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A Fully-Coupled Bedload Sediment Transport Model Based on a Two-Dimensional Modified Wave Propagation Algorithm

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ABSTRACT:

In this paper, an extension of a second-order Godunov-type wave propagation algorithm is presented for modelling two-dimensional morphodynamic problems using a coupled approach. In this solution, the two-dimensional shallow water equations (SWEs) and bedload sediment mass balance laws are expressed in a coupled form. The proposed numerical solver treats the source term including the bedload variations as well as the friction terms within the flux-differencing of the finite-volume neighbouring cells. In order to solve the morphodynamic system in two-dimensions, the dimensional-splitting method is utilised. To consider the bedload sediment discharge within the Exner equation, the Smart and Meyer-Peter & Müller formulae are adopted. To verify the capability of the extended wave propagation solver in dealing with different flow regimes several numerical test cases are investigated. The numerical results show that for all examined cases, excellent agreement is achieved between the numerical results and the exact solutions and experimental data, confirming the effectiveness of the method."

*Keywords***:** Two-dimensional morphodynamic system, Bedload sediment transport, Coupled solution, Wave propagation algorithm, Flux-wave approach.

1. Introduction

Accurate modelling of free-surface flows, sediment bed profiles and their interactions is essential for morphodynamic studies such as evaluating river bed variation, beach profile predictions and river restoration planning and design. Numerical models used for such purposes should be capable of accurately estimating bed bathymetry variations as well as fluid depth changes. To accomplish these attributes, morphodynamic models used for modelling bedload sediment are generally composed of a hydrodynamic component that encompasses the two-dimensional (2D) shallow water equations (SWEs), and a bedload transport model for

approximating sediment propagation. These set of equations form a nonlinear hyperbolic system which can be solved by two different approaches. The first method is the uncoupled solution, in which, the hydrodynamic and the bedload sediment transport equations are solved separately (Cordier, Le, & Morales de Luna, 2011; Wu, 2007). However, one major drawback of this scheme is that it only applicable in the cases of weak or mild sediment transport and surface wave interactions, which is mainly due to the assumption of a constant total depth(Canestrelli, Dumbser, Siviglia, & Toro, 2010; Hudson & Sweby, 2005). Commercial packages such as MIKE 21 (Danish Hydraulic Institute, 2017) Delf3D (Deltares, 2017), TELEMAC-MASCARET (TELEMAC, 2017) and SRH-2D (USBR, 2017) generally link the 2D shallow water solver to a bedload sediment model described by an advection-diffusion sediment model or the Exner equation using the uncoupled strategy. In SRH-2D the Exner equation is first solved and the resulting bed profile update is used as a source term for the 2D shallow water equations. In Delf3D-FLOW the 2D shallow water equations are utilised for the hydrodynamic part and a 2D advection-diffusion equation which mainly uses Van-Rijn's sediment formulation (Rijn, 1993) is used to calculate the bed profile update. TELEMAC-MASCARET employs the SISYPHE package to evaluate the morphological process, and the calculated results are then used into the depth-averaged shallow water component of TELEMAC-2D.

The second and alternative method is the coupled solution which solves the entire governing equations in a coupled form at each time step, and hence, is generally more stable (Cordier, Le, & Morales de Luna, 2011; Holly & Rahuel, 1990; Lyn & Altinakar, 2002; Saiedi, 1997). In order to solve the morphodynamic system, several numerical methods have been developed over the years which are mostly developed based on Godunov-type finite volume method (see reviews by LeVeque (2002) and Toro (1997)).

Godunov-type methods have recently been applied for the calculation of 2D morphodynamic systems based on upwind methods. For instance, Delis & Papoglou (2008) applied an upwind relaxation scheme for the solution of bedload sediment transport using the coupled solution. To achieve high-order accuracy, the MUSCL-TVD scheme was employed in their work. Serrano-Pacheco, Murillo,&Garcia-Navarro (2012) modelled 2D shallow water flow with a mobile bed using a partially coupled first-order Godunov-type finite volume method using the HLLC approximate Riemann solver (Toro, 2001). Soares-Frazão & Zech (2011) used a coupled system based on the HLLC scheme and chose a different pair of eigenvectors for approximating the flux. In another work, Rosatti & Fraccarollo (2006) employed a well-balanced method with a new strategy for the treatment of non-conservative fluxes. This work was later modified by Murillo & García-Navarro (2010) who defined a novel approximate coupled Jacobian matrix (CJM) method using a triangular mesh.

The main purpose of this paper is to extend a second-order Godunov-type wave propagation algorithm for solving 2D morphodynamic systems. To the best of the authors' knowledge, the wave propagation algorithm has yet to be extended for modelling bedload sediment transport dynamics. The proposed solver generalizes the 2D flux-wave formula introduced by Mahdizadeh, Stansby & Rogers (2011, 2012) for the solution of morphodynamic problems. To solve the problem in the 2D a dimensional-splitting method which solves each Riemann problem in one-dimension is used. In comparison to other accurate and novel coupled

morphodynamic solver such as CJM, the proposed the method provides identical results although use more straightforward and simpler formulations. The rest of this paper is structured as follows: In the next section, the 2D hydrodynamic system coupled to a bedload sediment equation is presented. Next the 2D morphodynamic system is defined and the associated eigenstructure calculations are explained in detail. Then, the 2D wave propagation algorithm and the flux-wave approach are introduced for 2D bedload sediment transport modelling. Finally, the performance of the proposed numerical solver is investigated over several 1D and 2D problems by comparing its results with exact solutions and/or experimental data.

2. Governing equations

The 2D hydrodynamic system coupled to the bedload sediment transport equation may be written as:

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(q_x)+\frac{\partial}{\partial y}(q_y)=0,$$

(1a)

$$\frac{\partial q_x}{\partial t}+\frac{\partial}{\partial x}\left(\frac{q_x^2}{h}+\frac{1}{2}gh^2\right)+\frac{\partial}{\partial y}\left(\frac{q_x q_y}{h}\right)=-gh\frac{\partial z_b}{\partial x}-\frac{\tau_{fx}}{\rho},$$

(1b)

$$\frac{\partial q_y}{\partial t}+\frac{\partial}{\partial y}\left(\frac{q_y^2}{h}+\frac{1}{2}gh^2\right)+\frac{\partial}{\partial x}\left(\frac{q_x q_y}{h}\right)=-gh\frac{\partial z_b}{\partial y}-\frac{\tau_{fy}}{\rho},$$

(1c)

$$\frac{\partial z_b}{\partial t}+\eta\frac{\partial q_{bx}}{\partial x}+\eta\frac{\partial q_{by}}{\partial y}=0.$$

(1d)

where h is the water depth and z_b shows the bedload sediment thickness as shown in Fig. 1, g is acceleration due to gravity, t is time, $q_x = hu$ and $q_y = hv$ are discharges per unit width, u and v are depth-average velocities in the x- and y directions, q_{bx} and q_{by} are the bedload sediment discharges per unit width in the x- and y- directions respectively, $\eta = (1-p)^{-1}$ where p is the porosity of the sediment layer, and finally τ_{fx} and τ_{fy} are bed shear stresses in the orthogonal and horizontal directions respectively which can be obtained as follows:

$$\tau_{fx}=\frac{1}{2}C_f\rho u\sqrt{u^2+v^2}\quad\text{and}\quad\tau_{fy}=\frac{1}{2}C_f\rho v\sqrt{u^2+v^2},$$

(2)

where ρ denotes the water density and C_f is the bed friction coefficient which can be expressed based on Manning's coefficient as $C_f = 2gn^2/h^{1/3}$ where n is Manning's roughness coefficient. The sediment discharge can be calculated by (Grass, 1981):

$$q_{bx}=A_g u(u^2+v^2),\quad q_{by}=A_g v(u^2+v^2),$$

(3)

where q_{bx} and q_{by} are sediment discharges in the x- and y-directions respectively, A_g is the interaction parameter which mainly depends on the sediment properties and is obtained through experimental data and can be expressed as $A_g = K.\chi$ where χ is described based on two different sediment transport formulae as summarized in Table (1) and :

$$K = \frac{g^{1/2} n^3}{(G_s - 1) h^{1/2}},$$ (4)

where $G_s = \rho_s / \rho$ is the relative density and ρ_s is the sediment density. In Table (1), θ_c is the critical Shields parameter and θ denotes the dimensionless bed shear stress defined as:

$$\theta = \frac{n^2}{(G_s - 1) d_m h^{1/3}} (u^2 + v^2),$$ (7)

where d_m is the average diameter of bed materials.

In Smart's formula (Smart, 1984), described in Eq. (6), d_{30} and d_{90} correspond to diameters where 30% and 90% of the bed material sample is finer by weight. S_0 is the bed slope obtained as $S_{0x} = -\partial z_b / \partial x$ and $S_{0y} = -\partial z_b / \partial y$ in x- and y-directions, respectively and finally, θ_s is Smart's critical Shield parameter (Juez, Murillo, & García-Navarro, 2013) which can be obtained from:

$$\theta_s = \theta_c \cos\varphi \left(1 - \frac{\tan\varphi}{\tan\psi} \right),$$ (8)

where ψ is the angle of repose for saturated bed materials, and φ is the angle of the bed slope.

It should be noted that for the Smart and Meyer-Peter & Müller sediment bedload formulations presented above, the effect of flow resistance which may slightly affect the sediment movement for uniform bedload profiles, has considered negligible. For more details see (Recking, Frey, Paquier, Belleudy, & Champagne, 2008).

3. The 2D coupled form of the morphodynamic system

The 2D coupled form of the morphodynamic system presented in Eq. (1) can be re-written in the following form:

$$\mathbf{U}_t + \mathbf{F}(U)_x + \mathbf{G}(U)_y = \mathbf{S}(\mathbf{U}, x),$$ (9)

where

$$\mathbf{U} = \begin{bmatrix} h \\ q_x \\ q_y \\ z_b \end{bmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{bmatrix} q_x \\ \frac{q_x^2}{h} + 1/2 g h^2 \\ \frac{q_x q_y}{h} \\ \eta q_{bx} \end{bmatrix}, \mathbf{G}(\mathbf{U}) = \begin{bmatrix} q_y \\ \frac{q_x q_y}{h} \\ \frac{q_y^2}{h} + 1/2 g h^2 \\ \eta q_{by} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 0 \\ -gh\frac{\partial z_b}{\partial x} - \frac{\tau_{fx}}{\rho} \\ -gh\frac{\partial z_b}{\partial y} - \frac{\tau_{fy}}{\rho} \\ 0 \end{bmatrix}.$$ (10)

If a sediment discharge formula in the form of Eq. (3) is used, the associated Jacobian matrices for the above system become:

$$\mathbf{F}'(\mathbf{U}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ gh - u^2 & 2u & 0 & 0 \\ -uv & v & u & 0 \\ \sigma_x & \delta_x & \zeta_x & 0 \end{bmatrix}, \quad \mathbf{G}'(\mathbf{U}) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -uv & v & u & 0 \\ gh - v^2 & 0 & 2v & 0 \\ \sigma_y & \delta_y & \zeta_y & 0 \end{bmatrix}, \tag{11}$$

where the values of σ_x, δ_x and ζ_x for the case of a constant interaction parameter, A_g, are computed as:

$$\sigma_x = -\frac{3A_g \eta u |V|^2}{h}, \; \delta_x = \frac{A_g \eta \left(3u^2 + v^2\right)}{h}, \quad \zeta_x = \frac{2A_g \eta uv}{h}. \tag{12}$$

where $|V| = \sqrt{u^2 + v^2}$ and consequently σ_y and δ_y and ζ_y for the Jacobian matrix in the y-direction become:

$$\sigma_y = -\frac{3A_g \eta v |V|^2}{h}, \; \delta_y = \frac{2A_g \eta uv}{h}, \quad \zeta_y = \frac{A_g \eta \left(3v^2 + u^2\right)}{h}. \tag{13}$$

As it can be observed, the obtained Jacobian matrices are singular, and so, cannot be used for eigenvector calculations. The following product rule can be implemented to revise the matrix singularity:

$$h\frac{\partial z}{\partial x} = \frac{\partial(z_b h)}{\partial x} - z_b \frac{\partial h}{\partial x}, \quad h\frac{\partial z}{\partial y} = \frac{\partial(z_b h)}{\partial y} - z_b \frac{\partial h}{\partial y}. \tag{14}$$

This implantation results in fluxes and source terms in the forms of:

$$\mathbf{F}(\mathbf{U}) = \begin{bmatrix} q_x \\ \dfrac{q_x^2}{h} + 1/2 gh^2 + ghz_b \\ \dfrac{q_x q_y}{h} \\ \eta q_{bx} \end{bmatrix}, \mathbf{G}(\mathbf{U}) = \begin{bmatrix} q_y \\ \dfrac{q_x q_y}{h} \\ \dfrac{q_y^2}{h} + 1/2 gh^2 + ghz_b \\ \eta q_{by} \end{bmatrix}, \mathbf{S} = \begin{bmatrix} 0 \\ gh_x z_b - \dfrac{\tau_{fx}}{\rho} \\ gh_y z_b - \dfrac{\tau_{fy}}{\rho} \\ 0 \end{bmatrix}. \tag{15}$$

and now the related Jacobian matrices become:

$$\mathbf{F}'(\mathbf{U}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ g(h + z_b) - u^2 & 2u & 0 & gh \\ -uv & v & u & 0 \\ \sigma_x & \delta_x & \zeta_x & 0 \end{bmatrix}, \quad \mathbf{G}'(\mathbf{U}) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -uv & v & u & 0 \\ g(h + z_b) - v^2 & 0 & 2v & gh \\ \sigma_y & \delta_y & \zeta_y & 0 \end{bmatrix}. \tag{16}$$

The first three associated eigenvalues for Jacobian matrices $\mathbf{F}'(\mathbf{U})$ and $\mathbf{G}'(\mathbf{U})$ are obtained by solving the following polynomial equation:

$$P(\lambda) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3. \tag{17}$$

where vector $[a_1, a_2, a_3]^T$ can be computed by the following equations in the x and y directions:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}^F = \begin{bmatrix} -2u \\ u^2 - g(h + z_b + h\delta_x) \\ -gh(\sigma_x + \zeta_x v) \end{bmatrix}, \quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}^G = \begin{bmatrix} -2v \\ v^2 - g(h + z_b + h\zeta_y) \\ -gh(\sigma_y + \delta_y u) \end{bmatrix}. \tag{18}$$

Therefore, the roots of the corresponding polynomials are:

$$\lambda_1 = 2\sqrt{-Q} \cos(\frac{1}{3}\mu) - \frac{1}{3}a_1,$$

$$\lambda_2 = 2\sqrt{-Q} \cos(\frac{1}{3}(\mu + 2\pi)) - \frac{1}{3}a_1, \tag{19}$$

$$\lambda_3 = 2\sqrt{-Q} \cos(\frac{1}{3}(\mu + 4\pi)) - \frac{1}{3}a_1.$$

where the values of Q, μ and R can be calculated as:

$$Q = \frac{1}{9}\left(3a_2 - a_1^2\right), \quad R = \frac{1}{54}\left(9a_1a_2 - 27a_3 - 2a_1^3\right), \quad \mu = a\cos\left(R/\sqrt{-Q^3}\right). \tag{20}$$

It can be proved that for the Grass-type equation (Grass, 1981) used in this work, the roots of the polynomials are always real (Castro Díaz, Fernández-Nieto, & Ferreiro, 2008; Hudson & Sweby, 2005). The fourth eigenvalues are given as:

$$\lambda_4^F = u, \quad \lambda_4^G = v \tag{21}$$

hence, the corresponding eigenvectors for the eigenvalues expressed in Eq. (16) can be found as:

$$\mathbf{r}_{k=1,2,3}^F = \begin{bmatrix} 1 \\ \lambda_k^F \\ v \\ \dfrac{u^2 - g(h + z_b) + (\lambda_k - 2u)\lambda_k}{gh} \end{bmatrix}, \quad \mathbf{r}_{k=1,2,3}^G = \begin{bmatrix} 1 \\ u \\ \lambda_k^G \\ \dfrac{v^2 - g(h + z_b) + (\lambda_k - 2v)\lambda_k}{gh} \end{bmatrix}. \tag{22}$$

and the eigenvectors related to the fourth eigenvalues where $\zeta_x \neq 0$ and $\zeta_y \neq 0$ is obtained as:

$$\mathbf{r}_{k=4}^F = \begin{bmatrix} 1 \\ u \\ -\dfrac{\sigma_x + \delta_x u}{\zeta_x} \\ -\dfrac{1}{h}(h + z_b) \end{bmatrix}, \quad \mathbf{r}_{k=4}^G = \begin{bmatrix} 1 \\ -\dfrac{\sigma_y + \delta_y v}{\zeta_y} \\ v \\ -\dfrac{1}{h}(h + z_b) \end{bmatrix}. \tag{23}$$

and in the case of $\zeta = 0$, the corresponding fourth eigenvector becomes:

6

$$\mathbf{r}_{k=4}^{F} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{r}_{k=4}^{G} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{24}$$

In order to accurately consider the effect of a non-constant interaction parameter, A_g, which is variable in time and space, into the eigenvector computations, the derivatives of the bedload sediment formula containing A_g should be also considered within the Jacobian matrix in both *x*- and *y*- directions. Table (2) shows the values of $(\sigma_x, \delta_x, \zeta_x)^T$ and $(\sigma_y, \delta_y, \zeta_y)^T$ for the Smart and Meyer-Peter & Müller bedload sediment discharges where the value of ω used in the Meyer-Peter & Müller bedload sediment formulation is given by:

$$\omega = \sqrt{1 - \frac{d_m (G_s - 1) h^{1/3} \theta_c}{n^2 |V|^2}}. \tag{25}$$

As can be observed in Table (2), theses derivatives contain rather complex, but necessary mathematical expressions for avoiding the miscalculation of the eigenvalues and eigenvectors of the Jacobian matrix, eventually leading to the accurate estimation of the bedload sediment movement. This is further discussed in the numerical results section.

4. The 2D wave propagation algorithm

To solve the 2D morphodynamic system described above, the Godunov-type wave propagation algorithm firstly introduced by LeVeque (1998, 2002) is used, which can be defined as:

$$\begin{aligned} \mathbf{U}_{ij}^{n+1} = \mathbf{U}_{ij}^{n} &- \frac{\Delta t}{\Delta x} \left(\mathcal{A}^{+} \Delta \mathbf{U}_{i-1/2,j} + \mathcal{A}^{-} \Delta \mathbf{U}_{i+1/2,j} \right) - \frac{\Delta t}{\Delta y} \left(\mathcal{B}^{+} \Delta \mathbf{U}_{i,j-1/2} + \mathcal{B}^{-} \Delta \mathbf{U}_{i,j+1/2} \right) \\ &- \frac{\Delta t}{\Delta x} \left(\tilde{\mathbf{F}}_{i+1/2,j} - \tilde{\mathbf{F}}_{i-1/2,j} \right) - \frac{\Delta t}{\Delta y} \left(\tilde{\mathbf{G}}_{i,j+1/2} - \tilde{\mathbf{G}}_{i,j-1/2} \right), \end{aligned} \tag{26}$$

where $\mathbf{U}_{ij}^{n}$ is the vector of unknowns at time $t = n\Delta t$ for cell $C_{ij} \in [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]$ in the finite volume method, and $\mathbf{U}_{ij}^{n+1}$ is the updated version of the vector of unknowns at the next time step. $\mathcal{A}^{\pm} \Delta \mathbf{U}_{i\pm1/2,j}$ and $\mathcal{B}^{\pm} \Delta \mathbf{U}_{i,j\pm1/2}$ are called the left- and right-going fluctuations for the *x*- and *y*-directions and can be obtained by solution of the Riemann problem at cell interfaces $x_{i\pm1/2}$ or $y_{i\pm1/2}$, respectively. The terms $\tilde{\mathbf{F}}_{i\pm1/2,j}$ and $\tilde{\mathbf{G}}_{i,j\pm1/2}$ are flux correction terms utilised to obtain second-order accuracies with different choice of total variation diminishing (TVD) limiters (LeVeque, 1998, 2002). If $\tilde{\mathbf{F}} = \tilde{\mathbf{G}} = 0$, then the first-order Godunov-type method is obtained. The right and left-going fluctuations, $\mathcal{A}^{\pm} \Delta \mathbf{U}_{i\pm1/2,j}$ and $\mathcal{B}^{\pm} \Delta \mathbf{U}_{i,j\pm1/2}$ at each cell-interface for the morphodynamic system can be calculated based on the flux-wave approach.

However, Eq. (26) is not fully second-order accurate as the cross-derivative terms U_{xy} are not considered into the calculations which will be addressed in the next section.

5. Flux-wave method for 2D morphodynamic systems

The Flux-wave formula was originally introduced by Bale, Leveque, Mitran,&Rossmanith (2002), and later developed by Mahdizadeh, Stansby & Rogers (2011,2012) for 1D and 2D propagation over dry-state. For solving morphodynamic problems, compared to 2D SWEs, another wave with respect to the sediment bed is added to the computations. The wave propagation algorithm for a 2D problem is generally obtained by the dimensional splitting scheme which solves the 1D problem, in each direction. For instance for the 2D morphodynamic system, $U_t + F(U)_x = S_1$ is solved using the flux-wave approach to compute $\mathcal{A}^{\pm}\Delta U_{i\pm1/2,j}$ (LeVeque, 2002). The flux-wave approach for a 1D problem can be expressed as:

$$\mathbf{F}(\mathbf{U}_i) - \mathbf{F}(\mathbf{U}_{i-1}) - \mathbf{S}_1 \Delta x = \sum_{k=1}^{M_w} \xi_{k,i-1/2}, \tag{27}$$

where $\xi_{k,i-1/2}$ is called the flux-wave, which is obtained by multiplying the constant coefficient $\beta_{k,i-1/2}$ by the eigenvector in the form of (23) or (24), say, $\xi_{k,i-1/2} = \beta_{i-1/2}\mathbf{r}_{k,i-1/2}^F$ and M_w denotes the number of waves and k implies the wave number, which for the current morphodynamic system (Eq. 1) is equal to four. The fluxes and the source term in the x-direction become:

$$\mathbf{F}(\mathbf{U}) = \left(q_x \quad \frac{q_x^2}{h} + 1/2gh^2 + ghz_b \quad \frac{q_x q_y}{h} \quad \eta q_{bx} \right)^T, \mathbf{S}_1 = \left(0 \quad gh_x z_b - \frac{\tau_{fx}}{\rho} \quad 0 \quad 0 \right). \tag{28}$$

To express the flux-wave formula, first, the differences between neighbouring fluxes for the cells and the source term, i.e. left-hand side of Eq. (27), are presented as a vector $\Delta \mathbf{F}$:

$$\Delta \mathbf{F} = \begin{bmatrix} \Delta F_1 \\ \Delta F_2 \\ \Delta F_3 \\ \Delta F_4 \end{bmatrix} = \begin{bmatrix} q_{x(i)} - q_{x(i-1)} \\ \left(\frac{q_{x(i)}^2}{h_i} + 1/2gh_i^2 + gh_i z_{bi} \right) - \left(\frac{q_{x(i-1)}^2}{h} + 1/2gh_{i-1}^2 + gh_{i-1}z_{bi-1} \right) - \left(\frac{z_{bi} + z_{bi-1}}{2} \right)(h_i - h_{i-1}) + \frac{\tau_{fx(i)} + \tau_{fx}}{2\rho} \\ \frac{q_{x(i)}q_{y(i)}}{h_i} - \frac{q_{x(i-1)}q_{y(i-1)}}{h_{i-1}} \\ \eta q_{bx(i)} - \eta q_{bx(i-1)} \end{bmatrix} \tag{29}$$

where i and i-1 are the left and right states of the cell interface i-1/2. The vector of $\Delta \mathbf{F}$ is then equated to the summation of flux-waves, $\sum_{k=1}^{M_w} \xi_{k,i-1/2}$, which leads to the following system of equations if $\zeta_x \neq 0$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & u \\ v & v & v & \kappa \\ s_1 & s_2 & s_3 & \varepsilon \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} \Delta F_1 \\ \Delta F_2 \\ \Delta F_3 \\ \Delta F_4 \end{bmatrix}, \tag{30}$$

and in the case $\zeta_x = 0$, the above system becomes:

$$
\begin{bmatrix} 1 & 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & \lambda_3 & 0 \\ v & v & v & 1 \\ s_1 & s_2 & s_3 & 0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} \Delta F_1 \\ \Delta F_2 \\ \Delta F_3 \\ \Delta F_4 \end{bmatrix},
\tag{31}
$$

where s_k, μ and σ can be calculated through the following equations:

$$
s_k = \frac{\bar{u}^2 - g(\bar{h} + z_b) + (\lambda_k - 2\bar{u})\lambda_k}{g\bar{h}}.
\tag{32}
$$

$$
\kappa = -\frac{\sigma_x + \delta\bar{u}}{\zeta_x}, \quad \varepsilon = -\frac{1}{\bar{h}}(\bar{h} + z_b).
\tag{33}
$$

The average velocities, $\bar{u}$ and $\bar{v}$, are then computed using a Roe speed (Roe, 1981):

$$
\bar{u} = \frac{\sqrt{h_{i-1}}u_{i-1} + \sqrt{h_i}u_i}{\sqrt{h_{i-1}} + \sqrt{h_i}} \text{ and } \bar{v} = \frac{\sqrt{h_{i-1}}v_{i-1} + \sqrt{h_i}v_i}{\sqrt{h_{i-1}} + \sqrt{h_i}}
\tag{34}
$$

where $\bar{h}$ is the average fluid depth, evaluated as $\bar{h} = (h_i + h_{i-1})/2$. By solving the system of equations given in (30) and (31), the obtained β_k coefficients can be used to calculate the left and right-going fluctuations $\mathcal{A}^{\pm}\Delta\mathbf{U}_{i\pm1/2,j}$ based on the following equations (Bale, Leveque, Mitran, & Rossmanith, 2002; Mahdizadeh, Stansby, & Rogers, 2012):

$$
\begin{aligned}
\mathcal{A}^-\Delta\mathbf{U}_{i-1/2,j} &= \sum_{k:\lambda_k<0} \xi_{k,i-1/2} = \beta_{k,i-1/2}\mathbf{r}^F_{k,i-1/2}, \\
\mathcal{A}^+\Delta\mathbf{U}_{i-1/2,j} &= \sum_{k:\lambda_k>0} \xi_{k,i-1/2} = \beta_{k,i-1/2}\mathbf{r}^F_{k,i-1/2}.
\end{aligned}
\tag{35}
$$

Equation systems (30) and (31) can be solved using any direct solver, in this work the LU decomposition with partial pivoting (Press, Teukolsky, Vetterling, & Flannery, 1992) is utilised. Similarly, the left and right-going fluctuations in the *y*-direction can be evaluated by solving $\mathbf{U}_t + \mathbf{G}(\mathbf{U})_y = \mathbf{S}_2$ where the flux and source terms are defined as:

$$
\mathbf{G}(\mathbf{U}) = \left(q_y \quad \frac{q_x q_y}{h} \quad \frac{q_y^2}{h} + 1/2gh^2 + ghz_b \quad \eta q_{by} \right)^T, \mathbf{S}_2 = \left(0 \quad 0 \quad gh_y z_b - \frac{\tau_{fy}}{\rho} \quad 0 \right).
\tag{36}
$$

To consider the effect of a non-constant sediment discharge coefficient, A_g, into the flux calculations for the defined modified flux wave approach, it is only required that each left and right states of the sediment discharges are given based on left and right states of the fluid depth:

$$q_{bx(i)} = \chi(h_i) K(h_i) u_i (u_i^2 + v_i^2),$$
$$q_{bx(i-1)} = \chi(h_{i-1}) K(h_{i-1}) u_{i-1} (u_{i-1}^2 + v_{i-1}^2). \tag{37}$$

Eq. (37) is an important condition for the sediment discharge formulation developed upon a non-constant value of A_g as it provides the accurate estimation for the bedload materials. If the above condition is not satisfied, then, an inaccurate prediction of the sediment bed is obtained, which leads to excessive numerical diffusion. This will be later addressed in the numerical results section.

As mentioned earlier, the wave propagation algorithm defined in Eq. (26) is not fully second-order accurate as the cross-derivative terms are not added into the second-order correction flux terms $\tilde{\boldsymbol{F}}_{i\pm1/2,j}$ and $\tilde{\boldsymbol{G}}_{i,j\pm1/2}$. To account for these terms, in this work, another Riemann problem in the orthogonal direction is also solved at each time step. The calculation of cross-derivative terms for the wave-propagation algorithm has been fully described in (LeVeque, 2002; Mahdizadeh, 2010). The numerical scheme explained in this paper employs a modified version of the flux-wave formula for 2D morphodynamic systems. Additionally, the method takes advantage of a non-constant A_g for the eigenvector calculations.

5.1. Stability Condition

The stability condition for the introduced coupled flux wave solver developed based on the flux-wave approach can be determined by the Courant-Friedrichs-Lewy condition (CFL) (Courant, Friedrichs, & H.Lewy, 1967) for time step Δt. For the dimensional-splitting approach used here, it is necessary to apply this condition for each 1D morphodynamic system in each direction:

$$\Delta t = \min(\Delta t_x, \Delta t_y), \qquad \text{where,} \qquad \begin{aligned} \Delta t_x &= \frac{\text{CFL}}{\max(\lambda_1^F, \lambda_2^F, \lambda_3^F, \lambda_4^F)}, \\ \Delta t_y &= \frac{\text{CFL}}{\max(\lambda_1^G, \lambda_2^G, \lambda_3^G, \lambda_4^G)}, \end{aligned} \tag{38}$$

where CFL number takes the values between zero and one.

6. Numerical results

This section discusses several test cases that were adopted from the literature, and used to examine the validity of the proposed coupled flux wave solver (CFW). First, a radial dam-break test over a fixed bed was examined. Second, the ability of CFW solver in simulating a test case containing shocks and rarefaction wave scenarios over a mobile bed was investigated. This was followed by studying dam break waves over a step type sediment hump. Then, a dam-failure by overtopping was simulated. Next, the 2D propagation of a sediment layer was modelled. Finally, the evolution of a canonical sediment dune was investigated. For all simulations demonstrated in this paper, a second-order high-resolution term with the choice of a Montonised-Centre (MC) limiter (LeVeque, 2002) was employed. Additionally, the numerical computations were performed using an in-house FORTRAN code on an Intel Core (i7-4790) 3.6 GHz processor with 16GB of

RAM. The Courant number and number of finite volume cells are reported separately for each individual test case.

6.1. *Radial dam-break test case*

The first considered test case, was the instantaneous failure of a radially symmetric dam break consisting of a circular water column 20m in diameter, initially filled with 2m of water. The water waves created by the dam failure propagated over a wet state outside the column with an initial depth of 0.5m. This test case is important as it examines the capability of the defined solver in preserving radial symmetry for 2D problems.

Figure (2) demonstrates the numerical results obtained by the proposed second-order CFW solver over a rectangular computational domain containing 256×256 computational cells with a CFL number equal to 0.9. After the dam's failure, the shock waves move radially outward whilst the rarefaction waves travel inside toward the centre of computational domain. Figures (2a & 2b) show the 3D water surface plots at times *t*=1s and 2.5s, where at the later time, the inward rarefaction wave and outward shock are reasonably recognizable. In Figures (2c & 2d) the corresponding contour plots are also illustrated. The corresponding numerical results show that the CFW approach can preserve the cylindrical symmetry containing radial shocks and rarefaction waves.

6.2. *Shock and rarefaction waves*

In this test case, three Riemann problem tests for moveable bed equations, originally proposed in (Murillo & García-Navarro, 2010), were adopted with the aim of validating the numerical suitability of the CFW method against the exact solution. In all of these test cases, the relevant friction terms were removed from the momentum equations. Three tests labelled *A*, *B* and *C* with different initial conditions containing left and right states of the Riemann problem for the fluid depth (h_L and h_R), velocity in both directions (u_L, u_R, v_L and v_R) and bedload (z_{bL} and z_{bR}) were considered (Table 3). These initial conditions cause different flow regimes and discontinuities involving shocks and rarefaction waves above a sediment type step. The exact solution was obtained by linking several waves from a left state until reaching the right state (Murillo & García-Navarro, 2010; USBR, 2017). The porosity of bed materials for this particular test was set to $p = 0.4$ and a frictionless condition was assumed. In order to better compare the accuracy of the proposed method, the result of all simulations were also compared with the results of the coupled Jacobian matrix (CJM) approach introduced in (Murillo & García-Navarro, 2010). For test cases *A* and *B*, a constant sediment discharge coefficient equal to $A_g = 0.01$ was considered, and in the final test, *C*, a non-constant sediment discharge parameter $A_g = 0.01 / h$ was chosen where *h* denotes the fluid depth. In this test case, the non-constant sediment discharge formula only affects parameters τ_x and τ_y, and their new values become $\sigma_x = -0.04 \eta u |V|^2 / h^2$ and $\sigma_y = -0.04 \eta v |V|^2 / h^2$, respectively. All simulations were performed using CFL=1 and cell distance $\Delta x = 0.1 \, \text{m}$. Figure (3) displays numerical results for test *A* calculated at time *t*=2s. As it can be seen, the CFW solver with the second-accurate terms can accurately capture the left and right-going rarefaction waves as well as the central shock that appears at the location of the step, which confirms that

Additionally, for both free-surface and sediment bed results, it gives a very good agreement with the exact solution

Figure (4) depicts the numerical solution for test B calculated at time t=2s. In this test case, two-rarefaction waves, a contact wave and a shock appear within the solution. As it is observed, the CFW method provides approximately the same results equal to the CJM approach in capturing the rarefaction waves and also the contact wave for all simulated variables (i.e. $h+z_b$, z_b, q_x and q_y). The only discrepancy appears at the left-going rarefaction wave in particular for the free-surface and sediment bed propagation where CFW produces smoother results. Additionally, the shock wave is precisely modelled by the CFW without any oscillation. Figure (5) shows the numerical results for test C, calculated using a non-constant sediment discharge parameter, A_g. The findings confirm that the CFW and the exact solution clearly give similar results for all conserved variables even with a variable interaction parameter. If sediment discharge wave decomposition is not considered based on left and right sediment discharge states, as illustrated in Figure (6), the numerical solver converges to a non-physical solution. The CPU time for the CFW approach for this test was 0.0624s.

6.3. *Dam break test case*

Using this test case, the effectiveness of the proposed numerical scheme was examined for modelling dam-break waves over a step by comparing the numerical results with experimental data obtained at the Civil Engineering laboratory of the Université catholique de Louvain (UCL) (Spinewine & Zech, 2007). The experiments were performed in a 6m long and 25cm wide channel with a central gate simulating a dam. The initial water depth at the left side and downstream of the gate was set at 25cm and 10cm, respectively. Additionally, the left side of the gate was filled with a sediment layer 10cm high, making a downward bed step, and no bedload sediment was considered for the right side. The channel bed profile was covered with uniform sand of $d_{50}=1.82\text{mm}$, density $\rho_s=2683\text{kg}\,\text{m}^{-3}$, porosity $p=0.47$, and a friction angle of $\varphi=30$, and a Manning's coefficient of $n_m=0.0165$ was considered. In order to examine the accuracy of the sediment discharge formulae defined in Table (1), the Smart and Meyer-Peter & Müller sediment formulations were utilised within the CFW approach.

Figure (7) shows a comparison between experimental data and the CFW simulation results, computed using a uniform cell distance $\Delta x=0.01\text{m}$ and CFL=1, together with CJM results obtained using the Smart bedload discharge. As it can be observed, the agreement of the numerical results with the experimental data is

remarkable, particularly in modelling the front shock and also left-going rarefaction waves using both sediment discharge models. This verifies that the effect of a variable interaction parameter for the Smart and Meyer-Peter & Müller has been accurately counted within the Jacobian matrix. The only exception is the hydraulic jump formed at the gate location where the Smart formula shows better agreement with experimental data. One reason behind this might be the Smart sediment discharge formula incorporates the bed slope variation into the sediment discharge calculation. Very similar results are also obtained with the CJM approach and the choice of Smart formula which verifies that the solver which seems unexpected for modelling bedload sediment transport due to its simplicity can produce very accurate results in particular at the place of shocks.

Figure (8) displays the obtained numerical simulations of the sediment profile against experimental data and CJM solutions. As shown, the CFW method with the Smart model, again, exhibits much better agreement with the experimental data and provides identical results to the CJM method for all of the computed times. The CFW method takes approximately 0.0621s to reach time t=1.5s.

6.4. Dam failure caused by overtopping

This problem was first introduced by Tingsanchali & Chinnarasri (2001) and was adopted here to evaluate the wet/dry front modelling capability of the defined CFW solver for situations where morphological variations occur over a dry state. The performance of the approach in modelling erosion processes due to overtopping over an initial irregular bed was examined by performing simulations of a channel 35m long, 1m deep and 1m wide, and a dam with a height of 0.8m and crest width of 0.3m. The downstream slope of the dam was set at $1V:3H$ whilst the downstream slope was varied but initially fixed at $1V:2.5H$. The downstream face of the dam was also covered with a material called Sand I with the Manning coefficient equal to $n_m = 0.018$, $d_{30} = 0.52$ mm and mean grain size of $d_m = 1.13$ mm and $d_{90} = 3.8$ mm corresponding to the test conditions of test C-2 in the original paper (Tingsanchali & Chinnarasri, 2001). As a boundary condition, an inflow discharge equal to $1.23 l/s$ was imposed onto the left boundary and an extrapolation boundary condition was utilised for the left boundary. The CFW results were calculated with a cell length of 0.02m and CFL=0.5.

Figure (8) displays the numerical results for the bed profiled obtained with the CFW approach with both choices of Meyer –Peter & Müller and Smart formula at time t=30 and 60s. As can be observed, the CFW approach with the Smart sediment discharge formula can accurately follow the experimental data in particular at time t=60s, and rather identical results were obtained. This indicates that the defined CFW approach can precisely model a rapid varying flow over a dry and erodible bed. For the Meyer-Peter & Müller bedload discharge, some discrepancy is still seen, in particular at the dam crest where the overtopping was initiated. This again, may be due to fact that in the Meyer-Peter & Müller formula the bed slope is not incorporated into sediment discharge computations. In terms of CPU time, the CFW approach takes 0.39s to reach time t=60s.

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6.5. Two-dimensional propagation of the parabolic bedload transport

A fifth test case was considered to examine the behaviour of the numerical scheme for 2D propagation of free-surface flow over a parabolic sediment layer based on Grass equation with $A_g = 0.01$ and porosity $p = 0.4$. The computational domain was set to $[0,1000m] \times [0,1000m]$ and the initial water depth and sediment layer topography were defined by the following equations:

$$h(x,y,0) = 10 - z_b(x,y,b),$$

$$z_b(x,y,0) = \begin{cases} \sin^2\left(\dfrac{\pi(x-300)}{200}\right) & \text{If} \quad 300 \leq x \leq 500, \\ 0 & \text{Otherwise.} \end{cases} \tag{39}$$

As for the boundary conditions, a sediment discharge equal to $q_{bx}(x,y,0) = 10 \text{ m}^2/\text{s}$ was imposed at the left boundary, and the extrapolation boundary condition was set for the downstream. It should be stressed that for this particular test case, the flow only propagates in one direction and the performance of the 2D numerical scheme was assessed to ensure it is not affected by the mesh topology. The 2D numerical results at time t=50,000s obtained by the CFW approach is shown in Figure (9). As it is observed, no transverse waves are created in the y-direction. Figure (10) demonstrates comparisons between the mid-section vertical plane bedload transportation results calculated based on 1st and second-order CFW approach at $y = 500m$ with the 1D analytical solution provided in (Castro Díaz, Fernández-Nieto, & Ferreiro, 2008; Hudson & Sweby, 2005). As shown, the second-order CFW approach with the dimensional-splitting method and the analytical solution give similar results, and even the top of the sediment layer shock is accurately captured by the proposed numerical model. However, the 1st order CFW provides rather diffusive results as expected for the first-order accurate schemes. For the CFW calculations 256×256 numerical cells with CFL=0.6 were used. In terms of CPU setup time the 2D CFW takes 372.85s to reach time t=50,000s.

6.6. Evolution of a canonical dune

The final test case was adopted from the work of Hudson and Sweby (Hudson & Sweby, 2005) with the aim of investigating the capability of the flux-wave solver in modelling a 2D canonical sediment dune under a subcritical regime. This test case is important as it contains the evolution of a canonical sediment layer in both x- and y-directions ultimately leading to a star-shaped pattern. The computational domain for this problem was chosen equal to $[0,1000m] \times [0,1000m]$ and the initial hydrodynamic conditions were defined by the following equations:

$$h(x,y,0) = 10 - z_b(x,y,b),$$

$$z_b(x,y,0) = \begin{cases} \sin^2\left(\dfrac{\pi(x-300)}{200}\right)\sin^2\left(\dfrac{\pi(y-400)}{200}\right) & \text{If} \quad 300 \leq x \leq 500, \quad 400 \leq y \leq 600 \\ 0 & \text{Otherwise} \end{cases} \tag{40}$$

To obtain the real initial condition for this particular test case, a discharge of $q_{bx}(x,y,0) = 10 \text{ m}^2/\text{s}$ was imposed at the left boundary and the problem was run similar to the 2D SWEs over the undeformable bed

until it reached a stationary state. In order to obtain the steady-state condition the following global relative error can be used (Mahdizadeh, 2010):

$$R_G = \sqrt{\sum \left(\frac{h_{i,j}^n - h_{i,j}^{n-1}}{h_{i,j}^n} \right)}, \tag{41}$$

The regime is considered to be steady-state where the value of R_G reaches approximately zero. Figure (11) illustrates the steady-state initial condition for the canonical sediment dune problem. The obtained results are in close agreement with the results presented in (Canestrelli, Dumbser, Siviglia, & Toro, 2010; Hudson & Sweby, 2005) which verifies that the proposed solver can accurately balance the effect of source term with the flux-differencing of the neighbouring cells (well-balanced scheme). Once the steady state is achieved, the CFW method with moveable bed is implemented using Grass formula (Grass, 1981) with $A_g = 0.001$. This value of A_g produces a rather weak interaction between the sediment bed and free-surface waves. For this problem, de Vriend (1987) derived an approximate solution for the angle of spread under the weak interaction when the value of $A_g \le 10^{-2}$ (Hudson & Sweby, 2005):

$$\tan \alpha = \frac{3\sqrt{3}(m_g - 1)}{9m_g - 1} \tag{42}$$

where for the defined Grass formula, the value of m_g is chosen equal to 3, and therefore, α becomes $21.7867^\bullet$ for the analytical solution. Figure (12) displays the numerical results obtained after t=100h which confirms that the CFW approach with second-order accurate terms produces very smooth results with no spurious oscillations, in particular at the sediment bump boundaries. The star-shaped pattern calculated at three different times together with the computed angle of spread line which is obtained equal to $\alpha = 23.0225^\bullet$ for the CFW method is shown in Figure (14). As shown, a good agreement between the numerical solver and the approximate solution is achieved. These results are also in qualitative agreement with the second-order accurate results provided in (Siviglia et al., 2013). For the CFW computations, the computational cells and the CFL number were chosen as 256×256 and 0.6, respectively. The total computation time for this test case to reach t=100h including steady-state initial condition calculations, was 6009s.

7. Conclusions

In this paper a generalization of the flux-wave formula was presented for the solution of coupled 2D morphodynamic systems. The numerical technique proposed here is well-balanced and incorporates the effect of flux differencing of finite volume neighbouring cells into the flux-waves propagating from each Riemann interface leading the proposed numerical scheme to be used in a less sophisticate way compared to other novel coupled morphodynamic solver whilst preserving its accuracy. To solve a 2D morphodynamic system, a dimensional splitting method is utilised which solves each Riemann problem in each direction. To

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obtain real second-order accurate results the cross-derivative terms are added into the solution by solving another Riemann problem in the orthogonal direction. A number of test cases were used to validate the proposed method. First, different shock waves and rarefaction wave propagations were modelled by the proposed sediment transport model and a good agreement was obtained with the experimental data. For dam-break propagation over a sediment step, the results from the CFW approach with Smart's formula matched closely with the experimental data. The solver was then used to model free-surface propagation over parabolic bedload sediment and the mid-section vertical plane results were compared with analytical solutions, and comparable results were obtained. The evolution of a conical sand dune was then demonstrated and a very close agreement with other qualitative sediment models was achieved.

Notation

$a_1, a_2, a_3 =$ coefficients for third degree polynomial in Eq. (17) (-)

$A_g =$ interaction parameter ($\mathrm{s^2 m^{-1}}$)

$\mathcal{A}^{\pm} \Delta \mathbf{U}_{i\pm1/2,j} =$ left- and right-going fluctuations in x-direction

$\mathcal{B}^{\pm} \Delta \mathbf{U}_{i,j\pm1/2} =$ left- and right-going fluctuations in x-direction

$C_f =$ friction coefficient(-)

CFL= Courant–Friedrichs–Lewy number (-)

$d_{30} =$ 30% of bed material (mm)

$d_{90} =$ 90% of bed material (mm)

$d_m =$ 50% of bed material (mm)

$\mathbf{F} =$ flux term vector in the x-direction (-)

$\mathbf{F}' =$ Jacobian matrix in the x-direction (-)

$\tilde{\mathbf{F}} =$ correction flux terms for the wave propagation algorithm in the x-direction (-)

$g =$ acceleration due to gravity ($\mathrm{ms^{-2}}$)

$\mathbf{G} =$ flux term vector in the y-direction (-)

$\mathbf{G}' =$ Jacobian matrix in the y-direction (-)

$\tilde{\mathbf{G}} =$ correction flux terms for the wave propagation algorithm in the y-direction (-)

$G_s =$ relative density (-)

$h =$ fluid depth (m)

$h_L =$ left-state Riemann initial fluid depth for the test case 6.2 (m)

$h_R =$ right-state Riemann initial fluid depth for the test case 6.2 (m)

$\bar{h}$ = average fluid depth (m)

k = flux wave number (-)

K = coefficient used for A_g calculation ($m^{-1/2}s$)

m_g = variable used for defining angle of spread Eq. (42) (-)

M_w = number of flux-waves (-)

n = number of time steps (-)

n_m = Manning's coefficient ($m^{-1/3}s$)

p = porosity (-)

$P(\lambda)$ = third degree polynomial used for calculation eigenvalues in Eq.(17) (-)

q_x = discharge per unit weight in the x-direction (m^2s^{-1})

q_y = discharge per unit weight in the y-direction (m^2s^{-1})

q_{bx} = bedload sediment discharge per unit weight in the x-direction (m^2s^{-1})

q_{by} = bedload sediment discharge per unit weight in the y-direction (m^2s^{-1})

Q = parameter used for calculation of eigenvalues in Eq. (20) (-)

$\mathbf{r}^F$ = eigenvector in the x-direction (-)

$\mathbf{r}^G$ = eigenvector in the y-direction (-)

R = parameter used for calculation of eigenvalues in Eq. (20) (-)

R_G = global relative error (-)

s = wave speed (ms^{-1})

S_0 = bed slope (-)

S_{0x} = bed slope in the x-direction (-)

S_{0y} = bed slope in the y-direction (-)

$\mathbf{S}$ = source term vector (-)

$\mathbf{S_1}$ = source term vector in the x-direction (-)

$\mathbf{S_2}$ = source term vector in the **y**-direction (-)

t = time (s)

u = velocity in the x-direction (ms^{-1})

u_L = left state Riemann initial velocity in the x-direction for test case 6.2 (ms^{-1})

u_R = left state Riemann initial velocity in the x-direction for test case 6.2 (ms^{-1})

$\bar{u}$ = Roe speed in the x-direction (ms^{-1})

$\mathbf{U}$ = vector of unknowns (-)

v = velocity in the y-direction ($\mathrm{ms^{-1}}$)

v_L = left state Riemann initial velocity in the y-direction for test case 6.2 ($\mathrm{ms^{-1}}$)

v_R = left state Riemann initial velocity in the x-direction for test case 6.2 ($\mathrm{ms^{-1}}$)

$\bar{v}$ = Roe speed in the y-direction ($\mathrm{ms^{-1}}$)

V = velocity ($\mathrm{ms^{-1}}$)

z_b = bedload sediment thickness (m)

z_{bL} = left state Riemann initial bedload height for test case 6.2 (m)

z_{bR} = right state Riemann initial bedload height for test case 6.2 (m)

α = angle of spread (-)

β = coefficient required to calculate flux-wave (-)

$\Delta \mathbf{F}$ = difference vector for the neighboring finite volume flux-waves (-)

Δt = time step(s)

Δt_x = time step in the x-direction (-)

Δt_y = time step in the y-direction (-)

Δx = finite volume cell distance (m)

ε = coefficient required for calculation of linear system appeared in Eq. (30) (-)

η = coefficient calculated based the porosity (-)

θ = dimensionless bed shear stress (-)

θ_c = critical Shields parameter (-)

θ_s = Smart's critical Shield parameter (-)

κ = coefficient required for calculation of linear system appeared in Eq. (30) (-)

λ_k^F = kth eigenvalue in the x-direction (-)

λ_k^G = kth eigenvalue in the y-direction (-)

μ = parameter used for calculation of eigenvalues in Eq. (20) (-)

ξ = flux-wave vector (-)

ρ = water density ($\mathrm{kgm^{-3}}$)

ρ_s = sediment density ($\mathrm{kgm^{-3}}$)

τ_{fx}, τ_{fy} = bed shear stress in the x and y-directions ($\mathrm{Nm^{-2}}$)

φ = angle of bed slope (-)

$\chi =$ coefficient used for A_g calculation (-)

$\psi =$ function used for the interaction parameter calculation (-)

$\omega =$ coefficient required for calculation of the Jacobian matrix for the variable A_g described in Eq. (25) (-)

$\sigma_x, \delta_x, \zeta_x =$ coefficients required for the Jacobian matrix with variable A_g in the x-direction (-)

$\sigma_y, \delta_y, \zeta_y =$ coefficients required for the Jacobian matrix with variable A_g in the in the y-direction (-)

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Table 1. Expression of function χ based on different sediment transport formulations.

Reference	Formula	
Meyer-Peter & Mueller (1984)	$\chi = 8(1 - \theta_c / \theta)^{3/2}$	(5)
(Smart, 1984)	$\chi = 4\left(\dfrac{d_{90}}{d_{30}}\right)^{0.2} \left(\dfrac{S_0^{0.6} h^{1/6}}{n}\right)(1 - \theta_s / \theta)$	(6)

Table 2.The derivatives of bedload sediment discharges for the for the Smart and Meyer-Peter & Müller bedload sediment formulae for the Jacobian matrix of (17) in the *x*- and *y*-directions with respect to the vector of unknowns.

	Meyer-Peter & Müller	**Smart**
σ_x	$-\dfrac{28\sqrt{g}n^3\eta u\omega\|V\|^2}{(G_s-1)h^{3/2}}$	$-\dfrac{4\left(\dfrac{d_{90}}{d_{30}}\right)^{0.2}\sqrt{g}S_0^{0.6}\eta u\left(-3dm(G_s-1)h^{1/3}\theta_s+10n^2\|V\|^2\right)}{3(G_s-1)h^{4/3}}$
δ_x	$\dfrac{8\sqrt{g}n\eta\omega\left(n^2\left(3u^4+4u^2v^2+v^4\right)-d_m(G_s-1)h^{1/3}v^2\theta_c\right)}{(G_s-1)h^{3/2}\|V\|^2}$	$\dfrac{4\left(\dfrac{d_{90}}{d_{30}}\right)^{0.2}\sqrt{g}S_0^{0.6}\eta\left(n^2\left(3u^2+v^2\right)-d_m(G_s-1)h^{1/3}\theta_s\right)}{(G_s-1)h^{4/3}}$
ζ_x	$\dfrac{8\sqrt{g}n\eta uv\omega\left(d_m(G_s-1)h^{1/3}\theta_c+2n^2\|V\|^2\right)}{(G_s-1)h^{3/2}\|V\|^2}$	$\dfrac{8\left(\dfrac{d_{90}}{d_{30}}\right)^{0.2}\sqrt{g}n^2\eta S_0^{0.6}uv}{(G_s-1)h^{4/3}}$
σ_y	$-\dfrac{28\sqrt{g}n^3\eta v\omega\|V\|^2}{(G_s-1)h^{3/2}}$	$-\dfrac{4\left(\dfrac{d_{90}}{d_{30}}\right)^{0.2}\sqrt{g}S_0^{0.6}\eta v\left(-3d_m(G_s-1)h^{1/3}\theta_s+10n^2\|V\|^2\right)}{3(G_s-1)h^{4/3}}$
δ_y	$\dfrac{8\sqrt{g}n\eta uv\omega\left(d_m(G_s-1)h^{1/3}\theta_c+2n^2\|V\|^2\right)}{(G_s-1)h^{3/2}\|V\|^2}$	$\dfrac{8\left(\dfrac{d_{90}}{d_{30}}\right)^{0.2}\sqrt{g}n^2\eta S_0^{0.6}uv}{(G_s-1)h^{4/3}}$
ζ_y	$\dfrac{8\sqrt{g}n\eta\omega\left(n^2\left(u^4+4u^2v^2+3v^4\right)-d_m(G_s-1)h^{1/3}u^2\theta_c\right)}{(G_s-1)h^{3/2}\|V\|^2}$	$\dfrac{4\left(\dfrac{d_{90}}{d_{30}}\right)^{0.2}\sqrt{g}S_0^{0.6}\eta\left(n^2\left(u^2+3v^2\right)-d_m(G_s-1)h^{1/3}\theta_s\right)}{(G_s-1)h^{4/3}}$

Table 3. Initial conditions for the tests *A*, *B* and *C* (Murillo & García-Navarro, 2010)

Test	h_L (m)	h_R (m)	u_L (ms^{-1})	u_R (ms^{-1})	v_L (ms^{-1})	v_R (ms^{-1})	z_{bL} (m)	z_{bR} (m)
A	2.0	2.0	0.25	2.3247449	0.05	0.04	3.0	2.846848
B	2.25	1.18868612	0.20	2.4321238	0.045	0.02	5.0	5.124685

| C | 6.0 | 5.2 | 0.3 | 15.167196 | 0.015 | 0.04 | 3.0 | 4.631165 |

Table 4. Euclidian error norm comparison computed between the CFW and CJM approaches with the analytical solution for test cases *A, B* and *C*.

Test	CFW			CJM		
	$h + z_b$	z_b	q_{bx}	$h + z_b$	z_b	q_{bx}
A	0.02298	0.00786	0.00148	0.06128	0.05619	0.09765
B	0.01285	0.00412	0.00622	0.04791	0.04733	0.07919
C	0.02771	0.01085	0.02419	0.02210	0.03817	0.05628