34),(56)(78),\\(13)(24)(57)(68)} \right\rangle & n \in \{8, 9\} \text{ or } \widetilde{KO_2(M)} \cong \mathrm{Alt}(10) \\ \left\langle \substack{(12)(34),(56)(78),\\(13)(24)(57)(68),(9,10)} \right\rangle & \widetilde{KO_2(M)} \cong \mathrm{Sym}(10) \end{cases}$$

which has order 8 in the first cases and 16 in the second. As $L_K \leq N_G(Q)$, the projection $\widetilde{Y_M}$ is contained in $\widetilde{J_1} \cap \widetilde{O_2(L_K)}$.

Suppose that $n \in \{8, 9\}$ or $\widetilde{KO_2(M)} \cong \mathrm{Alt}(10)$. Then we have $|C_{\widetilde{K}}(O^2(\widetilde{L_K}))| = 2$ and so, as $\widetilde{Y_M} \leq \widetilde{K}$, Lemma 5.26 (ii) applies to give

$$\widetilde{Y_M} = \widetilde{J_1} \cap \widetilde{O_2(L_K)}.$$

Pick $\rho \in L_K$ corresponding to $(1, 3, 5)(2, 4, 6)$. As $(13)(24)(57)(68) \in \widetilde{Y_M}$,

$$(3, 5)(4, 6)(1, 7)(2, 8) = ((13)(24)(57)(68))^\rho \in \widetilde{\langle Y_M^{L_K} \rangle}.$$

Since $(12)(34)$ and $(3, 5)(4, 6)(1, 7)(2, 8)$ do not commute, we have a contradiction to $\widetilde{\langle Y_M^{L_K} \rangle} \leq \widetilde{U_Q}$ being abelian. Hence $\widetilde{KO_2(M)} \cong \mathrm{Sym}(10)$.

Let $\widetilde{H} \leq \widetilde{KO_2(M)}$ be the subgroup that preserves the partition

$$\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}\}.$$

Then $\widetilde{H} \cong 2 \wr \mathrm{Sym}(5)$, and $\widetilde{Y_M} \leq \widetilde{O_2(H)}$ and we have

$$\widetilde{Y_M} \leq \widetilde{J_1} \cap \widetilde{O_2(H)} \leq \langle (12)(34), (56)(78), (9, 10) \rangle$$

which has order 8. Notice that $\widetilde{z}$ is the only $\widetilde{KO_2(M)}$-conjugate of $\widetilde{z}$ in $\widetilde{Y_M}$.

By the choice of z we have $z \in Y_M \cap O_2(M)'$. By Lemma 5.25, there exists $m \in M$ such that $\widetilde{z^m} \neq \widetilde{z}$. Obviously $z^m \in Y_M \cap O_2(M)'$ and so

$$\widetilde{z} \in \widetilde{O_2(M)' \cap Y_M} \leq \widetilde{K} \cong \mathrm{Alt}(10).$$

Hence $\widetilde{z^m}$ corresponds to an element of cycle type 2^2. However this means $C_K(z^m)$ contains a component isomorphic to $\mathrm{Alt}(6)$ and this contradicts Lemma 4.7 (v).

Assume now $n = 10$. Then $|\mathrm{supp}(z)| \neq 8$. By Lemma 9.3 this gives $|\mathrm{supp}(z)| = 10$, which contradicts $\widetilde{z} \in \widetilde{K}$. □

Proposition 9.5. *We have $K/Z(K)$ is not an alternating group.*

Proof. By Lemma 9.4 we have $n > 10$. Further Lemma 9.2 gives us $Z(K) = 1$. By the choice of z we have $\widetilde{K\langle z\rangle} = \widetilde{K}$.

Assume that $\widetilde{Y_M}$ covers the unique non-trivial irreducible $O^2(L_K)$-module in $\widetilde{O_2(L_K)}$. Then $C_{\widetilde{K}}(\widetilde{Y_M}) = \widetilde{O_2(L_K)}$. By Lemma 9.1 we have that $O_2(M)$ normalizes K and so $\widetilde{O_2(M)} \leq C_{\widetilde{K}}(\widetilde{Y_M})$ is elementary abelian. Therefore $z \in O_2(M)' \leq C_S(K)$, which is impossible.

Hence $\widetilde{Y_M}$ does not cover the non-trivial irreducible $O^2(L_K)$-module in $\widetilde{O_2(L_K)}$ and so any $O^2(L_K)$-invariant subgroup W of Y_M is centralized by $O^2(L_K)$. Therefore Lemma 5.20 yields $O^2(L_K)$ is normalized by Q. Since $O_2(O^2(L_K))$ is elementary abelian and contains exactly one non-central $O^2(L_K)$-chief factor, Lemma 5.24 yields

$$[O_2(O^2(L_K)), O^2(L_K)] \leq Z(Q).$$

We now notice

$$[O_2(O^2(L_K)), O^2(L_K)] = O_2(O^2(L_K))$$

contains an element w which is K-conjugate to the permutation $(12)(34)$. As $w \in Z(Q)$, $C_G(w)$ has characteristic 2, hence $C_{C_G(y)}(w)$ has characteristic 2 by Lemma 4.7 (vi) which it plainly does not. This contradiction shows that $K/Z(K)$ is not an alternating group. □

10. Groups of Lie type in characteristic 2 as components

In this section we tackle the possibility that $K/Z(K)$ is a group of Lie type in characteristic two. Some of these groups have been considered before under different names. For example $\mathrm{L}_2(4) \cong \mathrm{L}_2(5) \cong \mathrm{Alt}(5)$, $\mathrm{PSp}_4(2)' \cong \mathrm{Alt}(6)$, $\mathrm{L}_3(2) \cong \mathrm{L}_2(7)$, $\mathrm{G}_2(2)' \cong \mathrm{U}_3(3)$, $\mathrm{L}_4(2) \cong \Omega_6^+(2) \cong \mathrm{Alt}(8)$ and $\Omega_6^-(2) \cong \mathrm{U}_4(2) \cong \mathrm{PSp}_4(3)$.

We will start with the groups $\mathrm{SL}_2(2^a)$ and ${}^2\mathrm{B}_2(2^a)$, which then also handles the case of ${}^2\mathrm{G}_2(3)'$ which was left open in Proposition 8.5.

Lemma 10.1. *Suppose that $K/Z(K) \cong \mathrm{SL}_2(2^a)$ or ${}^2\mathrm{B}_2(2^a)$, $a \geq 3$. Then*

(i) *$Y_M \leq \Omega_1(S_y)T_y$;*
(ii) *K is simple; and*
(iii) *$S_y \leq O_2(M)$.*

Proof. By Lemmas 3.11 and 5.14, $N_G(S_yT_y)$ has characteristic 2. Lemma 5.12 yields $Y_M \leq O_2(N_G(S_yT_y))$ and Y_M normalizes every component of E_y. In particular, Y_M induces inner automorphisms on each of such component. It follows that $Y_M \leq \Omega_1(S_y)T_y$. This proves (i).

Assume that $Z(K) \neq 1$. By [6, Table 6.1.3], $K/Z(K) \cong {}^2\mathrm{B}_2(8)$, as $a \geq 3$. Furthermore by Lemma 3.19 $Z(S_y \cap K) = Z(K)$.

Hence $K\langle z\rangle = KC_{K\langle z\rangle}(K)$, $z \notin K$ and $z \notin C_{K\langle z\rangle}(K)$. Thus $z = ab$ where $a \in S \cap K$ and $b \in C_{K\langle z\rangle}(K)$. As z and $C_{K\langle z\rangle}(K)$ centralize $S \cap K$, so does a. Therefore $a \in Z(S \cap K) = Z(K)$ and we conclude $z \in C_{K\langle z\rangle}(K)$, a contradiction. This proves (ii).

By (ii) K is simple. Since S_y centralizes $\Omega_1(S_y)T_y \geq Y_M$, we have $S_y \leq S \cap C_M(Y_M) = O_2(M)$. This is (iii). □

Lemma 10.2. *We have that $K/Z(K) \ncong \mathrm{SL}_2(2^a)$ or ${}^2\mathrm{B}_2(2^a)$ with $a \geq 3$.*

Proof. Assume that $K/Z(K) \cong \mathrm{SL}_2(2^a)$ or ${}^2\mathrm{B}_2(2^a)$ with $a \geq 3$. By Lemma 10.1 (ii), K is simple.

We first prove that $J(O_2(M)) = \Omega_1(S_y) \times J(T_y)$. By Lemma 10.1 (iii), $S_y \leq O_2(M)$ and so we consider $X = E_yO_2(M)$. We have $O_2(M) \in \mathrm{Syl}_2(X)$. Using Lemmas 2.8 and 3.3, the fact that $a > 2$ yields

$$J(O_2(M)) = J(C_{O_2(M)}(E_y)) \times J(S_y).$$

Using Lemma 3.3 again gives $J(S_y) = \Omega_1(S_y) = Z(S_y)$. From the structure of $J(O_2(M))$, we see that $N_{E_y}(S_y)$ normalizes $J(O_2(M))$ and therefore $N_E(S_y) \leq M^\dagger$ by Lemma 4.3. Since Y_M is normalized by $M^\dagger$, it is also normalized by $N_{E_y}(S_y)$. Therefore $[Y_M, N_{E_y}(S_y)] \leq Y_M \cap E_y$. Since Y_M does not centralize $N_{E_y}(S_y)$, we deduce that $\Omega_1(S_y) = [Y_M, N_{E_y}(S_y)] < Y_M$. Now we have $J(O_2(M)) = J(C_S(E_y))Y_M$. Thus

$$\begin{aligned}[J(O_2(M)), O_2(M)] &= [J(C_{O_2(M)}(E_y))Y_M, O_2(M)] \\ &= [J(C_{O_2(M)}(E_y)), O_2(M)] \leq T_y.\end{aligned}$$

As $[J(O_2(M)), O_2(M)] \leq T_y$, Lemma 5.7 implies

$$[J(O_2(M)), O_2(M)] = 1.$$

Hence

$$Y_M \leq J(O_2(M)) \leq \Omega_1(Z(O_2(M))) = Y_M.$$

But then $Y_M = J(O_2(M))$ and this contradicts Lemma 4.11. The lemma is proved. □

Lemma 10.3. *We have $K/Z(K) \ncong \mathrm{PSU}_3(q)$ for $q = 2^a \geq 4$.*

Proof. Suppose that $K/Z(K) \cong \mathrm{PSU}_3(q)$ with $q \geq 4$. We have $Z(K) = 1$ by [6, Table 6.1.3] and Lemma 5.1. We take facts about K from [4, 5.4] and [8, II.10.12 Satz]. An important point is that $\Omega_1(S \cap K) = Z(S \cap K)$.

We have $|L_K| = q^3(q+1)/(q+1,3)$ and so L_K is not a 2-group. Therefore $C_G(Y_M)C_Q(y)$ normalizes K by Lemma 5.21.

We start with the following statement.

(10.3.1) Suppose that A is an elementary abelian normal subgroup of $O_2(M)C_Q(y)$. Then $A \le \Omega_1(S \cap K)C_{C_S(y)}(K) = Z(S \cap K)C_{C_S(y)}(K)$.

Since A normalizes K, is elementary abelian, and no outer automorphism of K centralizes $(S \cap K)/Z(S \cap K)$, $A \le (S \cap K)C_{C_S(y)}(K)$. Therefore

$$A \le \Omega_1(S \cap K)C_{C_S(y)}(K) = Z(S \cap K)C_{C_S(y)}(K).$$

■

By (10.3.1), $Y_M \le Z(S \cap K)C_{C_S(y)}(K)$. As $Z(S \cap K)$ is centralized by L_K, we obtain

$$O^2(L_K) \le C_G(Y_M).$$

Now, as $O_2(M)$ normalizes L_K, Lemma 5.22 (iii) yields $F^*(C_G(y)) = KO_2(C_G(y))$ (and part (ii) leads to $q = 8$, but we shall not use this).

Furthermore, (10.3.1) implies that $[S \cap K, Y_M] \le S \cap K \cap C_S(K) = 1$,

$$S \cap K \le C_S(Y_M) = O_2(M).$$

Assume $w \in O_2(M)$ has order 2 and induces an outer automorphism on K. Then, as $O_2(M)$ normalizes K and is contained in S, w acts on $N_K(S \cap K)/(S \cap K)$. By [6, Proposition 4.9.2 (b)(2) and (g)], w is conjugate in $\mathrm{Aut}(K)$ to a standard graph automorphism and so w centralizes $Z(S \cap K)$ (see [6, Theorem 2.5.1 d]) and $C_K(w) \cong \mathrm{SL}_2(q)$. Hence there is an element of $\nu \in N_{C_K(w)}(Z(S \cap K))$ of order $q-1$. Since the Sylow 2-subgroups of K are trivial intersection subgroups in K, ν normalizes $S \cap K$. As $N_K(S \cap K)/(S \cap K)$ is cyclic, $\langle \nu \rangle (S \cap K)$ is uniquely determined by its order in $N_K(S \cap K)/(S \cap K)$, and so ν normalizes $\Omega_1(O_2(M))(S \cap K)$. Therefore ν normalizes $\Omega_1(O_2(M))$ as $S \cap K \le O_2(M)$. It follows that $\nu \in N_G(\Omega_1(O_2(M))) \le M^\dagger$. Therefore ν normalizes $Y_{M^\dagger} = Y_M$. As $\langle \nu \rangle$ acts irreducibly on $Z(S \cap K)$ and normalizes Y_M, we have

$$Y_M C_{C_S(y)}(K) = Z(S \cap K)C_{C_S(y)}(K).$$

Recall that U_Q is elementary abelian. Hence $U_Q \le O_2(M)$ and (10.3.1) implies that

$$U_Q \le Z(S \cap K)C_{C_S(y)}(K) = Y_M C_{C_S(y)}(K) \le U_Q C_{C_S(y)}(K).$$

Therefore

$$U_Q C_{C_S(y)}(K) = Y_M C_{C_S(y)}(K).$$

Hence $[U_Q, O_2(M)] \leq C_{C_S(y)}(K)$ and $[U_Q, O_2(M)]$ is normalized by Q. We conclude from Lemma 5.7 that $[U_Q, O_2(M)] = 1$ and so $U_Q = Y_M$ and

$$N_G(Q) \leq N_G(Y_M) = M^\dagger.$$

Suppose that $J \in \mathcal{L}_G(S)$. Then, as Y_M is not characteristic 2-tall, $Y_M \leq O_2(J)$. Hence $Y_J \leq C_S(Y_M) = O_2(M)$ and so $Y_J \leq Y_M C_{C_S(y)}(K)$ by (10.3.1). Thus $[O_2(M), Y_J] \leq C_{C_S(y)}(K)$ and again this is normalized by Q. Hence $[O_2(M), Y_J] = 1$ and so $Y_J \leq Y_M$. This contradicts Lemma 4.10. Hence $K/Z(K) \not\cong \mathrm{PSU}_3(q)$. □

Lemma 10.4. *Suppose that $K/Z(K) \cong \mathrm{PSL}_3(q)$, $q = 2^a \geq 4$. Then $K/Z(K) \cong \mathrm{PSL}_3(4)$.*

Proof. We assume that $q \geq 8$ and seek a contradiction. As $q \neq 4$, [6, Table 6.1.3] and Lemma 5.1 combine to give $Z(K) = 1$. The structural information we require about $S \cap K$ is given in Lemma 3.10. From there we see that $S \cap K$ has exactly two elementary abelian subgroups of order q^2. Name these subgroups E_1 and E_2. Furthermore, every element of $(S \cap K) \setminus (E_1 \cup E_2)$ has order 4.

We have $|L_K| = q^3(q-1)/(q-1, 3)$ and so, as $q > 4$, L_K is not a 2-group. Thus by Lemma 5.21 $C_G(Y_M)C_Q(y)$ normalizes K and so therefore does Y_M. By Lemma 3.10(ii) $Y_M \leq (S_y \cap K)C_{C_S(y)}(K)$. If $Y_M \not\leq Z(S \cap K)C_{C_S(y)}(K)$, then we may assume $Y_M \leq E_1 C_{C_S(y)}(K)$ and $Y_M \not\leq E_2 C_{C_S(y)}(K)$. Since $O_2(M)$ centralizes Y_M, we infer that $O_2(M)$ normalizes E_1. Hence $O_2(M)$ normalizes $N_K(E_1)$ and also $N_K(E_2)$. Then Lemma 4.8 implies that $Y_M \leq E_2 C_{C_S(y)}(K)$, which is a contradiction. Hence

(10.4.1) $Y_M \leq Z(S_y \cap K)C_{C_S(y)}(K)$.

Since $[Z(S_y \cap K), L_K] = 1$, we now have $O^2(L_K) \leq C_G(Y_M)$ and so, as $O_2(M)$ normalizes L_K, $K = E(C_G(y))$ and M° normalizes $L_K = O^2(L_K)$ by Lemma 5.22. Therefore

(10.4.2) $E_y = K$ and M° normalizes $O_2(L_K) = S_y$.

By Lemma 3.12, K satisfies the Sylow centralizer property (Definition 5.13) with respect to $C_S(y)$. Furthermore all elements in $Z(S_y)$ are K-conjugate and, as Q normalizes $Z(S_y)$, they all have a centralizer which has characteristic 2. Thus Lemma 5.15 yields that

$$T_y \text{ is elementary abelian.}$$

Since $Y_M \le Z(S_y)T_y$, $[S_y, Y_M] \le T_y$. As $Q \le M^\circ$ normalizes S_y by (10.4.2), Lemma 5.7 implies that $S_y \le C_S(Y_M) = O_2(M)$. Combining Lemma 2.8 with Lemma 3.10(iii) yields

(10.4.3) $J(O_2(M)) = S_y T_y = E_1 E_2 T_y$.

In particular, (10.4.3) implies $N_K(S_y) \le N_G(J(O_2(M))) \le M^\dagger$ and so $N_K(S_y)$ normalizes Y_M. As $N_K(S_y)$ acts irreducibly on $Z(S_y)$, (10.4.1) produces $Y_M T_y = Z(S_y)T_y$. Therefore $[T_y Z(S_y), O_2(M)] \le T_y$. Because $T_y Z(S_y) = J(O_2(M))$ is normalized by S, we obtain $[T_y Z(S_y), O_2(M)] = 1$. Thus

(10.4.4) $Y_M = T_y Z(S_y)$.

Assume that some element in S conjugates $T_y E_1$ to $T_y E_2$. Let A be an elementary abelian normal subgroup of S contained in $O_2(M)$. Then $\widetilde{A} \le \widetilde{S_y}$ by Lemma 3.10(ii). Therefore, as S does not normalize $T_y E_1$,

$$A \le T_y E_1 \cap T_y E_2 = T_y Z(S_y) = Y_M$$

by (10.4.4). Applying Lemma 4.11 provides a contradiction. Thus we have

(10.4.5) $S \le N_G(T_y E_1)$.

By (10.4.3) and (10.4.5), $M \le N_G(E_1 T_y)$. Thus $N_G(E_1 T_y) \le M^\dagger$. As $N_K(E_1) \le N_G(E_1 T_y)$ does not normalize Y_M, this is impossible. $\square$

Lemma 10.5. *We have $K/Z(K) \not\cong \mathrm{Sp}_{2n}(q)$ with $n \ge 3$ and $q = 2^a$.*

Proof. We follow the notation from Lemma 3.5. Thus $Z(S_y \cap K) = \widetilde{R_1 R_2}$, and, for $i = 1, 2$, $L_i = C_K(R_i)$. We have $\widetilde{z}$ is centralized by $\widetilde{L_1 \cap L_2}$. Thus L_K contains a section isomorphic to $\mathrm{Sp}_{2n-4}(q)$, which is not a 2-group as $n \ge 3$. Thus Lemma 5.21 yields K is normalized by $C_G(Y_M)C_Q(y)$. Since L_1 and L_2 are not isomorphic, we have L_1 and L_2 are normalized by $O_2(M)$. Lemmas 4.8 and 5.27 imply that

$$Y_M C_{C_S(y)}(K) \le O_2(N_K(R_1))C_{C_S(y)}(K) \cap O_2(N_K(R_2))C_{C_S(y)}(K).$$

Furthermore, for $i = 1, 2$, $\langle Y_M^{N_K(R_i)} \rangle$ is elementary abelian. If $Y_M \not\le Z(O_2(N_K(R_2)))C_{C_S(y)}(K)$, then Lemma 3.5 (ii) implies that

$$\langle Y_M^{N_K(R_i)} \rangle \widetilde{Z(O_2(N_K(R_2)))} = \widetilde{O_2(L_2)}$$

which is non-abelian. As $\langle Y_M^{N_K(R_i)} \rangle$ is abelian, we conclude

$$\widetilde{Y_M} \le Z(O_2(\widetilde{N_K(R_2)})).$$

As $Z((S_y \cap K)/Z(K)) = (Z(O_2(N_K(R_2))) \cap O_2(N_K(R_1)))/Z(K)$, we have that
$$\widetilde{Y_M} \leq Z(\widetilde{S_y \cap K}).$$
Now Y_M is centralized by $O^2(L_1 \cap L_2)$ and so Lemma 5.20 gives $K = E_y$ and $S_y = S_y \cap K$.

Assume that $Z(K) \neq 1$. Then, by [6, Table 6.1.3], $K/Z(K) \cong \mathrm{Sp}_6(2)$. Further the preimage of $Z(S_y/Z(K))$ is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$. This contradicts Lemma 5.25. Hence $Z(K) = 1$ and K is simple.

Since $K = E_y$ is simple, $Y_M \leq Z(S_y)T_y$ which implies $[Y_M, S_y] \leq K \cap T_y = 1$. Thus $S_y \leq C_S(Y_M) = O_2(M)$. Using Lemmas 2.8 and 3.6 we obtain
$$J(O_2(M)) = J(S_y) \times J(T_y \cap O_2(M)).$$
Therefore $N_K(J(S_y)) \leq N_K(J(O_2(M))) \leq M^\dagger$. But $N_K(J(S_y))$ normalizes no non-trivial subgroup in $Z(S_y)$ by Lemma 3.6, which is a contradiction as $Y_M \leq Z(S_y)T_y$ and $Y_M \not\leq T_y$. We have proved $K/Z(K) \not\cong \mathrm{Sp}_{2n}(q)$ with $n \geq 3$ and $q = 2^a \geq 2$. □

Lemma 10.6. *We have $K/Z(K) \not\cong \mathrm{F}_4(q)$ with $q = 2^a$, $a \geq 1$.*

Proof. By [6, Table 6.1.3] we have $Z(K) = 1$ unless $q = 2$. We use Lemmas 3.15 and 3.16 for structural information about $K/Z(K)$. Let R_1 and R_2 be the preimages of root subgroups of K in $S_y \cap K$. Thus $|R_i| = q$ unless $Z(K) \neq 1$ in which case they are elementary abelian of order 4. We also let Z be the preimage of $Z((S \cap K)/Z(K))$.

By Lemma 3.11, $\widetilde{z} \in \widetilde{Z}$ and, by Lemma 5.1, $K = [K, z]$. We have $L_K \geq I_{12}$ (using the notation of Lemma 3.16) and so L_K is not a 2-group. By Lemma 5.21

(10.6.1) $C_G(Y_M)C_Q(y)$ normalizes K.

We next intend to show

(10.6.2) $\widetilde{Y_M} \leq \widetilde{Z}$.

Suppose that $\widetilde{Y_M} \cap \widetilde{R_i} \neq 1$ for some $i \in \{1, 2\}$. Without loss of generality we assume that $i = 1$. Let $w \in Y_M$ be such that $\widetilde{w} \in \widetilde{R_1}^{\#}$. Then $O^2(C_K(R_1))$ centralizes w. Hence $O_2(M)$ normalizes $O^2(C_K(R_1))$. It follows that $O_2(M)$ normalizes $N_K(R_1)$ and $N_K(R_2)$. By Lemma 5.27
$$\widetilde{Y_M} \leq O_2(\widetilde{N_K(R_1)}) \cap O_2(\widetilde{N_K(R_2)}).$$
For $i = 1, 2$, set $W_i = \langle Y_M^{N_K(R_i)} \rangle$. Then W_1 and W_2 are elementary abelian and Lemma 3.15 gives $\widetilde{W_i} \leq Z(O_2(\widetilde{N_K(R_i)}))$. By Lemma 3.16 (ii), $Z(O_2(N_K(R_1))) \cap Z(O_2(N_K(R_2))) = Z$. Therefore $\widetilde{Y_M} \leq \widetilde{Z}$ in this case.

To complete the proof of (10.6.2) we assume that

$$\widetilde{Y_M} \cap \widetilde{R_1} = \widetilde{Y_M} \cap \widetilde{R_2} = 1.$$

Since $O_2(M)$ normalizes I_{12}, Lemma 5.27 implies that $\widetilde{Y_M} \le \widetilde{O_2(I_{12})} \le \widetilde{K}$. Thus $\widetilde{Y_M}$ normalizes $O_2(\widetilde{N_K(R_i)})$ for $i = 1, 2$. If, for some i, we have $\widetilde{Y_M} \cap O_2(\widetilde{N_K(R_i)}) \not\le Z(O_2(\widetilde{N_K(R_i)}))$, then $1 \ne [O_2(\widetilde{N_K(R_i)}), \widetilde{Y_M}] \le \widetilde{R_i} \cap \widetilde{Y_M} = 1$, a contradiction. Hence $\widetilde{Y_M} \cap O_2(\widetilde{N_K(R_i)}) \le Z(O_2(\widetilde{N_K(R_i)}))$ and so Lemma 3.15 implies that first $\widetilde{Y_M} \le O_2(\widetilde{N_K(R_i)})$ for both $i = 1, 2$ and then that

$$\widetilde{Y_M} \le Z(O_2(\widetilde{N_K(R_1)})) \cap Z(O_2(\widetilde{N_K(R_2)})) = \widetilde{Z}.$$

This completes the proof of (10.6.2). ■

Since $O^2(I_{12})$ centralizes Z, we have $O^2(I_{12}) \le C_K(Y_M) \le C_G(Y_M)$. Hence $J = C_K(Y_M)$ is not a 2-group. Notice that $J = O^2(N_K(R_1))$ or $J = O^2(N_K(Z(S \cap K)))$, Lemmas 3.15 and 3.16 show that the non-central $O^2(J)$-chief factors in $O_2(O^2(J))/\Phi(O_2(O^2(J)))$ are pairwise non-isomorphic. In this way Lemma 5.22 provides a contradiction. This proves $K/Z(K) \not\cong \mathrm{F}_4(q)$. □

Lemma 10.7. *We have $K/Z(K) \not\cong {}^2\mathrm{F}_4(q)'$ with $q = 2^a \ge 2$.*

Proof. Suppose false. By Lemma 7.1 we have $q > 2$ and [6, Theorem 6.1.4] states $Z(K) = 1$. Furthermore [6, Theorem 2.5.12] gives $\mathrm{Out}(K)$ has odd order.

For the structure of the 2-local subgroups of K we shall use Lemma 3.17 in the arguments that follow, without specific reference. In particular, we know L_K is not a 2-group and so Lemma 5.21 yields

$$C_G(Y_M)C_Q(y) \text{ normalizes } K.$$

We introduce some notation. Let $R = Z(S \cap K)$ and $W_1 = Z_2(O_2(L_K))$. Then $L_K = C_K(R)$, $|W_1| = q^5$ and W_1/R is the natural $L_K/O_2(L_K)$-module where $L_K/O_2(L_K) \cong {}^2\mathrm{B}_2(q)$. Put $W_2 = C_K(W_1)$. Then $|W_2| = q^6$ and $W_1 = \Omega_1(W_2)$. Let P be the maximal parabolic subgroup of K containing $N_K(S \cap K)$ but not containing L_K.

Since L_K and P are normalized by $O_2(M)$, by Lemma 4.8 we have $Y_M \le O_2(L_K O_2(M)) \cap O_2(P O_2(M))$ and $\langle Y_M^{L_K} \rangle$ and $\langle Y_M^P \rangle$ are elementary abelian.

As $\widetilde{\langle Y_M^{L_K} \rangle}$ is elementary abelian and normal in $\widetilde{L_K}$, we have $\widetilde{Y_M} \le \widetilde{\langle Y_M^{L_K} \rangle} \le \widetilde{W_1}$. If $\widetilde{Y_M} = \widetilde{W_1}$, then $\widetilde{\langle Y_M^P \rangle}$ is not abelian, a contradiction.

Hence $\widetilde{Y_M} < \widetilde{W_1}$. In particular, any subgroup W of Y_M which is normalized by $O^2(L_K)$ has $\widetilde{W} \le \widetilde{R}$ and so W is centralized by $O^2(L_K)$. Lemma 5.21 implies that

$$E(C_G(y)) = K \text{ and } Q \text{ normalizes } O^2(L_K).$$

In particular, some element of R is centralized by Q. Since all root elements are conjugate, all the involutions in R have characteristic 2 centralizers. Therefore Lemma 5.15 yields T_y is elementary abelian of order at most q. Since $O_2(M) \le S_yT_y$, and $T_y \le Z(S_yT_y)$, $T_y \le Y_M$. Hence $Y_M = (Y_M \cap K)T_y$.

Assume that $\widetilde{Y_M} \le \widetilde{R}$. Then $L_K = O^2(L_K) \le C_G(Y_M)$. Hence Lemma 5.22 implies that

$$L_K = O^2(L_K) \text{ is normalized by } M^\circ.$$

Considering the action of Q on $V = O_2(L_K)/W_1$, which is an indecomposable 5-dimensional $\mathrm{GF}(q)$-module for $L_K/O_2(L_K)$, we see that $C_V(Q)$ has a non-trivial $O^2(L_K)$ chief factor by the $A \times B$-lemma. Hence Q centralizes V. As $Z_2(O_2(L_K)) = \Phi(O_2(L_K))$, $O^2(M^\circ)$ centralizes $O_2(L_K)$ which is normalized by Q and so this contradicts Lemma 4.5. Hence

$$\widetilde{Y_M} \not\le \widetilde{R}.$$

Because $\widetilde{Y_M} \not\le \widetilde{R}$, $\widetilde{Y_M} \cap Z_2(\widetilde{S_y}) \not\le \widetilde{R}$ and so as $Z(O_2(P))$ is a natural $P/O_2(P)$-module where $P/O_2(P) \cong \mathrm{SL}_2(q)$, $[\widetilde{Y_M} \cap Z_2(\widetilde{S_y}), \widetilde{S_y}] = \widetilde{R}$ and we deduce $[S_y, Y_M] \ge R$. Since $O_2(M)P = C_{O_2(M)P}(K)P$ and $\widetilde{O_2(M)}$ centralizes $\widetilde{Y_M \cap Z_2(S_y)}$ we conclude that P normalizes $O_2(M)T_y$. But then P normalizes $O_2(M)$ as $O_2(M)$ is weakly closed in S by Lemma 4.4. In particular, $Y_M \cap K$ is normalized by P and so $Y_M \cap K \ge Z_2(S_y)$.

Since $O_2(L_K)/C_{O_2(L_K)}(W_1)$ and W_1/R are irreducible $O^2(L_K)$-modules, $Q \le M^\circ$ normalizes L_K and $L_K \le C_G(z)$ normalizes Q, Q centralizes these sections. Therefore

$$[O_2(L_K), Q, W_1] = 1$$

and

$$[W_1, Q, O_2(L_K)] = 1.$$

The Three Subgroups Lemma implies that $[W_1, O_2(L_K), Q] = 1$. Since $[W_1, O_2(L_K)]$ is normalized by $N_K(R)$ and $N_K(R)$ permutes the involutions in R transitively, $R = [W_1, O_2(L_K)] \le Z(Q)$. Let $\omega \in O^2(L_K)$ have order $q + \sqrt{2q} + 1$. Then ω acts fixed-point-freely on the natural module for $L_K/O_2(L_K) \cong {}^2\mathrm{B}_2(q)$. Since ω normalizes Q, we have

$Q = [Q,\omega]C_Q(\omega)$ and $[Q,\omega] \le L_K$. Now $C_Q(\omega)$ normalizes $[W_1,\omega]$ and as $[W_1, Q] \le R = C_{W_1}(\omega)$, we have

$$[[W_1,\omega], C_Q(\omega)] \le [W_1,\omega] \cap C_{W_1}(\omega) = 1.$$

Hence $C_Q(\omega)$ centralizes $W_1 = R[W_1,\omega]$.

By Lemma 4.11, $O_2(M)$ is not abelian. Since $O_2(M) \le S_yT_y$ and T_y is abelian, we have $O_2(M)' \le S_y$ and $O_2(M)' \cap Y_M$ is normalized by P. In particular,

$$R < Z(O_2(P)) \le O_2(M)' \cap Y_M.$$

Choose $x \in C_Q(\omega) \setminus C_Q(y)$ such that $x^2 \in C_Q(y)$. Then

$$x^2 \in C_{S_yT_y}(\omega) = C_{S_y}(\omega)T_y \le C_{S_yT_y}(W_1T_y) \le C_{S_yT_y}(Y_M) \le O_2(M).$$

We consider the action of x on Y_M. As x centralizes W_1 and $T_y \cap T_y^x = 1$ by Lemma 5.10 and Lemma 5.9, $C_{Y_M}(x) = Y_M \cap W_1 = Y_M \cap K$. As $x^2 \in O_2(M)$,

$$[Y_M, x] \le C_{Y_M}(x) = Y_M \cap K.$$

We also have $[O_2(L_K), x] \le [O_2(L_K), Q] \le C_{O_2(L_K)}(W_1)$ and so

$$[O_2(L_K), x, Y_M] = 1.$$

As $[Y_M, O_2(L_K), x] \le [R, x] = 1$, the Three Subgroups Lemma implies that

$$[Y_M, x] \le C_{Y_M}(O_2(L_K)) = R \le O_2(M)' \cap Y_M \le K \cap Y_M = C_{Y_M}(x).$$

Hence x centralizes the sections of the M-invariant series

$$Y_M > Y_M \cap O_2(M)' > 1$$

which means that $x \in O_2(M/C_M) = 1$. Thus x centralizes Y_M. But then x centralizes y and this is impossible. Therefore $C_Q(\omega) \le C_Q(y)$ and $Q = [Q,\omega]C_Q(y) \le O_2(L_K)C_Q(y) \le C_G(y)$, a contradiction. □

Lemma 10.8. *Suppose that $K/Z(K) \cong \mathrm{PSL}_n(q)$, $n \ge 4$ and $q = 2^a$. Then $K/Z(K) \cong \mathrm{PSL}_3(4)$.*

Proof. Suppose false. By Lemmas 10.2 and 10.4, $n \ge 4$. Furthermore, as $\mathrm{PSL}_4(2) \cong \mathrm{Alt}(8)$, we have by Proposition 9.5 that $K/Z(K) \not\cong \mathrm{PSL}_4(2)$. Now by Lemmas 3.11 and 5.1 and [6, Theorem 6.1.4] we have K is simple. Let $R = Z(S \cap K)$. Then Lemma 3.11 implies that $\widetilde{z} \in \widetilde{R}$. Thus L_K is not a 2-group and so K is normalized by $C_G(Y_M)C_Q(y)$ by Lemma 5.21. Since $(n, q) \ne (4, 2)$, Lemmas 3.1 and 3.2 show that $O_2(L_K)/R$ is a direct sum of two non-isomorphic $C_K(R)$-modules. Let their preimages be E_1 and E_2. We have E_1 and E_2 are elementary abelian and they have order q^{n-1}.

If $Y_M \le RC_{C_S(y)}(K)$, then $O^2(L_K)$ centralizes Y_M and the aforementioned module structure of $O_2(L_K)$ provides a contradiction via Lemma

5.22 (ii). Therefore $Y_M \not\leq RC_{C_S(y)}(K)$. Since U_Q is elementary abelian and is normalized by L_K and $U_Q \not\leq RC_{C_S(y)}(K)$, we have, without loss of generality, $U_Q C_{C_S(y)}(K) = E_1$. Hence $O_2(M)$ normalizes E_1 and hence also E_2. But then $Y_M \leq O_2(O_2(M)N_K(E_2)) \leq E_2 C_{C_S(y)}(K)$ by Lemma 5.27. Therefore $Y_M \leq E_1 C_{C_S(y)}(K) \cap E_2 C_{C_S(y)}(K) = RC_{C_S(y)}(K)$ and we have a contradiction. □

Lemma 10.9. *We have $K/Z(K) \not\cong \mathrm{G}_2(4)$.*

Proof. By [6, Table 6.1.3] we have $|Z(K)| \leq 2$. Let R be the preimage of $Z((S \cap K)/Z(K))$. By Lemma 3.11, $\widetilde{z} \in \widetilde{R}^{\#}$. As U_Q is elementary abelian and normalized by L_K, application of Lemma 3.18 shows that $Y_M C_S(K)$ is centralized by $O^2(L_K)$. Therefore $K = E_y$ and M° normalizes $O^2(L_K)$ by Lemma 5.22. Since, by Lemma 5.25, $4 \leq |Y_M T_y/T_y| \leq |Z(S \cap K)T_Y/T_Y| \leq 4$, we have $Y_M T_y/T_Y = Z(S \cap K)T_Y/T_Y$ has order 4. Lemmas 4.12 and 5.11 imply that

$$|Y_M| = 16.$$

Furthermore by Lemma 5.15 we have that

$$T_y = Y_M \cap T_y.$$

Notice that $O_2(M) \leq C_{S_y O_2(M)}(R/Z(K))$ and so $O_2(M)$ is normalized by $N_K(S \cap K)$. This subgroup contains an element ρ of order 3 which operates non-trivially on $R/Z(K)$. It follows that $[Y_M, \rho]$ has order 4 and ρ centralizes $C_{Y_M}(K)$. Since $z \in O_2(M)' \leq K$ and we have $Y_M \cap O_2(M)'$ has order 4 or 8. Suppose first that the order is 8. Then $1 \neq Z(K) \leq R$. Since $O_2(M^\dagger/C_M) = 1$ and $M^\dagger$ is not transitive on R, we have $M^\dagger/C_M \cong \mathrm{Sym}(3)$ and $Z(K) = C_R(\rho)$ is centralized by Q, a contradiction. Hence $R = Y_M \cap O_2(M)'$ has order 4. Since ρ centralizes Y_M/R, we have $\langle \rho \rangle$ is normalized by S and hence so is $C_{Y_M}(\rho) = C_{Y_M}(K)$. But then $Z(Q) \cap T_y \neq 1$, and so we have a contradiction. □

Proposition 10.10. *If $K/Z(K)$ is a simple group of Lie type in characteristic 2, then $K/Z(K) \cong \mathrm{PSL}_3(4)$ or $\mathrm{Sp}_4(q)$, $q = 2^a \geq 4$.*

Proof. Suppose false and let $q = 2^a$. Then combining the lemmas of this section, we have $K/Z(K) \cong \mathrm{U}_n(q)$, $n \geq 4$, $\Omega^{\pm}_{2n}(q)$, $n \geq 4$, $\mathrm{G}_2(q)$, $q \geq 8$, ${}^3\mathrm{D}_4(q)$, ${}^2\mathrm{E}_6(q)$ or $\mathrm{E}_n(q)$, $n = 6, 7, 8$. Also, by Proposition 8.5, $K/Z(K) \not\cong \mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3)$. Lemma 3.11 implies that z acts on $K/Z(K)$ as a 2-central element. Set $L_K = C_K(z)$ and $J = C_{O^2(L_K)}(Z(O_2(O^2(L_K))))$. Then Lemma 3.1 implies $O_2(O^2(L_K))$ is non-abelian and $O^2(J)$ acts irreducibly on $O_2(O^2(L_K))/Z(O_2(O^2(L_K)))$. This contradicts Lemma 5.23 and proves the proposition. □

11. The groups $\mathrm{PSL}_3(4)$ and $\mathrm{PSp}_4(q)$ as components

In this section we assume that $K/Z(K) \cong \mathrm{PSL}_3(4)$ or $\mathrm{PSp}_4(q)$, $q = 2^a > 2$. We will show that this is not possible. By [6, Theorem 6.1.4] we have $Z(K) = 1$ if $K \cong \mathrm{PSp}_4(q)$. By Lemma 3.11, z acts as an inner automorphism on K and so $Z(K)$ is a 2-group by Lemma 5.1.

Lemma 11.1. *If $K/Z(K) \cong \mathrm{PSL}_3(4)$ and $Z(K) \neq 1$, then $Z(K)$ is elementary abelian of order at most 4.*

Proof. Assume that $Z(K)$ contains an element of order four. Then $Z(S \cap K) = Z(K)$ by Lemma 3.9 and z acts on K as an element of $Z(S \cap K)$ and so z centralizes K. This contradicts Lemma 5.1. Thus $Z(K)$ is elementary abelian of order at most 4 by [6, Theorem 6.1.4]. □

We now establish the notation which will be used throughout this section. We write

$$E_y = K_1 \dots K_r$$

where each K_i is a component of $C_G(y)$ with $K_i/Z(K_i) \cong K/Z(K)$ and $|K_i| = |K|$. For $1 \leq i \leq r$, we define

$$S_i = S_y \cap K_i.$$

If $K \cong \mathrm{PSp}_4(q)$, we let E_{ij}, $j = 1, 2$, be the maximal order elementary abelian subgroups of order q^3 in S_i described in Lemma 3.7. If $K/Z(K) \cong \mathrm{PSL}_3(4)$, then we let E_{ij}, $j = 1, 2$, be the elementary abelian subgroups of order $16|Z(K)|$ as described in Lemma 3.13 (i). In all cases we have that every elementary abelian subgroup of S_i is contained in E_{i1} or in E_{i2}. When discussing a fixed component K, we often abbreviate our notation using E_1 and E_2 in place of E_{i1} and E_{i2}.

Define

$$D_y = J(O_2(M) \cap T_y).$$

The proof takes different directions depending upon whether or not D_y is abelian.

Lemma 11.2. *Suppose that $K/Z(K) \cong \mathrm{Sp}_4(q)$, $q = 2^a \geq 4$. Then $Z(K) = 1$ and no element of $\Omega_1(Z(S))$ projects on to a root element of $\widetilde{K}$.*

Proof. By [6, Table 6.1.3] we have $Z(K) = 1$. Let $Z(S \cap K) = R_1 R_2$ with R_1 and R_2 root subgroups. Suppose that $z \in \Omega_1(Z(S))$ is such that $\widetilde{z}$ is a root element in $\widetilde{S \cap K}$. Then L_K is not a 2-group and so Lemma 5.21 implies $C_G(Y_M)C_Q(y)$ normalizes K and then $C_Q(y)$ normalizes $O^2(L_K)$.

Suppose $\widetilde{Y_M} = O_2(\widetilde{N_K(R_1)})$. Then $O_2(M)$ normalizes $N_K(R_1)$ and $N_K(R_2)$ and $Y_M \not\leq O_2(N_G(R_2))$. Employing Lemma 4.8 and Lemma 5.27 we have a contradiction. Now Lemma 3.7 shows that $O^2(L_K)$ centralizes any subgroup of Y_M, which is normalized by $O^2(L_K)$. Application of Lemma 5.20 shows that Q normalizes $O^2(L_K)$. Then, as $O_2(L_K)$ is elementary abelian and contains exactly one non-central $O^2(L_K)$-chief factor, Lemma 5.24 implies $O_2(L_K) \leq Z(Q)$. Hence $K = \langle C_K(R_1), C_K(R_2)\rangle \leq N_G(Q)$, a contradiction. This proves the lemma. □

We remark that the next lemma does not require that $|C_S(y)|$ is chosen to be maximal.

Lemma 11.3. *Suppose that $y \in \mathcal{Y}_S^*$. Then the following hold.*

(i) $N_{E_y}(S_y) \leq M^\dagger$.

(ii) $Y_M = (Y_M \cap T_y)(Y_M \cap S_y)$ *and* $(Y_M \cap K)Z(K) = Z(S_y \cap K)$.

(iii) $O_2(M)$ *normalizes* K *and* $J(O_2(M)) = S_y D_y$.

(iv) *Either* $O_2(M) = S_y(O_2(M) \cap T_y)$ *or* $\widetilde{O_2(M)}K$ *is isomorphic to* $\mathrm{PSL}_3(4)$ *extended by a graph automorphism.*

Proof. By Lemma 5.16, Y_M normalizes K. Thus $[S_y \cap K, Y_M] \leq Y_M \cap K$.

Assume that $\widetilde{Y_M} \cap \widetilde{K} \not\leq Z(\widetilde{S_y \cap K})$. Then $[Y_M, S_y \cap K] \not\leq Z(K)$ and so $Y_M \cap K \not\leq Z(K)$. Therefore, as $O_2(M)$ centralizes Y_M, $O_2(M)$ normalizes K. We may assume that $\widetilde{Y_M} \cap \widetilde{S_y \cap K}$ is contained in $\widetilde{E_1}$ but not in $Z(\widetilde{S_y \cap K})$. In particular $\widetilde{O_2(M)}$ normalizes $\widetilde{E_1}$. But then $O_2(M)$ normalizes E_1 and also normalizes E_2. We have that $J = N_K(E_2)$ is of characteristic 2 and is normalized by $O_2(M)$. However $\widetilde{Y_M} \not\leq \widetilde{E_2}$ and this contradicts Lemma 4.8. Thus

(11.3.1) $\widetilde{Y_M} \cap \widetilde{S_y \cap K} \leq Z(\widetilde{S_y \cap K})$.

Assume there exits $x \in Y_M^\#$, which induces an outer automorphism on K. Then $[\widetilde{x}, \widetilde{S_y \cap K}] \leq [\widetilde{Y_M}, \widetilde{S_y \cap K}] \leq \widetilde{Y_M} \cap \widetilde{S_y \cap K} \leq Z(\widetilde{S_y \cap K})$. In particular x cannot interchange E_1 and E_2. This yields that x induces a field automorphism on K. But such an automorphism is non-trivial on $E_1/Z(S_y \cap K)$. Therefore $\widetilde{Y_M} \leq \widetilde{S_y \cap K}$ which then means by (11.3.1)

(11.3.2) $\widetilde{Y_M} \leq Z(\widetilde{S_y \cap K})$.

That is

$$Y_M \leq Z(S_y \cap K)C_{C_S(y)}(K) \leq C_{C_S(y)}(S_y \cap K).$$

Since this is true for all the components of E_y, we have

(11.3.3) $S_y \le C_S(Y_M) = O_2(M)$.

Consider $\overline{E_y} = E_y O_2(M)/Z(E_y)$. Then $\overline{O_2(M)}$ is a Sylow 2-subgroup of $\overline{E_y}$. We have $\overline{E_1E_2} = J(\overline{S_y \cap K}) = J(\overline{O_2(M)})$ by Lemma 3.13 and Lemma 3.7. Therefore, by Proposition 2.7 and Lemmas 3.14 and 11.1 we get $J(O_2(M))$ normalizes K. It follows that $J(O_2(M)) \cap E_y = S_y$ and $J(O_2(M)) = S_yD_y$ by Lemma 2.8 and the definition of D_y. In particular, $J(O_2(M))$ is normalized by $N_{E_y}(S_y)$ and so $N_{E_y}(S_y) \le M^\dagger$ by Lemma 4.3. Hence (i) is true.

Using (i) and the fact that $1 \ne \widetilde{Y_M} \le Z(\widetilde{S_y \cap K})$ by (11.3.2), we have $\widetilde{Y_M} = Z(\widetilde{S_y \cap K})$, as by Lemma 11.2 $\widetilde{Y}_M$ is not contained in a root group when $K \cong \mathrm{Sp}_4(q)$. Further

$$1 \ne [Y_M, N_K(S_y \cap K)]Z(K)/Z(K) = Z((S \cap K)/Z(K)).$$

In particular, $Y_M \cap K \not\le Z(K)$ and so $O_2(M)$ normalizes K. Furthermore, letting C be a complement to S_y in $N_{E_y}(S_y)$, we have

$$Y_M = [Y_M, C]C_{Y_M}(C) = (Y_M \cap S_y)(Y_M \cap T_y)$$

and $[Y_M, C]Z(K) = Z(S_y)$. This is (ii).

We have just seen that $O_2(M)$ normalizes all the components of E_y and $S_y \le O_2(M)$. We have also proved $J(O_2(M)) = S_yD_y$. This is (iii).

Since $O_2(M)$ normalizes K and $\widetilde{O_2(M)}$ centralizes $Z(\widetilde{S_y \cap K})$ by (ii), either $\widetilde{O_2(M)} = \widetilde{S_y}$ or $\widetilde{O_2(M)}K$ is isomorphic to $\mathrm{PSL}_3(4)$ extended by a graph automorphism (see [7, Chapter 10, Lemma 2.1]). Hence (iv) holds. □

Lemma 11.4. *Suppose that D_y is abelian and $r \ge 2$. If $x \in (Y_M \cap K_i) \setminus Z(K_i)$ for some $1 \le i \le r$, then $E_x = \prod_{j \ne i} K_j < E_y$.*

Proof. By Lemma 5.5, $\prod_{j\ne i} K_j \le E_x$ and $\prod_{j \ne i} K_j$ is non-trivial as $r \ge 2$. Assume that $E_x > \prod_{j \ne i} K_j$. Then there is a component L of $C_G(x)$ with $L/Z(L) \cong K/Z(K)$, $|L| = |K|$ and $L \not\le \prod_{j\ne i} K_j$. By Lemma 5.5, the normal closure of $\prod_{j \ne i} K_j$ in E_x has at least $r-1$ components of $C_G(x)$ and only has $r-1$ components if every component of $\prod_{j\ne i} K_j$ is a component of $C_G(x)$. Therefore E_x has exactly r components.

Suppose that K is simple. Then $E_x \cong E_y$ and we can apply Lemma 11.3 to find $S_x \cap L = O_2(M) \cap L \in \mathrm{Syl}_2(L)$ and $S_x \cap L \le J(O_2(M)) = S_yD_y$. Consider $K_j \le E_x$ and let C be a complement to $N_{K_j}(S_j)$. Then

$[S_yD_y, C] = S_j$ and so, as C normalizes L,

$$[S_x \cap L, C] \le S_j \cap L \le K_j \cap L \le Z(K_j) = 1.$$

Hence, temporarily setting $\overline{E_x} = E_x/C_{E_x}(L)$, we have

$$\overline{C} \le C_{\overline{L}}(\overline{S_x \cap L}) = Z(\overline{S_x \cap L})$$

and this means that $C \le C_{E_x}(L)$. Hence $K_j \cap C_{E_x}(L) \not\le Z(K_j)$ which means that K_j centralizes L. Therefore L centralizes $\prod_{j\neq i} K_j$ and so

$$S_x \cap L \le C_{J(O_2(M))}(\prod_{j\neq i} K_j) = D_yS_i.$$

Since D_y is abelian, this shows that $(S \cap L)' = S_i'$. As $x \in S_i' \le L$ and x centralizes L, we deduce $Z(L) \neq 1$, a contradiction. Hence

$$Z(K) \neq 1.$$

As $O_2(M)$ normalizes K_j, $1 \le j \le r$, by Lemma 11.3 (iii), we can choose an involution $w \in Z(\prod_{j\neq i} K_j) \cap Y_M$. Then $w \in D_y$ and $E_w = E_y$ by Lemma 5.6. Since L is a component of $C_G(x)$ and K_j is quasisimple for $2 \le j \le r$, we have

$$L/Z(L) \cong LC_{E_x}(L)/C_{E_x}(L) \cong K_jC_{E_x}(L)/C_{E_x}(L) \cong K_j/Z(K_j)$$

and so $\prod_{i\neq j} Z(K_j) \le C_{E_x}(L)$. It follows that L centralizes w and so L normalizes $E_w = E_y$. Thus L normalizes $E(C_{E_y}(x)) = \prod_{j\neq i} K_j$ and so L normalizes $K_i = E(C_{E_y}(\prod_{j\neq i} K_j))$. Since L normalizes $\prod_{j\neq i} K_j$ and L is a component in E_x, L centralizes $\prod_{j\neq i} K_j$. Because L centralizes x and L is a component of $C_G(x)$, L centralizes $C_{K_i}(x)$. We know that $C_{K_i}(x) = S_i$. Hence L centralizes $S_i \prod_{j\neq i} K_j \ge S_y$. From Lemma 11.3(iii), $J(O_2(M)) = D_yS_y$. Therefore, as D_y is abelian, $1 \neq J(O_2(M))' = S_y'$. Thus L is a component in $C_G(J(O_2(M))')$ and this contradicts Lemma 4.7 (ii). Thus $E_x = \prod_{j\neq i} K_j$, a claimed □

Lemma 11.5. *D_y is non-abelian.*

Proof. Assume that D_y is abelian. By Lemma 11.3 (ii), $J(O_2(M)) = D_yS_y$. As $D_y = J(D_y)$, D_y is elementary abelian. Now $\Omega_1(Z(J(O_2(M)))) = D_yZ(S_y \cap K) = D_yY_M$ by Lemma 11.3(ii). Hence $[D_yY_M, O_2(M)] \le D_y$ and as $Z(S) \cap D_y = 1$, we see that $[D_y, O_2(M)] = 1$. Hence

$$D_y = Y_M \cap T_y = \Omega_1(T_y).$$

Recall that S_i contains exactly two maximal rank elementary abelian subgroups E_{i1}, E_{i2} of order $16|Z(K_i)|$ if $K/Z(K) \cong \mathrm{PSL}_3(4)$, and q^3

otherwise. Thus the set of maximal order elementary abelian subgroups in D_yS_y is

$$\mathfrak{A} = \left\{ D_y \prod_{i=1}^{r} E_{ii_j} \mid i_j \in \{1,2\} \right\}$$

and M permutes $\mathfrak{A}$ by conjugation. The group M also permutes the pairs $(F_1, F_2) \in \mathfrak{A} \times \mathfrak{A}$ which have the property that

$$|F_1 : F_1 \cap F_2| = |F_2 : F_1 \cap F_2| = \begin{cases} 4 & K/Z(K) \cong \mathrm{PSL}_3(4) \\ q & \text{otherwise} \end{cases}.$$

Then M permutes the set of commutators $[F_1, F_2]$ for all such pairs (F_1, F_2). Let the set of such commutators be Θ. Then, as $[E_{i1}, E_{i2}] = S_i'$,

$$\Theta = \{S_i' \mid 1 \le i \le r\}$$

and we have explained that

$$M \text{ permutes the groups in } \Theta.$$

Assume that $r > 1$. Then, as M permutes Θ, M normalizes

$$\begin{aligned} N = \langle L \quad | \quad & L \text{ a component of } C_G(Y_M \cap K_i), \\ & L/Z(L) \cong K/Z(K) \text{ and } |L| = |K|, i = 1, \dots, r\rangle. \end{aligned}$$

By Lemmas 11.3 and 11.4, $N = \langle \prod_{j \neq i} K_j \mid 1 \le i \le r\rangle = E_y$ is normalized by M. It follows that $D_y = C_{J(O_2(M))}(E_y)$ is normalized by M and this contradicts Lemma 5.7 as $y \in D_y$ shows that $D_y \neq 1$. Thus

$$r = 1 \text{ and } \mathfrak{A} = \{D_yE_{11}, D_yE_{12}\}.$$

If D_yE_{11} is normal in M, then $N_K(D_yE_{11}) = N_K(E_{11}) \le M^\dagger$, but by Lemma 11.3 Y_M is not normal in $N_K(E_{11})$. Hence D_yE_{11} and D_yE_{12} are conjugate in M and so in S. Suppose that $A \le O_2(M)$ is an elementary abelian normal subgroup of S. By Lemma 11.3, we either have $O_2(M) = (O_2(M) \cap T_y)J(O_2(M))$ or $K/Z(K) \cong \mathrm{PSL}_3(4)$ and $O_2(M)$ induces a graph automorphism. In the latter case, Lemma 3.13 implies $A \le S_yT_y = J(O_2(M))$. Hence always $A \le \Omega_1(T_yS_y) = J(O_2(M))$, and so, as A is normal in S, we obtain $A \le D_yE_{11} \cap D_yE_{12} = D_y(E_{11} \cap E_{12}) = Y_M$, as $D_y = Y_M \cap T_y$. This contradicts Lemma 4.11. □

Proposition 11.6. *Suppose that $y \in \mathcal{Y}_S^*$ and let $K \le E_y$ be a component of $C_G(y)$. Then $K/Z(K) \not\cong \mathrm{PSL}_3(4)$ or $\mathrm{Sp}_4(q)$, $q = 2^a > 2$.*

Proof. Assume the proposition is false. By Lemma 11.5 D_y is non-abelian. We first prove the following claim.

(11.6.1) *The component K is simple and there exists $N \le G$ normalized by M such that*

$$N = E(N) = E_y K_{r+1} = K_1 \cdots K_{r+1}$$

with $[E_y, K_{r+1}] = 1$ and $K \cong K_{r+1}$. Furthermore, $S \cap N = J(O_2(M))$, S permutes the components of N transitively by conjugation, and $M = SN_M(K_1)$.

Let $g \in N_S(N_S(T_y)) \setminus N_S(T_y)$ with $g^2 \in N_S(T_y)$. By Lemma 5.10, $D_y \cap D_y^g = 1$ and $[D_y, D_y^g] = 1$. As $D_y \le J(O_2(M))$ and g normalizes $O_2(M)$, $D_y^g \le J(O_2(M))$. As D_y is normal in $N_S(T_y)$ the same applies for D_y^g.

For $1 \le i \le r$, set

$$C_i = \prod_{j \ne i} S_j D_y.$$

As, for $i \ne j$, $S_i \cap S_j \le K_i \cap K_j \le Z(K_i) \cap Z(K_j) \le D_y$, the Modular Law implies $\bigcap_{i=1}^r C_i = D_y$. In addition, we also have $[S_i, C_i] \le [K_i, C_i] = 1$.

If $D_y^g C_i/C_i$ is abelian for all i, then $(D_y^g)' \le \bigcap_{i=1}^r C_i = D_y$ contrary to $(D_y^g)' \ne 1$ and $D_y \cap D_y^g = 1$. Thus we may fix notation so that $D_y^g C_1/C_1$ is not abelian. Set $\overline{S_y D_y} = S_y D_y / C_1$. Then $\overline{D_y^g} \le \overline{S_1}$ and $\overline{D_y^g} \not\le \overline{E_{1j}}$ for $j = 1, 2$ as $\overline{D_y^g}$ is not abelian. Let $\rho \in N_{K_1}(S_1)$ be arbitrary of maximal odd order and such that ρ acts fixed-point-freely on $S_1/Z(S_1)$. By Lemma 11.3 $\rho \in N_{M^\dagger}(O_2(M))$ and by Lemma 5.10 $T_y^g \cap O_2(M)$ is a trivial intersection group in $N_G(O_2(M))$. As $D_y^g = J(O_2(M) \cap T_y^g)$, we see that ρ normalizes D_y^g if and only if it normalizes $T_y^g \cap O_2(M)$. Suppose that ρ does not normalize D_y^g. Then $[D_y^g, D_y^{g\rho}] = 1$, as both groups are normal in $O_2(M)$. But then $[\overline{D_y^g}, \overline{D_y^{g\rho}}] = 1$, and this contradicts Lemmas 3.8 and 3.13 (iii). Hence ρ normalizes D_y^g. It also normalizes $\overline{D_y^g}$ and, as $D_y = J(D_y)$ is generated by involutions, it follows that $\overline{D_y^g} = \overline{S_1}$. In particular, $[S_1, D_y^g] = S_1' = Z(S_1)$. We have $D_y^g = C_{D_y^g}(\rho)[D_y^g, \rho]$. Now $[S_y D_y, \rho] = S_1$. Hence $[D_y^g, \rho] \le S_1$ and $C_{D_y^g}(\rho) \le Z(S_1)C_1$. We conclude that $S_1 \le D_y^g$ as

$$\overline{S_1} = \overline{D_y^g} = \overline{[D_y^g, \rho] C_{D_y^g}(\rho)} \le \overline{[D_y^g, \rho] Z(S_1)} \le \overline{[D_y^g, \rho](S_1 \cap D_y^g)} \le \overline{(D_y^g \cap S_1)}.$$

If K_1 is not simple, then $Z(K_1) \le S_1 \le D_y^g$ and so $D_y \cap D_y^g \ne 1$, a contradiction. Hence the components in E_y are simple groups. This proves the first statement in (11.6.1).

Since $D_y^g \ge S_1$ and $Y_M \cap S_1 = Z(S_1)$ by Lemma 11.3, using Lemma 5.6 we have $E_x = E_y^g$ for all $x \in (Y_M \cap S_1)^\#$.

If $K_1 \cong \mathrm{Sp}_4(q)$, then $Z(S_1)$ contains root subgroups R_1 and R_2 and so $K_1 = \langle C_{K_1}(R_1), C_{K_1}(R_2)\rangle$ normalizes E_y^g.

Suppose that $K_1 \cong \mathrm{PSL}_3(4)$. Then all the involutions in K_1 are K_1-conjugate. Thus for involutions $t \in S_1\backslash Z(S_1)$, the group E_t is conjugate to E_x. Since $t \in D_y^g$, we get $E_t = E_y^g$ by Lemma 5.4. Hence this time we see that $K_1 = \langle C_K(t) \mid t \in S_1\rangle$ normalizes E_y^g. Thus in all cases

$$K_1 \text{ normalizes } E_y^g.$$

Furthermore, by Lemma 5.5, $K_2 \dots K_r \le E_y^g$ (this is obviously true if $r = 1$). Hence

$$E_y \text{ normalizes } E_y^g.$$

Since $g^2 \in N_G(E_y)$, we also have E_y^g normalizes $E_y^{g^2} = E_y$. It follows that the components of E_y and the components of E_y^g are components of $E_yE_y^g$. It now follows that $E_y \cap E_y^g = K_2 \cdots K_r$ and that we can write

$$N = E_yE_y^g = K_1K_2 \cdots K_rK_{r+1}$$

where $E_y^g = K_2 \dots K_{r+1}$.

We have $J(O_2(M)) = S_y^gD_y^g$ and so $S_y^gD_y^g \cap N \in \mathrm{Syl}_2(N)$. Furthermore $C_{O_2(M)}(N) = 1$, as otherwise $C_{Y_M}(N) \ne 1$, but this is not possible as $y \in \mathcal{Y}^*$. Therefore

$$J(O_2(M)) \in \mathrm{Syl}_2(N).$$

This verifies the second and third statement in (11.6.1).

Define $S_{r+1} = O_2(M) \cap K_{r+1}$. Then

$$\prod_{i=1}^{r+1} S_i \in \mathrm{Syl}_2(N).$$

We now argue as in the case when D_y was abelian. The subgroups S_i contains exactly two maximal rank elementary abelian subgroups E_{i1}, E_{i2} of order 16 if $K \cong \mathrm{PSL}_3(4)$, and q^3 otherwise. Thus the set of maximal order elementary abelian subgroups in $J(O_2(M))$ is

$$\mathfrak{A} = \left\{ \prod_{i=1}^{r+1} E_{ii_j} \mid i_j \in \{1, 2\} \right\}$$

and M permutes $\mathfrak{A}$ by conjugation. The subgroup M also permutes the pairs $(F_1, F_2) \in \mathfrak{A} \times \mathfrak{A}$ which have the property that

$$|F_1 : F_1 \cap F_2| = |F_2 : F_1 \cap F_2| = \begin{cases} 4 & K \cong \mathrm{PSL}_4(3) \\ q & \text{otherwise} \end{cases}.$$

Thus M permutes the set of commutators $[F_1, F_2]$ for all such pairs (F_1, F_2). As $[F_1, F_2] = Y_M \cap S_i$ for some i, M permutates the set

$$\Theta = \{Y_M \cap S_i \mid 1 \leq i \leq r\}.$$

Hence M normalizes the subgroup

$$\begin{aligned} N^* = \langle L \quad \mid \quad & L \text{ a component of } C_G(Y_M \cap K_i), \\ & L/Z(L) \cong K/Z(K) \text{ and } |L| = |K|, i = 1, \ldots, r\rangle. \end{aligned}$$

Using the structure of N, we see that $N^* = N$. Hence M normalizes N. In particular, S permutes the set $\{K_1, \ldots K_{r+1}\}$ by conjugation. Suppose that $\{K_j \mid j \in J\}$ is an S-orbit. We get that $Z(S) \cap Y_M \cap \prod_{j \in J} K_j \neq 1$. Application of Lemma 4.7 (v) gives $J = \{1, \ldots, r+1\}$. Thus S acts transitively on $\{K_i \mid 1 \leq i \leq r+1\}$. Finally, as S is transitive on $\{K_i \mid 1 \leq i \leq r+1\}$, $M = N_M(K_1)S$ by the Frattini Argument. This completes the explanation of (11.6.1). ■

By (11.6.1), $M = N_M(K_1)S$ and so $S \cap N_M(K_1) \in \mathrm{Syl}_2(N_M(K_1))$. We also know $N_M(K_1)$ normalizes $J(O_2(M)) \cap K_1 = S_1$. Suppose that E_{11} is not conjugate to E_{12} in $N_M(K_1)$. Then E_{11} is normal in $N_M(K_1)$ and $F = \langle E_{11}^M \rangle$ is elementary abelian with $F \cap K_j \in \{E_{j1}, E_{j2}\}$. Thus F is normalized by $\langle M, N_{K_1}(E_{11})\rangle \leq M^\dagger$. Since $N_{K_1}(E_{11})$ does not normalize Y_M, this is impossible. Hence E_{11} is conjugate to E_{12} in $N_M(K_1)$. Since $S \cap N_M(K_1) \in \mathrm{Syl}_2(N_M(K_1))$, (11.6.1) implies S acts transitively on $\{E_{ij} \mid 1 \leq i \leq r+1, j = 1, 2\}$.

Let A be an elementary abelian normal subgroup of S contained in $O_2(M)$. Put $\overline{N_M(K_1)} = N_M(K_1)/C_M(K_1)$. Then $\overline{N_M(K_1)}$ normalizes $\overline{J(O_2(M)) \cap K_1} = \overline{S_1}$ and $\overline{A}$ is normal in $\overline{S \cap N_M(K_1)}$. It follows that

$$\overline{A} \leq \overline{E_{11}} \cap \overline{E_{12}} = \overline{Y_M}.$$

Hence $[A, O_2(M)] \leq C_M(K_1)$. Since S acts transitively on $\{K_1, \ldots, K_{r+1}\}$ and S normalizes $[A, O_2(M)]$, we have

$$[A, O_2(M)] \leq O_2(M) \cap \bigcap_{i=1}^{r+1} C_M(K_i) = C_{O_2(M)}(N) = 1.$$

Hence $A \leq Y_M$. Now application of Lemma 4.11 yields the contradiction. This proves the proposition. □

12. Proof of the Theorem

Let M and Y_M be as in the assumption of the theorem. That is Y_M is tall, asymmetric but not characteristic 2-tall. By Lemma 5.2 there is some $y \in Y_M^\#$ with $E(C_G(y)) \neq 1$. In particular $\mathcal{Y}_S^* \neq \emptyset$. For $y \in \mathcal{Y}_S^*$ we have $E_y \neq 1$. Let K be a component of E_y. By Proposition 7.2 $K/Z(K)$ is not a sporadic simple group. By Proposition 8.5 and Lemma 10.2 $K/Z(K)$ is not a group of Lie type in odd characteristic. Proposition 9.5 states that $K/Z(K)$ is not an alternating group. Hence $K/Z(K)$ a group of Lie type in characteristic 2. Proposition 10.10 shows that $K/Z(K) \cong \mathrm{PSL}_3(4)$ or $\mathrm{Sp}_4(q)$, $q \geq 4$. Finally Proposition 11.6 provides the contradiction which proves the theorem.

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