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# ON THE STABILITY OF SELF-SIMILAR SOLUTIONS OF 1D CUBIC SCHRÖDINGER EQUATIONS

S. GUTIERREZ[1] AND L. VEGA[2]

ABSTRACT. In this paper we will study the stability properties of self-similar solutions of 1-d cubic NLS equations with time-dependent coefficients of the form

$$iu_t + u_{xx} + \frac{u}{2}(|u|^2 - \frac{A}{t}) = 0, \qquad A \in \mathbb{R}. \qquad (0.1)$$

The study of the stability of these self-similar solutions is related, through the Hasimoto transformation, to the stability of some singular vortex dynamics in the setting of the *Localized Induction Equation* (LIE), an equation modeling the self-induced motion of vortex filaments in ideal fluids and superfluids. We follow the approach used by Banica and Vega that is based on the so-called pseudo-conformal transformation, which reduces the problem to the construction of modified wave operators for solutions of the equation

$$iv_t + v_{xx} + \frac{v}{2t}(|v|^2 - A) = 0.$$

As a by-product of our results we prove that equation (0.1) is well-posed in appropriate function spaces when the initial datum is given by $u(0,x) = z_0 \, \mathrm{p.\,v} \frac{1}{x}$ for some values of $z_0 \in \mathbb{C} \setminus \{0\}$, and $A$ is adequately chosen. This is in deep contrast with the case when the initial datum is the Dirac-delta distribution.

## 1. INTRODUCTION

In this paper, we study the stability properties of self-similar solutions of the form

$$u_f(t,x) = \frac{e^{\frac{ix^2}{4t}}}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right), \qquad x \in \mathbb{R}, \qquad t > 0 \qquad (1.1)$$

to the cubic nonlinear Schrödinger equations (NLS) in one dimension:

$$iu_t + u_{xx} + \frac{u}{2}\left(|u|^2 - \frac{A}{t}\right) = 0, \qquad A \in \mathbb{R}, \qquad (1.2)$$

that is solutions $u_f$ of the form (1.1) with $f$ a solution of the equation

$$f'' + i\frac{x}{2}f' + \frac{f}{2}(|f|^2 - A) = 0, \qquad A \in \mathbb{R}. \qquad (1.3)$$

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Our main motivation for the study of solutions of (1.2) of the form (1.1) comes from their connection to the singular vortex dynamics of what we refer to as "self-similar" solutions to the so-called *Localized Induction Approximation*, a geometric flow in $\mathbb{R}^3$ modeling the dynamics of a vortex filament in ideal fluids and superfluids.

The *Localized Induction Approximation*, often abbreviated LIA or LIE, is described by the following system of nonlinear equations:

$$\mathbf{X}_t = \mathbf{X}_x \times \mathbf{X}_{xx}, \tag{1.4}$$

where $\mathbf{X} = \mathbf{X}(t, x)$ represents a curve in $\mathbb{R}^3$ with $t$ and $x$ denoting time and arclength, respectively. Using the Frenet equations we can also write

$$\mathbf{X}_t = cb \tag{1.5}$$

with $c$ and $b$ denoting the curvature and the binormal vector respectively. For this reason the geometric PDE (1.4) is also referred to as *binormal flow*.

Equation (1.4) was first proposed by Da Rios in 1906, and rediscovered independently by Arms-Hamma and Betchov in the early 1960s (see [DaR], [AH] and [Be]), as an approximation model for the self-induced motion of a vortex filament in a 3D-incompressible inviscid fluid. The use of the localized induction equation to model the dynamical behaviour of a vortex in superfluids such as $^4$He started with the work by Schwarz in 1985 ([Sch]). In both, the classical and the superfluid settings, the term localized induction approximation is used to highlight the fact that this approximation only retains the local effects of the Biot-Savart integral. We refer the reader to [B], [S], [AKO] and [MB] for a detailed analysis of the model and its limitations, and to the two papers by T. Lipniacki in [Lip1] and [Lip2] for further background and references about the use of LIA in the setting of superfluid helium.

Cubic NLS equations of the type (1.2) are related to LIA through the so-called Hasimoto transformation (see [Has]). This connection is establised as follows: Let $\mathbf{X} = \mathbf{X}(t, x)$ be a regular solution of LIA with associated curvature $c(t, x)$, and torsion $\tau(t, x)$. Assuming that the curvature is strictly positive at all points $x$, define the *filament function*

$$u(t, x) = c(t, x)\exp\left(i \int_0^x \tau(t, x') \, dx'\right). \tag{1.6}$$

Then $u$ solves the nonlinear Schrödinger equation

$$iu_t + u_{xx} + \frac{u}{2}(|u|^2 - A(t)) = 0, \tag{1.7}$$

where $A(t)$ is a time-dependent function which depends on the values of $c(t, x)$ and $\tau(t, x)$ at $x = 0$. Precisely,

$$A(t) = \left(2\frac{c_{xx} - c\tau^2}{c} + c^2\right)(t, 0). \tag{1.8}$$

Our analytical study of solutions of LIA started in [GRV], and [GV], where the existence of solutions of LIA which develop a singularity in finite time was established[1]. The study of the stability properties of the singular dynamics leading to the formation of a corner singularity in finite time found in [GRV] was carried out by V. Banica and L. Vega in the papers [BV1], [BV2], and [BV3].

Here, we are concerned with the singular dynamics found in [GV]. In particular, in [GV], solutions of LIA of the form

$$\mathbf{X}(t,x) = e^{\frac{\mathcal{A}}{2}\log t}\sqrt{t}\mathbf{G}(x/\sqrt{t}), \qquad t > 0 \tag{1.9}$$

with $\mathcal{A}$ a real antisymmetric $3 \times 3$ matrix of the form

$$\mathcal{A} = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad a \in \mathbb{R} \tag{1.10}$$

are found to converge to a singular initial data $\mathbf{X}(0,x)$. The precise statement of the result is the following:

**Proposition 1.1.** *(See* [GV, Proposition 1 and 2]*) For any given $a \in \mathbb{R}$, and $\mathbf{G}$ solution of*

$$\mathbf{G}'' = \frac{1}{2}(\mathcal{I} + \mathcal{A})\mathbf{G} \times \mathbf{G}' \tag{1.11}$$

*associated to an initial data $(\mathbf{G}(0), \mathbf{G}'(0))$ satisfying*

$$|\mathbf{G}(0)| = 1 \qquad and \qquad (\mathcal{I} + \mathcal{A})\mathbf{G}(0) \cdot \mathbf{G}'(0) = 0, \tag{1.12}$$

*define*

$$\mathbf{X}_a(t,x) = e^{\frac{\mathcal{A}}{2}\log t}\sqrt{t}\mathbf{G}(x/\sqrt{t}), \qquad t > 0, \qquad with \qquad \mathcal{A} = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.13}$$

*Then, $\mathbf{X}_a(t,x)$ is an analytic solution of LIA for all $t > 0$, and there exist non-zero vectors $\mathbf{A}^+$ and $\mathbf{A}^- \in \mathbb{R}^3$ such that[2]*

$$\lim_{t \to 0^+} \mathbf{X}_a(t,x) = xe^{\mathcal{A}\log|x|}(\mathbf{A}^+\chi_{[0,+\infty)}(x) + \mathbf{A}^-\chi_{(-\infty,0]}(x)) := \mathbf{X}_a(0,x)$$

*with*

$$|\mathbf{X}_a(t,x) - xe^{\mathcal{A}\log|x|}\mathbf{A}^\pm| \leq 2\sqrt{t}\left(\sup_{x \in \mathbb{R}}|c(x)|\right).$$

*Here, $c(x)$ is the curvature of the curve $\mathbf{G}(x) = \mathbf{X}(1,x)$, which is always bounded.*

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[1]See also [Bu], [Lip1], and [Lip2].

[2]$\chi_E(x)$ denotes the characteristic function of a Lebesgue measurable set $E$.

Solutions of the form (1.9) have also been considered by T. Lipniacki (see [Lip1] and [Lip2]) in the setting of the flow defined by

$$\mathbf{X}(t, x) = \beta \mathbf{X}_x \times \mathbf{X}_{xx} + \alpha \mathbf{X}_{xx}, \qquad \alpha \neq 0$$

modeling the motion of a quantum vortex in superfluid helium.

Proposition 1.1 asserts that the evolution of the solution $\mathbf{G}(x)$ of (1.11)-(1.12) under the relation (1.13) leads to a solution of LIA which converges as $t \to 0^+$ to an initial curve $\mathbf{X}_a(0, x)$ given by

$$\mathbf{X}_a(0, x) = xe^{\mathcal{A} \log |x|}(\mathbf{A}^+ \chi_{[0,+\infty)}(x) + \mathbf{A}^- \chi_{(-\infty,0]}(x)).$$

The initial curve $\mathbf{X}_a(0, x)$ is the sum of two 3d-logarithmic spirals with a common origin. The rotation axis of these spirals is the OZ-axis under the condition that the matrix $\mathcal{A}$ is of the form (1.10). In the case when the parameter $a \neq 0$, the singularity of the initial curve $\mathbf{X}_a(0, x)$ comes from the non-existence of the limit as $x \to 0$ of its tangent vector $\mathbf{T}_a(0, x)$.

The properties of the "self-similar" solutions of LIA given by Proposition 1.1 rely, through the Hasimoto transformation, on the properties of their associated filament function defined by (1.6). This connection plays a fundamental role in the study of the properties of these solutions (see [GV]). In particular, and following the philosophy in [BV2], a first step to understand the stability properties of these solutions is to study the stability of their related filament function in the setting of the cubic Schrödinger equations (1.7). This will be our main interest here.

In order to find the filament function associated to the "self-similar" solutions of LIA given by Proposition 1.1, first notice that it is straightforward to verify that the curvature and torsion associated to solutions of LIA $\mathbf{X}_a(t, x)$ of the form (1.13) are of the self-similar form[3]

$$c(t, x) = \frac{1}{\sqrt{t}} c(x/\sqrt{t}) \qquad \text{and} \qquad \tau(t, x) = \frac{1}{\sqrt{t}} \tau(x/\sqrt{t}),$$

so their filament function is given by (see (1.6))

$$u(t, x) = \frac{1}{\sqrt{t}} u\left(\frac{x}{\sqrt{t}}\right) \qquad \text{and} \qquad A(t) = \frac{A}{t}, \tag{1.14}$$

with $A = A(1)$ defined by (1.8). Since $\mathbf{X}_a(t, x)$ is a solution of LIA, through the Hasimoto transform, we know that its filament function (1.14) solves the NLS

$$iu_t + u_{xx} + \frac{u}{2}(|u|^2 - \frac{A}{t}) = 0 \qquad \text{with} \qquad A = A(1). \tag{1.15}$$

Thus the function $u(s)$ in (1.14) is a solution of the complex ODE

$$u'' - \frac{i}{2}(u + xu') + \frac{u}{2}(|u|^2 - A) = 0. \tag{1.16}$$

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[3]This is the reason why we refer to solutions of LIA of the form (1.9) as "self-similar" solutions.

Notice that, by introducing a new variable $f$ defined by

$$u(x) = f(x)e^{i\frac{x^2}{4}}, \tag{1.17}$$

equation (1.16) becomes

$$f'' + i\frac{s}{2}f' + \frac{f}{2}(|f|^2 - A) = 0. \tag{1.18}$$

Previous lines show that the filament function associated to a solution of LIA $\mathbf{X}_a(t, x)$ given by Proposition 1.1 is of the form

$$u(t, x) = \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right),$$

with $f$ a solution of the second order ODE (1.18), and solves the nonlinear Schrödinger equation (1.15).

Furthermore, in [GV] it was proved that the constant $A$ in the above equation is given in terms of the initial conditions $(\mathbf{G}(0), \mathbf{G}'(0))$ and the parameter $a$ by the identity

$$A = aT_3(0) + \frac{|(\mathcal{I} + \mathcal{A})\mathbf{G}(0)|^2}{4}, \tag{1.19}$$

and a solution $\mathbf{X}_a(t, x)$ and its associated function $f$ are related through the following identities (see [GV, pp. 2114])

$$|f|^2(x) = -aT_3(x) + A, \qquad \text{and} \qquad |f'|^2(x) = \frac{1}{4}|\mathcal{A}\mathbf{T} \times \mathbf{T}|^2(x) = \frac{a^2}{4}(1 - T_3^2), \tag{1.20}$$

where $\mathbf{T} = (T_1, T_2, T_3)$ is the tangent unitary vector associated to $\mathbf{G}(x) = \mathbf{X}(1, x)$.

Among all the possible solutions of LIA $\mathbf{X}_a(t, x)$ of the form (1.13) given by Proposition 1.1, and in order to motivate further our main result in this paper, it is important to mention two special cases. In what follows, we will define what we refer to as *odd-solutions* and *mixed-symmetry solutions* of LIA.

The following cases come from the symmetry properties of the equation

$$\mathbf{G}'' = \frac{1}{2}(\mathcal{I} + \mathcal{A})\mathbf{G} \times \mathbf{G}', \qquad \mathcal{A} = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.21}$$

*Odd solutions:* For fixed $a \in \mathbb{R}$ and $-1 \le \lambda \le 1$, let $\mathbf{G}_{a,\lambda}$ the solution of (1.21) with the initial condition

$$\mathbf{G}_{a,\lambda}(0) = (0, 0, 0) \qquad \text{and} \qquad (\mathbf{G}_{a,\lambda})'(0) = (0, \sqrt{1 - \lambda^2}, \lambda). \tag{1.22}$$

Then,

$$\mathbf{G}_{a,\lambda}(x) = -\mathbf{G}_{a,\lambda}(-x) \tag{1.23}$$

(notice that if $\mathbf{G}(x)$ is a solution of (1.21)) with the initial condition (1.22), then the function $\tilde{\mathbf{G}}(x) = -\mathbf{G}(-x)$ is also a solution).

We refer the solutions of LIA of the form (1.13) with $\mathbf{G}_{a,\lambda}(x)$ the solution of (1.21)-(1.22) as *odd solutions*.

In Figure 1 and Figure 2, we display the graphics of different solutions $\mathbf{G}_{a,\lambda}$ of (1.21) associated to an initial data of the form (1.22). The right-handside pictures represent the solution near the point $x = 0$. The curvature of the curves $\mathbf{G}_{a,\lambda}$ at the point $x = 0$ is zero.



FIGURE 1. Odd solutions. The vortex line $\mathbf{G}_{a,\lambda}$ corresponding to the solution of the system (1.21)-(1.22) with $a = 10$ and $\lambda = 0.956$.
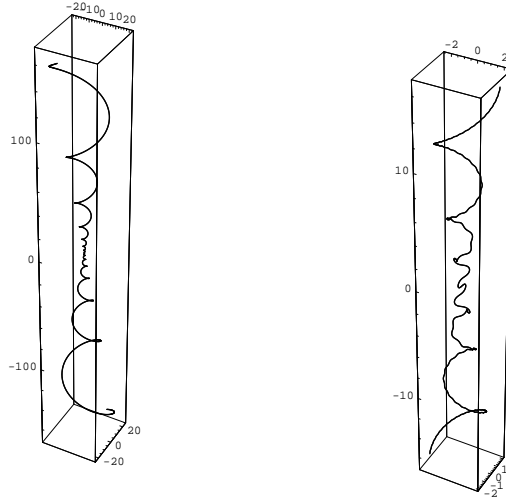


FIGURE 2. Odd solutions. The vortex line $\mathbf{G}_{a,\lambda}$ corresponding to the solution of the system (1.21)-(1.22) with $a = 10$ and $\lambda = -0.1$.

*Mixed-symmetry solutions:* For fixed $a \in \mathbb{R}$ and $c_0 > 0$, let $\mathbf{G}_{a,c_0}$ the solution of (1.21) with the initial condition

$$\mathbf{G}_{a,c_0}(0) = (\frac{2c_0}{\sqrt{1+a^2}}, 0, 0) \qquad \text{and} \qquad (\mathbf{G}_{a,c_0})'(0) = (0, 0, \pm 1). \tag{1.24}$$

Then, $\mathbf{G}_{a,c_0} = (G_1, G_2, G_3)$ satisfies

$$\begin{cases} G_1(x) = G_1(-x) \\ G_2(x) = G_2(-x) \\ G_3(x) = -G_3(-x). \end{cases} \tag{1.25}$$

This is a consequence of the fact that the equation (1.21) and the initial condition in (1.24) remain unchanged by the transformation $\mathbf{G}(x) = (G_1(x), G_2(x), G_3(x)) \rightsquigarrow (G_1(-x), G_2(-x), -G_3(-x))$.

We refer the solutions of LIA of the form (1.13) with $\mathbf{G}_{a,c_0}(x)$ the solution of (1.21)-(1.24) as *mixed-symmetry solutions*.

Two examples of solutions of (1.21) with initial data of the form (1.24) are plotted in Figures 3 and Figure 4. As before, the r.h.s figure represents the curve $\mathbf{G}_{a,c_0}$ near the point $x = 0$.
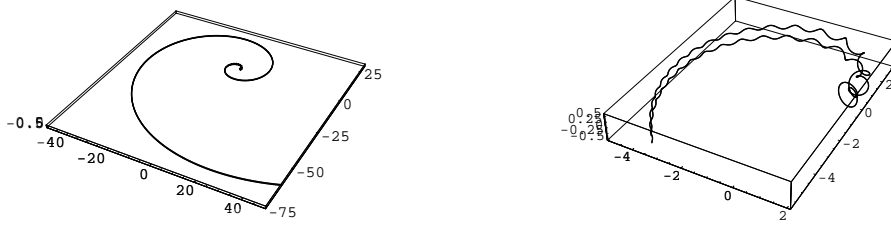


FIGURE 3. Mixed-symmetry solutions. The vortex line $\mathbf{G}_{a,c_0}$ corresponding to the solution of the system (1.21)-(1.24) with $a = 3$ and $c_0 = 1.8$.
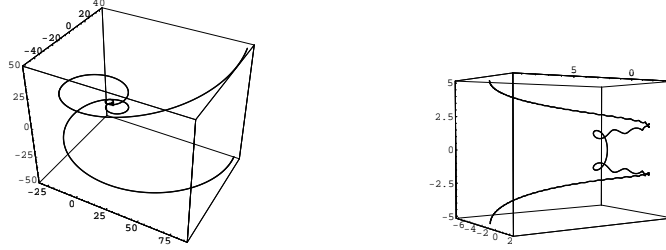


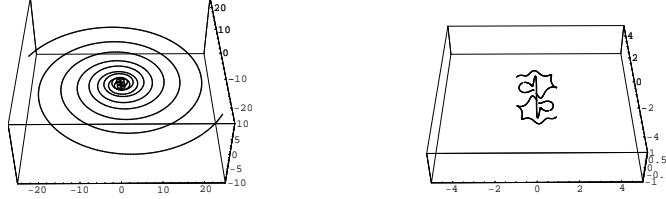FIGURE 4. Mixed-symmetry solutions. The vortex line $\mathbf{G}_{a,c_0}$ corresponding to the solution of the system (1.21)-(1.24) with $a = 3$ and $c_0 = 0.4$.

Finally, observe that if $\mathbf{G}(x) = (G_1(x), G_2(x), G_3(x))$ is a solution of (1.21), then the function $\tilde{\mathbf{G}}(x) = (G_1(-x), -G_2(-x), G_3(-x))$ is a solution of

$$\mathbf{G}'' = \frac{1}{2}(\mathcal{I} + \tilde{\mathcal{A}})\mathbf{G} \times \mathbf{G}' \qquad \text{with} \qquad \tilde{A} = \begin{pmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As a consequence, in what follows we will assume w.l.o.g that $a \geq 0$.

Using the formulae (1.19) and (1.20), from the initial conditions (1.22) it follows that the function $f$ associated to an odd solution of LIA is, through the Hasimoto transformation (1.6) and the change of variables (1.17), a (*odd*) solution of

$$f'' + i\frac{x}{2}f' + \frac{f}{2}(|f|^2 - A) = 0, \qquad A = a\lambda$$

with initial conditions $(f(0), f'(0))$ satisfying

$$|f(0)|^2 = 0 \qquad \text{and} \qquad |f'(0)|^2 = \frac{a^2}{4}(1 - \lambda^2), \qquad a > 0[4], \qquad -1 \le \lambda \le 1.$$

Analogously, from (1.19), (1.20), and (1.24) it follows that the function $f$ associated to a mixed-symmetry solution of LIA is a (*even*) solution of

$$f'' + i\frac{x}{2}f' + \frac{f}{2}(|f|^2 - A) = 0, \qquad A = \pm a + c_0^2$$

with initial conditions $(f(0), f'(0))$ satisfying

$$|f(0)|^2 = c_0^2 \qquad \text{and} \qquad |f'(0)|^2 = 0.$$

From (1.14), (1.15), and the above argument it follows that the filament function $u(t, x)$ associated to an odd solution of LIA (respectively mixed-symmetry solution) is of the form

$$u_f(t, x) = \frac{e^{\frac{ix^2}{4t}}}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right), \qquad x \in \mathbb{R}, \qquad t > 0 \tag{1.26}$$

$$f'' + i\frac{x}{2}f' + \frac{f}{2}(|f|^2 - A) = 0, \tag{1.27}$$

with $A = a\lambda$ (resp. $A = \pm a + c_0^2$), and solves the one dimensional cubic Schrödinger equation

$$iu_t + u_{xx} + \frac{u}{2}\left(|u|^2 - \frac{A}{t}\right) = 0, \tag{1.28}$$

with $A = a\lambda$ (resp. $A = \pm a + c_0^2$).

As we have already mentioned, this paper is devoted to the study the stability properties of certain self-similar solutions $u_f$ in (1.26) of the 1D cubic Shrödinger equation (1.28).

In order to give a precise statement of our results, we consider the so-called pseudo-conformal transformation of (1.28). Briefly, given any solution $u$ of (1.28), we define a new unknown $v$ as follows

$$u(t, x) = \mathcal{T}v(t, x) = \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} \bar{v}\left(\frac{1}{t}, \frac{x}{t}\right). \tag{1.29}$$

Here, and elsewhere, an overbar denotes complex conjugation. Then $v$ has to be a solution of

$$iv_t + v_{xx} + \frac{v}{2t}\left(|v|^2 - A\right) = 0. \tag{1.30}$$

In particular, solutions $u_f$ of (1.28) correspond to solutions $v_f$ of (1.30) of the form

$$v_f(t, x) = \bar{f}\left(\frac{x}{\sqrt{t}}\right). \tag{1.31}$$

Thus, we are reduced to prove the existence of appropriate perturbations (modified wave operator) around the solutions $v_f$,

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[4]Notice that if $a = 0$, then $|f(0)| = |f'(0)| = 0$, so that $f \equiv 0$.

The study of the stability properties of solutions of (1.30) of the form (1.31) (and, consequently, of solutions of (1.28) of the form (1.26)) started in [BV1], and carried on in [BV2] and [BV3]. Precisely, in [BV2], the authors studied the stability of the solution of (1.30) with $A = c_0^2$ given by

$$v_{c_0}(t, x) = c_0.$$

In [BV2, Theorem 1.2]), under the smallness assumption of the parameter $c_0 > 0$, the authors prove that for any $t_0 > 0$, and any given asymptotic state $u_+$ small in $L^1 \cap L^2$ (w.r.t $t_0$ and $c_0$) the equation (1.30) has a unique solution $v(t, x)$ in the interval $[t_0, \infty)$ which behaves like

$$v_1(t, x) = c_0 + e^{i\frac{c_0^2}{2}\log t}\left(e^{it\partial_x^2}u_+\right)(x)$$

as t goes to infinity, in the sense that

$$\|v(t) - v_1(t)\|_{L^2} = \mathcal{O}(t^{-\frac{1}{4}}), \qquad \text{as} \qquad t \longrightarrow \infty. \tag{1.32}$$

Here $e^{it\partial_x^2}$ denotes the free propagator (see notation below). In other words, they construct the so-called (modified) wave operators. In [BV3] this result is extended to first remove the smallness assumption on $c_0$ and moreover to consider also the asymptotic completeness of the scattering operators. One of the fundamental ingredients in this paper is the study of the linearized equation of (1.30) around the constant solution $v_{c_0}(t, x) = c_0$ $(A = c_0{}^2)$ given by

$$iz_t + z_{xx} + \frac{c_0^2}{2t}(z + \bar{z}) = 0. \tag{1.33}$$

Notice that the coefficients in the above equation only depend on $t$, and as a consequence this linearized equation can be analyzed by computing the Fourier transform in space. Unfortunately, in our case the linearized equation of (1.30) around solutions $v_f$ of the form (1.31) is given by (see (2.1) below)

$$iz_t + z_{xx} + \frac{1}{2t}[(2|v_f|^2 - A)z + v_f^2\bar{z}] = 0, \tag{1.34}$$

with coefficients that are also space dependent[5]. This makes the analysis of the linearized equation (1.34) to be much more delicate. Therefore, we put ourselves in the most simple situation. Firstly, and as in [BV2], we will just consider the construction of the wave operators. Secondly, we reduce our analysis to those self-similar solutions $v_f(t, x) = \bar{f}(x/\sqrt{t})$ that have the extra property that[6]

$$|f|_{+\infty} = |f|_{-\infty} \qquad (\text{that is} \quad |f|(+\infty) = |f|(-\infty)),$$

and in particular those that the function $f$ is an odd or even function. Even under this assumption the equation (1.34) is not so easy to handle. In fact, as we will see in the statement of the main theorem below, we can not consider the asymptotic state

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[5]See Proposition 2.1 below for the properties of $v_f(t, x) = \bar{f}(x/\sqrt{t})$.

[6]The existence of $|f|_{\pm\infty}$ was established in [GV], see Proposition 2.1 below.

$u_+$ to be in $L^1 \cap L^2$ as in [BV2] and some weighted $L^2$-spaces are necessary. This implies some loss in the rate of decay given in (1.32). The main difficulty comes from the appearance in the Duhamel term (2.13) of $v_f^2$, which depends on both the spatial and time variable. This differs from the situation in [BV2] where $v_f^2(t,x) = c_0^2$.

Before stating our results, we introduce some conventions and function spaces. We denote by $L^2(|x|^\gamma)$ and $L^2(\langle x \rangle^\gamma)$ the $L^2$-spaces with Lebesgue measure replaced by $|x|^\gamma \, dx$, and $\langle x \rangle^\gamma \, dx = (1+|x|^2)^{\gamma/2} \, dx$, respectively, i.e.,

$$L^2(|x|^\gamma) = \{\phi : \mathbb{R} \longrightarrow \mathbb{C} \; : \; \|\phi\|_{L^2(|x|^\gamma)} = \left( \int_\mathbb{R} |\phi(x)|^2 |x|^\gamma \, dx \right)^{1/2} < \infty\},$$

and

$$L^2(\langle x \rangle^\gamma) = \{\phi : \mathbb{R} \longrightarrow \mathbb{C} \; : \; \|\phi\|_{L^2(\langle x \rangle^\gamma)} = \left( \int_\mathbb{R} |\phi(x)|^2 (1+|x|^2)^{\gamma/2} \, dx \right)^{1/2} < \infty\}.$$

For $s \in \mathbb{N}^\star$, the Sobolev space $H^s$ is defined by

$$H^s = \{f \in \mathcal{S}(\mathbb{R}) \; : \; \nabla^k f \in L^2(\mathbb{R}), \; \forall \, 0 \leq k \leq s\}.$$

The Fourier transform of $v$, $\hat{v}$, is defined by

$$\hat{v}(\xi) = \frac{1}{2\pi} \int_\mathbb{R} e^{-ix \cdot \xi} v(x) \, dx,$$

$e^{it\partial_x^2} u_0$ denotes the solution to the initial value problem for the free 1D Schrödinger equation with initial data $u_0$, defined by

$$\left( e^{it\partial_x^2} u_0 \right)(x) := \int_\mathbb{R} e^{ix\xi} e^{-i\xi^2 t} \widehat{u_0}(\xi) \, d\xi, \tag{1.35}$$

or, equivalently,

$$\left( e^{it\partial_x^2} u_0 \right)(x) := \frac{1}{\sqrt{4\pi it}} \int_\mathbb{R} u_0(y) e^{i\frac{(x-y)^2}{4t}} \, dy. \tag{1.36}$$

For any $u_+$, and $f$ solution of (1.3) such that $|f|_{+\infty} = |f|_{-\infty}$, we define $\tilde{v}_f$ by

$$\tilde{v}_f(t,x) = v_f(t,x) + e^{i\frac{\alpha}{2} \log t} \left( e^{it\partial_x^2} u_+ \right)(x), \tag{1.37}$$

with

$$v_f(t,x) = \bar{f}\left( \frac{x}{\sqrt{t}} \right) \quad \text{and} \quad \alpha = 2|f|_\infty^2 - A.^7$$

Our main result is the following

**Theorem 1.2.** *Let $t_0 > 0$, and $0 < \gamma < 1$. There exists a (small) positive constant $B_0$, such that for any $A$ and any $f$ solution of*

$$f'' + i\frac{x}{2} f' + \frac{f}{2}(|f|^2 - A) = 0$$

---

[7] If $|f|_{+\infty} = |f|_{-\infty}$ ($|f'|_{+\infty} = |f'|_{-\infty}$), then we will denote $|f|_{\pm\infty}$ by $|f|_\infty$ (respectively, $|f'|_{\pm\infty}$ by $|f'|_\infty$).

such that $|f|_{-\infty} = |f|_{+\infty}$ with $\|f\|_{L^\infty} \leq B_0$, and $u_+$ small in $L^1 \cap L^2(\langle x \rangle^\gamma)$ with respect to $B_0$, $t_0$, and $f$, the equation

$$iv_t + v_{xx} + \frac{v}{2t}(|v|^2 - A) = 0 \tag{1.38}$$

has a unique solution $v(t,x)$ in the time interval $[t_0, \infty)$ such that

$$v - \tilde{v}_f \in \mathcal{C}\left([t_0, \infty), L^2(\mathbb{R})\right) \cap L^4\left([t_0, \infty), L^\infty(\mathbb{R})\right).$$

Moreover, the solution $v$ satisfies

$$\|v - \tilde{v}_f\|_{L^2(\mathbb{R})} + \|v - \tilde{v}_f\|_{L^4((t,\infty),L^\infty(\mathbb{R}))} = \mathcal{O}\left(\frac{1}{t^{\frac{\gamma}{4}}}\right), \tag{1.39}$$

as $t$ goes to infinity.

In addition, if $\partial_x u_+ \in L^1 \cap L^2(\langle x \rangle^\gamma)$, with $0 < \gamma < 1$, then $v - \tilde{v}_f \in H^1$ and

$$\|v - \tilde{v}_f\|_{H^1} = \mathcal{O}\left(\frac{1}{t^{\frac{\gamma}{4}}}\right), \qquad t \to \infty. \tag{1.40}$$

The above result asserts the existence of the modified wave operator in the time interval $[t_0, \infty)$ with $t_0 > 0$, for any given final data $u_+$ in $L^1 \cap L^2(\langle x \rangle^\gamma)$ with $0 < \gamma < 1$, and any $f$ solution of (1.3) such that $|f|_{+\infty} = |f|_{-\infty}$, under smallness conditions on $\|f\|_{L^\infty}$ and the data $u_+$.

**Remark 1.3.** *As we said before, the new difficulties in the proof of this result with respect to those in [BV2] come from the space dependence of the coefficients of linearized equation (1.34). There is a particular case where this equation is as simple as (1.33). This happens when the phase function $\phi_2(x) = (|f|_{\pm}^2 - A) \log |x|$ in Proposition 2.1 is identically zero, that is when $|f|_{\pm\infty}^2 = A$.*

*It turns out that in this particular case the corresponding curve $\mathbf{X}(t,x)$ is asymptotically flat at infinity, that is $T_3(\pm\infty) = 0$ with $T_3(x)$ being the third component of the tangent vector to the curve $\mathbf{X}(t,x)$ (see (1.20)). In this situation, one could expect that the stronger results proved in [BV3] could also be extended to this case. This will be studied elsewhere.*

Once $v$ has been constructed, we recover $u$ through the pseudo-conformal transformation (1.29). Precisely, defining $\tilde{u}_f$ as

$$\tilde{u}_f(t,x) = \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right) + \sqrt{\pi i}\, e^{i\frac{\alpha}{2}\log t}\, \widehat{\tilde{u}_+}\left(-\frac{x}{2}\right), \qquad \alpha = 2|f|_\infty^2 - A, \tag{1.41}$$

as a consequence of Theorem 1.2 we obtain the following:

**Theorem 1.4.** *Let $\tilde{t}_0 > 0$, and $0 < \gamma < 1$. There exist a (small) positive constant $B_0$, such that for any $A$ and any $f$ solution of*

$$f'' + i\frac{x}{2}f' + \frac{f}{2}(|f|^2 - A) = 0$$

*such that* $|f|_{+\infty} = |f|_{-\infty}$ *with* $\|f\|_{L^\infty} \leq B_0$, *and* $u_+$ *small in* $L^1 \cap L^2(\langle x \rangle^\gamma)$ *with respect to* $B_0$, $\tilde{t}_0$ *and* $f$, *the equation*

$$iu_t + u_{xx} + \frac{u}{2}\left(|u|^2 - \frac{A}{t}\right) = 0. \tag{1.42}$$

*has a unique solution* $u(t, x)$ *in the interval* $(0, \tilde{t}_0]$ *such that*

$$u - \tilde{u}_f \in \mathcal{C}\left((0, \tilde{t}_0], L^2(\mathbb{R})\right) \cap L^4\left((0, \tilde{t}_0], L^\infty(\mathbb{R})\right).$$

*Moreover, as* $t$ *goes to zero, the solution* $u$ *satisfies,*

$$\|u - \tilde{u}_f\|_{L^2(\mathbb{R})} + \|u - \tilde{u}_f\|_{L^4((0,t),L^\infty(\mathbb{R}))} = \mathcal{O}(t^{\frac{\gamma}{4}}). \tag{1.43}$$

*In particular, as* $t$ *goes to zero*

$$\left\| u(t, \cdot) - \frac{e^{i\frac{(\cdot)^2}{4t}}}{\sqrt{t}} f\left(\frac{\cdot}{\sqrt{t}}\right) \right\|_{L^2(\mathbb{R})} = \mathcal{O}(1), \qquad and \tag{1.44}$$

$$\left\| \left| u(t, \cdot) - \frac{e^{i\frac{(\cdot)^2}{4t}}}{\sqrt{t}} f\left(\frac{\cdot}{\sqrt{t}}\right)\right|^2 - \left|\sqrt{\pi}\,\widehat{u_+}\left(-\frac{\cdot}{2}\right)\right|^2 \right\|_{L^1(\mathbb{R})} = \mathcal{O}(t^{\frac{\gamma}{4}}), \tag{1.45}$$

*but the limit of* $u(t, x) - \dfrac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} f\left(\dfrac{x}{\sqrt{t}}\right)$ *does not exist in* $L^2(\mathbb{R})$ *as* $t$ *goes to zero unless* $\alpha = 2|f|_\infty^2 - A = 0$.

*Finally, if in addition* $\partial_x u_+ \in L^1 \cap L^2(\langle x \rangle^\gamma)$, *then*

$$|u(t, x)| \leq \frac{2}{\sqrt{t}}\left|f\left(\frac{x}{\sqrt{t}}\right)\right|, \tag{1.46}$$

*for all* $x \in \mathbb{R}$ *and* $0 < t < 1$, *and if* $x \neq 0$ *there exists* $t^*(x) > 0$ *such that for* $0 < t < t^*(x)$

$$\frac{1}{2\sqrt{t}}\left|f\left(\frac{x}{\sqrt{t}}\right)\right| \leq |u(t, x)|. \tag{1.47}$$

**Remark 1.5.** *The case when the solution* $f$ *satisfies the condition* $2|f|_\infty^2 - A = 0$ *(that is, when* $\alpha = 0$*) deserves a special attention. First of all, from (1.41) and (1.43), we observe that* $\tilde{u}_f(t, \cdot)$, *and then* $u(t, \cdot)$, *will have a limit as long as such a limit exists for*

$$u_f(t, x) = \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right).$$

*We will see in Section 3 that precisely under the same condition* $2|f|_\infty^2 - A = 0$, $u_f(t, \cdot)$ *converges in the distribution sense to* $z_0$ p. v $\frac{1}{x}$. *As a consequence, the initial value problem (IVP for short) given by (1.42) and*

$$u(0, x) = z_0 \text{ p. v} \frac{1}{x} + \sqrt{\pi i}\,\widehat{u_+}\left(-\frac{x}{2}\right)$$

*is well-posed in appropriate function spaces. See Theorem 3.2 in Section 3 for the precise statement.*

**Remark 1.6.** *Recall that, for the 1d-cubic NLS equations associated to solutions of LIA of the form (1.9)-(1.10), the coefficient $A$ is linked to the initial conditions and the parameter $a$ through the identity (1.19).*

*From the identity (1.19) and the conservation law for $f$ stated in Proposition 2.1, we conclude that the smallness assumption for $\|f\|_{L^\infty}$ can be achieved by considering initial data $(\mathbf{G}(0), \mathbf{G}'(0), a)$ sufficiently small.*

Notice that the solutions $u$ given by Theorem 1.4 do not have a trace at $t = 0$ (see comment after (1.45)). Nevertheless, associated to these solutions we are able to construct a family of curves $\mathbf{X}(t, x)$ solutions of LIA which do have a limit at $t = 0$. The precise statement of the result is the following:

**Corollary 1.7.** *Let $0 < \gamma < 1$ and $u_+, \ \partial_x u_+ \in L^1 \cap L^2(\langle x \rangle^\gamma)$. Then, under the smallness assumptions of Theorem 1.4, for $0 < t < \tilde{t}_0$ there exists a unique solution $\mathbf{X}(t, x)$ of LIA such that the filament function of $\mathbf{X}(t, x)$ is the function $u(t, x)$ given by Theorem 1.4, $\mathbf{X}(\tilde{t}_0, 0) = (0, 0, 0)$ and $\mathbf{X}_x(\tilde{t}_0, 0) = (1, 0, 0)$.*

*Moreover,*

*i) the curvature of the curve $\mathbf{X}(t, x)$, $c(t, x)$, satisfies*

$$|c(t, x)| \leq \frac{c_1}{\sqrt{t}}$$

*for all $x \in \mathbb{R}$ and $0 < t < 1$, and if $x \neq 0$, there exists $t^*(x) > 0$ such that for all $0 < t < t^*(x)$*

$$\frac{c_2}{\sqrt{t}} \leq |c(t, x)|.$$

*ii) In addition, there exists a unique $\mathbf{X}_0(x)$ such that*

$$|\mathbf{X}(t, x) - \mathbf{X}_0(x)| \leq c_3 \sqrt{t},$$

*uniformly on the interval $(-\infty, \infty)$, with $\mathbf{X}_0(x)$ a Lipschitz continuous function.*

*Here, $c_1, c_2$, and $c_3$ are non-negative constants.*

The proofs of all these results are given in Section 2. In Section 3 we state and prove Theorem 3.2 about the well-posedness of the IVP given by (1.2) and $u(0, x) = z_0 \, \text{p. v} \frac{1}{x}$ for some values of $z_0 \in \mathbb{C} \setminus \{0\}$. The question of well-posedness of the $1d$ cubic NLS for spaces that include $L^2$ was started in [VV], and then extended in [Gru] to all the range of subcritical scales. In fact it was proved in [KPV] that when the initial datum is given by Dirac-delta function, the IVP is ill-posed due to the appearance of a logarithmic correction in the phase. This phase can be canceled out by modifying the equation with an extra factor $A(t) = c_0/t$ for some constant $c_0$ as in equation (1.2). As we said before this modification naturally appears when the $1d$ cubic NLS is obtained from LIA through the Hasimoto transformation. However,

even with this modification it was proved in [BV2] and [BV3] that the problem is still ill-posed for the Dirac-delta. The reason is the same that the one pointed out in the statement of Theorem 1.4. In [KPV] the ill-posedness was obtained as a consequence of the invariance of NLS by galilean transformations. The same ideas imply that, if the notion of well-posedness includes the uniform continuity of the map datum-solution, then the class of Sobolev spaces of negative index has to be excluded, see also [KPV]. However the existence of a priori upper bounds for the Sobolev norm of the solution, in terms of the Sobolev norm of the datum, for arbitrarily large data, and for sufficiently short time can be proved, see [CCT] and [KT].

## 2. MODIFIED WAVE OPERATORS IN MIXED NORM SPACES.

In order to find the "appropriate" modified wave operators for $v - v_f$ in the setting of the equation (1.30), we follow the arguments given in [BV2]. Briefly, write

$$v = v_f + w,$$

with $v$ and $v_f$ solutions of the Schrödinger equation

$$iv_t + v_{xx} + \frac{v}{2t}\left(|v|^2 - A\right) = 0.$$

Then, $w$ has to be a solution of

$$iw_t + w_{xx} + \frac{1}{2t}\left[(|v_f|^2 - A)w + (v_f\bar{w} + \bar{v}_f w + |w|^2)(v_f + w)\right] = 0,$$

or,

$$iw_t + w_{xx} + \frac{1}{2t}\left[(2|v_f|^2 - A)w + v_f^2\bar{w} + 2v_f|w|^2 + \bar{v}_f w^2 + |w|^2 w\right] = 0. \qquad (2.1)$$

The linear term $(2|v_f|^2 - A)w/2t$ (in the above equation) is resonant and, as we will continue to show, it is the responsible for a logarithmic correction of the phase.

In order to deal with the resonant structure of this term, here and in what follows, we assume that $f$ is such that $|f|_{+\infty} = |f|_{-\infty}$, and we write the above equation equivalently as

$$iw_t + w_{xx} + \frac{1}{2t}\left[(2|f|_\infty^2 - A)w + 2(|v_f|^2 - |f|_\infty^2)w + v_f^2\bar{w} + \right.$$

$$\left. 2v_f|w|^2 + \bar{v}_f w^2 + |w|^2 w\right] = 0.$$

Observe that $|f|_{\pm\infty}^2$ is nothing but the limit of $|v_f(t,x)|^2$ as $x \to \pm\infty$, i.e.

$$|f|_\infty^2 = \lim_{x \to \pm\infty}\left|\bar{f}\left(\frac{x}{\sqrt{t}}\right)\right|^2 = \lim_{x \to \pm\infty}|v_f(t,x)|^2.$$

Then, if we define a new function $u$ as

$$u(t,x) = w(t,x)e^{-i\frac{\alpha}{2}\log t}, \quad t > 0, \quad \text{with} \quad \alpha = 2|f|_\infty^2 - A,$$

the function $u$ has to be a solution of

$$iu_t + u_{xx} + \frac{1}{t}(|v_f|^2 - |f|^2_\infty)u + \frac{v_f^2}{2t}e^{-i\alpha \log t}\bar{u}+$$

$$\frac{1}{2t}\left[2v_f e^{-i\frac{\alpha}{2}\log t}|u|^2 + \bar{v}_f e^{i\frac{\alpha}{2}\log t}u^2 + |u|^2 u\right] = 0.$$

Although the first linear term in (2.6), that is

$$(|v_f|^2 - |f|^2_\infty)\frac{u}{t}$$

is still resonant, the structure of $|v_f(t,x)|^2 - |f|^2_\infty$ allows us to treat this term as a perturbative term in the Duhamel formula for the solution and, to consider as initial guess

$$u(t,x) \approx \left(e^{it\partial_x^2}u_+\right)(x), \qquad \text{as} \qquad t \to +\infty.$$

Summing up, for any given asymptotic state $u_+$, we consider the following guess for the perturbation

$$e^{i\frac{\alpha}{2}\log t}\left(e^{it\partial_x^2}u_+\right)(x), \qquad \alpha = 2|f|^2_\infty - A,$$

and define $\tilde{v}_f$ to be

$$\tilde{v}_f(t,x) = v_f(t,x) + e^{i\frac{\alpha}{2}\log t}\left(e^{it\partial_x^2}u_+\right)(x), \quad \text{with} \quad \alpha = 2|f|^2_\infty - A.$$

2.1. **Preliminaries.** Using the notation introduced previously, given $u_+$ and $f$ solution of (1.3) such that $|f|_{+\infty} = |f|_{-\infty}$, we define

$$\tilde{v}_f(t,x) = v_f(t,x) + e^{i\frac{\alpha}{2}\log t}\left(e^{it\partial_x^2}u_+\right)(x), \tag{2.2}$$

where

$$v_f(t,x) = \bar{f}\left(\frac{x}{\sqrt{t}}\right), \qquad \text{and} \qquad \alpha = 2|f|^2_\infty - A \tag{2.3}$$

(recall that if $|f|_{+\infty} = |f|_{-\infty}$, then we write $|f|_\infty$ for $|f|_{\pm\infty}$).

In order to prove the existence of a solution $v$ of

$$iv_t + v_{xx} + \frac{v}{2t}(|v|^2 - A) = 0, \tag{2.4}$$

"close" to $\tilde{v}_f$, as $t$ goes to $\infty$, following the steps in the previous lines, we write

$$v = v_f + e^{i\frac{\alpha}{2}\log t}u, \tag{2.5}$$

so that the function $u$ has to be a solution of is

$$iu_t + u_{xx} + \frac{1}{t}(|v_f|^2 - |f|^2_\infty)u + \frac{v_f^2}{2t}e^{-i\alpha \log t}\bar{u}+$$

$$\frac{1}{2t}\left[2v_f e^{-i\frac{\alpha}{2}\log t}|u|^2 + \bar{v}_f e^{i\frac{\alpha}{2}\log t}u^2 + |u|^2 u\right] = 0. \tag{2.6}$$

Now, notice that under the change of variables (2.5) and the definition of $\tilde{v}_f$ in (2.2), we have that

$$
\begin{aligned}
v - \tilde{v}_f &= (v_f + e^{i\frac{\alpha}{2}\log t}u) - (v_f + e^{i\frac{\alpha}{2}\log t}(e^{it\partial_x^2}u_+)) \\
&= e^{i\frac{\alpha}{2}\log t}(u - e^{it\partial_x^2}u_+).
\end{aligned}
$$

Therefore, we are reduced to prove the existence of a solution of (2.6) "close" to $e^{it\partial_x^2}u_+$. To this end, it is convenient to perform a further change of variables. Precisely, we write

$$
u(t,x) = z(t,x) + z_+(t,x), \qquad \text{with} \qquad z_+(t,x) = e^{it\partial_x^2}u_+. \tag{2.7}
$$

Under the change of variable given by (2.7), equation (2.6) becomes

$$
iz_t + z_{xx} = \frac{1}{2t}\{F_0(z_+) - F_1(z) - NLT(z + z_+)\} \tag{2.8}
$$

where $F_0$, $F_1$ and $NLT$ are defined by

$$
F_0(z_+) = 2(|v_f|^2 - |f|_\infty^2)z_+ + v_f^2 e^{-i\alpha \log t}\bar{z}_+, \tag{2.9}
$$

$$
F_1(z) = 2(|v_f|^2 - |f|_\infty^2)z + v_f^2 e^{-i\alpha \log t}\bar{z}, \tag{2.10}
$$

and

$$
NLT(u) = 2v_f e^{-i\frac{\alpha}{2}\log t}|u|^2 + \bar{v}_f e^{i\frac{\alpha}{2}\log t}u^2 + |u|^2 u. \tag{2.11}
$$

Hence, it suffices to prove the existence of a fixed point of the operator

$$
\begin{aligned}
Bz(t) &= \frac{i}{2}\int_t^\infty e^{i(t-\tau)\partial_x^2}F_0(z_+)\,d\tau - \frac{i}{2}\int_t^\infty e^{i(t-\tau)\partial_x^2}F_1(z)\,\frac{d\tau}{\tau} \\
&\quad - \frac{i}{2}\int_t^\infty e^{i(t-\tau)\partial_x^2}NLT(z + z_+)\,\frac{d\tau}{\tau}
\end{aligned} \tag{2.12}
$$

in an appropriate space.

The Duhamel terms which determine the behaviour of the operator $B$ are the ones related to $F_0(z_+)$, the source term. Notice that in our case we are left to estimate two linear source terms (see (2.9)). Namely, we need to estimate in an appropriate space

$$
\int_t^\infty e^{i(t-\tau)\partial_x^2}\left((|v_f|^2 - |f|_\infty^2)z_+\right)\frac{d\tau}{\tau}
$$

and

$$
\int_t^\infty e^{i(t-\tau)\partial_x^2}(v_f^2 e^{-i\alpha \log t}\bar{z}_+)\frac{d\tau}{\tau}, \qquad \text{with} \qquad z_+ = e^{it\partial_x^2}u_+. \tag{2.13}
$$

The structure of $|v_f(t,x)| - |f|_\infty^2$ allows us to treat the first term as an "error" term. The second linear term has the extra difficulty of being dependent of the spatial variable $x$ through the function $v_f(t,x) = \bar{f}(x/\sqrt{t})$. In order to estimate the linear term involving $v_f^2$, we will use some known properties of the function $v_f$ (more precisely,

in our arguments we will make use of the asymptotic behaviour as $x \to \infty$ of $v_f$).
Recall that the function $v_f$ is defined by

$$v_f(t,x) = \bar{f}\left(\frac{x}{\sqrt{t}}\right),$$

where $f$ is any given solution of the equation

$$f'' + i\frac{x}{2}f' + \frac{f}{2}\left(|f|^2 - A\right) = 0, \qquad A \in \mathbb{R}. \tag{2.14}$$

Equation (2.14) was previously considered in [GV]. The following result summarizes
some of the properties of the solutions $f$ of (2.14) obtained in the latter paper.

**Proposition 2.1.** *Let $f$ be a solution of the equation (2.14). Then*

    *i) $f(x)$, and $f'(x)$ are bounded globally defined functions. Moreover, there exists $E(0) > 0$ such that the identity*

$$|f'|^2 + \frac{1}{4}(|f|^2 - A)^2 = E(0)$$

    *holds true for all $x \in \mathbb{R}$.*

    *ii) The limits $\lim_{x\to\pm\infty} |f|^2(x) = |f|^2_{\pm\infty}$ and $\lim_{x\to\pm\infty} |f'|^2(x) = |f'|^2_{\pm\infty}$ do exist and*

$$|f(x)|^2 - |f|^2_{\pm\infty} = O\left(\frac{1}{|x|}\right), \qquad as \qquad x \to \pm\infty.$$

    *iii) If $|f|_{+\infty} \neq 0$ or $|f|_{-\infty} \neq 0$, then*

$$f(x) = |f|_{\pm\infty}\, e^{ic_\pm}\, e^{i\phi_2(x)} + 2i\,|f'|_{\pm\infty}\frac{e^{id_\pm}}{x}\, e^{i\phi_3(x)} + O\left(\frac{1}{|x|^2}\right),$$

    *as $x \to \pm\infty$.*

*Here, $|f|_{\pm\infty}$, $|f'|_{\pm\infty} \geq 0$, and $c_\pm$ and $d_\pm$ are arbitrary constants in $[0, 2\pi)$,*

$$\phi_2(x) = (|f|^2_{\pm\infty} - A)\log|x|, \qquad and \qquad \phi_3(x) = -(x^2/4) - (2|f|^2_{\pm\infty} - A)\log|x|.$$

We continue to recall the one-dimensional dispersive and Strichartz estimates
which will used throughout this section (see [Caz]). In what follows, we call a pair
$(p,q)$ of exponents *admissible* if

$$p \geq 2, \qquad q \leq \infty, \qquad and \qquad \frac{2}{p} + \frac{1}{q} = \frac{1}{2}.$$

*i) 1d-Strichartz estimates.* Let $I$ be a time interval, then

$$\|e^{it\partial_x^2} f\|_{L^{p_1}(\mathbb{R}; L^{q_1})} \leq C\, \|f\|_{L^2(\mathbb{R})} \tag{2.15}$$

and

$$\left\|\int_{s\in I; s\leq t} e^{i(t-s)\partial_x^2} F(s)\, ds\right\|_{L^{p_1}(I; L^{q_1})} \leq C\, \|F\|_{L^{p'_2}\left(I; L^{q'_2}\right)}, \tag{2.16}$$

for any admissible exponents $(p_i, q_i)$, $i \in \{1, 2\}$.

*ii) Dispersion estimate.*

$$\|e^{it\partial_x^2}f\|_{L^\infty(\mathbb{R})} \le C\, t^{-1/2}\|f\|_{L^1(\mathbb{R})} \tag{2.17}$$

*iii) $L^2$-Conservation law.*

$$\|e^{it\partial_x^2}f\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})} \tag{2.18}$$

The constant $C$ in the above inequalities depend on the exponents involved in the estimates.

The lemmas below will be also used in the construction of the modified wave operators. It is immediate to prove the following:

**Lemma 2.2.** *Let $0 \le \beta \le 4$, $f \in L^2(|x|^\beta)$, and $t > 0$. Then*

$$\|f(\cdot)(e^{-i\frac{(\cdot)^2}{4t}} - 1)\|_{L^2} \le \frac{C}{t^{\beta/4}}\|f\|_{L^2(|x|^\beta)},$$

*for some positive constant $C$ independent of $f$ and $t$.*

**Lemma 2.3.** *Given $\delta \ne 0$, and $t > 0$, define*

$$A_t(\xi) = \int_t^\infty e^{2i\tau\xi^2}\frac{d\tau}{\tau^{1+i\delta}} = \int_t^\infty e^{2i\tau\xi^2}e^{-i\delta\log\tau}\frac{d\tau}{\tau}$$

*for $\xi \ne 0$. Then, there exists a constant $C > 0$ such that*

$$|A_t(\xi)| \le \frac{C}{1 + t\xi^2}, \qquad \forall\, \xi \ne 0. \tag{2.19}$$

*Proof.* For fixed $\xi \ne 0$ , and $t > 0$ such that $\xi^2 t \ge 1$, write

$$A_t(\xi) = \frac{1}{2i\xi^2}\int_t^\infty \frac{d}{d\tau}(e^{2i\tau\xi^2})\frac{d\tau}{\tau^{1+i\delta}},$$

and for $\xi$ and $t$ such that $\xi^2 t \le 1$, write

$$\begin{aligned}
A_t(\xi) &= \left(\int_t^{1/\xi^2} + \int_{1/\xi^2}^\infty\right)e^{2i\tau\xi^2}\,e^{-i\delta\log\tau},\frac{d\tau}{\tau} \\
&= \frac{i}{\delta}\int_t^{1/\xi^2}\frac{d}{d\tau}(e^{-i\delta\log\tau})e^{2i\tau\xi^2}\,d\tau + \frac{1}{2i\xi^2}\int_{1/\xi^2}^\infty\frac{d}{d\tau}(e^{2i\tau\xi^2})\frac{d\tau}{\tau^{1+i\delta}}.
\end{aligned}$$

Inequality (2.19) now follows by integrating by parts in the above identities. $\square$

**Lemma 2.4** (Pitt's inequality. See [Pitt]). *For $f \in \mathcal{S}(\mathbb{R}^d)$, and $0 \le \beta < d$,*

$$\int_{\mathbb{R}^d}|\xi|^{-\beta}|\hat{f}(\xi)|^2\,d\xi \le C_\beta\int_{\mathbb{R}^d}|x|^\beta|f(x)|^2\,dx,$$

*where*

$$C_\beta = \pi^\beta\left[\Gamma\left(\frac{d-\alpha}{4}\right)\Big/\Gamma\left(\frac{d+\alpha}{4}\right)\right]^2$$

We will continue to prove Theorem 1.2.

2.2. **Proof of Theorem 1.2.** Let $0 < \gamma < 1$, and $u_+ \in L^1 \cap L^2(\langle x \rangle^\gamma)$.

For $t_0 \geq 1$ and $\nu \in \mathbb{R}$. We denote $Y_{t_0}^\nu$ the space of functions $v(t, x)$ such that the norm

$$\|z\|_{Y_{t_0}^\nu} = \sup_{t \in [t_0, \infty)} \left( t^\nu \|z(t)\|_{L^2} + t^\nu \|z\|_{L^4((t,\infty), L^\infty(\mathbb{R}))} \right)$$

is finite. In order to prove Theorem 1.2, as we have already mentioned, we shall do a fixed point for the operator $B$ defined in (2.12) in the closed ball

$$\mathcal{B}_R = \{z \; : \; \|z\|_{Y_{t_0}^\nu} \leq R\}, \qquad R > 0$$

with $\nu > 0$ and $R > 0$ to be chosen later on.

For any given $z$ such that[8] $\|z\|_Y \leq R$, we want to estimate (see (2.9), (2.10), (2.11) and 2.12))

$$
\begin{aligned}
(Bz)(t) &= \frac{i}{2} \int_t^\infty e^{i(t-\tau)\partial_x^2} \left\{ 2(|v_f|^2 - |f|_\infty^2) z_+ + v_f^2 e^{-i\alpha \log t} \bar{z}_+ \right\} \frac{d\tau}{\tau} \\
&\quad - \frac{i}{2} \int_t^\infty e^{i(t-\tau)\partial_x^2} \left\{ 2(|v_f|^2 - |f|_\infty^2) z + v_f^2 e^{-i\alpha \log \tau} \bar{z} \right\} \frac{d\tau}{\tau} \\
&\quad - \frac{i}{2} \int_t^\infty e^{i(t-\tau)\partial_x^2} NLT(z + z_+) \frac{d\tau}{\tau}
\end{aligned}
\tag{2.20}
$$

in $Y$. Here, the non-linear term is given by (see (2.11))

$$NLT(u) = 2 v_f e^{-i \frac{\alpha}{2} \log t} |u|^2 + \bar{v}_f e^{i \frac{\alpha}{2} \log t} u^2 + |u|^2 u. \tag{2.21}$$

Recall that (see (2.3) and (2.7))

$$v_f(t, x) = \bar{f} \left( \frac{x}{\sqrt{t}} \right), \qquad z_+(t, x) = e^{it\partial_x^2} u_+, \qquad \alpha = 2|f|^2 - A, \tag{2.22}$$

and $u_+$ is a given function in $L^1 \cap L^2(\langle x \rangle^\gamma)$, with $0 < \gamma < 1$.

In what follows the constant $C$ may be different from an inequality to another in a chain of inequalities.

We begin by estimating the $Y$-norm of the source term in (2.20). To this end, observe that Strichartz estimates (2.16) with exponents $(4, \infty)$ and $(\infty, 2)$, and the dispersion

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[8]In order to simplify notation, in what follows we will write simply $Y$ to denote the space $Y_{t_0}^\nu$.

estimate (2.17) lead to

$$\left\| \int_t^\infty e^{i(t-\tau)\partial_x^2} \left( (|v_f|^2 - |f|_\infty^2) z_+ \right) \frac{d\tau}{\tau} \right\|_Y$$

$$\leq C \sup_{t \geq t_0} t^\nu \int_t^\infty \left\| (|v_f|^2 - |f|_\infty^2) z_+ \right\|_{L^2} \frac{d\tau}{\tau}$$

$$\leq C \sup_{t \geq t_0} t^\nu \int_t^\infty \left\| |v_f|^2 - |f|_\infty^2 \right\|_{L^2} \|z_+\|_{L^\infty} \frac{d\tau}{\tau}$$

$$\leq C \|u_+\|_{L^1} \| |f(\cdot)|^2 - |f|_\infty^2 \|_{L^2} \sup_{t \geq t_0} t^\nu \int_t^\infty \frac{d\tau}{\tau^{1+\frac{1}{4}}}$$

$$= C \|u_+\|_{L^1} \| |f(\cdot)|^2 - |f|_\infty^2 \|_{L^2} \frac{1}{t_0^{\frac{1}{4}-\nu}}, \tag{2.23}$$

for all $\nu$ such that $\nu \leq 1/4$.

In order to control the second source term in (2.20), we use the fact that $f$ is a solution of (2.14) satisfying $|f|_{+\infty} = |f|_{-\infty}$ (and as a consequence $|f'|_{+\infty} = |f'|_{-\infty}$, see part $i)$ in Proposition 2.1). Then, from the asymptotics of $f$ in Proposition 2.1, it follows that

$$(\bar{f})^2(x) = |f|_\infty^2 e^{-2ic_\pm} e^{-2i(|f|_\infty^2 - A) \log |x|}$$

$$- 4i|f|_\infty |f'|_\infty e^{-i(c_\pm + d_\pm)} \frac{e^{i\left(\frac{x^2}{4} + |f|_\infty^2 \log |x|\right)}}{x} + O\left(\frac{1}{|x|^2}\right), \tag{2.24}$$

as $x \to \pm\infty$, so that

$$v_f^2(t, x) = (\bar{f})^2\left(\frac{x}{\sqrt{t}}\right) = v_{f,\infty}^2(t, x) + O\left(\frac{\sqrt{t}}{|x|}\right), \tag{2.25}$$

for $|x| \gg \sqrt{t}$, where we denote by $v_{f,\infty}^2(t, x)$ the function defined for positive time by

$$v_{f,\infty}^2(t, x) = |f|_\infty^2 e^{-2i\delta \log \left|\frac{x}{\sqrt{t}}\right|} m(x), \qquad m(x) = e^{-2ic_+} \chi_{[0,\infty)}(x) + e^{-2ic_-} \chi_{(-\infty,0)}(x) \tag{2.26}$$

with $c_\pm \in [0, 2\pi)$, and $\delta = |f|_\infty^2 - A$.

Next notice that, from the representation of the solution of the free Schrödinger equation, $e^{-i\tau\partial_x^2} u_0$, as a convolution, it is easy to see that

$$\left( e^{-i\tau\partial_x^2} u_0 \right)(x) = \overline{\frac{1}{\sqrt{4\pi i\tau}}} \int u_0(y) e^{-i\frac{(x-y)^2}{4\tau}} \, dy$$

$$= \frac{c_1}{\sqrt{\tau}} e^{-i\frac{x^2}{4\tau}} (\widehat{u_0}) \left(-\frac{x}{2\tau}\right)$$

$$+ \frac{c_1}{\sqrt{\tau}} e^{-i\frac{x^2}{4\tau}} \widehat{\left( u_0(\cdot)(e^{-\frac{(\cdot)^2}{4\tau}} - 1) \right)} \left(-\frac{x}{2\tau}\right), \tag{2.27}$$

with $c_1 = \sqrt{\pi i}$.

Define the auxiliary function $\omega_+$ by

$$\omega_+ = e^{i\alpha \log t} u_+. \tag{2.28}$$

Then, from (2.22), (2.28), (2.25) and (2.27), it is straightforward to see that the second linear term in (2.20) can be rewritten as

$$
\int_t^\infty e^{i(t-\tau)\partial_x^2} \big( v_f^2 e^{-i\alpha \log t} \bar{z}_+ \big) \frac{d\tau}{\tau} = \int_t^\infty e^{i(t-\tau)\partial_x^2} \big( v_f^2 e^{-i\tau \partial_x^2} \overline{\omega}_+ \big) \frac{d\tau}{\tau}
$$

$$
= \int_t^\infty e^{i(t-\tau)\partial_x^2} \Big[ (v_f^2 - v_{f,\infty}^2) e^{-i\tau \partial_x^2} \overline{\omega}_+ \Big] \frac{d\tau}{\tau}
$$

$$
+ \int_t^\infty e^{i(t-\tau)\partial_x^2} \big( v_{f,\infty}^2 e^{-i\tau \partial_x^2} \overline{\omega}_+ \big) \frac{d\tau}{\tau}
$$

$$
= \int_t^\infty e^{i(t-\tau)\partial_x^2} \Big[ (v_f^2 - v_{f,\infty}^2) e^{-i\tau \partial_x^2} \overline{\omega}_+ \Big] \frac{d\tau}{\tau}
$$

$$
+ c_1 \int_t^\infty e^{i(t-\tau)\partial_x^2} \left[ v_{f,\infty}^2 \frac{e^{-i\frac{x^2}{4\tau}}}{\sqrt{\tau}} \widehat{\left( \overline{\omega}_+(\cdot)(e^{-i\frac{(\cdot)^2}{4\tau}} - 1) \right)} \left( -\frac{x}{2\tau} \right) \right] \frac{d\tau}{\tau}
$$

$$
+ c_1 \int_t^\infty e^{i(t-\tau)\partial_x^2} \left[ v_{f,\infty}^2 \frac{e^{-i\frac{x^2}{4\tau}}}{\sqrt{\tau}} (\widehat{\overline{\omega}_+}) \left( -\frac{x}{2\tau} \right) \right] \frac{d\tau}{\tau}. \tag{2.29}
$$

To control the first term on the r.h.s. in (2.29), we first observe that

$$
\big\| v_f^2(\tau, \cdot) - v_{f,\infty}^2(\tau, \cdot) \big\|_{L^2(\mathbb{R})} = \tau^{\frac{1}{4}} \big\| (\bar{f})^2(\cdot) - (\bar{f})_\infty^2(\cdot) \big\|_{L^2(\mathbb{R})},
$$

where

$$
(\bar{f})_\infty^2(x) = |f|_\infty^2 e^{-2i\delta \log |x|} m(x), \qquad \delta = |f|_\infty^2 - A,
$$

and $\|(\bar{f})^2(\cdot) - (\bar{f})_\infty^2(\cdot)\|_{L^2} < \infty$ (recall that $f$ is a bounded function and the asymptotics of $(\bar{f})^2(x)$ given in (2.24)). Then, arguing similarly to the control of the first linear term (see (2.23)), we obtain that

$$
\left\| \int_t^\infty e^{i(t-\tau)\partial_x^2} \Big( (v_f^2 - v_{f,\infty}^2) e^{-i\tau \partial_x^2} \overline{\omega}_+ \Big) \frac{d\tau}{\tau} \right\|_Y
$$

$$
\leq C \, \|u_+\|_{L^1} \|(\bar{f})^2(\cdot) - (\bar{f})_\infty^2(\cdot)\|_{L^2} \frac{1}{t_0^{\frac{1}{4}-\nu}}, \tag{2.30}
$$

for $\nu \leq 1/4$.

The second term in the r.h.s. in (2.29) is an error term. Strichartz estimates with exponents $(4, \infty)$ and $(\infty, 2)$, (2.26), Plancherel's identity, and lemma 2.2 lead to

$$\left\| \int_t^\infty e^{i(t-\tau)\partial_x^2} \left[ v_{f,\infty}^2(\cdot, \tau) \frac{e^{-i\frac{x^2}{4\tau}}}{\sqrt{\tau}} (\overline{\omega}_+(\cdot)(e^{-i\frac{(\cdot)^2}{4\tau}} - 1))\widehat{\phantom{x}} \left( -\frac{x}{2\tau} \right) \right] \frac{d\tau}{\tau} \right\|_Y$$

$$\leq C \sup_{t \geq t_0} t^\nu \int_t^\infty \left\| v_{f,\infty}^2(x, \tau) \frac{e^{-i\frac{x^2}{4\tau}}}{\sqrt{\tau}} \left( \overline{\omega}_+(\cdot)(e^{-i\frac{(\cdot)^2}{4\tau}} - 1) \right)\widehat{\phantom{x}} \left( -\frac{x}{2\tau} \right) \right\|_{L^2} \frac{d\tau}{\tau}$$

$$= C |f|_\infty^2 \sup_{t \geq t_0} t^\nu \int_t^\infty \left\| \left( \overline{\omega}_+(\cdot)(e^{-i\frac{(\cdot)^2}{4\tau}} - 1) \right)\widehat{\phantom{x}} \left( -\frac{x}{2\tau} \right) \right\|_{L^2} \frac{d\tau}{\tau^{\frac{3}{2}}}$$

$$= C |f|_\infty^2 \sup_{t \geq t_0} t^\nu \int_t^\infty \left\| \overline{\omega}_+ (e^{-i\frac{(\cdot)^2}{4\tau}} - 1) \right\|_{L^2} \frac{d\tau}{\tau}$$

$$\leq C \|f\|_{L^\infty}^2 \|u_+\|_{L^2(|x|^\gamma)} \frac{1}{t_0^{\frac{\gamma}{4} - \nu}}, \tag{2.31}$$

for $0 < \gamma \leq 4$ and $\nu \leq \gamma/4$, (recall that $\omega_+ = e^{i\alpha \log t} u_+$, see (2.28)).

In order to control the third term on the r.h.s. in (2.29), recall the definition of $v_{f,\infty}^2(t, x)$ in (2.26),

$$v_{f,\infty}^2(t, x) = |f|_\infty^2 e^{-2i\delta \log\left| \frac{x}{\sqrt{t}} \right|} m(x), \qquad m(x) = e^{-2ic_+} \chi_{[0,\infty)}(x) + e^{-2ic_-} \chi_{(-\infty,0)}(x)$$

or, equivalently,

$$v_{f,\infty}^2(t, x) = |f|_\infty^2 e^{-2i\delta \log 2} t^{-i\delta} e^{-2i\delta \log\left| \frac{x}{2t} \right|} m\left( \frac{x}{2t} \right), \qquad t > 0.$$

Then, using once again the expression for the free Shrödinger solution in (2.27), the latter term rewrites

$$c_1 \int_t^\infty e^{i(t-\tau)\partial_x^2} \left( v_{f,\infty}^2 \frac{e^{-i\frac{x^2}{4\tau}}}{\sqrt{\tau}} (\widehat{\overline{\omega}_+}) \left( -\frac{x}{2\tau} \right) \right) \frac{d\tau}{\tau}$$

$$= |f|_\infty^2 e^{-2i\delta \log 2} \int_t^\infty e^{i(t-\tau)\partial_x^2} \left( c_1 \frac{e^{-i\frac{x^2}{4\tau}}}{\sqrt{\tau}} e^{-2i\delta \log\left| \frac{x}{2\tau} \right|} m\left( \frac{x}{2\tau} \right) (\widehat{\overline{\omega}_+}) \left( -\frac{x}{2\tau} \right) \right) \frac{d\tau}{\tau^{1+i\delta}}$$

$$= |f|_\infty^2 e^{-2i\delta \log 2} \int_t^\infty e^{i(t-\tau)\partial_x^2} \left( c_1 \frac{e^{-i\frac{x^2}{4\tau}}}{\sqrt{\tau}} (\widehat{\overline{T_\delta \omega_+}}) \left( -\frac{x}{2\tau} \right) \right) \frac{d\tau}{\tau^{1+i\delta}}$$

$$= |f|_\infty^2 e^{-2i\delta \log 2} \int_t^\infty e^{i(t-2\tau)\partial_x^2} (\overline{T_\delta \omega_+}) \frac{d\tau}{\tau^{1+i\delta}}$$

$$- c_1 |f|_\infty^2 e^{-2i\delta \log 2} \int_t^\infty e^{i(t-\tau)\partial_x^2}$$

$$\left( e^{-i\frac{x^2}{4\tau}} \left( (\overline{T_\delta \omega_+})(\cdot)(e^{-i\frac{(\cdot)^2}{4\tau}} - 1) \right)\widehat{\phantom{x}} \left( -\frac{x}{2\tau} \right) \right) \frac{d\tau}{\tau^{\frac{3}{2}+i\delta}}$$

$$= I_1 + I_2. \tag{2.32}$$

Here $T_\delta$ is the operator defined (in the Fourier transform side) by

$$\widehat{T_\delta u}(\xi) = e^{2i\delta \log |\xi|} \bar{m}(\xi) \hat{u}(\xi). \tag{2.33}$$

$I_2$ is an "*error*" term. The same argument as the one given in obtaining (2.31) (that is using Strichartz estimates with exponents $(4, \infty)$ and $(\infty, 2)$, Plancherel's identity, and Lemma 2.2) leads to the following chain of inequalities

$$
\begin{aligned}
\|I_2\|_Y &\leq C|f|_\infty^2 \sup_{t \geq t_0} t^\nu \int_t^\infty \left\| (\overline{T_\delta \omega_+}(\cdot)(e^{-\frac{(\cdot)^2}{4\tau}} - 1))^\wedge \left( -\frac{x}{2\tau} \right) \right\|_{L^2} \frac{d\tau}{\tau^{1+\frac{1}{2}}} \\
&\leq C|f|_\infty^2 \sup_{t \geq t_0} t^\nu \int_t^\infty \left\| (\overline{T_\delta \omega_+})(e^{-\frac{(\cdot)^2}{4\tau}} - 1) \right\|_{L^2} \frac{d\tau}{\tau} \\
&\leq C|f|_\infty^2 \sup_{t \geq t_0} t^\nu \int_t^\infty \|T_\delta \omega_+\|_{L^2(|x|^\gamma)} \frac{d\tau}{\tau^{1+\frac{\gamma}{4}}}, \tag{2.34}
\end{aligned}
$$

for any $0 \leq \gamma \leq 4$.

Now, since $T_\delta = T_1 \circ T_2$, where $T_1$ and $T_2$ are defined by

$$\widehat{T_1 f}(\xi) = e^{2i\delta \log |x|} \hat{f}(\xi) \qquad \text{and}$$

$$\widehat{T_2 f}(\xi) = \bar{m}(\xi) \hat{f}(\xi) = \left( \frac{e^{2ic_+}}{2}(1 + \mathrm{sgn}(\xi)) + \frac{e^{2ic_-}}{2}(1 - \mathrm{sgn}(\xi)) \right) \hat{f}(\xi)$$

with $T_1$ and $T_2$ Calderón-Zygmund operators (see [Duo, pp. 97-98] and notice that $T_2$ is just a linear combination of the identity operator and the Hilbert transform), and $|x|^\gamma$ is an $A_2$-weight in the one-dimensional case, in particular, for any $0 \leq \gamma < 1$, from the known $L^2$-weighted inequalities for Calderón-Zygmund operators (see [Duo, pp. 144], or [Stein, pp. 204-205]), we have that

$$\|T_\delta u\|_{L^2(|x|^\gamma)} \leq C\|u\|_{L^2(|x|^\gamma)}, \tag{2.35}$$

for any $0 \leq \gamma < 1$.

From the inequalities (2.34) and (2.35), we conclude that

$$\|I_2\|_Y \leq \frac{C}{t_0^{\frac{\gamma}{4} - \nu}} \|f\|_{L^\infty}^2 \|\omega\|_{L^2(|x|^\gamma)}, \tag{2.36}$$

for any $0 < \gamma < 1$, and $\nu \leq \gamma/4$.

Only $I_1$ remains to be estimated. First, recall (1.35), and the definition of $T_\delta$ in (2.33). Then, $I_1$ in (2.32) rewrites equivalently as

$$
\begin{aligned}
I_1 &= |f|_\infty^2 e^{-2i\delta \log 2} \int_t^\infty e^{i(t-2\tau)\partial_x^2} (\overline{T_\delta \omega_+}) \frac{d\tau}{\tau^{1+i\delta}} \\
&= |f|_\infty^2 e^{-2i\delta \log 2} \int_t^\infty \left( \int_\mathbb{R} e^{ix\xi} e^{-i(t-2\tau)\xi^2} \overline{\widehat{T_\delta \omega_+}}(\xi) \, d\xi \right) \frac{d\tau}{\tau^{1+i\delta}} \\
&= |f|_\infty^2 e^{-2i\delta \log 2} \int_\mathbb{R} e^{ix\xi} e^{-it\xi^2} e^{-2i\delta \log |\xi|} m(-\xi) \overline{\hat{\omega}_+}(\xi) A_t(\xi) \, d\xi, \tag{2.37}
\end{aligned}
$$

where

$$A_t(\xi) = \int_t^\infty e^{2i\tau\xi^2} \frac{d\tau}{\tau^{1+i\delta}}.$$

On the one hand, from Plancherel's identity and Lemmma 2.3, it is easy to see that

$$\begin{aligned}
\|I_1\|_{L^2} &= |f|_\infty^2 \|m(-\cdot)\widehat{\overline{\omega}_+}(\cdot)\, A_t(\cdot)\|_{L^2} \\
&\leq 2|f|_\infty^2 \left( \|\widehat{\overline{\omega}_+}\, A_t\|_{L^2(t\xi^2\leq 1)} + \|\widehat{\overline{\omega}_+}\, A_t\|_{L^2(t\xi^2>1)} \right) \\
&\leq C \frac{|f|_\infty^2}{t^{\gamma/4}} \left( \int \frac{|\widehat{\overline{\omega}_+}(\xi)|^2}{|\xi|^\gamma}\, d\xi \right)^{1/2} \\
&= C \frac{\|f\|_{L^\infty}^2}{t^{\gamma/4}} \|\omega_+\|_{L^2(|x|^\gamma)},
\end{aligned}$$

for any $0 \leq \gamma < 1$. Here, we have used Pitt's inequality (see Lemma 2.4) to obtain the last inequality. Thus,

$$\sup_{t\geq t_0} t^\nu \|I_1\|_{L^2} \leq \frac{C}{t_0^{\frac{\gamma}{4}-\nu}} \|f\|_{L^\infty}^2 \|\omega_+\|_{L^2(|x|^\gamma)}, \tag{2.38}$$

for any $0 \leq \gamma < 1$ and $\nu \leq \gamma/4$.

In order to estimate the $L^4((t,\infty), L^\infty(\mathbb{R}))$-norm of $I_1$, consider $\theta$ a cut-off function with $\theta(x) = 0$ if $|x| \leq 1/2$, and $\theta(x) = 1$, if $|x| > 1$. We decompose $I_1$ in (2.37) as follows

$$\begin{aligned}
I_1 &= |f|_\infty^2 e^{-2i\delta\log 2} \left( \int (1-\theta)(t\xi^2) + \int \theta(t\xi^2) \right) \\
&\quad \left( e^{ix\xi} e^{-it\xi^2} e^{-2i\delta\log|\xi|} m(-\xi)\widehat{\overline{\omega}_+}(\xi)\, A_t(\xi) \right) d\xi, \\
&= I_{1,1} + I_{1,2}. \tag{2.39}
\end{aligned}$$

Using Lemma 2.3, and Cauchy-Schwarz inequality, we find that

$$\begin{aligned}
|I_{1,2}| &\leq 2|f|_\infty^2 \int_{t\xi^2\geq 1/2} |\widehat{\overline{\omega}_+}(\xi)|\, |A_t(\xi)|\, d\xi \leq C \frac{|f|_\infty^2}{t} \int_{t\xi^2\geq 1/2} |\widehat{\overline{\omega}_+}(\xi)| \frac{d\xi}{\xi^2} \\
&= C \frac{|f|_\infty^2}{t} \int_{t\xi^2\geq 1/2} \frac{|\widehat{\overline{\omega}_+}(\xi)|}{|\xi|^{\frac{\gamma}{2}}} \frac{|\xi|^{\frac{\gamma}{2}}}{\xi^2}\, d\xi \\
&\leq C \frac{|f|_\infty^2}{t^{\frac{\gamma}{4}+\frac{1}{4}}} \left( \int \frac{|\widehat{\overline{\omega}_+}(\xi)|^2}{|\xi|^\gamma}\, d\xi \right)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
|I_{1,1}| &\leq |f|_\infty^2 \int_{t\xi^2\leq 1} |\widehat{\overline{\omega}_+}(\xi)|\, |A_t(\xi)|\, d\xi \leq \frac{C}{t^{\frac{\gamma}{4}}} |f|_\infty^2 \int_{t\xi^2\leq 1} \frac{|\widehat{\overline{\omega}_+}(\xi)|}{|\xi|^{\frac{\gamma}{2}}}\, d\xi \\
&\leq \frac{C}{t^{\frac{\gamma}{4}+\frac{1}{4}}} |f|_\infty^2 \left( \int \frac{|\widehat{\overline{\omega}_+}(\xi)|^2}{|\xi|^\gamma}\, d\xi \right)^{1/2},
\end{aligned}$$

for any $0 \leq \gamma < 3$. Plugging the above inequalities into (2.39) and using Pitt's inequality (see Lemma 2.4) give

$$|I_1| \leq C \frac{|f|_\infty^2}{t^{\frac{\gamma}{4}+\frac{1}{4}}} \left( \int \frac{|\widehat{\overline{\omega}_+}(\xi)|^2}{|\xi|^\gamma} d\xi \right)^{1/2} \leq C \frac{|f|_\infty^2}{t^{\frac{\gamma}{4}+\frac{1}{4}}} \|\omega_+\|_{L^2(|x|^\gamma)},$$

for $0 \leq \gamma < 1$. Therefore,

$$\sup_{t \geq t_0} t^\nu \|I_1\|_{L^4((t,\infty),L^\infty)} \leq C \frac{|f|_\infty^2}{t_0^{\frac{\gamma}{4}-\nu}} \|\omega_+\|_{L^2(|x|^\gamma)}, \tag{2.40}$$

for $0 < \gamma < 1$, $\nu \leq \gamma/4$.

From (2.38) and (2.40), we get that

$$\|I_1\|_Y \leq C \frac{\|f\|_{L^\infty}^2}{t_0^{\frac{\gamma}{4}-\nu}} \|\omega_+\|_{L^2(|x|^\gamma)}, \tag{2.41}$$

for $0 < \gamma < 1$, and $\nu \leq \gamma/4$. Thus, from (2.32), (2.36), and (2.41), we conclude the following control for the last term on the r.h.s. in (2.29)

$$\left\| \int_t^\infty e^{i(t-\tau)\partial_x^2} \left( v_{f,\infty}^2 \frac{e^{-i\frac{x^2}{4\tau}}}{\sqrt{\tau}} (\widehat{\overline{\omega}_+}) \left( -\frac{x}{2\tau} \right) \right) \frac{d\tau}{\tau} \right\|_Y \leq \frac{C}{t_0^{\frac{\gamma}{4}-\nu}} \|f\|_{L^\infty}^2 \|u_+\|_{L^2(|x|^\gamma)}, \tag{2.42}$$

for any $\nu \leq \frac{\gamma}{4}$, and $0 < \gamma < 1$. Recall that $\omega_+ = e^{i\alpha \log t} u_+$ (see (2.28)).

Finally, the identity (2.29), and the inequalities (2.30), (2.31) and (2.42) give the following control of the second source term in (2.20)

$$\left\| \int_t^\infty e^{i(t-\tau)\partial_x^2} (v_f^2 e^{-i\alpha \log t} \overline{z}_+) \frac{d\tau}{\tau} \right\|_Y \leq \frac{C}{t_0^{\frac{1}{4}-\nu}} \|u_+\|_{L^1} \|(\bar{f})^2(\cdot) - (\bar{f})_\infty^2(\cdot)\|_{L^2}$$

$$+ \frac{C}{t_0^{\frac{\gamma}{4}-\nu}} \|f\|_{L^\infty}^2 \|u_+\|_{L^2(|x|^\gamma)}, \tag{2.43}$$

for any $0 < \gamma < 1$, and $\nu \leq \gamma/4$.

We continue to analize the non-source terms in (2.20). To this end, notice that for any $z \in Y$ the following inequalities hold true

$$\|z\|_{L^2} \leq \frac{\|z\|_Y}{t^\nu} \qquad \text{and} \qquad \|z\|_{L^4((t,\infty),L^\infty(\mathbb{R}))} \leq \frac{\|z\|_Y}{t^\nu} \qquad \forall t \geq t_0. \tag{2.44}$$

Also, recall that $z_+(t,x) = e^{it\partial_x^2} u_+$ (see (2.22)), so that from the well-known inequalities for the solution of the free Schrödinger equation in (2.17) and (2.18), we have that

$$\|z_+\|_{L^\infty} = \|e^{it\partial_x^2} u_+\|_{L^\infty} \leq C \frac{\|u_+\|_{L^1}}{\sqrt{t}} \quad \text{and} \quad \|z_+\|_{L^2} = \|e^{it\partial_x^2} u_+\|_{L^2} = \|u_+\|_{L^2}. \tag{2.45}$$

First, using (2.16) with exponents $(4, \infty)$ and $(\infty, 2)$, (2.44) and the fact that $v_f(t, x)$ is a bounded function (notice that $v_f(t, x) = \bar{f}(x/\sqrt{t})$, and $f$ is bounded by Proposition 2.1), we obtain the following control for the second integral term on the r.h.s in

(2.20)

$$\left\| \int_t^\infty e^{i(t-\tau)\partial_x^2} \{2(|v_f|^2 - |f|_\infty^2)z + v_f^2 e^{-i\alpha \log \tau} \bar{z}\} \frac{d\tau}{\tau} \right\|_Y \leq$$

$$C \sup_{t \geq t_0} t^\nu \int_t^\infty \|2(|v_f|^2 - |f|_\infty^2)z + v_f^2 e^{-i\alpha \log \tau} \bar{z}\|_{L^2} \frac{d\tau}{\tau} \leq$$

$$C \|f\|_{L^\infty}^2 \sup_{t \geq t_0} t^\nu \int_t^\infty \|z\|_{L^2} \frac{d\tau}{\tau} \leq$$

$$C \|f\|_{L^\infty}^2 \|z\|_Y \sup_{t \geq t_0} t^\nu \int_t^\infty \frac{d\tau}{\tau^{1+\nu}} = C \|f\|_{L^\infty}^2 \|z\|_Y, \qquad (2.46)$$

for any $\nu \geq 0$.

Only the Duhamel term in (2.20) related to the non-linear terms $NLT(z + z_+)$, where

$$NLT(z + z_+) = 2v_f e^{-i\frac{\alpha}{2}\log t}|z + z_+|^2 + \bar{v}_f e^{i\frac{\alpha}{2}\log t}(z + z_+)^2 + |z + z_+|^2(z + z_+).$$

(see (2.21)) remains to be estimated.

To control the terms associated to quadratic powers of $z + z_+$, we use as before the inequalities (2.16) with exponents $(4, \infty)$ and $(\infty, 2)$, the fact that $\|v_f\|_{L^\infty} = \|f\|_{L^\infty} < \infty$, and estimates (2.44) and (2.45) to obtain that

$$\left\| \int_t^\infty e^{i(t-\tau)\partial_x^2} \{2v_f e^{-i\frac{\alpha}{2}\log \tau}|z + z_+|^2 + \bar{v}_f e^{i\frac{\alpha}{2}}(z + z_+)^2\} \frac{d\tau}{\tau} \right\|_Y$$

$$\leq C \sup_{t \geq t_0} t^\nu \int_t^\infty \|2v_f e^{-i\frac{\alpha}{2}\log \tau}|z + z_+|^2 + \bar{v}_f e^{i\frac{\alpha}{2}}(z + z_+)^2\|_{L^2} \frac{d\tau}{\tau}$$

$$\leq C \|f\|_{L^\infty} \sup_{t \geq t_0} t^\nu \int_t^\infty \{\|z\|_{L^2}(\|z\|_{L^\infty} + \|z_+\|_{L^\infty}) + \|z_+\|_{L^\infty}\|z_+\|_{L^2}\} \frac{d\tau}{\tau}$$

$$\leq C \|f\|_{L^\infty} \|z\|_Y \left( \sup_{t \geq t_0} t^\nu \int_t^\infty (\|z\|_{L^\infty} + \frac{\|u_+\|_{L^1}}{\sqrt{\tau}}) \frac{d\tau}{\tau^{1+\nu}} \right) +$$

$$C \|f\|_{L^\infty} \|u_+\|_{L^1} \|u_+\|_{L^2} \sup_{t \geq t_0} t^\nu \int_t^\infty \frac{d\tau}{\tau^{3/2}}$$

$$\leq C \|f\|_{L^\infty} \|z\|_Y \left( \frac{\|z\|_Y}{t_0^{\frac{1}{4}+\nu}} + \frac{\|u_+\|_{L^1}}{t_0^{\frac{1}{2}}} \right) + C \|f\|_{L^\infty} \|u_+\|_{L^1} \|u_+\|_{L^2} \frac{1}{t_0^{\frac{1}{2}-\nu}}, \quad (2.47)$$

for all $0 \leq \nu \leq 1/2$. Here, we have also used Hölder's inequality in the $\tau$-variable to obtain the last inequality.

Next, notice that a straightforward computation gives

$$|z + z_+|^2(z + z_+) = |z|^2 z + |z|^2 z_+ + z^2 \bar{z}_+ + |z|^2 z_+ + \bar{z} z_+^2 + z|z_+|^2 + z_+|z_+|^2.$$

Then, similar arguments to the ones given to control the quadratic terms in $z + z_+$ (that is, using (2.16), pulling out of the $L^2$-norm $\|z\|_{L^\infty}$ or $\|z_+\|_{L^\infty}$, and using

the estimates (2.44) and (2.45)) give the following control of $Y$-norm of the term associated to the cubic term $|z + z_+|^2(z + z_+)$

$$\left\| \int_t^\infty e^{i(t-\tau)\partial_x^2}(|z + z_+|^2(z + z_+)) \frac{d\tau}{\tau} \right\|_Y$$

$$\leq C \sup_{t \geq t_0} t^\nu \int_t^\infty \| |z + z_+|^2(z + z_+)\|_{L^2} \frac{d\tau}{\tau}$$

$$\leq C \sup_{t \geq t_0} t^\nu \int_t^\infty \left( \|z\|_{L^\infty}^2 \|z\|_{L^2} + 3\|z_+\|_{L^\infty}\|z\|_{L^\infty}\|z\|_{L^2} + \right.$$

$$2\|z_+\|_{L^\infty}^2 \|z\|_{L^2} + \|z_+\|_{L^\infty}^2\|z_+\|_{L^2} \Big) \frac{d\tau}{\tau}$$

$$\leq C\|z\|_Y \sup_{t \geq t_0} t^\nu \int_t^\infty \left( \frac{\|z\|_{L^\infty}^2}{\tau^{1+\nu}} + 3\|u_+\|_{L^1} \frac{\|z\|_{L^\infty}}{\tau^{\frac{3}{2}+\nu}} + 2\frac{\|u_+\|_{L^1}^2}{\tau^{2+\nu}} + \right) d\tau +$$

$$C\|u_+\|_{L^1}^2\|u_+\|_{L^2} \sup_{t \geq t_0} t^\nu \int_t^\infty \frac{d\tau}{\tau^2}$$

$$\leq C\frac{\|z\|_Y^3}{t_0^{\frac{1}{2}+2\nu}} + \|u_+\|_{L^1}\|z\|_Y^2 \frac{1}{t_0^{\frac{3}{4}+\nu}} + C\|u_+\|_{L^1}^2\|z\|_Y \frac{1}{t_0} +$$

$$C\|u_+\|_{L^1}^2\|u_+\|_{L^2} \frac{1}{t_0^{1-\nu}} \tag{2.48}$$

for all $0 \leq \nu \leq 1$.

Therefore, in view of the identity (2.20), and the inequalities (2.23), (2.43) and (2.46)-(2.48), we have that

$$\|Bz\|_Y \leq \frac{c(u_+)}{t_0^{\frac{1}{4}-\nu}} \left( \| |f(\cdot)|^2 - |f|_\infty^2\|_{L^2} + \|(\bar{f})^2(\cdot) - (\bar{f})_\infty^2(\cdot)\|_{L^2} \right)$$

$$+ \frac{c(u_+)}{t_0^{\frac{2}{4}-\nu}}\|f\|_{L^\infty}^2 + \frac{c(u_+)}{t_0^{\frac{1}{2}-\nu}}\|f\|_{L^\infty} + \frac{c(u_+)}{t_0^{1-\nu}}$$

$$+ C\|z\|_Y \left\{ \|f\|_\infty^2 + \frac{\|f\|_{L^\infty}}{t_0^{\frac{1}{4}+\nu}}\|z\|_Y + \frac{c(u_+)}{t_0^{\frac{1}{2}}}\|f\|_{L^\infty} \right.$$

$$\left. \frac{\|z\|_Y^2}{t_0^{\frac{1}{2}+2\nu}} + \frac{c(u_+)}{t_0^{\frac{3}{4}+\nu}}\|z\|_Y + \frac{c(u_+)}{t_0} \right\} \tag{2.49}$$

for any $0 \leq \nu \leq \gamma/4$. Here $c(u_+)$ denotes a positive constant which depends on the norm of $u_+$ in $L^1 \cap L^2(\langle x\rangle^\gamma)$.

For any fixed $t_0 > 0$, and $0 < \gamma < 1$, by choosing $\nu = \gamma/4$, from (2.49), we conclude that there exists a (small) positive consatnt $B_0$, and a constant $R > 0$ small with respect to $B_0$ and $t_0$, such that for all $f$ solution of

$$f'' + i\frac{s}{2}f' + \frac{f}{2}(|f|^2 - A) = 0$$

satisfying $|f|_{+\infty} = |f|_{-\infty}$ and such that $\|f\|_{L^\infty} \le B_0$, and all $u_+$ small in $L^1 \cap L^2(\langle x \rangle^\gamma)$ w.r.t. $t_0$, $B_0$, $\||f(\cdot)|^2 - |f|_\infty^2\|_{L^2}$, $\|(\bar{f})^2(\cdot) - (\bar{f})_\infty^2(\cdot)\|_{L^2}$ and $R$, the operator $B$ maps $\mathcal{B}_R$ into $\mathcal{B}_R$. On the other hand, by bearing in mind that

$$\|z + z_+\|_{L^\infty} \le \|z\|_{L^\infty} + C\frac{\|u_+\|_{L^1}}{\sqrt{t}}, \qquad \text{for all} \qquad z \in Y,$$

similar arguments to the ones given in obtaining the estimates (2.46)-(2.48) shows that the operator $B$ defined by (2.20) is a contraction on $(\mathcal{B}_R, \|\cdot\|_Y)$. As a consequence, the application of the contraction mapping principle yields the existence of a unique solution $z$ of the equation (2.8) such that

$$z \in \mathcal{C}([t_0, \infty), L^2(\mathbb{R})) \cap L^4([t_0, \infty), L^\infty(\mathbb{R}))$$

satisfying

$$\|z(t)\|_{L^2(\mathbb{R})} + \|z\|_{L^4((t,\infty),L^\infty(\mathbb{R}))} = \mathcal{O}\left(\frac{1}{t^{\frac{\gamma}{4}}}\right)$$

as $t \to \infty$, for $0 < \gamma < 1$.

Performing the change of variables (2.7) and (2.5), that is the changes defined by

$$v(t, x) = v_f(t, x) + e^{i\frac{\alpha}{2}\log t}u \qquad \text{with} \qquad v_f(t, x) = \bar{f}\left(\frac{x}{\sqrt{t}}\right), \qquad \alpha = 2|f|_\infty^2 - A,$$

and

$$u(t, x) = z(t, x) + z_+(t, x), \qquad \text{with} \qquad z_+(t, x) = e^{it\partial_x^2}u_+$$

gives the existence of a unique solution of (1.38) such that

$$v - \tilde{v}_f \in \mathcal{C}([t_0, \infty), L^\infty(\mathbb{R})) \cap L^4([t_0, \infty), L^\infty(\mathbb{R})),$$

and satisfying (1.39). To this end, since $\tilde{v}_f$ is defined by (2.2), suffices to notice that

$$
\begin{aligned}
v - \tilde{v}_f &= (v_f + e^{i\frac{\alpha}{2}\log t}u) - (v_f + e^{i\frac{\alpha}{2}\log t}(e^{it\partial_x^2}u_+)(x)) \\
&= e^{i\frac{\alpha}{2}\log t}(u - e^{it\partial_x^2}u_+) = e^{i\frac{\alpha}{2}\log t}z
\end{aligned}
$$

so that

$$\|v - \tilde{v}_f\|_{L^2} + \|v - \tilde{v}_f\|_{L^4((t,\infty),L^2(\mathbb{R}))} = \|z\|_{L^2} + \|z\|_{L^4((t,\infty),L^2(\mathbb{R}))}.$$

Finally, we have to prove that under if the asymptotic state $u_+$ satisfies that both $u_+$ and $\partial_x u_+$ are in $L^1 \cap L^2(\langle x \rangle^\gamma)$, then the solution $v$ is such that $v - \tilde{v}_f \in H^1$ and (1.40) holds.

Recall that solutions of (2.4) are in correspondence with solutions $z$ of (2.8) through the changes of variables (2.5) and (2.7) (see subsection 2.1). Define the auxiliary functions $y = \partial_x u$ and $y_+ = \partial_x z_+$, where as before $z_+ = e^{it\partial_x^2}u_+$. Then, if $z$ is a solution of (2.8), we have that $y$ has to be a solution of

$$
\begin{aligned}
iy_t + y_{xx} = {}& \frac{1}{2t}\left(F_0(y_+) - F_1(y) + 2\partial_x(|v_f|^2)(z_+ - z) + \partial_x(v_f^2)(\overline{z_+ - z})e^{-i\alpha \log t}\right. \\
& \left. - \partial_x NLT(z + z_+)\right).
\end{aligned}
$$

Now, notice that from the fact that $v_f(t, x) = \bar{f}(x/\sqrt{t})$, the properties of $f$ and $f'$ given in Proposition 2.1, and those of $z$ already proved, we conclude that the term

$$2\partial_x(|v_f|^2)(z_+ - z) + \partial_x(v_f^2)(\overline{z_+ - z})e^{-i\alpha \log t}$$

is an integrable in time forcing term. As a consequence we can follow the same argument as the one used to solve the equation for $z$, and concude that (1.40) holds. This finishes the proof.

2.3. **Proof of Theorem 1.4.** Let $\tilde{t}_0 > 0$ and $0 < \gamma < 1$. Define $t_0 = \frac{1}{\tilde{t}_0}$, and denote by $v$ the associated solution of the equation (1.38) verifying

$$\|v - \tilde{v}_f\|_{L^2(\mathbb{R})} + \|v - \tilde{v}_f\|_{L^4((t,\infty),L^\infty(\mathbb{R}))} = \mathcal{O}\left(\frac{1}{t^{\frac{7}{4}}}\right), \qquad \text{as} \quad t \to \infty, \qquad (2.50)$$

given by Theorem 1.2.

Define $u$ to be the pseudo-conformal transformation of the solution $v$, i.e.

$$u = \mathcal{T}(v).$$

Then, $u$ satisfies equation (1.42). Next, notice that (recall the definition of $\tilde{u}_f$ and $\mathcal{T}$ in (1.41) and (1.29), respectively)

$$\tilde{u}_f := \mathcal{T}\left(v_f + (2\pi)e^{i\frac{\alpha}{2}\log t}\frac{e^{i\frac{x^2}{4t}}}{\sqrt{4\pi it}}\hat{u}_+\left(\frac{x}{2t}\right)\right), \qquad \alpha = 2|f|_\infty^2 - A,$$

so that

$$u - \tilde{u}_f = \mathcal{T}\left(v - \left(v_f + (2\pi)e^{i\frac{\alpha}{2}\log t}\frac{e^{i\frac{x^2}{4t}}}{\sqrt{4\pi it}}\hat{u}_+\left(\frac{x}{2t}\right)\right)\right)$$

where

$$v - \left(v_f + (2\pi)e^{i\frac{\alpha}{2}\log t}\frac{e^{i\frac{x^2}{4t}}}{\sqrt{4\pi it}}\hat{u}_+\left(\frac{x}{2t}\right)\right) =$$

$$(v - \tilde{v}_f) + e^{i\frac{\alpha}{2}\log t}\left(\left(e^{it\partial_x^2}u_+\right)(x) - (2\pi)\frac{e^{i\frac{x^2}{4t}}}{\sqrt{4\pi it}}\hat{u}_+\left(\frac{x}{2t}\right)\right),$$

recall the definition of $\tilde{v}_f$ in (1.37).

Due to the invariance of $L^2(\mathbb{R})$ and $L^4((0,t), L^\infty(\mathbb{R}))$ under the pseudo-conformal transformation $\mathcal{T}$, and the decay estimates (2.50), in order to prove (1.43) it suffices to study the behaviour of

$$\left(e^{it\partial_x^2}u_+\right)(x) - (2\pi)\frac{e^{i\frac{x^2}{4t}}}{\sqrt{4\pi it}}\hat{u}_+\left(\frac{x}{2t}\right)$$

in $L^2(\mathbb{R})$ and $L^4((t,\infty), L^\infty(\mathbb{R}))$, as $t$ goes to infinity.

On the one hand, using the expression of $e^{it\partial_x^2}u_+$ as a convolution (see (1.36)) and Plancherel's indentity, we have

$$\left\|\left(e^{it\partial_x^2}u_+\right)(x) - (2\pi)\frac{e^{i\frac{x^2}{4t}}}{\sqrt{4\pi it}}\,\hat{u}_+\left(\frac{x}{2t}\right)\right\|_{L^2} =$$

$$C\left\|(u_+(\cdot)(e^{i\frac{(\cdot)^2}{4t}} - 1))\widehat{\phantom{x}}\right\|_{L^2} = C\|u_+(\cdot)(e^{i\frac{(\cdot)^2}{4t}} - 1)\|_{L^2} \le \frac{C}{t^{\frac{\gamma}{4}}}\,\|u_+\|_{L^2(|x|^\gamma)}.$$

Here, we have used Lemma 2.2 in obtaining the last inequality.

On the other hand, from the decay estimate (2.17)

$$\|e^{it\partial_x^2}u_+\|_{L^4((t,\infty),L^\infty(\mathbb{R}))} \le \frac{C}{t^{\frac{1}{4}}}\|u_+\|_{L^1}$$

from which it follows that

$$\left\|\left(e^{it\partial_x^2}u_+\right)(x) - (2\pi)\frac{e^{i\frac{x^2}{4t}}}{\sqrt{4\pi it}}\,\hat{u}_+\left(\frac{x}{2t}\right)\right\|_{L^4((t,\infty),L^\infty(\mathbb{R}))} \le$$

$$\frac{C}{t^{\frac{1}{4}}}\,\|u_+\|_{L^1} + \frac{C}{t^{\frac{1}{4}}}\,\|\hat{u}_+\|_{L^\infty} = \mathcal{O}\left(\frac{1}{t^{\frac{1}{4}}}\right), \qquad \text{as} \quad t \to \infty.$$

From the above inequalities, we get

$$\left\|\left(e^{it\partial_x^2}u_+\right)(x) - (2\pi)\frac{e^{i\frac{x^2}{4t}}}{\sqrt{4\pi it}}\,\hat{u}_+\left(\frac{x}{2t}\right)\right\|_{L^2(\mathbb{R})} +$$

$$\left\|\left(e^{it\partial_x^2}u_+\right)(x) - (2\pi)\frac{e^{i\frac{x^2}{4t}}}{\sqrt{4\pi it}}\,\hat{u}_+\left(\frac{x}{2t}\right)\right\|_{L^4((t,\infty),L^\infty(\mathbb{R}))} = \mathcal{O}\left(\frac{1}{t^{\frac{\gamma}{4}}}\right), \qquad \text{as} \quad t \to \infty,$$

for any $0 < \gamma < 1$, and $u_+ \in L^1 \cap L^2(\langle x\rangle^\gamma)$.

Now, (1.44) is an immediate consequence of the triangle inequality, (1.43) and Plancherel's identity. Also, inequality (1.45) follows from (1.43) and (1.44), by using the general inequality

$$\||f|^2 - |g|^2\|_{L^2} \le (\|f\|_{L^2} + \|g\|_{L^2})\,\|f - g\|_{L^2},$$

for any functions $f$ and $g$ in $L^2$.

Finally, assume by contradiction that there exists $g(\cdot, t) \in L^2(\mathbb{R})$ defined in a time interval $(0, T_0 > 0]$ such that

$$\left\|u(t,x) - \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}}f\left(\frac{x}{\sqrt{t}}\right) - g(t,x)\right\|_{L^2} \to 0 \qquad \text{as} \qquad t \to 0. \qquad (2.51)$$

Then, using the definition of $\tilde{u}_f$ in (1.41) and the triangular inequality, we obtain that

$$\left\| \sqrt{\pi i} e^{i\frac{\alpha}{2}\log t} \widehat{\overline{u_+}}\left(-\frac{x}{2}\right) - g(t,x) \right\|_{L^2}$$

$$= \left\| \tilde{u}_f(t,x) - \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right) - g(t,x) \right\|_{L^2}$$

$$= \left\| (\tilde{u}_f - u)(t,x) + \left( u(t,x) - \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right) - g(t,x) \right) \right\|_{L^2}$$

$$\leq \|\tilde{u}_f - u\|_{L^2} + \left\| u(t,x) - \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right) - g(t,x) \right\|_{L^2}.$$

Thus, from (1.43) and (2.51) and the above identity we conclude that

$$g(t,x) = \sqrt{\pi i} e^{i\frac{\alpha}{2}\log t} \widehat{\overline{u_+}}\left(-\frac{x}{2}\right) \qquad \text{a.e} \quad x,$$

which does not have limit in $L^2$ as $t \to 0$, unless $\alpha = 2|f|_\infty^2 - A = 0$.

It remains to prove (1.46) and (1.47). Using (1.40) and the inequality $|g|^2 \leq \|g\|_{L^2}\|g\|_{L^2}$ in one dimension, we get that

$$\|v - \tilde{v}_f\|_{L^\infty} = \mathcal{O}\left(\frac{1}{t^{\frac{\gamma}{4}}}\right), \qquad t \to \infty, \qquad (0 < \gamma < 1). \tag{2.52}$$

From the definition of $u$ in terms of $v$, given by the pseudo-conformal transformation (1.29), we write

$$u(t,x) \;=\; \mathcal{T}v(t,x) = \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} \bar{v}\left(\frac{1}{t}, \frac{x}{t}\right)$$

$$= \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}}\left( \overline{v - \tilde{v}_f}\left(\frac{1}{t}, \frac{x}{t}\right) - \overline{\tilde{v}_f}\left(\frac{1}{t}, \frac{x}{t}\right) \right),$$

where (see (2.2) or (1.37))

$$\tilde{v}_f(t,x) = \bar{f}\left(\frac{x}{\sqrt{t}}\right) + e^{i\frac{\alpha}{2}\log t}\left( e^{it\partial_x^2} u_+ \right)(x), \qquad \alpha = 2|f|_\infty^2 - A.$$

Then (1.46)-(1.47) follow from the above identities, (2.52), and the decay estimate for $e^{it\partial_x^2} u_+$ given in (2.17). This finishes the proof of Theorem 1.4.

2.4. **Proof of Corollary 1.7.** Theorem 1.4 gives the existence of a filament function $u(t,x)$ which is regular and bounded for $0 < t < \tilde{t}_0$. From the filament function $u$ given by Theorem 1.4, one can construct a corresponding curve $\mathbf{X}$ solution of LIE.

First, notice that at least in the case of odd solutions the curve $\mathbf{X}_f(t,x)$ has a point of curvature 0 (the curvature of an odd solution vanishes at least at the point $x = 0$), and as a consequence here we need to consider a different parallel frame (other that the Serret-Frenet frame) to avoid the restriction that the curvature of the

curvature should not vanish. Precisely, one can consider the parallel frame of vectors $\{\mathbf{T}, \mathbf{e_1}, \mathbf{e_2}\}$ given by the system of equations

$$\begin{cases} \mathbf{T}_x = \quad \alpha\mathbf{e_1} + \beta\mathbf{e_2} \\[2mm] \mathbf{e_{1x}} = -\alpha\mathbf{T} \\[2mm] \mathbf{e_{2x}} = -\beta\mathbf{T}, \end{cases} \tag{2.53}$$

where the quantities $\alpha$ and $\beta$ are defined through the function $u$ by

$$u = \alpha + i\beta,$$

to construct the tangent vector $\mathbf{T}$ solution of $\mathbf{T}_t = \mathbf{T} \times \mathbf{T}_{xx}$. Then, using the regularity of $u$, and after integration with the initial conditions

$$\mathbf{X}(\tilde{t}_0, 0) = (0, 0, 0) \qquad \text{and} \qquad \mathbf{X}_x(\tilde{t}_0, 0) = (1, 0, 0),$$

we get a curve $\mathbf{X}(t, x)$ solution of LIE.[9] The details can be found for example in [BV3] and [BV4], see also [NSVZ].

Once $\mathbf{X}(t, x)$ has been constructed for $0 < t < \tilde{t}_0$, part $i)$ is an immediate consequence of $(1.46)$-$(1.47)$, the boundedness property of $f$, and the fact that $u$ is the filament function associated to $\mathbf{X}(t, x)$ (thus $|u(t, x)| = |c(t, x)|$, with $c$ the curvature of $\mathbf{X}$).

The existence of $\mathbf{X}_0(x)$, the trace of $\mathbf{X}(t, x)$ at time $t = 0$, follows from the integrability of $\mathbf{X}_t$ at $t = 0$ thanks to the uniform bound of the curvature in part $i)$. Indeed, since $\mathbf{X}(t, x)$ is a solution of LIA, from the system of equations $(2.53)$, and the fact that the vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ are unitary, it follows that

$$\begin{aligned} |\mathbf{X}_t(t, x)| &= |\mathbf{X}_x \times \mathbf{X}_{xx}| = |\mathbf{T} \times \mathbf{T}_x| = |\mathbf{T} \times (\alpha\mathbf{e_1} + \beta\mathbf{e_2})| \\ &= |\alpha\mathbf{e_2} - \beta\mathbf{e_1}| = \sqrt{\alpha^2 + \beta^2} = |u(t, x)| = |c(t, x)| \leq \frac{c_1}{\sqrt{t}}, \end{aligned}$$

uniformly on the interval $x \in (-\infty, \infty)$, since $u = \alpha + i\beta$.

Therefore, for any fixed positive times $t_1$, and $t_2$ with $t_1 < t_2$, we have that

$$\begin{aligned} |\mathbf{X}(t_1, x) - \mathbf{X}(t_2, x)| &= \left| \int_{t_1}^{t_2} \mathbf{X}_t(t', x) \, dt' \right| \leq \int_{t_1}^{t_2} |\mathbf{X}_t(t', x)| \, dt' \\ &\leq c_1 \int_{t_1}^{t_2} \frac{dt'}{\sqrt{t'}}, \end{aligned}$$

---

[9] Conversely, using the parallel frame defined by the system $(2.53)$, it can be also proved that if $\mathbf{X}(t, x)$ is a regular solution of LIE, and define the function $u = \alpha + i\beta$, then $u$ solves the 1d-cubic Schrödinger equation

$$iu_t + u_{xx} + \frac{u}{2}(|u|^2 - A(t)) = 0$$

with $A(t) = -|u|^2(0, t)/2 - \langle \partial_t \mathbf{e_1}, \mathbf{e_2} \rangle (0, t)$.

from which the existence of the limit $\lim_{t \to 0} \mathbf{X}(t, x) = \mathbf{X}_0(x)$ follows by taking $t_2 = t > 0$ and letting $t_1$ go to zero in the above inequality. Moreover, we have

$$|\mathbf{X}(t, x) - \mathbf{X}_0(x)| \leq 2c_1\sqrt{t}.$$

Finally, the regularity property of $\mathbf{X}_0$ easily follows from the above inequality, and the identity

$$\mathbf{X}_0(x) - \mathbf{X}_0(y) = [\mathbf{X}_0(x) - \mathbf{X}(t, x)] - [\mathbf{X}_0(y) - \mathbf{X}(t, y)] + [\mathbf{X}(t, x) - \mathbf{X}(t, y)].$$

To this end, if suffices to observe that

$$|\mathbf{X}(t, x) - \mathbf{X}(t, y)| = \left| \int_x^y \mathbf{T}(t, z)\, dz \right| \leq |x - y|$$

since the tangent vector to the curve, $\mathbf{T}$, is unitary. As a consequence,

$$|\mathbf{X}_0(x) - \mathbf{X}_0(y)| \leq 2c_1\sqrt{t} + |x - y| \leq c_3|x - y|$$

for some non-negative constant $c_3$, whenever $t$ is sufficiently small. Therefore, we conclude that $\mathbf{X}_0(x)$ is a Lipschitz continuous function.

## 3. The initial value problem for the principal value distribution

We begin this section proving the existence of non-trivial solutions

$$u_f(t, x) = \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right)$$

of

$$iu_t + u_{xx} + \frac{u}{2}(|u|^2 - \frac{A}{t}) = 0$$

such that $u_f(t, \cdot)$ converges as a distribution to

$$u_f(0, x) = z_0 \text{ p.\,v} \frac{1}{x},$$

for some $z_0 \in \mathbb{C} \setminus \{0\}$, and appropriate values of $A$. Moreover, these solutions are characterized by the property that $2|f|_\infty^2 - A = 0$, so that the solution $u$ constructed in Theorem 1.4 has a trace at $t = 0$. We have the following lemma:

**Lemma 3.1.** *For any $a \neq 0$, there exist $A_a$ and a non-trivial odd solution $f$ of*

$$f'' + i\frac{x}{2}f' + \frac{f}{2}(|f|^2 - \frac{A_a}{t}) = 0, \tag{3.1}$$

*such that*

$$\lim_{t \to 0^+} \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right) = z_0 \text{ p.\,v} \frac{1}{x}, \quad z_0 \neq 0$$

*in the distributional sense. Moreover,*

$$|z_0| = 2|f'|_\infty \qquad with \qquad \frac{\sqrt{3}}{2}|a| \leq |z_0| < |a|.$$

*In addition, $f$ satisfies $\|f\|_{L^\infty} \leq 2|a|$.*

*Proof.* First, assume $f$ is an odd solution of (3.1) such that

$$2|f|_\infty^2 - A = 0.$$

Then, from the asymptotic behaviour for odd solutions of (3.1) established in Proposition 2.1, it easily follows that

$$f(x) = |f|_\infty e^{ic_+} e^{i\phi_2(x)} \operatorname{sgn}(x) + 2i|f'|_\infty e^{id_+} \frac{e^{i\phi_3(x)}}{x} + \mathcal{O}\left(\frac{1}{|x|}\right)$$

as $|x| \longrightarrow \infty$, with

$$\phi_2(x) = (|f|_\infty^2 - A)\log|x| \qquad \text{and} \qquad \phi_3(x) = -(x^2/4) - (2|f|_\infty^2 - A)\log|x|,$$

and $d_+ \in [0, 2\pi)$.

Recall that $f$ is regular and odd, then, by using the dominated convergence theorem, we have that

$$\frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right) \chi_{|x| \le M\sqrt{t}}(x), \qquad M \gg 1$$

goes to zero in $\mathcal{S}'(\mathbb{R})$ as $t \to 0^+$. On the other hand, the function $g(x) = e^{i(\frac{x^2}{4} + \phi_2(x))} \operatorname{sgn}(x)$ is a bounded, odd and has a continuous Fourier transform that is zero at zero. Hence, by Parseval theorem $\frac{1}{\sqrt{t}} g(x/\sqrt{t})$ also tends to zero as $t \downarrow 0$. Finally, the error term is integrable and odd, therefore arguing as we did before the convergence of the error term to zero follows by using the dominated convergence theorem.

The convergence of

$$u_f(t, x) = \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right)$$

as $t \downarrow 0$ to $z_0 \,\mathrm{p.\,v}(1/x)$, with $z_0$ such that $|z_0| = 2|f'|_\infty$ easily follows from previous remarks and the hypothesis that $2|f|_\infty^2 - A = 0$.

Now, we continue to prove that, for any given $a \ne 0$, there exists $A_a \in \mathbb{R}$, and an odd solution $f$ of (3.1) satisfying the condition $2|f|_\infty^2 - A_a = 0$.

Indeed, for fixed $a \ne 0$, and $-1 \le \lambda \le 1$, let $\mathbf{X}_{a,\lambda}(t, x)$ be an odd solution of LIA, that is a solution of LIA the form $\mathbf{X}_{a,\lambda}(t, x) = e^{\frac{A}{2}\log t} \sqrt{t} \mathbf{G}_{a,\lambda}(x\sqrt{t})$ with $\mathbf{G}_{a,\lambda}$ the solution of (1.21) with the initial conditions

$$\mathbf{G}_{a,\lambda}(0) = (0, 0, 0) \qquad \text{and} \qquad (\mathbf{G}_{a,\lambda})'(0) = (0, \sqrt{1 - \lambda^2}, \lambda). \tag{3.2}$$

Define the function $F_a$ as follows

$$F_a(\lambda) = 2T_{3,a,\lambda}(\infty) - T_{3,a,\lambda}(0),$$

where, as before $T_{3,a,\lambda}$ denotes the third component of the tangent vector to the curve $\mathbf{X}_{a,\lambda}$[10]. Notice that, for $\lambda = 1$, $\mathbf{G}_{a,1}(x) = (0,0,x)$, and therefore $F_a(1) = 2 - 1 > 0$. Also, for $\lambda = -1$, $\mathbf{G}_{a,-1}(x) = (0,0,-x)$, and therefore $F_a(-1) = -2 - (-1) < 0$. Since the map $(\mathbf{G}(0), \mathbf{G}'(0), a) \longrightarrow T_3(\infty)$ is continuous (see [GV, Proposition 2, pp. 2101]), we conclude that there exists $\lambda_a \in (-1,1)$ such that $F_a(\lambda_a) = 0$, that is $2T_{3,a,\lambda_a} - T_{3,a,\lambda_a} = 0$.

Notice that the associated function $f$ (through the Hasimoto transfom and the change of variables (1.17)) is an odd solution of

$$f'' + i\frac{x}{2}f' + \frac{f}{2}(|f|^2 - A_a) = 0$$

with $A_a = a\lambda_a$ (recall that $A_a$ is given in terms of the initial conditions (3.2) by the identity (1.19)), and from (1.20) we have that

$$
\begin{aligned}
2|f|_\infty^2 - A_a &= 2(-aT_{3,a,\lambda_a}(\infty) + A_a) - A_a = -2aT_{3,a,\lambda_a} + A_a \\
&= -a(2T_{3,a,\lambda_a}(\infty) - T_{3,a,\lambda_a(0)}) = 0,
\end{aligned}
$$

since $T_{3,a,\lambda_a}(0) = \lambda_a$ (see (3.2)).

Finally, for odd solutions of LIA, notice that the conservation law in Proposition 2.1 becomes

$$|f'|^2(x) + \frac{1}{4}(|f|^2(x) - A_a)^2 = \frac{a^2}{4}$$

(the value of the constant on the r.h.s of the above identity follows from the identities (1.20) and the initial conditions (3.2)). From which we get that

$$|f(x)| \leq |a| + |A_a| = |a|(1 + |\lambda_a|), \qquad \forall\, x \in \mathbb{R}$$

and

$$|f'|_\infty^2 = \frac{a^2}{4} - \frac{1}{4}(|f|_\infty^2 - A_a)^2 = \frac{a^2}{4} - \frac{1}{4}\left(\frac{A_a}{2} - A_a\right)^2 = \frac{a^2}{4}\left(1 - \frac{\lambda_a^2}{4}\right)$$

by using the condition $2|f|_\infty^2 - A_a = 0$, and that $A_a = a\lambda_a$. From the above formulae we conclude that

$$\frac{3}{16}a^2 < |f'|_\infty^2 \leq \frac{a^2}{4} \qquad \text{and} \qquad \|f\|_{L^\infty} \leq 2|a|$$

since $\lambda_a \in (-1,1)$. This concludes the proof of the lemma. $\qquad\square$

As a consequence of Theorem 1.4 and Lemma 3.1, we obtain the following result:

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[10]Recall that for odd solutions of LIA, the third component of the associated tangent vector, $T_3$, is an even function. Thus, in particular $T_3(+\infty) = T_3(-\infty)$

**Theorem 3.2.** *Let $a \neq 0$ sufficiently small, and consider $A_a$, $f$, and $z_0$ as in Lemma 3.1. Then, there exists $\varepsilon > 0$ such that for any given $u_+$ with $\|u_+\|_{L^1 \cap L^2(\langle x \rangle^\gamma)} \leq \varepsilon$ and $0 < \gamma < 1$, the initial value problem:*

$$\begin{cases} iu_t + u_{xx} + \dfrac{u}{2}(|u|^2 - \dfrac{A_a}{t}) = 0 \\ u(0,x) = z_0 \, \text{p. v} \, \frac{1}{x} + \sqrt{\pi i} \widehat{u_+}\left(-\frac{x}{2}\right) \end{cases}$$

*has a unique solution $u(t,x)$ such that*

$$u - \tilde{u}_f \in \mathcal{C}((0,1], L^2(\mathbb{R})) \cap L^4((0,1], L^\infty(\mathbb{R}))$$

*where*

$$\tilde{u}_f(t,x) = \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right) + \sqrt{\pi i} \widehat{u_+}(-x/2).$$

Theorem 3.2 represents a well-posedness result for the initial value problem

$$\begin{cases} iu_t + u_{xx} + \dfrac{u}{2}(|u|^2 - \dfrac{A}{t}) = 0 \\ u(0,x) = z_0 \, \text{p. v} \, \frac{1}{x}, \end{cases} \tag{3.3}$$

for some values of $z_0$ and adequate constants $A$ in (3.3): If we denote by $u_f(t,x)$ the solution of the IVP (3.3), we have proved that there exist appropriate (small) perturbations $u$ of the solution $u_f$ such that

$$\lim_{t \to 0} u(t,x) = z_0 \, \text{p. v} \, \frac{1}{x} + \sqrt{\pi i} \widehat{u_+}\left(-\frac{x}{2}\right).$$

In particular, $u - u_f$ has a trace in $L^2$, i.e. there exists the limit in $L^2$ of $u - u_f$ as $t \to 0^+$. This is in contrast with the situation in which one considers as initial datum the delta distribution. In the latter case, it was shown in [BV2] (see also [BV1]) that when considering the IVP

$$\begin{cases} iu_t + u_{xx} + \dfrac{u}{2}(|u|^2 - \dfrac{c_0^2}{t}) = 0 \\ u(0,x) = \sqrt{4\pi i} \, c_0 \, \delta_{x=0}, \qquad c_0 \neq 0 \end{cases} \tag{3.4}$$

there exist (small) perturbations $u$ of the solution $u_{c_0}(t,x) = c_0 \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}}$ of the IVP (3.4) such that the limit of $u - u_{c_0}$ as $t \to 0^+$ does not exist in $L^2$. As a consequence the IVP for the Dirac-delta (3.4) is ill-posed.

## 4. ACKNOWLEDGMENTS

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