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ON THE NON-EXISTENCE OF SRG$(\mathbf{76, 21, 2, 7})$

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Abstract. We present a new non-existence proof for the strongly regular graph G with parameters $(76, 21, 2, 7)$, using the unit vector representation of the graph.

Keywords: Strongly regular graph, distance regular graph, unit vector representation.
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1. Introduction

A graph G is said to be strongly regular with parameters (v, k, λ, μ) if the following condition holds: G has v vertices (i.e., $|V(G)| = v$) and, for $u, w \in V(G)$, the number of common neighbours of u and w in G is k if $u = w$ (so G is regular of valency k), λ if u and w are adjacent, and μ if u and w are non-adjacent. Strongly regular graphs are among the central objects in graph theory and its applications. We write $srg(v, k, \lambda, \mu)$ for any strongly regular graph with parameters (v, k, λ, μ).

Haemers [4] proved non-existence of $srg(76, 21, 2, 7)$. His proof is very efficient, and it relies on edge counting to establish that such G must locally be a union of 3-cliques. This means that G is the collinearity graph of a point-line geometry $pg(3, 6, 1)$ (a generalized quadrangle of order $(3, 6)$). At this point Haemers quotes the non-existence result for $pg(3, 6, 1)$ by Dixmier and Zara [2]. (A shorter proof for non-existence of $pg(3, 6, 1)$ was provided by van Lint and Brouwer [6][1].)

In this note, we give an alternative proof of Haemers' theorem based on the well-known fact that every distance regular graph (and in particular, every strongly regular graph) admits a Euclidean realization as a set of unit vectors in an eigenspace of the adjacency matrix of G. In this realization, the value of the inner product of two vectors (the cosine of the angle between them) is fully determined by the mutual distance of the corresponding vertices. This is encoded in the so-called cosine sequence. Note that the eigenvalues of the adjacency matrix, dimension of each eigenspace, and the cosine sequence can be easily deduced from the parameters of G via the readily available formulas (for example, see [3]).

There are many open cases of strongly regular graphs even for relatively small values of v (see the table of feasible parameters up to $v = 100$ in [1]). Of course, the aim of our project is to contribute to one of the open cases. In this sense, the proof in this note is just a sample of things to come. However, we think that even this taster proof demonstrates efficiency of the method and it exhibits interesting features, such as the

[1]We thank the referee for pointing this out to us.

relation to root systems, which arise in our proof not once, but twice. (We refer the reader to [5] for the definition and classification of root systems.)

Just like Haemers, we aim to show that G is locally a union of cliques. However, once we arrive there, we do not stop, but rather use our unit vector setup to achieve an outright contradiction. In this sense, we also provide an alternative proof of the result of Dixmier and Zara.

2. Starting point

Suppose G is $srg(76, 21, 2, 7)$. Then the adjacency matrix of G has eigenvalues 21, 2, and -7 with multiplicities 1, 56, and 19, respectively. We focus on the 19-dimensional eigenspace corresponding to the eigenvalue -7. The cosine sequence for this eigenspace is $(1, -\frac{1}{3}, \frac{1}{9})$. This means that our graph G can be realized as a set of 76 unit vectors x_v, $v \in V(G)$, in the Euclidean space $\mathbb{R}^{19}$ such that $(x_u, x_v) = -\frac{1}{3}$ if the distinct vertices u and v are adjacent, and $(x_u, x_v) = \frac{1}{9}$, if they are not.

From now on we identify vertices of G with the corresponding unit vectors. Hence we simply write u and v in place of x_u and x_v.

3. Neighbourhood

Fix an arbitrary $u \in V(G)$. The subgraph induced on the 21 vertices in $G_1(u)$ is a union of cycles $C_1, \ldots, C_r$, since its degree λ is 2, where $G_l(u) = \{x \in V(G) : d(x, u) = l\}$, $l = 1, 2$. Let us slightly alter vectors $v \in G_1(u)$ to make them perpendicular to u. Namely, we set $\hat{v} := \frac{3}{2}v + \frac{1}{2}u$ for each $v \in G_1(u)$. Clearly, $(u, \hat{v}) = \frac{3}{2}(u, v) + \frac{1}{2}(u, u) = \frac{3}{2}(-\frac{1}{3}) + \frac{1}{2}1 = 0$, as desired. Also, for $v, w \in G_1(u)$, we have $(\hat{v}, \hat{w}) = \frac{9}{4}(v, w) + \frac{3}{4}(u, w) + \frac{3}{4}(v, u) + \frac{1}{4}(u, u) = \frac{9}{4}(v, w) - \frac{1}{4}$. Therefore,

$$(\hat{v}, \hat{w}) = \begin{cases} 2, & \text{if } v = w, \\ -1, & \text{if } v \text{ and } w \text{ are adjacent,} \\ 0, & \text{if they are not adjacent.} \end{cases}$$

Let $V_i := \langle \hat{v} \mid v \in V(C_i) \rangle$ be the subspace of $\mathbb{R}^{19}$ spanned by the vectors corresponding to the vertices of the ith cycle C_i in $G_1(u)$. It follows from the above inner product values that $u \perp V_i$ for all i and that $V_i \perp V_j$ for all $i \neq j$.

Lemma 3.1. *Let $V(C_i) = \{v_1, v_2, \ldots, v_t\}$. Then we have:*

(i) $\hat{v}_1 + \hat{v}_2 + \cdots + \hat{v}_t = 0$*; and*
(ii) $\dim V_i = t - 1$.

Proof. (i) Let $\hat{v} = \hat{v}_1 + \hat{v}_2 + \cdots + \hat{v}_t$. Then, for each j, we have that $(\hat{v}, \hat{v}_j) = 0$, since $\hat{v}_j$ itself contributes 2 to the sum, and its two neighbours contribute -1 each, while all the other vertices of C_i contribute naught. Therefore, $(\hat{v}, \hat{v}) = \sum_{j=1}^{t} (\hat{v}, \hat{v}_j) = 0$, proving that $\hat{v} = 0$.

(ii) Assuming that the vertices $v_1, v_2, \ldots, v_t$ appear in this order on the cycle C_i, let A_{t-1} be the Gram matrix of the vectors $\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{t-1}$. Then

$$A_{t-1} = \begin{pmatrix} 2 & -1 & 0 & 0 & . & 0 \\ -1 & 2 & -1 & 0 & . & 0 \\ 0 & -1 & 2 & -1 & . & 0 \\ . & . & . & . & . & . \\ 0 & . & . & -1 & 2 & -1 \\ 0 & . & . & . & -1 & 2 \end{pmatrix}.$$

Let d_{t-1} be the determinant of A_{t-1}. Viewing $r = t - 1$ as variable, we obtain the recursive relation $d_r = 2d_{r-1} - d_{r-2}$ by expanding the determinant along the bottom row. Taking into account that $d_1 = 2$ and $d_2 = 3$, we easily deduce that $d_r = r + 1 \neq 0$, and so $\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{t-1}$ are linearly independent. □

We included this proof for completeness; however, we need to mention that these facts are well known. Indeed, the matrix above is the Gram matrix of a basis from the root system of type A_{t-1}, and if we add the missing vector $\hat{v}_t$ then this gives the basis of the affine root system $\tilde{A}_{t-1}$.

We now focus on a vertex w from $G_2(u)$ and study the $\mu = 7$ neighbours of w in $G_1(u)$. Let s_i be the number of such neighbours on the cycle C_i.

Lemma 3.2. *The length t_i of C_i is a multiple of 3; namely, $t_i = 3s_i$.*

Proof. Let again $V(C_i) = \{v_1, v_2, \ldots, v_t\}$, where $t = t_i$, and $\hat{v} = \hat{v}_1 + \hat{v}_2 + \cdots + \hat{v}_t$.

Note that $(\hat{v}_j, w) = (\frac{3}{2}v_j + \frac{1}{2}u, w) = \frac{3}{2}(v_j, w) + \frac{1}{2}(u, w)$. If w is adjacent to v_j, this results in $-\frac{1}{2} + \frac{1}{18} = -\frac{4}{9}$, and otherwise, the result is $\frac{1}{6} + \frac{1}{18} = \frac{2}{9}$. Now consider the equality

$$0 = (0, w) = (\hat{v}, w) = (\hat{v}_1, w) + (\hat{v}_2, v) + \cdots + (\hat{v}_t, x).$$

Since w is adjacent to $s = s_i$ vertices and non-adjacent to $t - s$ vertices, we obtain from here that

$$0 = -\frac{4}{9}s + \frac{2}{9}(t - s),$$

which gives $t = 3s$, as claimed. □

4. Second layer

We alter the vertices in $G_2(u)$ in a similar way to make them perpendicular to u. For $w \in G_2(u)$, we set $\hat{w} := \frac{9}{4}w - \frac{1}{4}u$. Then $(\hat{w}, u) = \frac{9}{4}\frac{1}{9} - \frac{1}{4}1 = 0$, as claimed. Similarly, we compute, for $v, w \in G_2(u)$,

$$(\hat{v}, \hat{w}) = \begin{cases} 5, & \text{if } v = w, \\ -\frac{7}{4}, & \text{if } v \text{ adjacent to } w, \\ \frac{1}{2}, & \text{if } v \text{ is not adjacent to } w. \end{cases}$$

Finally, we also compute, and also in a very similar way, the inner products $(\hat{v}, \hat{w})$ for $v \in G_1(u)$ and $w \in G_2(u)$. These are:

$$(\hat{v}, \hat{w}) = \begin{cases} -1, & \text{if } v \text{ is adjacent to } w, \\ \frac{1}{2}, & \text{if } v \text{ is not adjacent to } w. \end{cases}$$

Recall that every vertex $w \in G_2(u)$ has seven neighbours in $G_1(u)$. Let us first describe the subgraph $M = M_w$ induced on these seven vertices.

Lemma 4.1. *Each connected component of M is of size 1 or 2.*

Proof. If xyz is a 2-path in M, with $x \neq z$, then $uxwz$ is a 4-cycle in $G_1(y)$, a contradiction with Lemma 3.2 with y in place of u. $\square$

If x is a size 1 component of M then the projection p_x of $\hat{w}$ to the 1-space spanned by $\hat{x}$ coincides with $\frac{(\hat{w},\hat{x})}{(\hat{x},\hat{x})}\hat{x} = -\frac{1}{2}\hat{x}$. Hence $(p_x, p_x) = \frac{1}{4}(\hat{x}, \hat{x}) = \frac{1}{2}$. If xy is a size 2 component of M then by symmetry the projection p_{xy} of $\hat{w}$ to the subspace spanned by $\hat{x}$ and $\hat{y}$ is a multiple of $d = \hat{x} + \hat{y}$. Note that $(d, d) = (\hat{x} + \hat{y}, \hat{x} + \hat{y}) = 2 - 1 - 1 + 2 = 2$ and $(\hat{w}, d) = (\hat{w}, \hat{x} + \hat{y}) = -1 - 1 = -2$. Hence $p_{xy} = \frac{(\hat{w},d)}{(d,d)} d = -d$, and so $(p_{xy}, p_{xy}) = (-d, -d) = (d, d) = 2$.

Projections corresponding to different components of M are orthogonal. Hence, if we have k components of size 2 and, correspondingly, $7 - 2k$ components of size 1 then the length of the projection of $\hat{w}$ to the subspace of V spanned by all $\hat{x}$ for $x \in M$, is $2k + \frac{1}{2}(7 - 2k) = 2k - k + \frac{7}{2} = k + \frac{7}{2}$. Since this must be at most $(\hat{w}, \hat{w}) = 5$, we conclude that $k = 0$ or 1.

Consider one of the cycles $C = C_i$ of length $t = t_i$ and $N = C \cap M$ consisting of $s = s_i$ vertices. We know that $t = 3s$. Let $U = V_i$, the subspace spanned by the vectors $\hat{v}$ for $v \in C$, and let p be the projection of $\hat{w}$ onto U.

Lemma 4.2. *We have $(p, p) \geq \frac{s}{2}$. Furthermore, $(p, p) = \frac{s}{2}$ if and only if the vertices of N are evenly spaced in C, containing every third vertex along C, and $p = -\frac{1}{2}\sum_{v \in N} \hat{v}$.*

Proof. Let p' be the projection of $\hat{w}$ to the subspace spanned by $\hat{v}$ for $v \in N$. Then certainly $(p, p) \geq (p', p')$ with equality holding if and only if $p = p'$. The graph induced on N consists of several components of M. The computation before the lemma shows that a component of size 1 contributes $\frac{1}{2}$ to (p', p') and a component of size 2 contributes $2 > 2 \cdot \frac{1}{2}$. Hence $(p, p) \geq (p', p') \geq s \cdot \frac{1}{2} = \frac{s}{2}$, as claimed. Furthermore, the equality only holds when every component is of size 1 and also $p = p' = \sum_{v \in N} p_v = -\frac{1}{2}\sum_{v \in N} \hat{v}$. In particular, N is an independent subset of C. Taking a vertex $x \in C \setminus N$, we see that $\frac{1}{2} = (\hat{w}, \hat{x}) = (p, \hat{x}) = (p', \hat{x}) = -\frac{1}{2}(\sum_{v \in N} \hat{v}, \hat{x}) = -\frac{1}{2}(-m)$, where m is the numbers of vertices of N adjacent to x. This shows that $m = 1$ for every $x \in C \setminus N$, and hence the graph induced by N is evenly spaced in C.

Conversely, if N is evenly spaced then $(p', \hat{x}) = \frac{1}{2} = (\hat{w}, \hat{x})$ for every $x \in C \setminus N$ and $(p', \hat{x}) = -\frac{1}{2}(\hat{x}, \hat{x}) = -\frac{1}{2} \cdot 2 = -1 = (\hat{w}, \hat{x})$ for every $x \in N$. Hence $p = p'$. $\square$

We now assume that $k = 1$ and focus on the cycle $C = C_i$ containing the only component of M of size 2. Without loss of generality, we may assume that $C = v_1 v_2 \cdots v_t$ and $v_1 v_2$ is the size 2 component of $N = C \cap M$.

Clearly, $t \geq 6$ and it follows from Lemma 3.1 that $t \leq 15$, since $\dim V \leq 18$. Also, it follows from Lemma 4.2 that the length (p, p) of the projection of $\hat{w}$ onto the subspace U corresponding to C is at most $5 - \frac{1}{2}(7 - s) = \frac{s+3}{2}$.

Lemma 4.3. *If w is adjacent to v_1 and v_2 then $\hat{w} \in V$, $p = -(\hat{v}_1 + \hat{v}_2) + \sum_{m=1}^{s-1} \frac{1}{2}(\hat{v}_{3m+1} + \hat{v}_{3m+2})$, and $(p, p) = \frac{s+3}{2}$. Furthermore, $N_m = C_m \cap M$ is evenly spaced in C_m for each $m \neq i$.*

Proof. The set of $t - 4 = 3s - 4$ vertices $\{v_4, v_5, \ldots, v_{t-1}\}$ consists of $s - 2$ vertices from N (neighbours of w) and $(3s - 4) - (s - 2) = 2(s - 1)$ other vertices (non-neighbours). Let us view the $s - 2$ neighbours as dividers splitting the non-neighbours into $s - 1$ connected parts R_j of size $r_1, r_2, \ldots, r_{s-1}$. Let $d = \hat{v}_1 + \hat{v}_2$ and, for $j = 1, \ldots, s - 1$, we set $d_j = \sum_{v \in R_j} \hat{v}$. We let U' be the subspace of U spanned by $d, d_1, \ldots, d_{s-1}$ and let p' be the projection of $\hat{w}$ onto U'. Then, clearly, $(p, p) \geq (p', p')$ and, since the vectors $d, d_1, \ldots, d_{s-1}$ are pairwise orthogonal and of length 2, we have that $p' = -d + \frac{1}{4}\sum_{j=1}^{s-1} r_j d_j$, which means that $(p', p') = 2 + \frac{1}{8}\sum_{j=1}^{s-1} r_j^2$. Hence, to find the minimum of the latter, we need to minimize $\sum_{j=1}^{s-1} r_j^2$ under the restriction that $\sum_{j=1}^{s-1} r_j = 2(s-1)$. Clearly, the minimum is achieved when all r_j are equal, that is when all r_j are equal to $\frac{2(s-1)}{s-1} = 2$. The minimum value (p', p') is, therefore, $2 + \frac{1}{8}(s-1)2^2 = 2 + \frac{s-1}{2} = \frac{s+3}{2}$.

Hence $\frac{s+3}{2} \geq (p, p) = (p', p') \geq \frac{s+3}{2}$. Clearly, this means that $p = p'$ is of length $\frac{s+3}{2}$, and so every part R_j is of size 2, which leads to the vectors in the statement of the lemma. Also for every cycle C_m other than C we must have the minimum length value $\frac{s_m}{2}$ and so the vertices $C_m \cap M$ must be evenly spaced in C_m. □

Let us adopt the following terminology: the vectors $d = \hat{v}_i + \hat{v}_{i+1}$ will be called *pairs*, while the edge $v_i v_{i+1}$ will be called the *base* of the pair d. Using these terms, p in the lemma above is the sum of the unique *minus-pair* $-(\hat{v}_1 + \hat{v}_2)$ and $s-1$ *half-pairs* $\frac{1}{2}(\hat{v}_{3m+1} + \hat{v}_{3m+2})$.

Lemma 4.4. *Every cycle C_i in $G_1(u)$ has length* 3.

Proof. Suppose, by contradiction, that $C = C_i = v_1 v_2 \ldots v_t$ has length $t \geq 6$. In $G_1(v_1)$, u is adjacent to v_t and v_2, which are not adjacent to each other. Hence $v_t u v_2$ is part of a cycle D in $G_1(v_1)$ of length at least 6. Let $w \neq u$ be the second neighbour of v_2 in D, and let $w' \neq v_2$ be the second neighbour of w in D. Note that w is adjacent to v_1 and v_2, and hence $\hat{w}$ is as in Lemma 4.3. In particular, $p = -(\hat{v}_1 + \hat{v}_2) + \sum_{j=1}^{s-1} \frac{1}{2}(\hat{v}_{3j+1} + \hat{v}_{3j+2})$ is the projection of $\hat{w}$ to the subspace $U = V_i$.

We obtain a contradiction by computing $(\hat{w}, \hat{w}')$. Since w and w' are adjacent vertices in $G_2(u)$ (note that w and w' are not adjacent to u, since D has length at least 6), the value of the inner product must be $-\frac{7}{4}$. On the other hand, we can estimate the value as follows. Recall that $\hat{w} \in V$ by Lemma 4.3 and so $\hat{w} = \sum_{j=1}^{r} p_j$, where p_j is the projection of $\hat{w}$ to the subspace V_j corresponding to the cycle C_j in $G_1(u)$. We already know $p = p_i$ and, by Lemmas 4.3 and 4.2, if $j \neq i$ then $p_j = -\frac{1}{2}\sum_{v \in N_j} \hat{v}$, where $N_j = M \cap C_j$ is evenly spaced in C_j. It follows that $(\hat{w}, \hat{w}') = \sum_{j=1}^{r} (p_j, \hat{w}')$. Clearly, w' is adjacent to v_1 but not to v_2. Hence $(-(\hat{v}_1 + \hat{v}_2), \hat{w}') = -(-1 + \frac{1}{2}) = \frac{1}{2}$. Consider now a half-pair $\frac{1}{2}(\hat{v}_{3m+1} + \hat{v}_{3m+2})$. If w' is adjacent to both v_{3m+1} and v_{3m+2} then $\hat{w}'$ is described as in Lemma 4.3 with the minus-pair base $-\ v_{3m+1}v_{3m+2}$. This means, however, that $v_1 v_2$ is the base of a half-pair for w'. Hence w' cannot be adjacent to v_1, a contradiction. Therefore, w' is adjacent to at most one of v_{3m+1} and v_{3m+2}. If w' is adjacent to one of these then $(\frac{1}{2}(\hat{v}_{3m+1} + \hat{v}_{3m+2}), \hat{w}') = \frac{1}{2}(-1 + \frac{1}{2}) = -\frac{1}{4}$. If w' is adjacent to neither of them then $(\frac{1}{2}(\hat{v}_{3m+1} + \hat{v}_{3m+2}), \hat{w}') = \frac{1}{2}(\frac{1}{2} + \frac{1}{2}) = \frac{1}{2}$. Hence the smallest possible value of $(p, \hat{w}')$ is $\frac{1}{2} + (s-1)(-\frac{1}{4}) = \frac{3}{4} - \frac{s}{4}$. In all $C_j \neq C$, w is adjacent to $7 - s$ vertices v, and for each such vertex, $\hat{v}$ appears in $\hat{w}$ with coefficient $-\frac{1}{2}$. If w' is adjacent to v then this gives contribution $-\frac{1}{2}(-1) = \frac{1}{2}$ to the value of $(\hat{w}, \hat{w}')$. If w' and v are not adjacent then the contribution is $-\frac{1}{2}\frac{1}{2} = -\frac{1}{4}$. Hence the smallest possible contribution from all vectors $\hat{v}$ appearing in $\hat{w}$, where $v \notin C$, is $(7-s)(-\frac{1}{4}) = -\frac{7}{4} + \frac{s}{4}$. Putting all of the above together, we conclude that $(\hat{w}, \hat{w}') \geq (\frac{3}{4} - \frac{s}{4}) + (-\frac{7}{4} + \frac{s}{4}) = -1$. This clearly is a contradiction since $(\hat{w}, \hat{w}') = -\frac{7}{4}$. □

5. Contradiction

Vertices and 4-cliques of G form a point-line geometry. It follows from Lemma 4.4 that every point lies in seven lines and then, using the parameters of G, it is easy to deduce that this geometry is a generalized quadrangle of order $(3,6)$, which cannot exist due to a theorem of Dixmier and Zara [2]. However, with the wealth of information that we have collected, we can achieve a quick contradiction without using [2].

Let $T = S^{\perp}$, where S is the span of the vectors in $\{u\} \cup G_1(u)$. That is, $S = \langle u \rangle \oplus V$. Since all cycles in $G_1(u)$ are of length 3, Lemma 3.1 shows that $\dim S = 1 + 7 \cdot 2 = 15$, and so $\dim T = 19 - 15 = 4$.

If $w \in G_2(u)$ then the projection of $\hat{w}$ onto V coincides with $-\frac{1}{2}\sum_{v \in M} \hat{v}$ and it has length $\frac{7}{2}$. It follows that the projection of $\hat{w}$ onto T has length $5 - \frac{7}{2} = \frac{3}{2}$. Let w° denote $\frac{2}{\sqrt{3}}$ times the projection of $\hat{w}$ onto T. Then $(w^{\circ}, w^{\circ}) = 2$. We now compute $(w^{\circ}, (w')^{\circ})$ for distinct $w, w' \in G_2(u)$.

If w and w' are adjacent then $(\hat{w}, \hat{w}') = -\frac{7}{4}$. Note that the edge ww' lies in a unique 4-clique and so w and w' have a unique common neighbour in $G_1(u)$. It follows that if p and p' are the projections of $\hat{w}$ and $\hat{w}'$ onto V then $(p, p') = \frac{1}{2} + 6(-\frac{1}{4}) = -1$. Hence $(w^{\circ}, (w')^{\circ}) = \frac{4}{3}(-\frac{7}{4} + 1) = -1$.

If w and w' are non adjacent then $(\hat{w}, \hat{w}') = \frac{1}{2}$. Let β the the number of common neighbours of w and w' in $G_1(u)$. Then $(p, p') = \beta\frac{1}{2} + (7 - \beta)(-\frac{1}{4}) = -\frac{7}{4} + \frac{3\beta}{4}$ and $(w^{\circ}, (w')^{\circ}) = \frac{4}{3}(\frac{1}{2} - (-\frac{7}{4} + \frac{3\beta}{4})) = 3 - \beta$.

To summarize, if $w, w' \in G_2(u)$ and $\beta = |G_1(u) \cap G_1(w) \cap G_1(w')|$ then

$$(w^{\circ}, (w')^{\circ}) = \begin{cases} 2, & \text{if } w = w', \\ -1, & \text{if } w \text{ and } w' \text{ are adjacent}, \\ 3 - \beta, & \text{if and } w \text{ and } w' \text{ are non-adjacent}. \end{cases}$$

Clearly, it follows that $1 \leq \beta \leq 5$.

Notice that the above values of inner products mean that all vectors w°, $w \in G_2(u)$ are contained in a root system in T. Indeed, since all values are integers, the vectors w° span an integral lattice and all vectors of length 2 from that lattice form a simply laced root system.

The largest simply laced root system in dimension 4 is D_4 having 24 vectors splitting into 12 pairs of opposite roots. Since $|G_2(u)| = 54 > 4 \cdot 12$, we must have five vertices $\{w_1, \ldots, w_5\}$ such that all vectors $(w_i)^{\circ}$ belong to the same pair of opposite roots.

Lemma 5.1. *There is no strongly regular graph with parameters* $(76, 21, 2, 7)$.

Proof. Consider the five vertices $\{w_1, \ldots, w_5\}$ such that all vectors $(w_i)^{\circ}$ are in the same pair of opposite roots $\{r, -r\}$. Without loss of generality, let the first $s \geq 3$ of the vectors $(w_i)^{\circ}$ be r and the remaining $5 - s$ be $-r$.

From the calculations above, the vertices w_i are pairwise non-adjacent. Furthermore, if $(w_i)^{\circ} = (w_j)^{\circ}$ then w and w' have exactly one common neighbour in $G_1(u)$, and if $(w_i)^{\circ} = -(w_j)^{\circ}$ then w_i and w_j have exactly five common neighbours in $G_1(u)$.

If $s \neq 5$ then $(w_5)^{\circ} = -r$ and so both w_1 and w_2 must have five neighbours among the seven vertices from $M = G_1(u) \cap G_1(w_5)$. However, this means that w_1 and w_2 have at least two common neighbours in M; a contradiction. Therefore, $s = 5$ and any two vectors w_i and w_j share a unique common neighbour in $G_1(u)$.

For the final contradiction, note that there are at most three 3-cycles in $G_1(u)$ where w_1, w_2, and w_3 may have common neighbours. It follows that there are at least four

3-cycles C where w_1, w_2 and w_3 are adjacent to the three distinct vertices of C. This means that, in each of these 3-cycles C, the vertex w_4 would have the same neighbour as one of the vertices w_1, w_2, and w_3. Clearly, this means that w_4 must share at least two common neighbours with one of the vectors w_1, w_2, or w_3; a contradiction. □

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