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Landini, Gabriel; Galton, Antony; Randell, David; Fouad, Shereen

DOI:
10.1016/j.image.2019.04.018

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## Document Version

Peer reviewed version
Citation for published version (Harvard):
Landini, G, Galton, A, Randell, D \& Fouad, S 2019, 'Novel applications of discrete mereotopology to mathematical morphology', Signal Processing: Image Communication, vol. 76, pp. 109-117.
https://doi.org/10.1016/j.image.2019.04.018

Link to publication on Research at Birmingham portal

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Checked for eligibility: 30/04/2019
https://doi.org/10.1016/j.image.2019.04.018

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## Accepted Manuscript

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PII: S0923-5965(18)31121-4
DOI: https://doi.org/10.1016/j.image.2019.04.018
Reference: IMAGE 15552
To appear in: Signal Processing: Image Communication
Received date: 4 December 2018
Revised date: 26 March 2019
Accepted date: 22 April 2019

Please cite this article as: G. Landini, A. Galton, D. Randell et al., Novel applications of discrete mereotopology to mathematical morphology, Signal Processing: Image Communication (2019), https://doi.org/10.1016/j.image.2019.04.018

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# Novel applications of Discrete Mereotopelngy ${ }^{\text {n }}$ Mathematical Morphology 

Gabriel Landini ${ }^{\text {a }}$, Antony Galton ${ }^{\mathrm{b}}$, David Randell ${ }^{\mathrm{a}}$ Sheret \ Fouad ${ }^{\mathrm{a}, \mathrm{c}}$<br>${ }^{a}$ School of Dentistry, University of Birminnham, ${ }^{\text {rIV }}$<br>${ }^{b}$ Department of Computer Science, Universit, of $F{ }^{{ }^{~}{ }^{\circ} \text { or, UK }}$<br>${ }^{c}$ School of Computing, Engineering and the Built Envi, $n^{2}$ _nt, ;irmingham City University, UK


#### Abstract

This paper shows how the Discrete Mereotop - ${ }^{\text {logy notions of adjacency and }}$  phology to accept or reject changes resu. "ng from traditional morphological operations such as closing and ope. is r. ' ' his leads to a set of six morphological operations (here referred to ge, orrually as minimal opening and minimal closing) where minimal ciu nges 'i ifil specific spatial constraints. We also present an algorithm to compute the RCC5D and RCC8D relation sets across multiple regions resul ${ }^{1 \cdot} \times$ in a performance improvement of over three orders of magnitude over our pr viously published algorithm for Discrete Mereotopology.


Keywords: mathema ical nornhology, discrete mereotopology, image processing, spatial roasu ng

## 1. Introduction

This pape. ^er res on the processing of spatial relationships between discrete regior using '/athematical Morphology (MM). There has been a longstanding $j$ ter st ir formal definitions of adjacency and containment between image regio. as nose types of relations can form a basis for model building in ima ee contenıs retrieval and analysis. This has applications to problems where the des ription of hierarchical structure is important, for instance, in biologic. ${ }^{1} \mathrm{im}$.ging numerous problems revolve around the characterisation of re ations of diverse nature, for instance molecules in organelles, organelles in ct 'ls, cell in tissue compartments and tissues in organs. The subject has
been approached from a variety of points of view, including 1 sitaı olygon geometry [1], digital topology [2, 3], hierarchical modelling [t] a' a nnnected filtering operators $[5,6,7]$.

Bloch $[8,9,10,11]$ has provided an extensive body $r$ - work on spatial relations in computer vision and identified ways to symb lically nd programmatically harness and represent the inherent imprecisio. aris' g from image formation, post-processing, perception and the se rant related to certain spatial relationships sought. In [11] Bloch shows ho, M] I can function as unifying framework for spatial knowledge repre ntat: $\cdots$ and provides connections to formal logics, in particular raising the pos 'bility of implementing Region Connection Calculus (RCC) [12] operat © (as ell as providing a MM definition for the RCC's tangential proper pa, '(TrP) relation). In [9], it is proposed to construct modal logics usine var me notion of adjunction [13] to define modal operators that can be tilised for symbolic representation and interpretation of spatial rel a...chins. In [14] the notion of fuzzy adjacency between image objects was $\cdot \mathrm{v}$ stigated and formally defined so the concept of adjacency can extenc ${ }^{1}$ (e.g using fuzzy MM formulations) to accommodate degrees of adjacency $\mathrm{b}_{-}$. n. ans of admissible transformations that lead to strict adjacency a. . llow consistent representations and the management of imprecision men 'ned earlier.

Research has also focused nn applying MM and spatial reasoning to discrete spaces with the purp se of a plying spatial reasoning to digital images. In this context, Galton [ $\left.2,{ }^{n}\right]$ intr duced the notion of Discrete Mereotopology (DM) where he de elons va ious mereotopological concepts for discrete spaces. Our work in 15,16$]$ shows that a subset of DM functions (closure and interior) map irecu. $\dagger$, the MM dilation and erosion operators [17] respectively, comr aly used in image processing. In [2] that mapping was exploited to implement $\therefore$ full spatial relation set given by the RCC5D and RCC8D logics ${ }_{1} 1<\jmath_{2}$ in terms of MM. Briefly, the relation sets RCC5D and RCC8D enco - fi e and eight set of relations respectively that capture various notion of par hood, overlap and contact. After mechanically verifying DM theor ms adorted in the imaging algorithms (using the theorem prover SPASS [18], ve i $\iota$ slemented the RCC5/8D relation sets and exploited several $D^{\text { }}$ \& thecrenıs as short-cuts in imaging algorithms to compute operations on pa rs of re ions. DM can therefore be used to perform certain types of segmenu +in and model-testing analyses based on MM procedures. Those a' alyses have applications in histological imaging, where segmented histologic 1 comf ments regions of interest (those corresponding to, e.g., nuclei and
cell bodies) represent valid theoretical models of histological … ${ }^{\text {lity }}$ hat are related in specific ways in terms of their spatial relations $[.5,1 ; j$. This logical, model-based approach to image interpretation proviau. clean formal semantic framework in which to interpret image segm ruation . esults and, furthermore, guarantees that the imaging algorithms ncodin theorems in DM are provably sound. It also enables development ot olgor ihms that explicitly encode and 'reason' about spatial relations and lncal structure (e.g., cell and tissue organisation) as well as facilitating t. a _ acoc ng of other structural data of interest, such as the spatial locali +ion $\sim^{\boldsymbol{s}}$ molecular markers in cells and tissues.

Next we report new applications of DM tha. ${ }^{\text {n }}$ nric' MM operations. The paper is organised as follows. First, we visit tı detmitions of adjacency, connection and region neighbourhood in DM $n d^{11}$ MM counterparts. Next we present a new, more efficient version oi the RCC5D and RCC8D algorithm that that outlined in a previous hliration [16]. Finally, we discuss a novel application of DM that extends $\mathrm{N}, V$ with a the notions of morphological minimal closing and minimal C. ning, where DM is used to restrict the changes of the traditional MM closing anc opening operations so the original region shape is minimally mod. $-\dot{u}, \cdots \cdot$ le still achieving a desirable result. The paper concludes with a discussın.

## 2. Methods

The convention adr sted he. is that images consist of 2D square pixel arrays with 8 -adjacer :y, sear ng every non-boundary pixel of the array is surrounded by 8 nf ghb •••rs iorming a $3 \times 3$ pixel matrix. Image regions are sets of pixels lc. 1 ly-connected under 8-neighbour adjacency, representing objects of interes in the : mage. We assume that these regions exist in binary images but can .nc ide multiple planes or slices representing the same spatial reality, so tha reg' ins can share the same image space without being merged.

### 2.1. Adjar ner

The $\mathrm{aa}_{\mathrm{J}}$ ncy relation between pixels is captured by a reflexive and symmetric eration $\rightarrow(x, y)$, meaning that pixel $x$ is adjacent to or equal to pixel $y$. $\mathrm{A}(:, y)$ is atisfied if $d(x, y) \leq \sqrt{2}$, where $d: \mathbb{Z}^{2} \times \mathbb{Z}^{2} \rightarrow \mathbb{R}$ is the twodimens. nal Juclidean distance function defined on pixel coordinates in $\mathbb{Z}^{2}$. Ir $D \mathrm{Mt} r \mathrm{rms}$ [15], the adjacency relation between regions $X$ and $Y$ is referred
to as external contact and is denoted $\mathrm{EC}(X, Y)$. It is built ${ }^{〔} \mathrm{~m} \mathrm{~m}$. other relations, namely contact.$^{1}$

$$
\begin{equation*}
\mathrm{C}(X, Y) \equiv_{\operatorname{def}} \exists x, y[x \in X \& y \in Y \& \mathrm{~A}(,, y)] \tag{1}
\end{equation*}
$$

and overlap:

$$
\begin{equation*}
\mathrm{O}(X, Y) \equiv_{\operatorname{def}} X \cap Y \neq \emptyset \tag{2}
\end{equation*}
$$

$$
\begin{equation*}
\mathrm{EC}(X, Y) \equiv_{\operatorname{def}} \mathrm{C}(X, Y) \& \neg \mathrm{O}(\Omega, Y) \tag{3}
\end{equation*}
$$

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89

In [14] Bloch et al. showed that the ad, ency relation (or external contact in DM [15]) reworked in MM is equivalent to.

$$
\begin{equation*}
\mathrm{EC}(X, Y) \equiv(X \cap Y=\emptyset)^{\circ} \cdot(X \oplus B) \cap Y \neq \emptyset \tag{4}
\end{equation*}
$$ ( $\mathrm{c}_{\mathrm{D}}$ ), instead of connec. $\mathfrak{\mathrm { n }}$, as follows:

$$
\begin{equation*}
\mathrm{DC}(X, Y) \equiv \mathrm{cl}_{\mathrm{D}}(X) \cap Y=\emptyset \tag{5}
\end{equation*}
$$

[^0]\[

$$
\begin{equation*}
\mathrm{cl}_{\mathrm{D}}(X)={ }_{\operatorname{def}}\{x \mid \mathrm{O}(N(x), X)\} . \tag{6}
\end{equation*}
$$

\]

In the case of our assumed 8-connected square grid, $\mathrm{cl}_{\mathrm{D}}(X)$ ic 'quivalent to the dilation of $X$ using a structuring element $B$, which n our nodel consists of an arbitrary pixel and its immediate neighbourhooc so

$$
\begin{equation*}
\mathrm{cl}_{\mathrm{D}}(X)=X \oplus B \tag{7}
\end{equation*}
$$

Therefore definition (5) translates into MM as:

$$
\begin{equation*}
\mathrm{DC}(X, Y) \equiv(X \oplus B) \vdash^{\top}=\emptyset \tag{8}
\end{equation*}
$$

We also define a special type of neighboun: nod relation between pairs of regions that is not part of the RCCE, ע seis but is particularly useful when considering binary regions residing in $\therefore$ same image: region $Y$ is a neighbour of $X$ and separated from ic by ... pixel. We name this relation NC (for neighbourhood connection) ${ }^{2}$ and efine it as:

$$
\begin{equation*}
\mathrm{NC}(X, Y) \equiv_{\text {def }}{ }^{\circ} \mathrm{C}\left(\mathrm{cl}_{\mathrm{D}}(X), Y\right), \tag{9}
\end{equation*}
$$

which in MM terms corresponds to

$$
\begin{equation*}
\mathrm{NC}^{\prime} X, Y, \equiv \mathrm{EC}((X \oplus B), Y) . \tag{10}
\end{equation*}
$$

These formulae allow imı ${ }^{\prime}{ }^{\prime}$ 'mf tation of the extended MM functions that follow in Section 4. $\mathrm{Fi}^{\prime}$, ure 1 shows examples of the RCC8D relation set and the special cases of $\mathrm{N}^{\cdot} \cdot \mathrm{a}^{r} \mathrm{dPr}, *$.

### 2.3. Region Conn ¿ $^{\prime}$ in Calculus via Mathematical Morphology

In [15] we intrnduced quivalences between DM and MM allowing DM to be implementr $d$ ar $d$ understood in terms of MM procedures. Those equivalences make $\leadsto$ - nvenient to develop DM using standard image processing application, sunpoı ing basic MM operations (erosion, dilation, reconstruction). In 16] an 「 M algorithm implementation was presented which made use of the or ${ }^{n} l_{\rho}$ of binary regions in images. That algorithm computes the sp itial r ' 'ations between two regions (self-connected or not) residing in differe it imag ss. For many applications, however, it is required to find the

[^1]

Figure 1: The five and eight spatial relations that hold $u$ ' ween regions in the RCC5D and RCC8D sets in the discrete domain. The blue , run resent region $X$ and the yellow regions $Y$, the intersection $X \cap Y$ being shown in bı wn. The names in RCC5D stand for disjoint (DR), partial overlap (PO), equal (E ${ }^{\prime}$ ’ nrnver part (PP) and proper part inverted (PPi). The RCC8D set makes additional dist nct ons: disconnection (DC), external connection (EC), partial overlap (PO), tangrntial \& oper part (TPP), non-tangential proper part (NTPP). TPPi and NTPPi are the in $\imath^{\prime}$ rela ions, e.g., $\operatorname{TPPi}(X, Y)$ means the same as $\operatorname{TPP}(Y, X)$. The extensions considered he e a a 2 NC for 'neighbourhood connection' (a case of a DC relation where the regio. $\quad \cdots$ tilation away from adjacency) and PO* (a case of EC occurring on 'crossing objects $\therefore$ at do not share any overlapping pixels), which while possible in the discrete domain, is counter-intuitive with real-world objects.
relations held between mut nle re, $10 n{ }^{3}$ contained in pairs of images (e.g., biological objects across .fferent .onfocal microscopy imaging planes, or stain channels). In such ( ses the computation can be decomposed into a sequence of analyses ${ }^{1}$ etwe $\sim r$ dirs of self-connected regions: first extract two given regions into $\ldots$ wv empty images (maintaining their relative positions), next compute the relatic held between them using the said algorithm, and repeat this for all maining region pairs. That implementation exploits the 'start pixels' - rer ons (the first pixel in a given region encountered in raster scan order) ind us morphological reconstruction [19] to extract each region separately and apply the RCC test to the extracted pair. Such an approach, however, qu. kly ',ecomes computationally expensive; when dealing with ei-

[^2]ther large images (for which morphological reconstruction is ${ }^{-1} \sim \mathrm{w}$ ) $u_{\text {. images }}$ featuring many regions (the number of tests is given by $t^{1} e$ pr,$u t{ }^{-1}$ of the number of regions across the images, complexity $\mathrm{O}(n m)$ ). ©, me shortcuts have been identified, for instance in RCC5D, the disjoir selation DR can be assumed by default for all region pairs and other spati l relatı ns only computed in cases of overlap, avoiding a considerable numbe of $t$ sts. Similarly, EQ can be identified in those regions pairs whose $m$ nimיm pixel value is 3 in the sum of image $X$ (labelled as 0 and 1 ) and ima っr $^{\text {r }}$ (lal elled as 0 and 2). However, the distribution of DM relations varies ith th image content and therefore such shortcuts do not necessarily lead to no. 'ceable execution time improvements. The next section presents a n ne eff ient algorithm which avoids the decomposition of the computation into an exhaustive sequence of region pairs. The procedure shows a pnn: ie advantage in execution time compared to our previous algorithm ¿ , nd it enables DM analysis to be more efficient and therefore applicabl hioh-throughput workflows.

## 3. Fast RCC5/8 Algorithm

We assume $n$ binary region: : :ma. $\quad X$ and $m$ binary regions in image $Y$. The aim is to identify the spatı ${ }^{1}$ relations of the regions in $X$ with the regions in $Y$. Those relations can be stored in an $n \times m$ matrix, here called the 'RCC table' (stored $\varepsilon$, an ı age) where the $x$ and $y$ coordinates are indices pointing to the $x t_{\perp}$ and $y^{\dagger} \_$regions in $X$ and $Y$ respectively.

### 3.1. Computing RCC $D$

First, two image are と $\urcorner \downarrow$ ated using connected component-labelling, one where all regions ; . $X$ have unique labels (according to their raster scan order) and the other simı' $\frown$ rly with the labels of the regions in $Y$. We call these images $Y_{\text {labe ed }}$ and $Y_{\text {labelled }}$ respectively. Two additional images are computed, or • wh se pixels belonging to regions in $X$ are labelled as 1 (or foreground) and $u$ therwise (background) and the other where pixels belonging to regi, ns ; $1 Y$ are labelled as 2 (foreground) and 0 otherwise. These two images are mm d to produce a third image $X Y$, where pixels now have values $1 \perp$ (the pixel is in $X$ but not $Y$ ), 2 (it is in $Y$ but not $X$ ), 3 (a region of $X$ verlaps a region of $Y$ at that location) or 0 (image background). A further 'inar image $O$ is computed as the intersection (overlap) of $X$ and $Y$ The overlaps arise in the case of RCC5D relations PO, EQ, PP and

PPi. Inspection of the values of the pixels of the overlaps in $\cap$ (by . directing to $X_{\text {labelled }}$ and $Y_{\text {labelled }}$ ) reveals which two regions forn ag $g$ overlap. We store the label values of the regions of $O$ in arrays ovi- ${ }_{-}^{-}$and ovY[ ]. The regions in images $X$ and $Y$ involved in overlappi relatu as are also inspected by redirection to image $X Y$, and their min num pi el values are stored in arrays $\min X[$ ] and $\min Y[$ ]. These arrays : rere ; formation on whether a given region contains non-overlapping pi .el valיํes of 1 or 2 (which occur in PPi and PO cases) or whether all the $\mathrm{p}_{\mathrm{r}} \mathrm{r}^{1}$, in region are overlapping (value 3, which occurs in PP and EQ cai -). A ${ }^{\text {Jing minX and minY }}$ provides enough information to compute four of the five RCC5D relations (i.e., all those that involve region overlaps)

| Relation | minX | minV | min |
| :--- | :---: | :---: | :---: |
| $\operatorname{rminY}$ |  |  |  |
| $\mathrm{PO}(x, y)$ | 1 | 2 | 3 |
| $\mathrm{EQ}(x, y)$ | 3 | 2 | 6 |
| $\mathrm{PP}(x, y)$ | 3 | 2 | 5 |
| $\mathrm{PPi}(x, y)$ | 1 | 3 | 4 |

Table 1: Minimum values of pixel combosith $\eta$ of overlapping regions $X$ and $Y$. Region labels are: background=$=0, X=1, Y=\left\llcorner, X \vdash_{1}=3\right.$. The columns $\min X$ and $\min Y$ indicate the minimum value in regions $X$ and $Y$ res, ctively when a given relation holds.

From this scheme, it an be worked out that the relation $R$ between regions $X_{\mathrm{ovX}[i]}$ and $Y_{\mathrm{ovY}[i]}$ н. image X and Y (given by the overlap region $O_{i}$ ) is:

$$
\begin{equation*}
\left.\left.R_{i} i\right]=\quad \operatorname{ut}^{\ulcorner } \operatorname{nin} \mathrm{X}[\mathrm{ovX}[\mathrm{i}]]+\operatorname{minY}[\mathrm{ovY}[\mathrm{i}]]\right], \tag{11}
\end{equation*}
$$

where out [ ] is \& $1 \mathrm{o}^{\prime}$ - up table holding labels for relations $\mathrm{PO}=3, \mathrm{EQ}=6$, $\mathrm{PP}=5$ and $\mathrm{PPi}-4$ (see Iable 1 , rightmost column). Since the only remaining RCC5D rf atic 1 , DR, does not involve an overlap, DR can conveniently be assumed by ${ }^{1}$ rault for all possible region pairs and during the analysis the values in the $\hbar^{2} \mathrm{C}$ table are only updated for those regions involved in overlappi or re atic is using the procedure described. The procedure is shown in pseudncou in Algorithm 1.

### 3.2. Fi.m. $R$, C5D to $R C C 8 D$

$\mathrm{RCC} \mathrm{D}^{\mathrm{D}}$ introduces the notion of contact between regions, covering both o rrlap a d adjacency [1] and resulting in eight spatial relations which pro-

## Algorithm 1 Pseudocode for RCC5D computat $\overline{\text { n aucuss multiple regions }}$ in images $X$ and $Y$.

1. Default all relations between regions in $X$ an $\therefore Y$ to DR.
2. Compute labelled images $X_{\text {labelled }}$ a $\cdot 1$ I labelled where each region has a unique label.
3. Compute image $X Y$, coded as 1 - pixel of a region in $X$ but not $Y$, $2 \rightarrow$ pixel of a region in $Y$ bu $n t \quad 3 \rightarrow$ pixel of a region in both $X$ and $Y$.
4. Compute binary image $O$, cu' $\cap d$ as $0 \rightarrow$ background, $1 \rightarrow X \cap Y$.
5. Create arrays ovX[7 an ' ovY[] holding the information of which regions in $X$ and $Y$ rorm ov rlaps in $O$, by inspecting region labels in $X_{\text {labelled }}$ and $Y_{\text {label ' d. }}$.
6. Create arrays $m . n X\ulcorner$ ] a $d \min Y[]$ by inspecting for each region in $O$ the minimum pixel . ${ }^{1}$.e for that region in image $X Y$.
7. For each resion $\ldots \cap, \min X+\operatorname{minY}$ gives the RCC5D relation: $3 \rightarrow$ $\mathrm{PO}, 4 \rightarrow$ г $5 \rightarrow \mathrm{PP}, 6 \rightarrow \mathrm{EQ}$.
vide a more fine-grained spatial description than RCC5D. T1 $\sim R c^{\wedge} 5 \mathrm{D}$ relation DR is split into the RCC8D relations EC (external con . . inn) and DC (disconnection), the RCC5D relation PP is split into ${ }^{T} P$ and NTPP (tangential and non-tangential proper part respectivelv,, whe tor mer occurring when the proper part abuts the background regic 1s, the atter when it does not; the same thing happens, mutatis mutandis, $h^{\circ}+\mathrm{th} \mathrm{t}_{\mathrm{t}}$ inverse relations. The RCC5D results obtained by the metho . des $\quad$ - ibed earlier can be reprocessed to capture the RCC8D relations of t . a ame set of regions by performing single forward image scans testing fo $\frown$ djac $\sim$ - y patterns (rather than processing region-pairs one at a time). The ci mputation of RCC8D could be seen as a decomposition of the probic ~ ints a set of sub-problems (first compute RCC5D, then re-process the inı ${ }^{\text {re without having to consider }}$ all region pairs, while exploiting the previnained results), similar to the type of problem reduction sought in u namic programming [20]. We search for the presence/absence of ce ... natterns of adjacent pixels occupancy which, in conjunction with the $\mathrm{kr}_{\mathrm{s}} \mathrm{w}$ a RCC5D relations, are indicative of specific RCC8D relations. The $\leadsto \leadsto P_{L}$ and EQ are the same in RCC5D and $\operatorname{RCC} 8 \mathrm{D}$. Of the remaining cases, su $\mu_{\mu_{1}}$ ose that we know the RCC5D relation between regions $X$ and : $\quad$ no Then the RCC8D relation can only be either DC or EC. For it to be EC is re must be at least one instance where a pixel of $X$ is adjacent to a nixel of $Y$. The relation is DC is assumed by default and then we scan $\dagger$ ィe ima ${ }_{2}$ e looking for the adjacency pattern; if it is found, EC is returned, if $t_{1}$ nattf n is not found, then the default DC holds good.

The following not tior is 1 sed to describe the two-pixel patterns. Consider a pixel $p$ and let . b one of its immediate neighbours. Set $p(X)$ to be 1 or 0 acc ${ }^{\text {ling }}$ as $p$ does or does not belong to region $X$; and likewise with $p(Y), n\left(\Lambda^{-}\right)$, and $n(Y)$. Then the two-pixel pattern exhibited by the pa $-p, \imath$ with respect to $X$ and $Y$ is denoted by the quadruple $\left(p(X), p(Y), n^{\prime} X^{\prime} n(Y)\right)$.

From tl $\stackrel{a b o}{ }$ we can say that a DR relation between $X$ and $Y$ will be DC un' ss $r$ ae of the quadruple patterns $(0,1,1,0)$ or $(1,0,0,1)$ is exhibited for some $p$, nai in the image, in which case the relation is EC. Similarly, a case $s \mathrm{PP}(X, \Gamma)$ will be $\operatorname{NTPP}(X, Y)$ unless patterns $(0,0,1,1)$ or $(1,1,0,0)$ occur, in whi h case it will be $\operatorname{TPP}(X, Y)$; and likewise with PPi, NTPPi, and TH: $\mathrm{T}_{\mathrm{J}}$ perform these tests, the image is scanned using the 'forward m ask' of nixel $p$, shown in Figure 2.

At ea h $p$ we determine the two-pixel patterns formed by $p$ with each


Figure 2: The forward mask of pixel $p$. The pixel ratterns fc occupancy of regions in image $X$ and $Y$ are tested between the central pixel $p \mathrm{~h}^{\text {n }}$ he n ghbours $n$ in the 'forward mask' (shaded pixels). The pixels in the 'backward man' ', do not need to be tested because the patterns have already been visited during the ractnn $n$.
of the shaded elements of the mask $\ldots \cdots n$ As the scan progresses, an accumulator records whether these patt $r_{i}$ s have arisen, and the relabelling of the region relations is done aft 'he $=$ an is finished. The form of the mask is dictated by the fact that the inı..ge is scanned top-to-bottom and left-to-right (no need to look at a pun in for, e.g., $p=(x, y), n=(x, y-1)$, since this will already have been dew thed when $(x, y-1)$ played the role of $p$, with the pattern $p=(x, \gamma, 1), n=(x, y))$.

### 3.3. Extended relations $\mathbf{N - ~ a n d ~}^{\mathrm{P}}$ )*

The NC relation (d'sinition ${ }^{2}$ ) describes two regions separated by a onepixel gap (Figure 1) T' is c curs when a region is detected as DC and the pixel patterns r rer $\mathrm{t}_{\mathrm{L}}, \mathrm{r} \mathrm{jxt}$-nearest neighbours (the external shell of a $5 \times 5$ neighbourhor $\cdots$, show that pixels $p$ (in the neighbourhood centre) and $n$ (in the shell) are occu ied by pixels of regions of $X$ and $Y$, or $Y$ and $X$, respectively. The 'O* relation arises when two 8 -connected regions 'cross' each other in ${ }^{\circ} \mathrm{rr}$ „r-connected regions, without overlapping or sharing any pixels (Figr e 1). ${ }^{1} 1$ ch a pattern can commonly arise in the square lattice and it is ir verp eter as EC in RCC8D. In practical applications, however such results can. un atuitive (e.g. a linear object crosses another without ever "passi s chrougn" it) and it might be useful to identify these occurrences. This : done 1 y inspecting $2 \times 2 n$ and $p$ pixel patterns for exclusive cornerconnecu. ${ }^{\mathrm{d}}$ n; .el pairs in relations that have been identified as EC.

254 I3 4. Cor ) lexity analysis and performance

The old algorithm in [16] uses morphological binary re- ${ }^{-}$stru tion to extract every pair of regions before calculating the relat onal in del held between them. It has been shown in [21] that morphologica' econstruction is a computationally expensive, highly non-linear proce iuie. Its complexity depends on the number of component/pixels to be re onstruc ed. Even for the efficient/best-compromise algorithms [22] it is recu niser that a mean case complexity analysis would be extremely diff jult + + compute because of the variety of input images that may be used $\Gamma_{1}$ ad ition to utilizing reconstruction, the computational complexity of $\because \mathrm{CO}$ - it the old algorithm (in the worst case scenario) is quadratic, $\mathrm{O}(\mathrm{nm})$ (whr $\mathrm{m} \approx \mathrm{n}$ ) because the relations are computed between all possible re $\epsilon_{\delta}$ คn pa rs ( n and m ) one at a time, or subquadratic when $\mathrm{m} \neq \mathrm{n}$. While somı hortcuts were identified (e.g. to avoid computing relations between thons that are further away than two dilations, guaranteed to be DC), a important bottleneck remains with the binary reconstruction steps : _ norv to extract the region pairs.

The new Algorithm 1, first, avoids $\mathrm{t} \dagger$ acting individual pairs of regions into new images to compute the 1 ...ioni thus avoiding morphological reconstruction altogether. Secondly, it orı putes the RCC5D relation from a sequence of steps that reduce ts wan wity from quadratic to linear yielding to an average case complexity oı ' $\cap(\mathrm{n}+\mathrm{m})$ ). In particular, steps 1,3 and 4 in Algorithm 1 have a conctant-time algorithm of order $1(\mathrm{O}(1))$. Step 2 (image labelling) requires a max num time complexity of $\mathrm{O}(\mathrm{n}+\mathrm{m})$. Steps 5,6 and 7 process the ove.' $\quad$ npin', subregions that occur across the two images. It should be notf 1 that iv ations PP, PPi and EQ are one to one, and result in one overlap ing segr ent per region pair. A worst case scenario where all the relatic is ht. ${ }^{1}$ a e any of the above (therefore $n=m$ ) would lead to a scaling of th steps to $\mathrm{O}(\mathrm{n})$ which is still less than $\mathrm{O}(\mathrm{n}+\mathrm{m})$. The PO relation, however, is a special case in the sense that a region can have more than one ove lapping subregion (with one or multiple other regions). For instance i. "oe and convoluted regions could potentially lead to a scaling higher thar $\mathrm{O}(\mathrm{n})$. While is not possible to foresee what regions configurations may be $f$, und in segmented images, it is nevertheless possible to clarify the impact - this unknown, experimentally. In a series of performance tests on rar aom hinary images (detailed below) we found that on average, the numb r of ovt lapping subregions across 500 tests (average 7152, maximum $19431 \mathrm{r}_{\mathrm{c}} \mathrm{inn}^{2}$, was smaller than the number of regions $\mathrm{n}+\mathrm{m}$ (average 11168, m sximu: 38934 ). The running time of the proposed algorithm would there$\mathrm{fc}_{\mathrm{c}} \cdot \mathrm{e}$, on a erage, increase linearly with the total number of regions $\mathrm{O}(\mathrm{n}+\mathrm{m})$,
with some exceptional configurations where it could be high $\quad$ dept ding of the number and nature of the PO relations. As examine . ex $\because$ mentally, situations where this is above the quadratic complexity of $u_{\mathrm{i}}$ ? sld algorithm appear to be unlikely. The successive forward passes or une labulled images to compute the RCC8D relation set, as well as the $\epsilon$ stendea relations NC and $\mathrm{PO} *$, are of the order $\mathrm{O}(1)$ and therefore do not in "ase ihe algorithm complexity.

Figure 3 shows the difference in performance, in seco ids, of the previously published [16] and the new algorithms u. $512{ }^{\circ}+12$ pixels, random binary images with varying probabilities, p , of foreg. ${ }_{1}$ und pixels. The tests were performed on the ImageJ platform, versın $1.5^{1}$ [23] under the Linux operating system on an Intel Xeon CPU (E31 $\mathrm{s}^{\wedge}$ ) at 3.1 GHz . The plot shows the average of 5 runs at each $p$ in steps $r^{f} \cap n 1$, le average difference over all p was an improvement of 491 times fas $\cdot \boldsymbol{r}$ than the previous algorithm, while largest difference was found at: $\cap 12$ where the new algorithm was 1684 times faster than the old one. The y cution times appear to be dependent not only on the number of reg. - a pe. image but also on the proportion of the different types of relations the $t$ c cur at various p (not shown). A slight advantage was noticed $\mathrm{f}_{\mathrm{c}}$ vin n ? algorithm implementation on images with the highest p , (where only -ry few regions exist, the images being mostly occupied by one large region), but this difference, in practical terms, becomes negligible as the xecutı $n$ times in those cases are all at a fraction of a second.

## 4. New morpholog ${ }^{\prime} \mathfrak{a}^{1}$ filt rs: Minimal closing and opening

In addition to he applıcations of DM in histological imaging [15, 16, 24], the fast algurithnı ${ }^{\text {nables new }}$ MM operators with reasonable speed performance ts ve designed, exploiting the relations between image regions and the chan ${ }_{c}$ 's $t$ ' ey undergo after other morphological operations.

In MM, he op ration closing $\phi$ with a kernel $B$ is defined as the dilation of a regior, fol swed by an erosion [17]:

$$
\begin{equation*}
\phi_{B}(X)==_{\operatorname{def}}(X \oplus B) \ominus B . \tag{12}
\end{equation*}
$$

Cl sing is an extensive transformation, where voids in regions, and detail that unnnot contain the translations of kernel $B$, are filled. Note that n ereotop logical closure, which refers to a topological operator defined on


Figure 3: Differences between the a men cution times of the old and new RCC8D algorithms. The tests were done, on raı ' 1 m binary $512 \times 512$ pixel images with varying foreground pixel probabilities. Each point 1 s the average of 5 runs and the vertical bars indicate one standard deviation ${ }^{〔}$. . . . the mean.
a discrete space, does rot cou ons to closing but to dilation in MM, despite the similarity in chei names. When closing binary regions, voids are filled with the foregro a va' ie. While the rest of this section only deals with closings, ther ${ }^{\text {r }}$ is dua. MM operation with respect to the set complement, namely op nins an anti-extensive transform, which instead of filling, removes those $\mathrm{c}^{1}$ :oct pixels that cannot be fully covered by the translations of kernel $B$.

Closing is cu. monly used to 'fill in' gaps between nearby regions desired to be joinf 1 , tr fill small holes in regions and to reduce the complexity of region bc nd ries ('shorelines' from now on). These actions are, however, not indonondt.+ gaps, holes and shoreline irregularities are processed concurrer ily as he operation does not differentiate between them. In certain circun. tances, however, it might be desirable to achieve only one of those rec aus, e.y., Joining nearby regions while avoiding major modifications of the s. oreline details that do not yield a connection to another region. Using

DM this can be modelled as follows. In MM, the set of pixf ${ }^{1-}$ adat ${ }^{\text { }}$ to the original region by the closing operator is known as the blar $\varepsilon$ to $\mu^{+}+$):

$$
\begin{equation*}
\operatorname{BTH}(X)=\phi_{B}(X)-X \tag{13}
\end{equation*}
$$

The black top-hat often consists of a set of discc nected components, the elements of which we call black top-hat segment- In $\mathbf{N a}^{-1}$ this is defined immediately following, where $\mathrm{BTH}_{\text {seg }}(Y, X)$ is read as ' 1 i a connected component of the black top-hat of $X^{\prime}$ and $\mathrm{BTH}_{\text {segs }}(X)$ to the $\mathrm{s} t$ of black top-hat segments of X . The relation $\mathrm{CC}(Y, X)$ which we nere, read as ' $Y$ is a connected component of $X^{\prime}$, is also defined iv DM-s e [15], p 572:

$$
\begin{gather*}
\mathrm{BTH}_{\text {seg }}(Y, X) \equiv_{\text {def }} \quad \mathrm{CC}\left(Y, \mathrm{~L}^{\top} \mathrm{H}(X)\right),  \tag{14}\\
\mathrm{BTH}_{\text {segs }}(Y, X)=_{\text {def }}\{\times \quad\ulcorner\mathrm{C}(Y, \mathrm{BTH}(X))\} . \tag{15}
\end{gather*}
$$

The set-builder notation used to define i action $\mathrm{BTH}_{\text {segs }}(X)$ in definition 15 returns a set of regions, namely th ' ''s. in DM, however, where a region rather than a set of regions is requirei as the output of a function ${ }^{4}$, the set union operator is added, i.e.,

$$
\begin{equation*}
\mathrm{BTH}_{\text {segs }}\left(\text { ri, } \quad \mid \int\{Y \mid \mathrm{CC}(Y, \operatorname{BTH}(X))\},\right. \tag{16}
\end{equation*}
$$

and the same principle $a_{r}$ lies fr: equations 19, 20 and 25 . The relation between $Y$ and $X$, giv in $\mathrm{B}_{\mathrm{r}}^{\mathrm{r}} \mathrm{g}(Y, X)$, is always EC, that is to say they are adjacent (externa' v cr aner 'jed) regions-remembering here that the segments are the result ff alı vtf usive transformation. In addition, the segments could also be adjar _ t to other regions in their neighbourhood; if the distance $\epsilon$ separating pairs of res $\mathrm{r}_{5}$ ns is no more than half the width of kernel $B$, the black top-hat s, ints create 'bridges' between originally disconnected (i.e., DC) regions. The 1 considering region X and all other regions Y in the segmented imf se, tw black top-hat segment types can arise. First we have what we $c$ dl $s$ ',orelines where $\mathrm{BTH}_{\text {shoreline }}(Y, X)$ is read as ' Y is a connected shoreline cu non of the black top-hat of X '. In this case the black top-hat segmer $\sim$ I adjo..ıs exactly one connected component of X:

[^3]\[

$$
\begin{align*}
& \mathrm{BTH}_{\text {shoreline }}(Y, X) \equiv_{\text {def }} \\
& \quad \mathrm{CC}(Y, \operatorname{BTH}(X)) \& \exists Z[\forall U[\mathrm{CC}(U, X) \& \mathrm{EC}(U .)] \leftrightarrow U=Z], \tag{17}
\end{align*}
$$
\]

i.e., $Y$ is a connected shoreline component of the black nn-r it of $X$ if and only if $Y$ is a connected component of the black cop- - of $X$, and there exists exactly one connected component of $X$ that ic $二 \mathrm{Ct} Y$.

The second case is where we have a black tor hat - oment that forms a bridge between two regions. $\mathrm{BTH}_{\text {bridge }}(Y, X)$ is read a. " $Y$ is a black top-hat bridge of $X^{\prime \prime}$ :

$$
\begin{align*}
& \mathrm{BTH}_{\text {bridge }}(Y, X) \equiv \text { def } \mathrm{CC}(Y, \mathrm{BTH}(X)) \& \\
& \quad \exists Z, U[\mathrm{CC}(Z, X) \& \mathrm{CC}(U, X) \& \& \mp \cdot \mathrm{EC}(Z, Y) \& \mathrm{EC}(U, Y)], \tag{18}
\end{align*}
$$

which is similar to definition (17) eג $\because \downarrow$ th re are now at least two connected components of $X$ externally connectei to ve black top-hat segment $Y$ of $X$, not one.

The spatial relations that hold u 'ween the black top-hat segments and the original regions provid $\sim$ - means for identifying those which act as bridges (between DC regic \& pairs, and those which do not (and consequently only modify a region short $\cap$ e). 'rom this it follows that the black top-hat segments adjacent to c sly nne ıugion are shoreline modifiers (including hole filling, when the holes san se fi led by the kernel), and those adjacent to more than one region are oridg - Retaining one or another type, (e.g., by means of binary reconstr $\mu$ : $\cap$ [19]), gives rise to two types of conditional minimal closing, shoreline smootı ag without region merging:

$$
\begin{equation*}
\phi_{B}^{\text {shoreline }}(X)=\left\{Y \mid \mathrm{BTH}_{\text {shoreline }}(Y, X)\right\} \tag{19}
\end{equation*}
$$

and region mer sing without boundary smoothing:

$$
\begin{equation*}
\phi_{B}^{\text {bridge }}(X)=\left\{Y \mid \operatorname{BTH}_{\text {bridge }}(Y, X)\right\} . \tag{20}
\end{equation*}
$$

384 Note hat in CCC 8 D , the notion of shoreline or boundary of a region does 3 not dific nnt' ite between the 'outside' boundary and the boundary with an ir ernal 'ole. The DM treatment of region holes is dealt with later in this p. ner.

With regards to implementation, the different black top- ${ }^{-}+$vaı, $n$ ts are sorted by an exhaustive analysis of the relations between $a^{\prime}$, ori ,,$^{-1}$ regions versus all black top-hat segments generated after an MM in ng (i.e., $X \oplus$ $B) \ominus B)$ ). Those results, arranged in an $m \times n$ mat is or ta. le indexed by regions and black top-hat segments in scan order ( here na led the RCC table), provide a convenient way to search for those sp rial elations. The DM relation between a given region and a black tc $ر$-hat agment can be one of two, out of the eight possible outcomes of the ${ }^{\supset} \mathrm{C} \mu 8 \mathrm{D}$;egion set: either DC or EC. To identify 'bridge' black top-hat stc ' nent - we use indexing of the original regions and black top-hat segments in ı. e $x$ and $y$ axis of the RCC table respectively: the number of EC ins. $\vee$ nces.$n$ a row indicates the number of different regions a given black to $\mathrm{u}_{1}$ hat segment is adjacent to. Black top-hat segments with total EC coint............. equal to 1 are therefore shoreline modifiers (i.e., they are adjacent • only one region), while those instances with counts exceeding 1 art, ~manteed to be bridges. As will be seen comparing Figures 4 f and 4 g , blacı $\dagger$ ر p-hat shoreline segments include those completely surrounded by a $r_{\bullet} \cdot \neg \eta$; $n \cdot$ call these segments lakes. In DM this can be defined as follows, where $\mathrm{B}_{1}$.' lake $(Y, X)$ is read as ' Y is a black top-hat lake of X ; the definiti. $\quad \omega^{\sim}$ he DM definition of a hole defined later:

$$
\begin{equation*}
\mathrm{BTH}_{\text {lake }}\left(Y, \Upsilon^{\prime}\right) \equiv_{\text {def }} \mathrm{STH}_{\text {seg }}(Y, X) \& \operatorname{Hole}(Y, X) \tag{21}
\end{equation*}
$$

The crucial distinctir $n$ be vee 1 a shoreline and lake black top-hat segment of a given region is th it a iake also satisfies what it is to be a hole in that region which again is $n$ ode . in another RCC table indexing regions and holes. Examples of sinary 1. jion merging with minimal shoreline smoothing and shoreline sm otır $r$ without region merging are given in Figure 4.

While black ${ }^{\circ} \mathrm{D}$-hat segments have the same connectivity as the original regions (r.g., ,-connected) the minimal closing can be minimised further by considt. Ig only the adjacency relations of their 4 -connected subcomponen + , The rationale for this is that retaining a given black top-hat segment i. sir lar $o$ adding some background pixels to the foreground. Since the 8-connecu ' 'Jreground convention implies a 4 -connected background, it is pos ible to "estrict minimal closing to the 4 -connected sub-components of a givei black op-hat segment that satisfies the bridge or shoreline properties de wibed earlier and not including the whole black top-hat region. Figure 5 shows t ie effect of retaining such 4 -connected components in cases of pro-


Figure 4: Closing and minimal closing of $\quad \ldots \quad$ images with a disc of radius 3. The original greyscale image of lymphocytes stainec $\mathrm{wi}^{+}$. 1 silver nitrate for detection of nucleolar organising regions (dark spots)(a) was $s{ }^{r}$ ment. 1 with the minimum error thresholding algorithm [25] (b). In (c) the traditional c. $\mathrm{s}_{\mathrm{s} .} \sim$ ( n, th the added pixels in grey that make the 'black top-hat' (d). Panel (e) shows in $n$. genva the black top-hat segments that have an adjacency relation with more thaı лue ... $\quad$ n in (b), acting as bridges. We call this operation 'minimal closing bridges'. Panc' (f) shows those black top-hat segments that have adjacency to only one other region in (b) (minimal closing shorelines), while in (g) are shown the lakes which are $r$ ack us -hat segments that have no connection to the rest of the background's subset $\mathrm{t}^{1}$ at inters ts the image boundaries. Panel (h) shows the traditional opening (with addea, 'vels n grey that make the 'white top-hat' (i). Panels ( $\mathrm{j}-\mathrm{l}$ ) shown the minimal of ninc of bı.dges, shorelines and islands respectively.
cessing regions wit ${ }^{1}$ null 1 .. ior.
Finally, the $d^{\prime} d_{1}$ - neration of the closing is opening, $\Upsilon$ :

$$
\begin{equation*}
\Upsilon_{B}(X)=_{\text {def }}(X \ominus B) \oplus B \tag{22}
\end{equation*}
$$

and the correspu fing top-hat transformation for the opening is called white top-hat:

$$
\begin{equation*}
\mathrm{WTH}(X)=_{\operatorname{def}} X-\Upsilon_{B}(X) \tag{23}
\end{equation*}
$$

which identifi s the segments that were removed from the original after the opening no ation. As before and mirroring definitions for the black top-hat w defin a white top-hat segment $Y$ of region $X$, and the set of white top-hat st rments of $X$ :


Figure 5: Closing versus minimal closing. (a` 4 digitisc : version of a 4th order von Koch curve with discontinuities, resulting in 30 fr . rme $\omega$. The target is to merge all pieces into a single region. (b) The classical result ui g morphological closing with a circular kernel of radius 4 (the smallest kernel $\mathrm{t} . \mathrm{a}$. lost all gaps). Note the loss of detail in the result. (c) A minimal closing where the gat between any two fragments were filled independently with the smallest ker : : $\rightarrow \infty$ issible until a single region was obtained. (d) The detail of the minimal closing ( $\quad$ 'ock is the original set, magenta (dark grey in B/W version) represents the black top-hat segments. (e) and (f) show the same example, but this time retaining only th $\quad$. $\quad$ nnected subregions of each black top-hat segment $\left(\mathrm{BTH}_{\text {segs }}\right.$ in the text) that $\mathrm{ac}^{\prime}$; as a $\mathrm{r}_{1}$ reotopological bridge between fragments. Note that this closing modifies the $c$ :rinal ven less than in (c). (g) shows in green (bright grey) the part sub-regions $r$. the blan iop-hat segments that were not necessary to retain to achieve the minimal cl sing i.e., the difference between (d) and (f).

$$
\begin{gather*}
\mathrm{W}_{\mathrm{I}} \sum_{\mathrm{gg}}(Y, X) \equiv_{\operatorname{def}} \mathrm{CC}(Y, \mathrm{WTH}(X)),  \tag{24}\\
\mathrm{WTH}_{\text {segs }}(X)=\{Y \mid \operatorname{CC}(Y, \operatorname{WTH}(X))\} . \tag{25}
\end{gather*}
$$

It is th sref re possible to implement minimal opening operations as the dual of mul al cosing. Note that while opening is an anti-extensive transformat ' In, the waite top-hat segments are in relation EC to the regions in the opent I image that is: $\mathrm{WTH}_{\text {seg }}(Y, X) \rightarrow \mathrm{EC}\left(Y, \Upsilon_{B}(X)\right)$. The two new dual minima ne ing operations are open shorelines and open bridges, dependir $b_{3}$ on which type of white top-hat segments are retained or removed. It is a. o possi , le to define an additional minimal opening operation that removes
those white top-hat segments that are DC to all other region - $\eta$ th - pened image. We call this procedure opening islands, and its dual clos oy ${ }^{1}$ nkes. Interestingly, these opening islands and closing lakes are equir ${ }^{1}$ nt to opening and closing by reconstruction, respectively [26]. This wuence of morphological operations combining MM with the explicit rl lations of DM shows the potential for defining a variety of fine-grained $\mathrm{mol}_{1}$ holor, $^{\text {ctal operators }}$ that target a particular goal. It also highlights tl ᄅ imritance of securing computationally efficient ways to compute and stc •o flati ns between pairs of regions when processing segmented images su ${ }^{h}$ as ${ }^{〔 1}$ sse assumed in the RCC table, where these relations are explicitly used , these new operators. An example of the advantage of these new op ators is shown in Figure 5. Here connecting fragments in a discontinued c. "ve can be restricted to places where the closing leads to fragment conrnnti...ithout interference at locations where the connection is not necessa.: By so doing we preserve the original as much as possible with a l Jmamatic loss of global detail than traditional closing.

Similarly to minimal binary of - ing c ad closing, the procedures above can directly be applied to process grivsc. le images via threshold decomposition (although the threshold a cedure). Figure 6 shows examples $\iota^{\circ}$ the greyscale versions of the minimal closing and opening respectivnlv.

## 5. Discussion

Bloch [11, 9] orig; iallv sug jested that RCC relations can be defined in MM, and specificall pro. res the translation forthe $T P P(X, Y)$ relation [11], which is equivalen ${ }^{+}$o ours in [15]. It should be noted that while MM is not specific about discrete $\cdot$ continuous space, that is not an exact translation of the RCC8 $T-1$ relation on discrete space, because RCC presupposes an infinitely divi ble ,ne. Instead, for the case of discrete space, the connections drawn are $v$ ith $t_{\iota}$ RCC8D relation set of discrete mereotopology.

The ir pler entation of RCC5D, RCC8D and additional DM relations as MM proce ${ }^{\text {a }}$ es pens a range of new opportunities to extend some operations 'eyond tıeir original design by means of exploiting spatial relations held $k$ atween egions. This is specially useful when designing analytical procedures that an benefit from mechanically reasoning about image contents.

The pproach presented here allows the results of closing and openings tc be ma le conditional on certain types of modifications which might not


Figure 6: Greyscale minimal closing and op ning. The examples were computed using a $3 \times 3$ kernel on a greyscale image ' texu. Note (second row) how minimal closing bridges connect nearby regions without mu 'ifying the shorelines or filling lakes and how the closing of lakes does not affert the shoreline features. The open bridges procedure leads to fragmentation of regiors in tı original without affecting other shoreline features (compared to open shorelines, while t e opening of islands removes the white top-hat segments with no adjacency relatı $\mathfrak{s} \dagger$, any other patterns at a given grey level.
be straightforward to 'ac' ievf otherwise or might require more complex approaches such as $m$ ltiscale uperators and directional information [27]. While conditional filteri.1g is . ${ }^{\text {t }}$ new, traditional conditional morphological operators apply their u straints in a given local sub-image (given by the kernel). To replicate $t^{\prime}$ is t pe of filtering region-wise is challenging because classical methods reruire ditional processing to account for relations between regions and ion' itions on these to be met, whereas in DM it is built into its very founu. ${ }^{+;}$رns.

The widges, ooundaries, island and lakes regions in relation to opening and c. $\operatorname{sing}$ (1. . the white top-hat and black top-hat segments) have similarities to what joille and Vogt call 'binary patterns' [28] for which they iden$\mathrm{tif}^{f}$ ed formulae for their computation (and include some additional patterns: c re, per s rations, branches and loops). For minimal closing and opening,
however DM has the advantage of being able to relate, via ${ }^{+1} \mathrm{Rc}^{\top}$ table, which original regions are adjacent to those segments and $\dagger^{\prime}$ eret $\not \approx$ คpen the possibility to control algorithmically whether segments are : cluded or removed from particular configurations of regions. That wuld wu ald require further computation in the approach presented in [28]

There has been interest in other types of conditiona' nder tions, for example homotopic sequential filtering to preserve ne trinlogy of an image [29, 30] or multiscale top-hat transforms to improve im ıge s ontrast [31]. Here we described how processing can be applied to ${ }^{\text {'`ange of regions or across }}$ regions. A number of new uses for DM via MM has s een recently identified in applications that require dealing with mous ${ }^{\prime}$ whe e image regions fulfil specific spatial relation between their parts $\left[\left\llcorner^{2} 24,32\right]\right.$. Such models commonly arise in histological imagery, whern re....... regions represent regions with special biological meanings (such as.$^{11} \mathrm{~s}$, nuclei, tissues, organs) that not only can be distinctly detected, $b, \quad 1 \sim n$ exist in specific spatial relations and hierarchies. Such relations need tc $b$, fulfilled if the extraction of biologically relevant information from - ages 's to be related to a given context in terms of ontological levels of orgar sai. on [33]. On a different kind of application, Cointepas [34] propo ••• ; of MM combined with adjacency relations to construct homotopically deformable cellular models and resolve complex problems, such as $3 \boldsymbol{N}$ rerebral cortex segmentation, where topology preservation is essential to yield $\mathrm{n}_{\mathrm{n}}$ t only accurate but anatomically plausible results.

The procedures pre ented hu e stem from our work in histological imaging using digital ima as $r$. 2 D tissue sections, and as as such are based on a 2D Cartesian grid repre on ation. It would be desirable to further develop these concepts ar a lgorithms in n-dimensions so they can be applied to e.g. temporal, volumet. and higher dimensional data sets. Furthermore, alternative sch me such as simplicial complexes (used to represent multidimensional da.-1 「‘,5], graphs [36] and hypergraphs [37, 38] (for non-lattice implemente ions u. MM) might be advantageous for such generalisation to higher dir ens: Jns.

## 6. Ar know'? dgements

The reser ch reported in this paper was supported by the Engineering ar a Phy ical Sciences Research Council (EPSRC), UK through funding und r grant EP/M023869/1 'Novel context-based segmentation algorithms for
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## AUTHOR DECLARATION

We wish to confirm that there are no known conflicts of interest ass cciated with this publication and there has been no significant financial suppor' for i. is work that could have influenced its outcome.

We confirm that the manuscript has been read and approved $ニ ン$ aıı $\mathfrak{\exists m e d}$ authors and that there are no other persons who satisfied th = rrite. ' $\exists$ for authorship but are not listed. We further confirm that the o der ו _uthors listed in the manuscript has been approved by all of us.

We confirm that we have given due consideration to th prote stion of intellectual property associated with this work and th -2 the, are no impediments to publication, including the timing of , ur'.ca ion, with respect to intellectual property. In so doing we confirm that $v$ 'o nave rollowed the regulations of our institutions concerning intellectual $\uparrow$ "operty.

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Signed by all authors as follow :

| Gabriel Landini |  | 26/March/2019 |
| :---: | :---: | :---: |
| David Randell | David | 26/March/2019 |
| Antony Galton | futere"1llon | 26/March/2019 |
| ereen Foud | Shere troual |  |

## Novel applications of Discrete Mereotopology to Mathematical Morphology.

## Highlights

- Six new mathematical morphology operators using mereotopological conc $\epsilon \mathrm{Nts}$.
- Novel "minimal closing" and "minimal opening" morphological operations
- A new discrete region connection calculus algorithm with improved execu:- I speed.



[^0]:    Ine svmbols $\exists, \&, \in, \cap, \neg$ and $\equiv$ are read "there exists", "and", "is a member of",
    " itersectic 1", "not", and "if and only if", respectively; $\emptyset$ denotes the empty set.

[^1]:    ${ }^{2}$ In DM the relation NC is symmetric, i.e., $\mathrm{NC}(X, Y) \rightarrow \mathrm{NC}(Y, X)$.

[^2]:    ${ }^{3} \mathrm{WI}$ ¹e a nor null region in DM is simply the union of an arbitrary set of pixels, the algn $\cdots^{\circ} \mathrm{hm} \quad$ ulipulation of regions being assumed here is typically restricted to connected er nponer. , or simple regions.

[^3]:    ${ }^{4 n}$ melnn $\quad$ ing that in DM, a region can comprise several disjoint, region-parts as well as being a imple region.

