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# On the Spherical Quasi-Convexity of Quadratic Functions<sup>☆</sup>

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## Abstract

In this paper the spherical quasi-convexity of quadratic functions on spherically convex sets is studied. Several conditions characterizing the spherical quasi-convexity of quadratic functions are presented. In particular, conditions implying spherical quasi-convexity of quadratic functions on the spherical positive orthant are given. Some examples are provided as applications of the obtained results.

*Keywords:* sphere, spherical quasi-convexity, quadratic functions, positive orthant.

*2010 MSC:* 26B25, 90C25

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## 1. Introduction

In this paper we study the spherical quasi-convexity of quadratic functions on spherically convex sets, which is related to the problem of finding their minimizer. This problem of minimizing a quadratic function on the sphere has arisen to S. Z. Németh by trying to make certain fixed point theorems, surjectivity theorems, and existence theorems for complementarity problems and variational inequalities more explicit (see [1] and the related references therein). In particular, some existence theorems could be reduced to the optimization of a

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quadratic function on the intersection of the sphere and a cone. Indeed, consider a closed convex cone  $K \subseteq \mathbb{R}^n$  with dual  $K^*$ . Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a continuous mapping such that  $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined by  $G(x) = \|x\|^2 F(x/\|x\|^2)$  and  $G(0) = 0$  is differentiable at 0. Denote by  $DG(0)$  the Jacobian matrix of  $G$  at 0. By [2, Corollary 8.1] and [3, Theorem18], if  $\min_{\|u\|=1, u \in K} \langle DG(0)u, u \rangle > 0$ , then the nonlinear complementarity problem defined by  $K \ni x \perp F(x) \in K^*$  has a solution. Thus, we need to minimize a quadratic form on the intersection between a cone and the sphere. These sets are exactly the spherically convex sets; see [4]. Therefore, this leads to minimizing quadratic functions on spherically convex sets. In fact the optimization problem above reduces to the problem of calculating the scalar derivative, along cones introduced by S. Z. Németh (see [1] and the related references therein). Similar minimizations of quadratic functions on spherically convex sets are needed in the other settings (see [1] and the related references therein). Apart from the above, the motivation of this study is much wider. For instance, the quadratic constrained optimization problem on the sphere

$$\min\{\langle Qx, x \rangle : x \in C\}, \quad C \subseteq \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}, \quad (1)$$

for a symmetric matrix  $Q$ , is a minimum eigenvalue problem in  $C$ , which includes the problem of finding the spectral norm of the matrix  $-Q$  when  $C = \mathbb{S}^{n-1}$  (see, e.g., [5]). It is important to highlight that the special case when  $C$  is the nonnegative orthant is of particular interest because the nonnegativity of the minimum value is equivalent to the copositivity of the matrix  $Q$  [6, Proposition 1.3] and to the nonnegativity of all Pareto eigenvalues of  $Q$  [6, Theorem 4.3]. As far as we are aware there are no methods for finding the Pareto spectra by using the intrinsic geometrical properties of the sphere, hence our study is expected to open new perspectives for detecting the copositivity of a symmetric matrix. More problems that deal with “spherical” constraints can be found in [7].

Optimization problems posed on the sphere have a specific underlying algebraic structure that could be exploited to greatly reduce the cost of obtaining the solutions; see [8, 9, 5, 10, 11, 12]. It is worth to point out that when a quadratic

15 function is spherically quasi-convex, then a spherical strict local minimizer is  
equal to a spherical strict global minimizer. Therefore, it is natural to consider  
the problem of determining the spherically quasi-convex quadratic functions on  
spherically convex sets. The goal of the paper is to present necessary conditions  
and sufficient conditions for quadratic functions which are spherically quasi-  
20 convex on spherical convex sets. As a particular case, we exhibit several such  
results for the spherical positive orthant.

The paper can be considered as a first spherical analogue for the study of  
quasi-convexity of quadratic functions. Without the aim of completeness, we list  
here some of the main papers about the quasi-convexity of quadratic functions:  
25 [13, 14, 15, 16, 17]

The remainder of this paper is organized as follows. In Section 2, we re-  
call some notations and basic results used throughout the paper. In Section 3  
we present some general properties of spherically quasi-convex functions on  
spherically convex sets. In Section 4 we present some conditions characteriz-  
30 ing quadratic spherically quasi-convex functions on a general spherically convex  
set. In Section 4.1 we present some properties of quadratic functions defined  
in the spherical positive orthant. We conclude this paper by making some final  
remarks in Section 5.

## 2. Basics results

In this section we present the notations and the auxiliary results used through-  
out the paper. Let  $\mathbb{R}^n$  be the  $n$ -dimensional Euclidean space with the canonical  
inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\|$ . The set of all  $m \times n$  matrices with real entries  
is denoted by  $\mathbb{R}^{m \times n}$  and  $\mathbb{R}^n \equiv \mathbb{R}^{n \times 1}$ . Denote by  $\mathbb{R}_+^n$  the nonnegative orthant  
and by  $\mathbb{R}_{++}^n$  the positive orthant, that is,

$$\mathbb{R}_+^n = \{x = (x_1, \dots, x_n)^\top : x_1 \geq 0, \dots, x_n \geq 0\}$$

and

$$\mathbb{R}_{++}^n = \{x = (x_1, \dots, x_n)^\top : x_1 > 0, \dots, x_n > 0\}.$$

Denote by  $e^i$  the  $i$ -th canonical unit vector in  $\mathbb{R}^n$ . A set  $\mathcal{K}$  is called a *cone* if it is invariant under the multiplication with positive scalars and a *convex cone* if it is a cone which is also a convex set. The *dual cone* of a cone  $\mathcal{K} \subset \mathbb{R}^n$  is the cone  $\mathcal{K}^* := \{x \in \mathbb{R}^n : \langle x, y \rangle \geq 0, \forall y \in \mathcal{K}\}$ . A cone  $\mathcal{K} \subset \mathbb{R}^n$  is called *pointed* if  $\mathcal{K} \cap \{-\mathcal{K}\} \subseteq \{0\}$ , or equivalently, if  $\mathcal{K}$  does not contain straight lines through the origin. Any pointed closed convex cone with nonempty interior will be called *proper cone*. The cone  $\mathcal{K}$  is called *subdual* if  $\mathcal{K} \subseteq \mathcal{K}^*$ , *superdual* if  $\mathcal{K}^* \subseteq \mathcal{K}$  and *self-dual* if  $\mathcal{K}^* = \mathcal{K}$ . The matrix  $I_n$  denotes the  $n \times n$  identity matrix. Recall that  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  is *positive* if  $a_{ij} > 0$  and *nonnegative* if  $a_{ij} \geq 0$  for all  $i, j = 1, \dots, n$ . A matrix  $A \in \mathbb{R}^{n \times n}$  is *reducible* if there is permutation matrix  $P \in \mathbb{R}^{n \times n}$  so that

$$P^T A P = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix},$$

$$B_{11} \in \mathbb{R}^{m \times m}, B_{22} \in \mathbb{R}^{(n-m) \times (n-m)}, B_{12} \in \mathbb{R}^{m \times (n-m)}, \quad m < n.$$

35 A matrix  $A \in \mathbb{R}^{n \times n}$  is *irreducible* if it not reducible. In the following we state a version of *Perron-Frobenius theorem* for both positive matrices and nonnegative irreducible matrices, its proof can be found in [18, Theorem 8.2.11] and [18, Theorem 8.4.4], respectively.

**Theorem 1.** *Let  $A \in \mathbb{R}^{n \times n}$  be either nonnegative and irreducible or positive.*

40 *Then  $A$  has a dominant eigenvalue  $\lambda_{\max}(A) \in \mathbb{R}$  with associated eigenvector  $v \in \mathbb{R}^n$  which satisfies the following properties:*

*i) The eigenvalue  $\lambda_{\max}(A) > 0$  and its associated eigenvector  $v \in \mathbb{R}_{++}^n$ ;*

*ii) The eigenvalue  $\lambda_{\max}(A) > 0$  has multiplicity one;*

*iii) Every other eigenvalue  $\lambda$  of  $A$  is less than  $\lambda_{\max}(A)$  in absolute value, i.e.,*  
45  $|\lambda| < \lambda_{\max}(A)$ ;

*iii) There are no other positive or non-negative eigenvectors of  $A$  except positive multiples of  $v$ .*

Recall that  $A \in \mathbb{R}^{n \times n}$  is *copositive* if  $\langle Ax, x \rangle \geq 0$  for all  $x \in \mathbb{R}_+^n$  and a *Z-matrix* is a matrix with nonpositive off-diagonal elements. Let  $\mathcal{K} \subset \mathbb{R}^n$  be a pointed closed convex cone with nonempty interior, the  *$\mathcal{K}$ -Z-property* of a matrix  $A \in \mathbb{R}^{n \times n}$  means that  $\langle Ax, y \rangle \leq 0$  for all  $(x, y) \in C(\mathcal{K})$ , where  $C(\mathcal{K}) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \in \mathcal{K}, y \in \mathcal{K}^*, \langle x, y \rangle = 0\}$ . If  $x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$ , then  $\text{diag}(x)$  will denote an  $n \times n$  diagonal matrix with  $(i, i)$ -th entry equal to  $x_i$ , for  $i = 1, \dots, n$ . Throughout the paper the *tangent hyperplane of the  $n$ -dimensional Euclidean sphere  $\mathbb{S}^{n-1}$  at a point  $x \in \mathbb{S}^{n-1}$*  is denoted by

$$T_x \mathbb{S}^{n-1} := \{v \in \mathbb{R}^n : \langle x, v \rangle = 0\},$$

The *intrinsic distance on the sphere* between two arbitrary points  $x, y \in \mathbb{S}^{n-1}$  is defined by

$$d(x, y) := \arccos \langle x, y \rangle. \quad (2)$$

It can be shown that  $(\mathbb{S}^{n-1}, d)$  is a complete metric space, so that  $d(x, y) \geq 0$  for all  $x, y \in \mathbb{S}^{n-1}$ , and  $d(x, y) = 0$  if and only if  $x = y$ . It is also easy to check that  $d(x, y) \leq \pi$  for all  $x, y \in \mathbb{S}^{n-1}$ , and  $d(x, y) = \pi$  if and only if  $x = -y$ . The intersection curve of a plane through the origin of  $\mathbb{R}^n$  with the sphere  $\mathbb{S}^{n-1}$  is called a *geodesic*. If  $x, y \in \mathbb{S}^{n-1}$  are such that  $y \neq x$  and  $y \neq -x$ , then the unique *segment of minimal geodesic from  $x$  to  $y$*  is

$$\gamma_{xy}(t) = \left( \cos(td(x, y)) - \frac{\langle x, y \rangle \sin(td(x, y))}{\sqrt{1 - \langle x, y \rangle^2}} \right) x + \frac{\sin(td(x, y))}{\sqrt{1 - \langle x, y \rangle^2}} y, \quad t \in [0, 1]. \quad (3)$$

Let  $x \in \mathbb{S}^{n-1}$  and  $v \in T_x \mathbb{S}^{n-1}$  such that  $\|v\| = 1$ . The minimal segment of geodesic connecting  $x$  to  $-x$ , starting at  $x$  with velocity  $v$  at  $x$  is given by

$$\gamma_{x\{-x\}}(t) := \cos(t) x + \sin(t) v, \quad t \in [0, \pi]. \quad (4)$$

Let  $\Omega \subset \mathbb{S}^{n-1}$  be a spherically open set (i.e., a set open with respect to the induced topology in  $\mathbb{S}^{n-1}$ ). The *gradient on the sphere* of a differentiable function  $f : \Omega \rightarrow \mathbb{R}$  at a point  $x \in \Omega$  is the vector defined by

$$\text{grad } f(x) := [I_n - xx^T] Df(x) = Df(x) - \langle Df(x), x \rangle x, \quad (5)$$

where  $Df(x) \in \mathbb{R}^n$  is the usual gradient of  $f$  at  $x \in \Omega$ . Let  $\mathcal{D} \subseteq \mathbb{R}^n$  be an open set,  $I \subset \mathbb{R}$  an open interval,  $\Omega \subset \mathbb{S}^{n-1}$  a spherically open set and  $\gamma : I \rightarrow \Omega$  a geodesic segment. If  $f : \mathcal{D} \rightarrow \mathbb{R}$  is a differentiable function, then, since  $\gamma'(t) \in T_{\gamma(t)}\mathbb{S}^{n-1}$  for all  $t \in I$ , the equality (5) implies

$$\frac{d}{dt}f(\gamma(t)) = \langle \text{grad } f(\gamma(t)), \gamma'(t) \rangle = \langle Df(\gamma(t)), \gamma'(t) \rangle, \quad \forall t \in I. \quad (6)$$

**Definition 2.** The set  $\mathcal{C} \subseteq \mathbb{S}^{n-1}$  is said to be spherically convex if for all  $x, y \in \mathcal{C}$  all the minimal geodesic segments joining  $x$  to  $y$  are contained in  $\mathcal{C}$ .

50 **Example 3.** The set  $S_+ = \{(x_1, \dots, x_n) \in \mathbb{S}^{n-1} : x_1 \geq 0, \dots, x_n \geq 0\}$  is a closed spherically convex set.

We assume for convenience that from now on all spherically convex sets are nonempty proper subsets of the sphere. For each closed set  $A \subset \mathbb{S}^{n-1}$ , let  $K_A \subset \mathbb{R}^n$  be the cone spanned by  $A$ , namely,

$$K_A := \{tx : x \in A, t \in [0, +\infty)\}. \quad (7)$$

Clearly,  $K_A$  is the smallest closed cone which contains  $A$ . In the next proposition we exhibit a relationship of spherically convex sets with the cones spanned by them; for the proof see [19].

55 **Proposition 4.** The set  $\mathcal{C}$  is spherically convex if and only if the cone  $K_{\mathcal{C}}$  is convex and pointed.

Let  $\mathcal{C} \subset \mathbb{S}^{n-1}$  be a spherically convex set. A function  $f : \mathcal{C} \rightarrow \mathbb{R}$  is said to be spherically convex (respectively, strictly spherically convex) if for all minimal geodesic segment  $\gamma : [0, 1] \rightarrow \mathcal{C}$ , the composition  $f \circ \gamma : [0, 1] \rightarrow \mathbb{R}$  is convex  
60 (respectively, strictly convex) in the usual sense.

We end this section by stating some standard notations. We denote the spherically open and the spherically closed ball with radius  $\delta > 0$  and center in  $x \in \mathbb{S}^{n-1}$  by  $B_\delta(x) := \{y \in \mathbb{S}^{n-1} : d(x, y) < \delta\}$  and  $\bar{B}_\delta(x) := \{y \in \mathbb{S}^{n-1} : d(x, y) \leq \delta\}$ , respectively. The sub-level sets of a function  $f : \mathbb{R}^n \supseteq \mathcal{D} \rightarrow \mathbb{R}$  are denoted by

$$[f \leq c] := \{x \in \mathcal{D} : f(x) \leq c\}, \quad c \in \mathbb{R}. \quad (8)$$

### 3. Spherically quasi-convex functions on spherically convex sets

In this section we study general properties of quasi-convex functions on the sphere. In particular, we present first order characterizations of differentiable quasi-convex functions on the sphere. Several results of this section have already  
65 appeared in [20], but here these results have more explicit statements and proofs. It is worth to remark that the quasi-convexity concept generalizes the convexity one, which was extensively studied in [4]. Let us start by defining this concept.

**Definition 5.** *Let  $\mathcal{C} \subset \mathbb{S}^{n-1}$  be a spherically convex set. A function  $f : \mathcal{C} \rightarrow \mathbb{R}$  is said to be spherically quasi-convex (respectively, strictly spherically quasi-convex) if for all minimal geodesic segment  $\gamma : [0, 1] \rightarrow \mathcal{C}$ , the composition  
70  $f \circ \gamma : [0, 1] \rightarrow \mathbb{R}$  is quasi-convex (respectively, strictly quasi-convex) in the usual sense, i.e.,  $f(\gamma(t)) \leq \max\{f(\gamma(0)), f(\gamma(1))\}$  for all  $t \in [0, 1]$ , (respectively,  $f(\gamma(t)) < \max\{f(\gamma(0)), f(\gamma(1))\}$  for all  $t \in [0, 1]$ ).*

Naturally, from the above definition, it follows that spherically convex (re-  
75 spectively, strictly spherically convex) functions are spherically quasi-convex (respectively, strictly spherically quasi-convex), but the converse is not true; see [4].

**Proposition 6.** *Let  $\mathcal{C} \subset \mathbb{S}^{n-1}$  be a spherically convex set. A function  $f : \mathcal{C} \rightarrow \mathbb{R}$  is spherically quasi-convex if and only if the sub-level sets  $[f \leq c]$  are spherically  
80 convex for all  $c \in \mathbb{R}$ .*

*Proof.* Assume that  $f$  is spherically quasi-convex and  $c \in \mathbb{R}$ . Take  $x, y \in [f \leq c]$  and  $\gamma : [0, 1] \rightarrow \mathbb{S}^{n-1}$  the minimal geodesic such that  $\gamma(0) = x$  to  $\gamma(1) = y$ , see (3) and (4). Since  $f$  is a spherically quasi-convex function and  $x, y \in [f \leq c]$  we have  $f(\gamma(t)) \leq \max\{f(\gamma(0)), f(\gamma(1))\} \leq c$  for all  $t \in [0, 1]$ , which implies that  
85  $\gamma(t) \in [f \leq c]$  for all  $t \in [0, 1]$ . Hence we conclude that  $[f \leq c]$  is a spherically convex set for all  $c \in \mathbb{R}$ . Conversely, we assume that  $[f \leq c]$  is spherically convex for all  $c \in \mathbb{R}$ . Let  $\gamma : [0, 1] \rightarrow \mathcal{C}$  be a minimal geodesic segment. Since  $[f \leq c]$  is a spherically convex set, we have  $\gamma(t) \in [f \leq c]$  for all  $t \in [0, 1]$ , which implies



$f(\gamma(t)) \leq \max\{f(\gamma(0)), f(\gamma(1))\}$  for all  $t \in [0, 1]$ . Therefore,  $f$  is a spherically  
90 quasi-convex function and the proof is concluded.  $\square$

**Proposition 7.** *Let  $\mathcal{C} \subset \mathbb{S}^{n-1}$  be spherically convex and  $f : \mathcal{C} \rightarrow \mathbb{R}$  be spherically quasi-convex. If  $x^* \in \mathcal{C}$  is a strict local minimizer of  $f$ , then  $x^*$  is also a strict global minimizer of  $f$  in  $\mathcal{C}$ .*

*Proof.* If  $x^*$  is a strict local minimizer of  $f$ , then there exists a number  $\delta > 0$  such that

$$f(x) > f(x^*), \quad \forall x \in B_\delta(x^*) \setminus \{x^*\} = \{y \in \mathcal{C} : 0 < d(y, x^*) < \delta\}. \quad (9)$$

Assume by contradiction that  $x^*$  is not a strict global minimizer of  $f$  in  $\mathcal{C}$ . Thus,  
95 there exists  $\bar{x} \in \mathcal{C}$  with  $\bar{x} \neq x^*$  such that  $f(\bar{x}) \leq f(x^*)$ . Since  $\mathcal{C}$  is spherically convex, we can take a minimal geodesic segment  $\gamma : [0, 1] \rightarrow \mathcal{C}$  joining  $x^*$  and  $\bar{x}$ , let's say,  $\gamma(0) = x^*$  and  $\gamma(1) = \bar{x}$ . Considering that  $f$  is spherically quasi-convex we have  $f(\gamma(t)) \leq \max\{f(x^*), f(\bar{x})\} = f(x^*)$  for all  $t \in [0, 1]$ . On the other hand, for  $t$  sufficiently small we have  $\gamma(t) \in B_\delta(x^*)$ . Therefore, the last  
100 inequality contradicts (9) and the proof is concluded.  $\square$

**Proposition 8.** *Let  $\mathcal{C} \subset \mathbb{S}^{n-1}$  be a spherically convex set and  $f : \mathcal{C} \rightarrow \mathbb{R}$  be a strictly spherically quasi-convex function. Then  $f$  has at most one local minimizer which is also a global minimizer of  $f$ .*

*Proof.* Assume by contradiction that  $f$  has two local minimizers  $x^*, \bar{x} \in \mathcal{C}$  with  
105  $\bar{x} \neq x^*$ . Thus, we can take a minimal geodesic segment  $\gamma : [0, 1] \rightarrow \mathcal{C}$  joining  $x^*$  and  $\bar{x}$ , let's say,  $\gamma(0) = x^*$  and  $\gamma(1) = \bar{x}$ . Due to  $f$  being strictly spherically quasi-convex  $f(\gamma(t)) < \max\{f(x^*), f(\bar{x})\}$  for all  $t \in [0, 1]$ . Since we can take  $t$  sufficiently close to 0 or 1, the last inequality contradicts the assumption that  $x^*, \bar{x}$  are two distinct local minimizers. Thus,  $f$  has at most one local  
110 minimizer. Since  $f$  is strictly quasi-convex, the local minimizer is strict. Therefore, Proposition 7 implies that the local minimizer is global and the proof is concluded.  $\square$

**Proposition 9.** *Let  $\mathcal{C} \subset \mathbb{S}^{n-1}$  be an open spherically convex set and  $f : \mathcal{C} \rightarrow \mathbb{R}$  be a differentiable function. Then  $f$  is spherically quasi-convex if and only if*

$$f(x) \leq f(y) \implies \langle Df(y), x \rangle - \langle x, y \rangle \langle Df(y), y \rangle \leq 0, \quad \forall x, y \in \mathcal{C}. \quad (10)$$

*Proof.* Let  $\gamma : I \rightarrow \mathcal{C}$  be a geodesic segment and consider the composition  $f \circ \gamma : I \rightarrow \mathbb{R}$ . The usual characterization of scalar quasi-convex functions implies that  $f \circ \gamma$  is quasi-convex if and only if

$$f(\gamma(t_1)) \leq f(\gamma(t_2)) \implies \frac{d}{dt} (f(\gamma(t))) (t_1 - t_2) \leq 0, \quad \forall t_2, t_1 \in I. \quad (11)$$

On the other hand, for each  $x, y \in \mathcal{C}$  with  $y \neq x$  we have from (3) that  $\gamma_{xy}$  is the minimal geodesic segment from  $x = \gamma_{xy}(0)$  to  $y = \gamma_{xy}(1)$  and

$$\gamma'_{xy}(1) = \frac{\arccos \langle x, y \rangle}{\sqrt{1 - \langle x, y \rangle^2}} (yy^T - I_n) x \in T_y \mathbb{S}^{n-1}, \quad y \neq -x.$$

Note that letting  $x = \gamma(t_1)$  and  $y = \gamma(t_2)$  we have that  $\gamma_{xy}(t) = \gamma(t_1 + t(t_2 - t_1))$ .

Therefore, by using (6) we conclude that the condition in (11) is equivalent to

115 (10) and the proof of the proposition follows.  $\square$

#### 4. Spherically quasi-convex quadratic functions on spherically convex sets

In this section our aim is to present some conditions characterizing quadratic spherically quasi-convex functions on a general spherically convex set. For that we need some definitions: *From now on we assume that  $\mathcal{K} \subset \mathbb{R}^n$  is a proper subdual cone,  $\mathcal{C} = \mathbb{S}^{n-1} \cap \text{int}(\mathcal{K})$  is an open spherically convex set and  $A = A^T \in \mathbb{R}^{n \times n}$  with the associated quadratic function  $q_A : \mathcal{C} \rightarrow \mathbb{R}$  defined by*

$$q_A(x) := \langle Ax, x \rangle. \quad (12)$$

We also need the *restriction on  $\text{int } \mathcal{K}$  of the Rayleigh quotient function  $\varphi_A : \text{int } \mathcal{K} \rightarrow \mathbb{R}$  defined by*

$$\varphi_A(x) := \frac{\langle Ax, x \rangle}{\|x\|^2}. \quad (13)$$

In the following propositions we present some equivalent characterizations of the convexity of  $q_A$  defined by (12) on spherically convex sets. Our first result  
120 is the following proposition.

**Proposition 10.** *Let  $q_A$  and  $\varphi_A$  be the functions defined in (12) and (13), respectively. The following statements are equivalent:*

- (a) *The quadratic function  $q_A$  is spherically quasi-convex;*
- (b)  *$\langle Ax, y \rangle \leq \langle x, y \rangle \max \{q_A(x), q_A(y)\}$  for all  $x, y \in \mathbb{S}^{n-1} \cap \mathcal{K}$ ;*
- 125 (c)  *$\frac{\langle Ax, y \rangle}{\langle x, y \rangle} \leq \max \{\varphi_A(x), \varphi_A(y)\}$  for all  $x, y \in \mathcal{K}$  with  $\langle x, y \rangle \neq 0$ .*

*Proof.* First of all, we assume that item (a) holds. Let  $x, y \in \mathcal{C}$ . Thus, either  $q_A(x) \leq q_A(y)$  or  $q_A(y) \leq q_A(x)$ . Hence, by using Proposition 9 we conclude that either  $\langle Ay, x \rangle \leq \langle x, y \rangle q_A(y)$  or  $\langle Ax, y \rangle \leq \langle x, y \rangle q_A(x)$ . Thus, since  $A = A^T$  implies  $\langle Ax, y \rangle = \langle Ay, x \rangle$ , taking into account that  $\mathcal{K}$  is a subdual cone and hence  $\langle x, y \rangle \geq 0$ , we have

$$\langle Ax, y \rangle \leq \max \{ \langle x, y \rangle q_A(x), \langle x, y \rangle q_A(y) \} = \langle x, y \rangle \max \{ q_A(x), q_A(y) \}, \forall x, y \in \mathcal{C}.$$

Therefore, by continuity we extend the above inequality to all  $x, y \in \mathbb{S}^{n-1} \cap \mathcal{K}$  and, then item (b) holds. Conversely, we assume that item (b) holds. Let  $x, y \in \mathcal{C}$  satisfying  $q_A(x) \leq q_A(y)$ . Then, by the inequality in item (b) and considering that  $\mathcal{K}$  is a subdual cone, we have  $\langle Ax, y \rangle \leq \langle x, y \rangle q_A(y)$ . Hence,  
130 by using Proposition 9 we conclude that  $q_A$  is spherically quasi-convex and the proof of the equivalence between (a) and (b) is complete.

To establish the equivalence between (b) and (c), we assume first that item (b) holds. Let  $x, y \in \mathcal{K}$  with  $\langle x, y \rangle \neq 0$ . Then,  $x \neq 0$  and  $y \neq 0$ . Moreover, we have

$$u := \frac{x}{\|x\|} \in \mathbb{S}^{n-1} \cap \mathcal{K}, \quad v := \frac{y}{\|y\|} \in \mathbb{S}^{n-1} \cap \mathcal{K}.$$

Hence, by using the inequality in item (b) with  $x = u$  and  $y = v$ , we obtain the inequality in item (c). Conversely, suppose that (c) holds. Let  $x, y \in \mathbb{S}^{n-1} \cap \mathcal{K}$ . First assume that  $\langle x, y \rangle \neq 0$ . Since,  $\|x\| = \|y\| = 1$ , from the inequality in item (c) we conclude that

$$\frac{\langle Ax, y \rangle}{\langle x, y \rangle} \leq \max \{ q_A(x), q_A(y) \}.$$

Due to  $\mathcal{K}$  being a subdual cone, we have  $\langle x, y \rangle \geq 0$ , and hence the last inequality is equivalent to the inequality in item (b). Now, assume that  $\langle x, y \rangle = 0$ . Then, take two sequences  $\{x^k\}, \{y^k\} \subset \mathcal{C}$  such that  $\lim_{k \rightarrow +\infty} x^k = x$ ,  $\lim_{k \rightarrow +\infty} y^k = y$  and  $\langle x^k, y^k \rangle \neq 0$ . Since  $\mathcal{K}$  is a subdual cone, we have  $\langle x^k, y^k \rangle > 0$  for all  $k = 1, 2, \dots$ . Therefore, considering that  $\|x^k\| = \|y^k\| = 1$  for all  $k = 1, 2, \dots$ , we can apply again the inequality in item (c) to conclude

$$\langle Ax^k, y^k \rangle \leq \langle x^k, y^k \rangle \max \{q_A(x^k), q_A(y^k)\}, \quad k = 1, 2, \dots$$

By tending with  $k$  to infinity, we conclude that the inequality in item (b) also holds for  $\langle x, y \rangle = 0$  and the proof of the equivalence between (b) and (c) is complete.  $\square$

135 **Corollary 11.** *Assume that  $\mathcal{K}$  is a self-dual cone. If the quadratic function  $q_A$  is spherically quasi-convex, then  $A$  has the  $\mathcal{K}$ -Z-property.*

*Proof.* Let  $x, y \in \mathbb{R}^n \times \mathbb{R}^n$  such that  $x \in \mathcal{K}$ ,  $y \in \mathcal{K}^*$  and  $\langle x, y \rangle = 0$ . If  $x = 0$  or  $y = 0$  we have  $\langle Ax, y \rangle = 0$ . Thus, assume that  $x \neq 0$  and  $y \neq 0$ . Considering that  $\mathcal{K}$  is self-dual we have  $x/\|x\|, y/\|y\| \in \mathbb{S}^{n-1} \cap \mathcal{K}$ . Thus, since  $q_A$  is spherically  
140 quasi-convex and  $\langle x/\|x\|, y/\|y\| \rangle = 0$ , we obtain, from items (a) and (b) of Proposition 10, that  $\langle Ax, y \rangle \leq 0$ . Therefore,  $A$  has the  $\mathcal{K}$ -Z-property and the proof is concluded.  $\square$

**Theorem 12.** *The function  $q_A$  defined in (12) is spherically quasi-convex if and only if  $\varphi_A$  defined in (13) is quasi-convex.*

*Proof.* For  $c \in \mathbb{R}$ , let  $[q_A \leq c] := \{y \in \mathcal{C} : q_A(x) \leq c\}$  and  $[\varphi_A \leq c] := \{x \in \text{int}(\mathcal{K}) : \varphi_A(x) \leq c\}$  be the sublevel sets of  $q_A$  and  $\varphi_A$ , respectively, where  $c \in \mathbb{R}$ . Let  $\mathcal{K}_{[q_A \leq c]}$  be the cone spanned by  $[q_A \leq c]$ . Since  $\mathcal{C} = \mathbb{S}^{n-1} \cap \text{int}(\mathcal{K})$ , we conclude that  $x \in \text{int} \mathcal{K}$  if and only if  $x/\|x\| \in \mathcal{C}$ . Hence, the definitions of  $[q_A \leq c]$  and  $[\varphi_A \leq c]$  imply that

$$\mathcal{K}_{[q_A \leq c]} = [\varphi_A \leq c]. \quad (14)$$

145 Now, we assume that  $q_A$  is spherically quasi-convex. Thus, from Proposition 6 we conclude that  $[q_A \leq c]$  is spherically convex for all  $c \in \mathbb{R}$ . Hence, it follows

from Proposition 4 that the cone  $\mathcal{K}_{[q_A \leq c]}$  is convex and pointed, which implies from (14) that  $[\varphi_A \leq c]$  is convex for all  $c \in \mathbb{R}$ . Therefore,  $\varphi_A$  is quasi-convex. Conversely, assume that  $\varphi_A$  is quasi-convex. Thus,  $[\varphi_A \leq c]$  is convex for  
150 all  $c \in \mathbb{R}$ . On the other hand, since  $\mathcal{K}$  is a proper subdual cone,  $\text{int } \mathcal{K}$  is pointed. Thus, considering that  $[\varphi_A \leq c] \subset \text{int } \mathcal{K}$  is a cone, it is also a pointed cone. Hence, from (14) it follows that  $\mathcal{K}_{[q_A \leq c]}$  is a pointed convex cone. Hence, Proposition 4 implies that  $[q_A \leq c]$  is spherically convex for all  $c \in \mathbb{R}$ . Therefore, by using Proposition 6, we conclude that  $q_A$  is spherically quasi-convex and the  
155 proof is completed.  $\square$

**Corollary 13.** *Assume that  $\{x \in \text{int}(\mathcal{K}) : \langle A_c x, x \rangle < 0\} \neq \emptyset$ , where  $c \in \mathbb{R}$  and  $A_c := A - cI_n$ . If  $q_A$  defined in (12) is spherically quasi-convex, then the cone*

$$\{x \in \mathcal{K} : \langle A_c x, x \rangle \leq 0\}, \quad (15)$$

*is convex.*

*Proof.* Assume that  $q_A$  is spherically quasi-convex. Hence Theorem 12 implies that  $\varphi_A$  is quasi-convex and then  $[\varphi_A \leq c]$  is convex for all  $c \in \mathbb{R}$ . Since  $\{x \in \text{int}(\mathcal{K}) : \langle A_c x, x \rangle < 0\} \neq \emptyset$  we conclude that

$$\text{closure}(\{x \in \text{int}(\mathcal{K}) : \langle A_c x, x \rangle \leq 0\}) = \{x \in \mathcal{K} : \langle A_c x, x \rangle \leq 0\}.$$

where “closure” is the topological closure operator of a set. Thus, considering that  $[\varphi_A \leq c] = \{x \in \text{int}(\mathcal{K}) : \langle A_c x, x \rangle \leq 0\}$ , we obtain that

$$\text{closure}([\varphi_A \leq c]) = \{x \in \mathcal{K} : \langle A_c x, x \rangle \leq 0\},$$

Taking into account that  $[\varphi_A \leq c]$  is convex, the set  $\text{closure}([\varphi_A \leq c])$  is also convex. Therefore, last equality implies that the set in (15) is convex.  $\square$

#### 4.1. Spherically quasi-convex quadratic functions on the spherical positive or- 160 thant

In this section we present some properties of a quadratic function defined in the spherical positive orthant, which corresponds to  $\mathcal{K} = \mathbb{R}_+^n$ . We know that if  $A$

has only one eigenvalue, then  $q_A$  is constant and, consequently, it is spherically quasi-convex. Therefore, *throughout this section we assume that  $A$  has at least two distinct eigenvalues. The domains  $\mathcal{C}$  and  $\text{int}(\mathcal{K})$  of  $q_A$  and  $\varphi_A$ , respectively are given by*

$$\mathcal{C} := \mathbb{S}^{n-1} \cap \mathbb{R}_{++}^n, \quad \text{int}(\mathcal{K}) := \mathbb{R}_{++}^n, \quad (16)$$

We recall that  $q_A$  and  $\varphi_A$  are defined in (12) and (13), respectively. Next we present a technical lemma which will be useful in the sequel.

**Lemma 14.** *Let  $n \geq 2$  and  $V = [v^1 \ v^2 \ v^3 \ \dots \ v^n] \in \mathbb{R}^{n \times n}$  be an orthogonal matrix,  $A = V^\top \Lambda V$  and  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ . Assume that  $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_n$ . If  $v^1 \in \mathbb{R}_+^n$  and  $c \notin [\lambda_2, \lambda_n]$  then the sublevel set  $[\varphi_A \leq c]$  is convex.*

*Proof.* By using that  $VV^\top = I_n$  and  $A = V^\top \Lambda V$  we obtain from the definition (13) that

$$[\varphi_A \leq c] = \left\{ x \in \mathbb{R}_{++}^n : \sum_{i=1}^n (\lambda_i - c) \langle v^i, x \rangle^2 \leq 0 \right\}. \quad (17)$$

We will show that  $[\varphi_A \leq c]$  is convex for all  $c \notin [\lambda_2, \lambda_n]$ . If  $c < \lambda_1$ , then since  $v^1, v^2, \dots, v^n$  are linearly independent, we conclude from (17) that  $[\varphi_A \leq c] = \{0\}$  and therefore it is convex. If  $c = \lambda_1$ , then from (17) we conclude that  $[\varphi_A \leq c] = \mathcal{S} \cap \mathbb{R}_{++}^n$ , where  $\mathcal{S} := \{x \in \mathbb{R}^n : \langle v^2, x \rangle = 0, \dots, \langle v^n, x \rangle = 0\}$ , and hence  $[\varphi_A \leq c]$  is convex. Assume that  $\lambda_1 < c < \lambda_2$ . By letting  $y = V^\top x$ , i.e.,  $y_i = \langle v^i, x \rangle$ , for  $i = 1, \dots, n$ , and since  $v^1 \in \mathbb{R}_{++}^n$  and  $x \in \mathbb{R}_{++}^n$ , we have  $y_1 > 0$  and from (17) we obtain that  $[\varphi_A \leq c] = \mathcal{L} \cap V^\top \mathbb{R}_{++}^n$ , where

$$\mathcal{L} := \left\{ y = (y_1, \dots, y_n) \in \mathbb{R}^n : y_1 \geq \sqrt{\theta_2 y_2^2 + \dots + \theta_n y_n^2} \right\},$$

$$\theta_i = \frac{\lambda_i - c}{c - \lambda_1}, \quad i = 2, \dots, n.$$

Since  $\mathcal{L}$  and  $V^\top \mathbb{R}_{++}^n$  are convex sets, we conclude that  $[\varphi_A \leq c]$  is convex. If  $c \geq \lambda_n$ , then  $[\varphi_A \leq c] = \mathbb{R}_{++}^n$  is convex, which concludes the proof.  $\square$

**Lemma 15.** *Let  $\lambda$  be an eigenvalue of  $A$ . If  $\lambda I_n - A$  is copositive and  $\lambda \leq c$ , then*

$$[\varphi_A \leq c] = \mathbb{R}_{++}^n$$

and consequently it is a convex set.

*Proof.* Let  $c \in \mathbb{R}$  and  $[\varphi_A \leq c] = \{x \in \mathbb{R}_{++}^n : \langle Ax, x \rangle - c\|x\|^2 \leq 0\}$ . Since  
 170  $\lambda \leq c$  and  $\lambda I_n - A$  is copositive, we have  $\langle Ax, x \rangle - c\|x\|^2 \leq \langle Ax, x \rangle - \lambda\|x\|^2 =$   
 $\langle (A - \lambda I_n)x, x \rangle \leq 0$  for all  $x \in \mathbb{R}_{++}^n$ , which implies that  $[\varphi_A \leq c] = \mathbb{R}_{++}^n$ .  $\square$

The next theorem exhibits a series of implications and, in particular, conditions which imply that the quadratic function  $q_A$  is spherically quasi-convex.

**Theorem 16.** Let  $A \in \mathbb{R}^{n \times n}$  be a symmetric matrix and let  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$   
 175 its eigenvalues. Consider the following statements:

- (i)  $q_A$  is spherically quasi-convex.
- (ii)  $A$  is a Z-matrix.
- (iii)  $\lambda_2 I_n - A$  is copositive and there exists an eigenvector  $v^1 \in \mathbb{R}_+^n$  of  $A$  corresponding to the eigenvalue  $\lambda_1$  of  $A$ .
- 180 (iv)  $A$  is a Z-matrix and  $\lambda_2 \geq a_{ii}$  for any  $i \in \{1, 2, \dots, n\}$ .
- (v)  $A$  is a Z-matrix,  $\lambda_1 < \lambda_2$  and  $\lambda_2 \geq a_{ii}$  for all  $i \in \{1, 2, \dots, n\}$ .
- (vi)  $A$  is an irreducible Z-matrix and  $\lambda_2 \geq a_{ii}$  for all  $i \in \{1, 2, \dots, n\}$ .

Then the following implications hold:

$$\begin{array}{c}
 (v) \\
 \Downarrow \\
 (iv) \Leftarrow (iii) \Rightarrow (i) \Rightarrow (ii) \\
 \Uparrow \\
 (vi)
 \end{array}$$

*Proof.*

(v) $\Rightarrow$ (iii) $\Leftarrow$ (vi): It is easy to verify that  $\lambda_2 I_n - A$  is nonnegative and hence  
 185 copositive. Moreover, Perron-Frobenius theorem applied to the matrix  $\lambda_2 I_n - A$   
 implies that there exists an eigenvector  $v^1 \in \mathbb{R}_+^n$  corresponding to the largest  
 eigenvalue  $\lambda_2 - \lambda_1$  of  $\lambda_2 I_n - A$ , which is also the eigenvector of  $A$  corresponding  
 to  $\lambda_1$ .

(iii) $\Rightarrow$ (i): If  $c \leq \lambda_2$ , then Lemma 14 implies that  $[\varphi_A \leq c]$  is convex. If  
190  $c \geq \lambda_2$ , then from Lemma 15 we have  $[\varphi_A \leq c] = \mathbb{R}_{++}^n$ , which is convex. Hence,  
 $[\varphi_A \leq c]$  is convex for all  $c \in \mathbb{R}$ . Therefore, by using Theorem 12, we conclude  
that  $q_A$  is spherically quasi-convex function.

(i) $\Rightarrow$ (ii): From Corollary 1, it follows that  $A$  has the  $\mathbb{R}_+^n$ -Z-property. It is  
easy to check that this is equivalent to  $A$  being a Z-matrix.

195 (iii) $\Rightarrow$ (iv): Since (iii)  $\Rightarrow$  (i)  $\Rightarrow$  (ii), it follows that  $A$  is a Z-matrix. Since  
 $\lambda_2 I_n - A$  is copositive it follows that its diagonal elements are nonnegative.  
Hence,  $\lambda_2 \geq a_{ii}$  for all  $i \in \{1, 2, \dots, n\}$ .

□

**Corollary 17.** *Let  $n \geq 2$  and  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$  be the eigenvalues of  $A$ . Assume  
200 that  $-A$  is a positive matrix,  $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_n$  and  $0 < \lambda_2$ . Then  $q_A$  is  
spherically quasi-convex.*

*Proof.* First note that the matrix  $\lambda_2 I_n - A$  is a positive matrix and  $\lambda_2 - \lambda_1 > 0$   
is its largest eigenvalue. Thus, Theorem 1 implies that the eigenvalue  $\lambda_2 - \lambda_1$   
has the associated eigenvector  $v^1 \in \mathbb{R}_{++}^n$ . Since  $(\lambda_2 I_n - A)v^1 = (\lambda_2 - \lambda_1)v^1$   
205 we have  $Av^1 = \lambda_1 v^1$ . Hence  $v^1$  is also an eigenvector of  $A$  associated to  $\lambda_1$ .  
Therefore, considering that  $A$  is a Z-matrix,  $v^1 \in \mathbb{R}_+^n$ ,  $\lambda_1 < \lambda_2$  and  $\lambda_2 \geq a_{ii}$  for  
all  $i \in \{1, 2, \dots, n\}$ , it follows from Theorem 16 (v) $\Rightarrow$ (i) that  $q_A$  is spherically  
quasi-convex. □

In the following two examples we use Theorem 16 (iii) $\Rightarrow$ (i) to present a class  
210 of quadratic quasi-convex functions defined in the spherical positive orthant.

**Example 18.** *Let  $n \geq 3$  and  $V = [v^1 \ v^2 \ v^3 \ \dots \ v^n] \in \mathbb{R}^{n \times n}$  be an orthogonal  
matrix,  $A = V^\top \Lambda V$  and  $\Lambda := \text{diag}(\lambda, \mu, \dots, \mu, \nu)$ , where  $\lambda, \mu, \nu \in \mathbb{R}$ . Then  $q_A$   
is a spherically quasi-convex, whenever*

$$v^1 - \sqrt{\frac{\nu - \mu}{\mu - \lambda}} |v^n| \in \mathbb{R}_+^n, \quad \lambda < \mu < \nu, \quad (18)$$

where  $|v^n| := (|v_1^n|, \dots, |v_n^n|)$ . Indeed, by using that  $VV^\top = I_n$  and  $A = V^\top \Lambda V$ ,



after some calculations we conclude that

$$\langle Ax, x \rangle - \mu \|x\|^2 = (\mu - \lambda) \left[ -\langle v^1, x \rangle^2 + \frac{\nu - \mu}{\mu - \lambda} \langle v^n, x \rangle^2 \right]. \quad (19)$$

Thus, using the condition in (18) and considering that  $x \in \mathbb{R}_{++}^n$ , we have

$$-\langle v^1, x \rangle^2 + \frac{\nu - \mu}{\mu - \lambda} \langle v^n, x \rangle^2 \leq \frac{\nu - \mu}{\mu - \lambda} [-\langle |v^n|, x \rangle^2 + \langle v^n, x \rangle^2] \leq 0.$$

Hence, by combining the last inequality with (19), we conclude that  $\mu I_n - A$  is copositive. Therefore, since  $v^1 \in \mathbb{R}_+^n$  we can apply Theorem 16 (iii)  $\Rightarrow$  (i) with  $\lambda_2 = \mu$  to conclude that  $q_A$  is a spherically quasi-convex function. For instance, taking  $\lambda < (\lambda + \nu)/2 < \mu < \nu$  the vectors  $v^1 = (e^1 + e^n)/\sqrt{2}$ ,  $v^2 = e^2$ ,  $\dots$ ,  $v^{n-1} = e^{n-1}$ ,  $v^n = (e^1 - e^n)/\sqrt{2}$ , satisfy (18).

**Example 19.** Let  $n \geq 3$  and  $V = [v^1 \ v^2 \ v^3 \ \dots \ v^n] \in \mathbb{R}^{n \times n}$  be an orthogonal matrix,  $A = V^\top \Lambda V$  and  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ . Then  $q_A$  is a spherically quasi-convex, whenever

$$v^1 = (v_1^1, \dots, v_n^1) \in \mathbb{R}_{++}^n, \quad \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n \leq \lambda_2 + \frac{\alpha^2}{(n-2)}(\lambda_2 - \lambda_1), \quad (20)$$

where  $\alpha := \min\{v_i^1 \neq 0 : i = 1, \dots, n\}$ . Indeed, by using that  $VV^\top = I_n$  and the definition of the matrix  $A$ , we obtain that

$$\langle Ax, x \rangle - \lambda_2 \|x\|^2 = (\lambda_1 - \lambda_2) \langle v^1, x \rangle^2 + (\lambda_3 - \lambda_2) \langle v^3, x \rangle^2 + \dots + (\lambda_n - \lambda_2) \langle v^n, x \rangle^2.$$

Since  $\lambda_2 - \lambda_1 > 0$  and  $0 \leq \lambda_j - \lambda_2 \leq \lambda_n - \lambda_2$  for all  $j = 3, \dots, n$ , the last equality becomes

$$\langle Ax, x \rangle - \lambda_2 \|x\|^2 \leq (\lambda_2 - \lambda_1) \left[ -\langle v^1, x \rangle^2 + \frac{\lambda_n - \lambda_2}{\lambda_2 - \lambda_1} (\langle v^3, x \rangle^2 + \dots + \langle v^n, x \rangle^2) \right]. \quad (21)$$

On the other hand, by using that  $v_i^1 \in \mathbb{R}_{++}$  and  $v_i^1 \geq \alpha$  for all  $i = 1, \dots, n$ , we obtain that

$$\begin{aligned} \langle v^1, x \rangle^2 &= (v_1^1 x_1 + \dots + v_n^1 x_n)^2 \\ &\geq \alpha^2 (x_1 + \dots + x_n)^2 \geq \alpha^2 (x_1^2 + \dots + x_n^2) = \alpha^2 \|x\|^2 \end{aligned} \quad (22)$$

for all  $x \in \mathbb{R}_+^n$ . Moreover, taking into account that  $\|v^j\| = 1$  for all  $j = 3, \dots, n$ , it follows from the Cauchy-Schwarz inequality, that

$$\langle v^3, x \rangle^2 + \dots + \langle v^n, x \rangle^2 \leq \|v^3\|^2 \|x\|^2 + \dots + \|v^n\|^2 \|x\|^2 \leq (n-2) \|x\|^2$$

for all  $x \in \mathbb{R}_+^n$ . Thus, combining the last inequalities with (21) and (22) and considering that the last inequality in (20) is equivalent to  $-\alpha^2 + (n-2)(\lambda_n - \lambda_2)/(\lambda_2 - \lambda_1) \leq 0$ , we have

$$\langle Ax, x \rangle - \lambda_2 \|x\|^2 \leq (\lambda_2 - \lambda_1) \left[ -\alpha^2 + (n-2) \frac{\lambda_n - \lambda_2}{\lambda_2 - \lambda_1} \right] \|x\|^2 \leq 0$$

for all  $x \in \mathbb{R}_+^n$ . Hence, we conclude that  $\lambda_2 I_n - A$  is copositive. Therefore, since  $v^1 \in \mathbb{R}_+^n$  is the eigenvector of  $A$  corresponding to the eigenvalue  $\lambda_1$ , we apply Theorem 16 (iii) $\Rightarrow$ (i), to conclude that  $q_A$  is spherically quasi-convex function. For instance,  $n \geq 3$ ,  $A = V^\top \Lambda V$ ,  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$  and  $V = [v^1 \ v^2 \ v^3 \ \dots \ v^n] \in \mathbb{R}^{n \times n}$ , where  $\alpha = 1/\sqrt{n}$ ,

$$v^1 := \frac{1}{\sqrt{n}} \sum_{i=1}^n e^i, \quad v^j := \frac{1}{\sqrt{(n+1-j) + (n+1-j)^2}} \left[ e^1 - (n+1-j)e^j + \sum_{i>j}^n e^i \right],$$

for  $j = 2, \dots, n$  and  $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_n < \lambda_2 + (1/[n(n-2)])(\lambda_2 - \lambda_1)$ , satisfy the orthogonality of  $V$  and the condition (20).

In the next theorem we establish the characterization for quasi-convex quadratic functions  $q_A$  on the spherical positive orthant where  $A$  is symmetric having only two distinct eigenvalues.

220

**Theorem 20.** *Let  $n \geq 3$  and  $A \in \mathbb{R}^{n \times n}$  be a symmetric matrix with only two distinct eigenvalues, such that its smallest one has multiplicity one. Then,  $q_A$  is spherically quasi-convex if and only if there is an eigenvector of  $A$  corresponding to the smallest eigenvalue with all components nonnegative.*

*Proof.* Let  $A := (a_{ij}) \in \mathbb{R}^{n \times n}$ ,  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalues of  $A$  corresponding to an orthonormal set of eigenvectors  $v^1, v^2, \dots, v^n$ , respectively. Then, we can assume without loss of generality that  $\lambda_1 =: \lambda < \mu := \lambda_2 = \dots = \lambda_n$ . Thus, we have

$$A = V \Lambda V^T, \quad V := [v^1 \ v^2 \ \dots \ v^n] \in \mathbb{R}^{n \times n}, \quad \Lambda := \text{diag}(\lambda, \mu, \dots, \mu) \in \mathbb{R}^{n \times n}. \quad (23)$$

First we assume that  $q_A$  is a spherically quasi-convex function. The matrix  $\Lambda$  can be equivalently written as follows

$$\Lambda = \mu I_n + (\lambda - \mu)D, \quad (24)$$

where  $D := (d_{ij}) \in \mathbb{R}^{n \times n}$  has all entries 0 except the  $d_{11}$  entry which is 1. Then (24) and (23) imply

$$a_{ij} = (\lambda - \mu)v_i^1 v_j^1 \quad i \neq j. \quad (25)$$

225 Since  $q_A$  is spherically quasi-convex and  $e^i \in \mathcal{C} = \mathbb{S}^{n-1} \cap \mathbb{R}_{++}^n$  for all  $i = 1, \dots, n$ , by using item (b) of Proposition 10 we conclude that  $a_{ij} \leq 0$  for all  $i, j = 1, \dots, n$  with  $i \neq j$ . Thus, since  $\lambda < \mu$ , we obtain from (25) that  $0 \leq v_i^1 v_j^1$  for all  $i \neq j$ , which implies  $v^1 \in \mathbb{R}_+^n$  or  $-v^1 \in \mathbb{R}_+^n$ . Therefore, there is an eigenvector corresponding to the smallest eigenvalue with all components  
230 nonnegative. Conversely, assume that  $v^1 \in \mathbb{R}_+^n$ . Then, applying Lemma 14 with  $\lambda = \lambda_1 < \mu = \lambda_2 = \dots = \lambda_n$  we conclude that  $[\varphi_A \leq c]$  is convex for all  $c \in \mathbb{R}$ , and hence  $\varphi_A$  is quasi-convex. Therefore, by using Theorem 12, we conclude that  $q_A$  is spherically quasi-convex.  $\square$

In the following example we present a class of matrices satisfying the as-  
235 sumptions of Theorem 20.

**Example 21.** Let  $v \in \mathbb{R}_+^n$  and define the Householder matrix  $H := I_n - 2vv^T/\|v\|^2$ . The matrix  $H$  is nonsingular and symmetric. Moreover,  $Hv = -v$  and letting  $E := \{u \in \mathbb{R}^n : \langle v, u \rangle = 0\}$  we have  $Hu = u$  for all  $u \in E$ . Since the dimension of  $E$  is  $n - 1$ , we conclude that  $-1$  and  $1$  are eigenvalues of  $H$   
240 with multiplicities one and  $n - 1$ , respectively. Furthermore, the eigenvector corresponding to the smallest eigenvalue of  $H$  has all components nonnegative. Therefore, Theorem 20 implies that  $q_H(x) = \langle Hx, x \rangle$  is spherically quasi-convex.

In order to give a complete characterization of the spherical quasi-convexity of  $q_A$  for the case when  $A$  is diagonal, in the following result we start with  
245 a necessary condition for  $q_A$  to be spherically quasi-convex on the spherical positive orthant.

**Lemma 22.** *Let  $n \geq 3$ ,  $\mathcal{C} = \mathbb{S}^{n-1} \cap \mathbb{R}_{++}^n$  and  $A \in \mathbb{R}^{n \times n}$  be a nonsingular diagonal matrix. If  $q_A$  is spherically quasi-convex, then  $A$  has only two distinct eigenvalues, such that its smallest one has multiplicity one.*

*Proof.* The proof will be made by contradiction. First we assume that  $A$  has at least three distinct eigenvalues, among which exactly two are negative, or at least two distinct eigenvalues, among which exactly one is negative and has multiplicity greater than one, i.e.,

$$Ae^1 = -\lambda_1 e^1, \quad Ae^2 = -\lambda_2 e^2, \quad Ae^3 = \lambda_3 e^3, \quad \lambda_1, \lambda_2, \lambda_3 > 0 \quad (26)$$

with  $-\lambda_1 < -\lambda_2 < 0 < \lambda_3$  or  $-\lambda_1 = -\lambda_2 < 0 < \lambda_3$  and  $e^1, e^2, e^3$  are canonical vectors of  $\mathbb{R}^n$ . Define the following two auxiliary vectors

$$v^1 := e^1 + t_1 e^3, \quad v^2 := e^2 + t_2 e^3, \quad t_i = \sqrt{\frac{\lambda_i}{\lambda_3}}, \quad i = 1, 2. \quad (27)$$

Hence, (26) and (27) imply that  $\langle Av^1, v^1 \rangle = 0$  and  $\langle Av^2, v^2 \rangle = 0$  and since  $v^1, v^2 \in \mathbb{R}_+^n$ , we conclude that  $v^1, v^2 \in \{x \in \mathbb{R}_+^n : \langle Ax, x \rangle \leq 0\}$ . However, using again (26) and (27) we obtain that

$$\langle A(v^1 + v^2), v^1 + v^2 \rangle = 2\sqrt{\lambda_1 \lambda_2} > 0,$$

250 and then  $v^1 + v^2 \notin \{x \in \mathbb{R}_+^n : \langle Ax, x \rangle \leq 0\}$ . Thus,  $\{x \in \mathbb{R}_+^n : \langle Ax, x \rangle \leq 0\}$  is not convex. Finally, assume that  $A$  has at least three distinct eigenvalues or at least two distinct ones with the smallest one having multiplicity greater than one. Let  $\lambda, \mu, \nu$  be eigenvalues of  $A$  such that  $\lambda < \mu < \nu$  or  $\lambda = \mu < \nu$ . Take a constant  $c \in \mathbb{R}$  such that  $\mu < c < \nu$ . Letting  $A_c := A - cI_n$  we conclude that  
255  $\lambda - c, \mu - c, \nu - c$  are eigenvalues of  $A_c$  and satisfy  $\lambda - c < \mu - c < 0 < \nu - c$  or  $\lambda - c = \mu - c < 0 < \nu - c$ . Hence, by the first part of the proof, with  $A_c$  in the role of  $A$ , we conclude that  $\{x \in \mathbb{R}_+^n : \langle A_c x, x \rangle \leq 0\}$  is not convex. On the other hand, due to  $e^i \in \mathbb{R}_+^n$  and  $\langle Ae^i, e^i \rangle = \lambda - c < 0$ , for some  $i$ , we obtain that  $\{x \in \mathbb{R}_{++}^n : \langle A_c x, x \rangle < 0\} \neq \emptyset$ . Thus, applying Corollary 13 with  
260  $\mathcal{K} = \mathbb{R}_+^n$  and taking into account that  $\{x \in \mathbb{R}_+^n : \langle A_c x, x \rangle \leq 0\}$  is not convex, we conclude that  $q_A$  is not spherically quasi-convex, which is absurd and the proof is complete.  $\square$

To make the paper self-contained we state the result of [21, Theorem 1] explicitly here:

**Theorem 23.** *Let  $\mathcal{C} = \mathbb{S}^{n-1} \cap \mathbb{R}_{++}^n$  and  $A \in \mathbb{R}^{n \times n}$  be a symmetric matrix. Then  $q_A$  is spherically convex if and only if there exists  $\lambda \in \mathbb{R}$  such that  $A = \lambda I_n$ . In this case  $q_A$  is a constant function.*

The next result gives a full characterization for  $q_A$  to be spherically quasi-convex quadratic function on the spherical positive orthant, where  $A$  is a diagonal matrix. The proof of this result is a combination of Theorem 20, Lemma 22 and Theorem 23. Before presenting the result we need the following definition: A function is called *merely spherically quasi-convex* if it is spherically quasi-convex, but it is not spherically convex.

**Theorem 24.** *Let  $n \geq 3$  and  $A \in \mathbb{R}^{n \times n}$  be a nonsingular diagonal matrix. Then  $q_A$  is merely spherically quasi-convex if and only if  $A$  has only two eigenvalues, such that its smallest one has multiplicity one and has a corresponding eigenvector with all components nonnegative.*

We end this section by showing that, if a symmetric matrix  $A$  has three eigenvectors in the nonnegative orthant associated to at least two distinct eigenvalues, then the associated quadratic function  $q_A$  cannot be spherically quasi-convex.

**Lemma 25.** *Let  $n \geq 3$  and  $v^1, v^2, v^3 \in \mathbb{R}^n$  be distinct eigenvectors of a symmetric matrix  $A$  associated to the eigenvalues  $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ , respectively, among which at least two are distinct. If  $q_A$  is spherically quasi-convex, then  $v^i \notin \mathbb{R}_+^n$  for some  $i = 1, 2, 3$ .*

*Proof.* Assume by contradiction that  $v^1, v^2, v^3 \in \mathbb{R}_+^n$ . Without loss of generality we can also assume that  $\|v^i\| = 1$ , for  $i = 1, 2, 3$ . We have three possibilities:  $\lambda_1 < \lambda_2 < \lambda_3$ ,  $\lambda_1 = \lambda_2 < \lambda_3$  or  $\lambda_1 < \lambda_2 = \lambda_3$ . We start by analyzing the possibilities  $\lambda_1 < \lambda_2 < \lambda_3$  or  $\lambda_1 = \lambda_2 < \lambda_3$ . First we assume that  $\lambda_1 < \lambda_2 < 0 < \lambda_3$  or  $\lambda_1 = \lambda_2 < 0 < \lambda_3$ . Define the following vectors

$$w^1 := v^1 + t_1 v^3, \quad w^2 := v^2 + t_2 v^3, \quad t_1 := \sqrt{\frac{-\lambda_1}{\lambda_3}}, \quad t_2 := \sqrt{\frac{-\lambda_2}{\lambda_3}}. \quad (28)$$

We have  $\langle v^i, v^j \rangle = 0$  for all  $i, j = 1, 2, 3$  with  $i \neq j$ , and since

$$Av^1 = \lambda_1 v^1, \quad Av^2 = \lambda_2 v^2, \quad Av^3 = \lambda_3 v^3, \quad v^1, v^2, v^3 \in \mathbb{R}_+^n, \quad (29)$$

we conclude from (28) that  $\langle Aw^1, w^1 \rangle = 0$  and  $\langle Aw^2, w^2 \rangle = 0$ . Moreover, since  $v^1, v^2, v^3 \in \mathbb{R}_+^n$  we conclude that  $w^1, w^2 \in \{x \in \mathbb{R}_+^n : \langle Ax, x \rangle \leq 0\}$ . On the other hand, by using (29) and (28), we obtain that

$$\langle A(w^1 + w^2), w^1 + w^2 \rangle = 2\sqrt{\lambda_1 \lambda_2} > 0,$$

and then  $w^1 + w^2 \notin \{x \in \mathbb{R}_+^n : \langle Ax, x \rangle \leq 0\}$ . Thus,  $\{x \in \mathbb{R}_+^n : \langle Ax, x \rangle \leq 0\}$  is not a convex cone. For the general case, take  $c \in \mathbb{R}$  such that  $\lambda_2 < c < \lambda_3$ . Letting  $A_c := A - cI_n$  we conclude that  $\lambda_1 - c, \lambda_2 - c, \lambda_3 - c$  are eigenvalues of  $A_c$  and satisfying  $\lambda_1 - c < \lambda_2 - c < 0 < \lambda_3 - c$  or  $\lambda_1 - c = \lambda_2 - c < 0 < \lambda_3 - c$  with the three corresponding orthonormal eigenvectors  $v^1, v^2, v^3 \in \mathbb{R}_+^n$ . Hence, by the first part of the proof, with  $A_c$  in the role of  $A$ , we conclude that the cone  $\{x \in \mathbb{R}_+^n : \langle A_c x, x \rangle \leq 0\}$  is not convex. On the other hand, due to  $v^1 \in \mathbb{R}_+^n$  and  $\langle Av^1, v^1 \rangle = \lambda_1 - c < 0$ , we have  $\{x \in \mathbb{R}_{++}^n : \langle A_c x, x \rangle < 0\} \neq \emptyset$ . Thus, applying Corollary 13 with  $\mathcal{K} = \mathbb{R}_+^n$  and taking into account that  $\{x \in \mathbb{R}_+^n : \langle A_c x, x \rangle \leq 0\}$  is not convex, we conclude that  $q_A$  is not spherically quasi-convex, which is absurd. To analyze the possibility  $\lambda_1 < \lambda_2 = \lambda_3$ , first assume that  $\lambda_1 < 0 < \lambda_2 = \lambda_3$  and define the vectors

$$w^1 := t_1 v^1 + v^3, \quad w^2 := t_2 v^1 + v^3, \quad t_1 = \sqrt{\frac{\lambda_2}{-\lambda_1}}, \quad t_2 = \sqrt{\frac{\lambda_3}{-\lambda_1}},$$

285 and then proceed as above to obtain again a contradiction. Therefore,  $v^i \notin \mathbb{R}_+^n$   
for some  $i = 1, 2, 3$  and the proof is complete.  $\square$

## 5. Final remarks

This paper is a continuation of [4, 19, 21], where intrinsic properties of the spherically convex sets and functions were studied. As far as we know this is  
290 the first study of spherically quasi-convex quadratic functions. As an even more

challenging problem, we will work towards developing efficient algorithms for constrained optimization on spherically convex sets. Minimizing a quadratic function on the spherical nonnegative orthant is of particular interest because the nonnegativity of the minimum value is equivalent to the copositivity of the corresponding matrix [6, Proposition 1.3] and to the nonnegativity of its Pareto eigenvalues [6, Theorem 4.3]. Considering the intrinsic geometrical properties of the sphere will open interesting perspectives for detecting the copositivity of a matrix. We foresee further progress in these topics in the near future.

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