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LOCAL CONVERGENCE AND STABILITY OF TIGHT BRIDGE-ADDABLE GRAPH CLASSES

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Abstract. A class of graphs is *bridge-addable* if given a graph G in the class, any graph obtained by adding an edge between two connected components of G is also in the class. The authors recently proved a conjecture of McDiarmid, Steger, and Welsh stating that if $\mathcal{G}$ is bridge-addable and G_n is a uniform n-vertex graph from $\mathcal{G}$, then G_n is connected with probability at least $(1 + o_n(1))e^{-1/2}$. The constant $e^{-1/2}$ is best possible since it is reached for the class of all forests.

In this paper we prove a form of uniqueness in this statement: if $\mathcal{G}$ is a bridge-addable class and the random graph G_n is connected with probability close to $e^{-1/2}$, then G_n is asymptotically close to a uniform n-vertex random forest in a local sense. For example, if the probability converges to $e^{-1/2}$, then G_n converges in the sense of Benjamini-Schramm to the uniformly infinite random forest F_∞. This result is reminiscent of so-called "stability results" in extremal graph theory, with the difference that here the stable extremum is not a graph but a graph class.

1. Introduction and Main Results

In this paper all graphs are simple. A graph is labeled if its vertex set is of the form $\{1, \ldots, n\}$ for some $n \geq 1$. An unlabeled graph is an equivalence class of labeled graphs by relabeling. Unless mentioned otherwise, graphs in this paper are labeled. A class of (labeled) graphs $\mathcal{G}$ is *bridge-addable* if given a graph G in the class, and an edge e of G whose endpoints belong to two distinct connected components, then $G \cup \{e\}$ is also in the class. Examples of bridge-addable classes include planar graphs, graphs that admit a perfect matching, forests, or H-free graphs where H is any 2-edge connected graph (see many more examples in [ABMR12, CP15]).

McDiarmid, Steger and Welsh [MSW06] conjectured that every bridge-addable class of graphs with n vertices contains at least a proportion $(1 + o_n(1))e^{-1/2}$ of connected graphs. This has recently been proved by the authors. In the next statement and later, we denote by $\mathcal{G}_n$ the set of graphs in $\mathcal{G}$ with n vertices, and by G_n a uniformly random element of $\mathcal{G}_n$. Herein, we will refer to statements and equations from [CP15] as they are numbered in there.

Theorem A. [[CP15, Theorem 2]] For every $\epsilon > 0$, there exists n_0 such that for every bridge-addable class $\mathcal{G}$ and every $n \geq n_0$, we have

$$\mathbf{Pr}\,(G_n \text{ is connected}) \geq (1 - \epsilon)e^{-1/2}. \tag{1.1}$$

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If $\mathcal{G}$ is the class of all forests, then Theorem A is asymptotically tight, since, as shown in [Rén59], if F_n is a uniformly random forest on n vertices, then, as n tends to infinity,

$$\mathbf{Pr}\left(F_n \text{ is connected}\right) \longrightarrow e^{-1/2}. \tag{1.2}$$

The aim of this paper is to show that any bridge-addable class of graphs that comes close to achieving the constant $e^{-1/2}$ is "close" to a uniformly random forest in a local sense.

Definition 1.1. For any $\zeta > 0$, we say that a bridge-addable class of graphs $\mathcal{G}$ is *ζ-tight with respect to connectivity* (or simply *ζ-tight*) if there exists n_0 such that for every $n \geq n_0$ we have

$$\mathbf{Pr}\left(G_n \text{ is connected}\right) \leq (1+\zeta)e^{-1/2} \ ,$$

where we recall that G_n is chosen uniformly at random from $\mathcal{G}_n$.

If H is a graph we let $|H|$ be its number of vertices. We denote by $\mathcal{U}$ the set of *unlabeled, unrooted* trees and by $\mathcal{T}$ the set of *unlabeled, rooted* trees, *i.e.* trees with a marked vertex called the root. For every unrooted tree $U \in \mathcal{U}$, we denote by $\mathrm{Aut}_u(U)$ the number of automorphisms of U, and for every rooted tree $T \in \mathcal{T}$, we denote by $\mathrm{Aut}_r(T)$ the number of automorphisms of T that fix its root. Moreover given k unrooted trees $U_1, \dots, U_k$ in $\mathcal{U}$, we denote by $\mathrm{Aut}_u(U_1, \dots, U_k)$ the number of automorphisms of the forest formed by disjoint copies of $U_1, \dots, U_k$.

Given a graph H, we let $\mathit{Small}(H)$ denote the graph formed by all the components of H that are not the largest one (in case of a tie, we say that the largest component of the graph is the one with the largest vertex label among all candidates). In what follows, we will always see $\mathit{Small}(H)$ as an unlabeled graph. Given a graph G and a rooted tree $T \in \mathcal{T}$, we let $\alpha^G(T)$ be the number of pendant copies of the tree T in G. More precisely, $\alpha^G(T)$ is the number of vertices v of G having the following property: there is at least one cut-edge e incident to v, and if we remove the such cut-edge that separates v from the largest possible component, the vertex v lies in a component of the graph that is a tree, rooted at v, which is isomorphic to T. The following is classical and the proof is omitted.

Theorem B. Let F_n be a uniformly random forest with n vertices. Then, for any fixed unlabeled unrooted forest $\mathbf{f}$ we have as n tends to infinity,

$$\mathbf{Pr}\big(\mathit{Small}(F_n) \equiv \mathbf{f}\big) \longrightarrow p_\infty(\mathbf{f}) := e^{-1/2} \frac{e^{-|\mathbf{f}|}}{\mathrm{Aut}_u(\mathbf{f})} \ , \tag{1.3}$$

where $\equiv$ denotes unlabeled graph isomorphism. Moreover, p_∞ is a probability distribution on the set of unlabeled unrooted forests.

For any fixed rooted tree $T \in \mathcal{T}$ we have as n tends to infinity,

$$\frac{\alpha^{F_n}(T)}{n} \xrightarrow{(p)} a_\infty(T) := \frac{e^{-|T|}}{\mathrm{Aut}_r(T)} \ . \tag{1.4}$$

where (p) indicates convergence in probability. Moreover a_∞ is a probability measure on $\mathcal{T}$.

Our main result states that, if $\mathcal{G}$ is bridge-addable and G_n satisfies an approximate version of (1.2), then it also satisfies an approximate version of (1.3) and (1.4).

Theorem 1.2 (**Main result**). *For every $\epsilon, \eta > 0$, there exist $\zeta > 0$ and n_0 such that for every ζ-tight bridge-addable class $\mathcal{G}$ and every $n \geq n_0$, the following holds.*

i) For every unlabeled unrooted forest $\mathbf{f}$,

$$\left|\mathbf{Pr}\big(\,Small(G_n) \equiv \mathbf{f}\big) - p_\infty(\mathbf{f})\right| < \epsilon\ .$$

ii) If $\mathcal{T}$ is the set of unlabeled rooted trees,

$$\mathbf{Pr}\left(\forall T \in \mathcal{T} : \left|\frac{\alpha^{G_n}(T)}{n} - a_\infty(T)\right| < \eta\right) > 1 - \epsilon\ .$$

Remark 1.3. Theorem 1.2, can be viewed both as a *uniqueness result* (since it states that in the limit, and through the lens of local observables, the class of forests is the only one to reach the optimum value $e^{-1/2}$) and as a *stability result* (since it also states that the only classes than come *close* to the extremal value $e^{-1/2}$ are *close* to forests, again through local observables of random graphs). Here we use the terminology "stability result" on purpose, by analogy with the field of extremal graph theory.

Our main result suggests that the question of stability of extremal graph classes, with respect to appropriate graph limit topologies (here, local convergence), should be further examined.

A bridge-addable class $\mathcal{G}$ is *tight* if it is ζ-tight for any $\zeta > 0$, that is to say, as n tends to infinity,

$$\mathbf{Pr}(G_n \text{ is connected}) \to e^{-1/2}\ .$$

Theorem 1.2 has the following consequence for tight bridge-addable classes.

Corollary 1.4. *Let $\mathcal{G}$ be a tight bridge-addable class of graphs. Then,*

$$Small(G_n) \xrightarrow{(d)} p_\infty\ . \tag{1.5}$$

and, for any unlabelled rooted tree $T \in \mathcal{T}$,

$$\frac{\alpha^{G_n}(T)}{n} \xrightarrow{(p)} a_\infty(T)\ . \tag{1.6}$$

Let V_n be a uniformly random vertex in G_n. Then for a given $T \in \mathcal{T}$, conditionally to G_n, the quantity $\alpha^{G_n}(T)/n$ is the probability that there is a copy of T hanging from V_n. Readers familiar with the Benjamini-Schramm (BS) convergence of rooted graphs will note the similarity with this notion (see [BS01, Lov12]).

It easily follows from a similar statement for random trees proved in [Ald98] that if F_n is a uniformly random forest on n vertices rooted at a uniformly random vertex V_n, then

$$(F_n, V_n) \to (F_\infty, V_\infty)\ ,$$

in distribution in the BS-sense, where (F_∞, V_∞) is the "uniformly random infinite rooted forest" (which we could also have called "uniformly random infinite rooted tree", since it is almost surely a tree). Namely, (F_∞, V_∞) can be constructed as follows. Consider a semi-infinite path, starting at a vertex V_∞, and identify each vertex of this path with the root of an independent Galton-Watson tree with offspring distribution $Poisson(1)$. In our context, passing from pendant trees to balls is an easy task, and one can deduce the following from Corollary 1.4.

Corollary 1.5. *Let $\mathcal{G}$ be a tight bridge-addable graph class. Let G_n be a uniformly random graph from $\mathcal{G}_n$ and let V_n be a uniformly random vertex of G_n. Then (G_n, V_n) converges to (F_∞, V_∞) in distribution in the Benjamini-Schramm sense.*

Remark 1.6. Our main theorem asserts that ζ-tight bridge-addable classes are "locally similar" to random forests in some precise sense. However, they can be very different from other perspectives. For example, consider the class $\mathcal{G}$ of graphs defined as follows. $\mathcal{G}_n$ is the smallest bridge-addable class containing the graph on $\{1, \dots, n\}$ in which all edges between vertices in $\{1, \dots, \lfloor n^{2/3} \rfloor\}$ are present and all other vertices are isolated. Then $\mathcal{G} = \cup_{n\geq 1}\mathcal{G}_n$ is a bridge-addable class, and it is easy to see that it is tight (see Appendix A for more details). However a uniformly random element of $\mathcal{G}_n$ is very different from a random forest. In particular, almost all edges of G_n belong to a clique of size $\lfloor n^{2/3} \rfloor$.

Remark 1.7. Our results do not imply that random graphs from tight bride-addable classes look like random forests in a "global" sense. Following the lines of the example of Remark 1.6, let $\mathcal{G}_n$ be the smallest bridge-addable class on $\{1, \dots, n\}$ containing the graph where the vertices in $\{1, \dots, \lfloor n^{2/3} \rfloor\}$ induce a path and all the other ones are isolated. Then $\mathcal{G} = \cup_{n\geq 1}\mathcal{G}_n$ is a tight bridge-addable class. Nevertheless, the diameter of the random graph G_n is at least $\lfloor n^{2/3} \rfloor$, while the diameter of the largest tree in a uniformly random n-vertex forest is of order $\sqrt{n}$. Moreover, when renormalized by a scaling factor of $n^{-2/3}$, G_n converges for the Gromov-Hausdorff topology to a real interval and not to the CRT (Continuum Random Tree, see [Ald93]). However, it may be true in general that *typical* distances in tight bridge-addable classes are of order $\sqrt{n}$. We leave this question open.

We conclude this list of results with a simpler statement that does not require the full strength of our main theorems (it is a relatively easy consequence of the results of [CP15], and we will prove it in Section 2).

Theorem 1.8. *Let $\mathcal{G}$ be a tight bridge-addable class and G_n a uniformly random graph from $\mathcal{G}_n$. Then for any $k \geq 0$, we have*

$$\mathbf{Pr}\left(G_n \text{ has } k+1 \text{ connected components}\right) \longrightarrow e^{-1/2}\,\frac{2^{-k}}{k!}\ .$$

In other words, the number of connected components of G_n converges in distribution to $1 + Poisson(1/2)$.

Structure of the paper. The proof of our main result roughly follows the one of Theorem A, which we proved in [CP15]. Very loosely speaking we show that for a class to be ζ-tight, some form of tightness has to occur in each intermediate inequality proven in [CP15]. As the length of the present paper shows, there is however quite an important amount of work to be done to achieve this goal.

We start in Section 2 by proving elementary results about the number of components (including Theorem 1.8) and we introduce some notions that will play a crucial role in the rest of the proof. Importantly, in Section 2.2, we introduce the partitioning of the space that underlies our technique of local double-counting from [CP15]. In particular we define the notion of "box" that we use in order to partition each graph class according to the local structure of the graphs it contains.

Sections 3 and 4 occupy the most important part of the paper. In Section 3, we prove an analogue of Theorem 1.2 under the assumption that all elements of $\mathcal{G}$ are forests. This is done in several steps. In 3.1 we define the notion of "good boxes" and we prove that most of the mass in tight bridge-addable graph classes is localized inside good boxes. These good boxes have the property that they locally realize the extremal value of the optimization problem introduced in [CP15]. This optimization problem expresses some ratios inherited from a double-counting strategy in terms of parameters that record the local structure of the graphs. In 3.2 we study the stability of this problem and deduce that for good boxes, all parameters have to be close to the unique extremum value (closely related to the quantities a_∞ and p_∞ appearing in Theorem 1.2). In 3.3 we use these facts to prove a version of our main result when the graph G_n has one or two components. In 3.4 we use an induction on the number of components to conclude the proof, in the case of forests.

In Section 4, we address the case of general bridge-addable graph classes. In 4.1 we prove that ζ-tight bridge-addable classes tend to have many removable edges (edges that when deleted from a graph in the class, give rise to a graph in the class), and in 4.2 we use this property and the results of Section 3 to conclude the proof of Theorem 1.2. We conclude with the proof of Corollary 1.5. Finally, Appendix A gives more details about the example of Remark 1.6.

2. First results and set-up for the proof

In this section, we obtain our first results and we introduce important notions and notation used in the whole paper. In 2.1 we study the number of connected components and we prove Theorem 1.8. In 2.2, we define the partitioning of the space that underlies our technique of local double-counting. Finally in 2.3, we give a few precisions for the use of quantifiers in the rest of the paper.

2.1. **Number of components in bridge-addable graph classes.** Through the rest of the paper, for a bridge-addable class of graphs $\mathcal{G}$ and for $i \geq 1$, we denote by $\mathcal{G}_n^{(i)}$ the set of n-vertex graphs in $\mathcal{G}$ having i connected components. An elegant double-counting argument going back to [MSW06] asserts that for all $i \geq 1$, and $n \geq 1$ we have

$$i \cdot \left|\mathcal{G}_n^{(i+1)}\right| \leq \left|\mathcal{G}_n^{(i)}\right| . \tag{2.1}$$

The main achievement of [CP15] was to improve this bound by a factor $\frac{1}{2}$, asymptotically.

Lemma C ([CP15, Proposition 5])**.** For every η and every m, if $\mathcal{G}$ is a bridge-addable class and n is large enough, we have for every $i \leq m$,

$$i|\mathcal{G}_n^{(i+1)}| \leq \left(\frac{1}{2} + \eta\right) |\mathcal{G}_n^{(i)}| . \tag{2.2}$$

The following lemma provides a converse inequality to (2.2) for ζ-tight classes. It directly implies Theorem 1.8.

Lemma 2.1. *For every η and every m there exists ζ such that for every ζ-tight bridge-addable class $\mathcal{G}$ and provided n is large enough, we have for every $i \leq m$,*

$$\left(\frac{1}{2} - \eta\right) |\mathcal{G}_n^{(i)}| \leq i|\mathcal{G}_n^{(i+1)}| \leq \left(\frac{1}{2} + \eta\right) |\mathcal{G}_n^{(i)}| .$$

Proof. The second inequality is precisely Lemma C.

To prove the first inequality, we proceed by contradiction. Fix η and m and assume that for every $\zeta > 0$ there exist a ζ-tight bridge-addable class $\mathcal{G}$, a large enough n_* and an $i_* \leq m$ such that

$$i_* |\mathcal{G}_{n_*}^{(i_*+1)}| \leq \left(\frac{1}{2} - \eta\right) |\mathcal{G}_{n_*}^{(i_*)}| \ . \tag{2.3}$$

Let $i_0 \geq m$ be an integer that we will choose later. By Lemma C, if n is large enough, (2.2) holds with $\eta = \zeta$ for any $i \leq i_0$. Also, since $\mathcal{G}$ is ζ-tight, provided that n_* is large enough, we have

$$\frac{|\mathcal{G}_{n_*}^{(1)}|}{|\mathcal{G}_{n_*}|} \leq (1+\zeta) e^{-1/2} \ . \tag{2.4}$$

Noting $f_i(x) := \sum_{j>i} \frac{x^j}{j!}$, we can now bound the inverse of the probability that G_{n_*} is connected as follows

$$\begin{aligned}
\frac{|\mathcal{G}_{n_*}|}{|\mathcal{G}_{n_*}^{(1)}|} &\leq \sum_{i=1}^{i^*-1} \frac{|\mathcal{G}_{n_*}^{(i)}|}{|\mathcal{G}_{n_*}^{(1)}|} + \sum_{i=i_*}^{i_0} \frac{|\mathcal{G}_{n_*}^{(i)}|}{|\mathcal{G}_{n_*}^{(1)}|} + \sum_{i \geq i_0+1} \frac{|\mathcal{G}_{n_*}^{(i)}|}{|\mathcal{G}_{n_*}^{(1)}|} \\
&\leq \sum_{i=1}^{i^*-1} \frac{1}{i!} \left(\frac{1}{2} + \zeta\right)^i + \sum_{i=i_*}^{i_0} \frac{1}{i!} \left(\frac{1}{2} + \zeta\right)^i \frac{\frac{1}{2} - \eta}{\frac{1}{2} + \zeta} + f_{i_0}(1) \ ,
\end{aligned}$$

where for the last term we used the bound (2.1). Thus,

$$\begin{aligned}
\frac{|\mathcal{G}_{n_*}|}{|\mathcal{G}_{n_*}^{(1)}|} &\leq e^{\frac{1}{2}+\zeta} - f_{i_0}(1/2+\zeta) + \left(\frac{\frac{1}{2} - \eta}{\frac{1}{2} + \zeta} - 1\right)(f_{i_*-1}(1/2+\zeta) - f_{i_0}(1/2+\zeta)) + f_{i_0}(1) \\
&\leq e^{\frac{1}{2}+\zeta} - \frac{\eta + \zeta}{1/2 + \zeta} \cdot f_{i_*-1}(1/2) + f_{i_0}(1) \\
&\leq e^{1/2} + (e^{\zeta} - 1) e^{1/2} - \eta f_m(1/2) + f_{i_0}(1) \ .
\end{aligned}$$

We now choose ζ small enough with respect to η and m such that $\frac{\eta}{2} f_m(1/2) \geq (e^{\zeta} - 1 + 2\zeta) e^{1/2}$, and we choose i_0 large enough with respect to m, in such a way that $\frac{\eta}{2} f_m(1/2) \geq f_{i_0}(1)$. These choices fix the value n_* as above, and we finally get the bound

$$\frac{|\mathcal{G}_{n_*}^{(1)}|}{|\mathcal{G}_{n_*}|} \geq (1 - 2\zeta)^{-1} e^{-1/2} \geq (1 + 2\zeta) e^{-1/2} \ ,$$

However, since n_* is arbitrarily large, we obtain a contradiction with (2.4). □

2.2. **Partitioning the graph class into highly structured subclasses.** We now introduce a partitioning of $\mathcal{G}_n$ in terms of some local statistics, which requires the following set-up modeled on [CP15, proof of Prop 4].

For $\ell \geq 1$, we let $\mathcal{T}_{\leq \ell}$ (resp., $\mathcal{U}_{\leq \ell}$) to denote the set of rooted (resp., unrooted) trees of order at most ℓ. An important role will be played by the two sets

$$\mathcal{U}_\epsilon := \mathcal{U}_{\leq \lceil \epsilon^{-1} \rceil} \quad , \quad \mathcal{T}_* := \mathcal{T}_{\leq k_*},$$

where the two constants ϵ and k_*, whose value may vary along the course of the paper, will *in fine* be chosen very small and very large, respectively. We will use the elements of $\mathcal{U}_\epsilon$ and $\mathcal{T}_*$ as "test graphs" to measure the shape of small components and the statistics of pendant subtrees in G_n.

For $\ell \geq 1$, we write $\mathcal{E}_\ell = \{0, \ldots, n-1\}^{\mathcal{T}_{\leq \ell}}$. For $\alpha \in \mathcal{E}_{k_*}$ and $w \geq 1$ (width), we define the *box* $[\alpha]^w \subset \mathcal{E}_{k_*}$ and its *q-neighborhood* $[\alpha]^w_q$ as the parallelepipeds:

$$
\begin{aligned}
[\alpha]^w &:= \{\alpha' \in \mathcal{E}_{k_*} : \ \forall T \in \mathcal{T}_*, \ \alpha(T) \leq \alpha'(T) < \alpha(T) + w\} \ , \\
[\alpha]^w_q &:= \{\alpha' \in \mathcal{E}_{k_*} : \ \forall T \in \mathcal{T}_*, \ \alpha(T) - q \leq \alpha'(T) < \alpha(T) + w + q\} \ .
\end{aligned}
$$

Note that here, and elsewhere in the paper, we slightly abuse notation by using both the letter α to denote an element of $\mathcal{E}_\ell$ and the notation α^G to denote the function $\alpha^G : \mathcal{T} \to \mathcal{E}_\ell$ that counts the number of pendant trees of a given shape in the graph G.

If $\mathcal{S}_n$ denotes a set of graphs (where the letter $\mathcal{S}$ could carry other decorations), we let $\mathcal{S}_{n,[\alpha]^w}$ be the set of graphs G in $\mathcal{S}_n$ such that $(\alpha^G(T))_{T\in\mathcal{T}_*} \in [\alpha]^w$, and we use the same notation with $[\alpha]^w_q$.

Also, for every forest $\{U_1, \ldots, U_k\}$, we denote by ${\mathcal{S}_n}^{\{U_1,\ldots,U_k\}}$ the set of graphs G in $\mathcal{S}_n$ such that $Small(G)$ is isomorphic to $\{U_1, \ldots U_k\}$. While we denote a forest by $\{U_1, \ldots, U_k\}$, one should understand it as an *unordered multiset* of unrooted trees. We use the notation $\mathcal{S}_n^U$ for $\mathcal{S}_n^{\{U\}}$, where $U \in \mathcal{U}$.

2.3. **Notation and quantifiers in the proof.** Each statement in Sections 3 and 4 involves several variables and the relative dependency between them plays a subtle role in the proof. We have carefully made all quantifiers explicit in all the statements. However, the reader can use the following inequalities to clarify the hierarchy of (small) parameters used in Sections 3 and 4,

$$
\begin{aligned}
&\frac{1}{n} \ll \zeta \ll \frac{1}{w} \ll \frac{1}{k_*} \ll \xi \ll \epsilon = \frac{1}{q} \ll \gamma \ll \rho \ll \nu \\
&\nu \ll \vartheta \ll \eta \ll \theta_1 \ll \cdots \ll \theta_k, \delta \ll 1/\ell, 1/k, 1/u \leq 1 \ ,
\end{aligned}
\tag{2.5}
$$

where the notation $a \ll b \leq 1$ has to be read as: In each statement involving both variables a and b, there exists a non-decreasing function $f : (0,1] \to (0,1]$ such that the statement holds for every $0 < a \leq b \leq 1$ such that $a \leq f(b)$. For example, the order in which the quantifiers appear in the statement of Lemma 2.1 above correspond to the notation

$$\frac{1}{n} \ll \zeta \ll \eta, \frac{1}{m} \ .$$

Note that $1/n$ is the leftmost quantity appearing in (2.5). Indeed, throughout the paper, n will be taken arbitrarily large with respect to all the other constants.

During the proof, we will use the notation $a = b \pm \mu$ to denote that $b - \mu \leq a \leq b + \mu$.

2.4. **Evaluation of generating functions of trees and forests.** In this subsection we recall two classical evaluations of generating functions of trees and forests that we will use several times in our proofs. Let $T(z) = \sum_{n\geq 1} \frac{t_n}{n!} z^n$ be the exponential generating function of rooted labelled trees by the number of vertices, so $t_n = n^{n-1}$. Let $F(z) = \sum_{n\geq 0} \frac{f_n}{n!} z^n$ be the exponential generating function of unrooted labelled forests by the number of vertices; by convention $f_0 = 1$.

Lemma 2.2. *Both $T(z)$ and $F(z)$ have radius of convergence e^{-1}, and both are finite at their main singularity $z = e^{-1}$, where we have $T(e^{-1}) = 1$ and $F(e^{-1}) = e^{1/2}$. Moreover for z in a slit neighbourhood of e^{-1} we have*

$$T(z) = 1 + O(\sqrt{1 - ze}) \ . \tag{2.6}$$

The proof is a classical exercise in analytic combinatorics.

3. Theorem 1.2 for bridge-addable classes of forests

Balister, Bollobás and Gerke [BBG08, Lemma 2.1] proposed an elegant argument that reduces the proof of Theorem A to the case where all graphs in $\mathcal{G}$ are forests. As we will see in the next section, their idea can be adapted to the present context. We will therefore start by proving Theorem 1.2 for classes $\mathcal{G}$ composed of forests.

Throughout the rest of Section 3, we will assume that all graphs in $\mathcal{G}$ are forests.

3.1. **Good and bad boxes.** The main concern of the paper [CP15] was to obtain a version of the double-counting argument of Section 2.1 that is *local* in the sense that it relates cardinalities of graphs corresponding to fixed boxes.

In order to select a collection of boxes, we will focus on the graphs in $\mathcal{G}_n$ that have either one or two connected components. We will use the notation $\mathcal{A}_n := \mathcal{G}_n^{(1)}$ and $\mathcal{B}_n := \mathcal{G}_n^{(2)}$.

Given ϵ and k_*, [CP15, Lemma 17] asserts that there exist integers K and w (independent of $\mathcal{G}$ and of n) and a set of K disjoint boxes of width w in $\mathcal{E}_{k_*}$, denoted $\{[\beta_i]^w,\ 1 \le i \le K\}$, such that if $q = q_\epsilon := \lceil \epsilon^{-1} \rceil$ and if n is large enough, then the q-neighbourhoods of boxes form a partition of $\mathcal{E}_{k_*}$,

$$\biguplus_{i=1}^{K} [\beta_i]_q^w = \mathcal{E}_{k_*} \ , \tag{3.1}$$

where $\biguplus$ denotes disjoint union, such that for each $U \in \mathcal{U}_\epsilon$, we have

$$\sum_{i=1}^{K} |\mathcal{B}^U_{n,[\beta_i]^w}| \ge (1-\epsilon)|\mathcal{B}^U_n|. \tag{3.2}$$

Note that from (3.1), the boxes $[\beta_i]^w$ are $2q$-apart from each other, and yet (3.2) ensures that they capture a proportion at least $(1-\epsilon)$ of the set $\mathcal{B}^U_n$ for each $U \in \mathcal{U}_\epsilon$. We now fix such a set of boxes $([\beta_i]^w)_{1\le i\le K}$ and we will use them through Section 3, keeping in mind that $K = K(\epsilon, k_*)$ and $w = w(\epsilon, k_*)$, depend on ϵ and k_* but neither on $\mathcal{G}$ nor on n.

In the present paper, one of the main tasks consists in showing that the global estimates obtained in [CP15], such as Lemma C, can be "lowered" down to boxes for ζ-tight classes. This is not true for every box in $\mathcal{E}_{k_*}$, but it will be for certain boxes that contain most of the graphs in the class. For every γ and every ϵ, we say that a box $[\alpha]^w$ is (γ,ϵ)-*good* (or simply *good*) if the two following conditions hold.

i) $|\mathcal{B}_{n,[\alpha]^w}| \ge \left(\frac{1}{2} - \gamma\right) \cdot |\mathcal{A}_{n,[\alpha]^w_q}|$, and
ii) $\sum_{U \notin \mathcal{U}_\epsilon} |\mathcal{B}^U_{n,[\alpha]^w}| < \gamma |\mathcal{B}_{n,[\alpha]^w}|$.

Note that Property i) is a local version of the first inequality of Lemma 2.1 for $i = 1$, while Property ii) ensures that the number of graphs in sets that we do not control, is small.

We will be interested in the boxes among the $[\beta_i]^w$ that are (γ,ϵ)-good,

$$\mathrm{Good}_{\gamma,\epsilon} := \{i \in \{1,\ldots,K\} : [\beta_i]^w \text{ is } (\gamma,\epsilon)\text{-good}\} \ .$$

An important step in the proof of Theorem 1.2 is the following result.

Lemma 3.1. *For every γ and every η, if $\epsilon < \epsilon_0(\gamma,\eta)$ and if $k_* \geq k_0(\epsilon)$, then there exists ζ such that for every ζ-tight bridge-addable class $\mathcal{G}$ and every large enough n, we have*

$$\frac{\sum_{i\notin Good_{\gamma,\epsilon}} |\mathcal{A}_{n,[\beta_i]^w_q}|}{|\mathcal{A}_n|} < \eta\ ,$$

and

$$\frac{\sum_{i\notin Good_{\gamma,\epsilon}} |\mathcal{B}_{n,[\beta_i]^w}|}{|\mathcal{B}_n|} < \eta\ .$$

Proof. Let $\epsilon > 0$ (to be fixed later). Up to setting k_* and n large enough, we can use Equation (16) in [CP15] for each $1 \leq i \leq K$,

$$\sum_{U\in\mathcal{U}_\epsilon} |\mathcal{B}^U_{n,[\beta_i]^w}| \leq \frac{1}{2}\cdot|\mathcal{A}_{n,[\beta_i]^w_q}|(1+3\epsilon) \leq \left(\frac{1}{2}+2\epsilon\right)\cdot|\mathcal{A}_{n,[\beta_i]^w_q}|\ .$$

Moreover, provided that n is large enough, we have (Equation (17) in [CP15])

$$\sum_{U\notin\mathcal{U}_\epsilon} |\mathcal{B}^U_n| \leq 2\epsilon\,|\mathcal{A}_n|\ . \tag{3.3}$$

From the last two inequalities, we have

$$\sum_{i\in \mathrm{Good}_{\gamma,\epsilon}} |\mathcal{B}_{n,[\beta_i]^w}| \leq 2\epsilon\,|\mathcal{A}_n| + \left(\frac{1}{2}+2\epsilon\right)\sum_{i\in \mathrm{Good}_{\gamma,\epsilon}} |\mathcal{A}_{n,[\beta_i]^w_q}|\ . \tag{3.4}$$

Let S and T be the sets of indices $i \notin \mathrm{Good}_{\gamma,\epsilon}$ such that $[\beta_i]^w$ violates i) and ii) respectively. Using (3.3), we have

$$\sum_{i\in T}|\mathcal{B}_{n,[\beta_i]^w}| \leq \sum_{i\in T}\frac{1}{\gamma}\sum_{U\notin\mathcal{U}_\epsilon}|\mathcal{B}^U_{n,[\beta_i]^w}| \leq \frac{1}{\gamma}\sum_{U\notin\mathcal{U}_\epsilon}|\mathcal{B}^U_n| \leq \frac{2\epsilon}{\gamma}|\mathcal{A}_n|\ .$$

From the previous equation it follows that

$$\begin{aligned}\sum_{i\notin \mathrm{Good}_{\gamma,\epsilon}} |\mathcal{B}_{n,[\beta_i]^w}| &\leq \sum_{i\in S}|\mathcal{B}_{n,[\beta_i]^w}| + \sum_{i\in T}|\mathcal{B}_{n,[\beta_i]^w_q}| \\ &\leq \left(\frac{1}{2}-\gamma\right)\sum_{i\notin \mathrm{Good}_{\gamma,\epsilon}} |\mathcal{A}_{n,[\beta_i]^w_q}| + \frac{2\epsilon}{\gamma}\,|\mathcal{A}_n|\ . \qquad (3.5)\end{aligned}$$

Using (3.4) and (3.5), we get

$$\begin{aligned}(\gamma+2\epsilon)\sum_{i\notin \mathrm{Good}_{\gamma,\epsilon}} |\mathcal{A}_{n,[\beta_i]^w_q}| &\leq (\gamma+2\epsilon)\sum_{i\notin \mathrm{Good}_{\gamma,\epsilon}} |\mathcal{A}_{n,[\beta_i]^w_q}| + \sum_{i\notin \mathrm{Good}_{\gamma,\epsilon}} |\mathcal{B}_{n,[\beta_i]^w}| \\ &\quad + \sum_{i\in \mathrm{Good}_{\gamma,\epsilon}} |\mathcal{B}_{n,[\beta_i]^w}| - \sum_{i=1}^K |\mathcal{B}_{n,[\beta_i]^w}| \\ &\leq (\gamma+2\epsilon)\sum_{i\notin \mathrm{Good}_{\gamma,\epsilon}} |\mathcal{A}_{n,[\beta_i]^w_q}| + \left(\frac{1}{2}-\gamma\right)\sum_{i\notin \mathrm{Good}_{\gamma,\epsilon}} |\mathcal{A}_{n,[\beta_i]^w_q}| \\ &\quad + \left(\frac{1}{2}+2\epsilon\right)\sum_{i\in \mathrm{Good}_{\gamma,\epsilon}} |\mathcal{A}_{n,[\beta_i]^w_q}| + \frac{4\epsilon}{\gamma}\,|\mathcal{A}_n| - \sum_{i=1}^K |\mathcal{B}_{n,[\beta_i]^w}|\ .\end{aligned}$$

The last inequality can be simplified as

$$(\gamma+2\epsilon)\sum_{i\notin \mathrm{Good}_{\gamma,\epsilon}} |\mathcal{A}_{n,[\beta_i]_q^w}| \le \left(\frac{1}{2}+2\epsilon\right)\sum_{i=1}^{K}|\mathcal{A}_{n,[\beta_i]_q^w}| - \sum_{i=1}^{K}|\mathcal{B}_{n,[\beta_i]^w}| + \frac{4\epsilon}{\gamma}\,|\mathcal{A}_n|$$

$$\le \left(\frac{1}{2}+\frac{6\epsilon}{\gamma}\right)|\mathcal{A}_n| - \sum_{i=1}^{K}|\mathcal{B}_{n,[\beta_i]^w}|\,. \tag{3.6}$$

where we used that the $[\beta_i]_q^w$ are disjoint. Using (3.2) and (3.3), we have

$$\begin{aligned}\sum_{i=1}^{K}|\mathcal{B}_{n,[\beta_i]^w}| &\ge \sum_{i=1}^{K}\sum_{U\in\mathcal{U}_\epsilon}|\mathcal{B}^U_{n,[\beta_i]^w}|\\ &\ge (1-\epsilon)(|\mathcal{B}_n|-2\epsilon|\mathcal{A}_n|)\\ &\ge (1-\epsilon)|\mathcal{B}_n|-2\epsilon|\mathcal{A}_n|\,.\end{aligned}$$

Finally, Lemma 2.1 with $i=1$ and η replaced by ϵ, implies that if ζ is small enough, $\mathcal{G}$ is ζ-tight and n is large enough, then the last quantity is larger than $(1/2-4\epsilon)\,|\mathcal{A}_n|$.

We now choose $\epsilon_0=\frac{\eta\gamma}{20}$. Going back to (3.6), if $\epsilon<\epsilon_0$, we get

$$\sum_{i\notin \mathrm{Good}_{\gamma,\epsilon}}|\mathcal{A}_{n,[\beta_i]_q^w}| \le \frac{10\epsilon}{\gamma(\gamma+2\epsilon)}\,|\mathcal{A}_n| \le \frac{\eta}{2}|\mathcal{A}_n|\,, \tag{3.7}$$

which proves the first part of the lemma.

For the second part of the lemma, we use (3.5) and Lemma 2.1 with η replaced by ϵ, to get

$$\frac{\sum_{i\notin \mathrm{Good}_{\gamma,\epsilon}}|\mathcal{B}_{n,[\beta_i]^w}|}{|\mathcal{B}_n|} \le \frac{\left(\frac{1}{2}-\gamma\right)\sum_{i\notin \mathrm{Good}_{\gamma,\epsilon}}|\mathcal{A}_{n,[\beta_i]_q^w}| + \frac{2\epsilon}{\gamma}|A_n|}{\left(\frac{1}{2}-\epsilon\right)|\mathcal{A}_n|}\,.$$

By (3.7), we conclude

$$\frac{\sum_{i\notin \mathrm{Good}_{\gamma,\epsilon}}|\mathcal{B}_{n,[\beta_i]^w}|}{|\mathcal{B}_n|} \le \frac{\left(\frac{1}{2}-\gamma\right)\frac{\eta}{2}|\mathcal{A}_n| + \frac{2\epsilon}{\gamma}|A_n|}{\left(\frac{1}{2}-\epsilon\right)|\mathcal{A}_n|} \le \eta\,. \qquad \square$$

3.2. **Stability of the extremum for the optimization problem.** The goal of this subsection is to estimate the ratio between $|\mathcal{B}^U_{n,[\alpha]^w}|$ and $|\mathcal{A}_{n,[\alpha]_q^w}|$, when $[\alpha]^w$ is a good box and $U\in\mathcal{U}_\epsilon$.

In order to do that, we will need to return to the original "optimization problem" introduced in [CP15]. Namely, we will study certain functionals of the ratios $|\mathcal{B}^U_{n,[\alpha]^w}|/|\mathcal{A}_{n,[\alpha]_q^w}|$, or more precisely of the variables $(z^U_{n,\alpha})_{U\in\mathcal{U}_\epsilon}$, defined by (3.8) below. We will proceed as follows. Lemma 3.2 gives the "constraints" of the optimization problem, by showing that the variables $z^U_{n,\alpha}$ have to be close to a certain domain D; Lemma 3.3 shows that if $[\alpha]^w$ is good, then the "objective function" of the optimization problem has to be close to its optimal value given these constraints (which was proved to be $\frac{1}{2}$ in [CP15]). Then Lemma 3.4 proves a form of uniqueness of the extremum. From these three lemmas we deduce the main results of this subsection: if $[\alpha]^w$ is good, then $(z^U_{n,\alpha})_{U\in\mathcal{U}_\epsilon}$ is close to $p_\infty(U)$ for each unrooted tree U of bounded size (Proposition 3.5) and if $[\alpha]^w$ is good, then $\alpha(T)/n$ is close to $a_\infty(T)$ for each rooted tree T of bounded size (Proposition 3.6)

Apart from the proof of Lemma 3.1 already given, the proofs of Lemmas 3.2–3.3–3.4 are the part of the present paper that rely the most on [CP15]. Indeed, we will refer to several technical statements therein in our proofs. This will no longer be the case in the next sections.

Following [CP15], given ϵ (hence $\mathcal{U}_\epsilon$) we define a *$\mathcal{U}_\epsilon$-admissible decomposition of* T as an increasing sequence $\mathbf{T} = (T_i)_{i\leq \ell}$ of labeled trees

$$T_1 \subset \cdots \subset T_\ell = T \ ,$$

for some $\ell \geq 1$ called the *length*, such that $T_1 \in \mathcal{U}_\epsilon$ and, for each $2 \leq i \leq \ell$, T_i is obtained by joining T_{i-1} by an edge e_i to some tree $U_i \in \mathcal{U}_\epsilon$. The *weight* of $\mathbf{T}$ with respect to $\mathbf{z} = (z^U)_{U\in\mathcal{U}_\epsilon} \in (\mathbb{R}_+)^{\mathcal{U}_\epsilon}$ is defined as $\omega(\mathbf{T}, \mathbf{z}) = \prod_{i=1}^{\ell} z^{U_i}$, where $U_i = T_i \setminus T_{i-1}$ as an unrooted tree (here we use the convention $T_0 = \emptyset$). The *maximum weight* of T with respect to $\mathbf{z}$, denoted by $\omega(T, \mathbf{z})$, is defined as the maximum of $\omega(\mathbf{T}, \mathbf{z})$ over all the $\mathcal{U}_\epsilon$-admissible decompositions $\mathbf{T}$ of T.

We now use $\omega(T, \mathbf{z})$ to define the following partition functions,

$$Y(\mathbf{z}) := \sum_{T\in\mathcal{T}} \frac{\omega(T,\mathbf{z})}{\mathrm{Aut}_r(T)} \ , \qquad Y^u(\mathbf{z}) := \sum_{U\in\mathcal{U}} \frac{\omega(U,\mathbf{z})}{\mathrm{Aut}_u(U)} \ ,$$

$$Y_{\mathcal{T}_*}(\mathbf{z}) := \sum_{T\in\mathcal{T}_*} \frac{\omega(T,\mathbf{z})}{\mathrm{Aut}_r(T)} \ , \qquad Y^u_{\mathcal{U}_\epsilon}(\mathbf{z}) := \sum_{U\in\mathcal{U}_\epsilon} \frac{\omega(U,\mathbf{z})}{\mathrm{Aut}_u(U)} \ ,$$

$$Y_{\leq k}(\mathbf{z}) := \sum_{T\in\mathcal{T}_{\leq k}} \frac{\omega(T,\mathbf{z})}{\mathrm{Aut}_r(T)} \ , \qquad \tilde{Y}^u_{\mathcal{U}_\epsilon}(\mathbf{z}) := \sum_{U\in\mathcal{U}_\epsilon} \frac{z^U}{\mathrm{Aut}_u(U)} \ .$$

Furthermore, we define the domain of convergence of $Y(\mathbf{z})$ as follows,

$$D := \{\mathbf{z} \in (\mathbb{R}_+)^{\mathcal{U}_\epsilon}, Y(\mathbf{z}) < \infty\} \ .$$

It is important to note that there is an implicit dependence of $\omega(T, \mathbf{z})$ on ϵ (via $\mathcal{U}_\epsilon$-admissible decompositions). Hence, all the partition functions defined above (and their respective domains) also depend on ϵ. In order to keep the notation light we do not make this dependence explicit.

Let $\mathbf{j} := (1)_{U\in\mathcal{U}_\epsilon}$ be the all-one vector of length $|\mathcal{U}_\epsilon|$. Given a choice of n, to each $\alpha \in \mathcal{E}_{k_*}$ we assign a vector $\mathbf{z}_{n,\alpha} = (z^U_{n,\alpha})_{U\in\mathcal{U}_\epsilon} \in (\mathbb{R}_+)^{\mathcal{U}_\epsilon}$, where

$$z^U_{n,\alpha} := \mathrm{Aut}_u(U) \frac{|\mathcal{B}^U_{n,[\alpha]^w}|}{|\mathcal{A}_{n,[\alpha]^w_q}|} \left(1 - \frac{|U|}{n}\right) \ , \tag{3.8}$$

where $q = \lceil \epsilon^{-1} \rceil$ as before and $w = w(\epsilon, k_*)$ is chosen as in Section 3.1.

Lemma 3.2. *For every ξ and every ϵ, if $k_* \geq k_0(\epsilon, \xi)$ and n is large enough, then for every $\alpha \in \mathcal{E}_{k_*}$ we have that $\mathbf{z}_{n,\alpha} - \xi\mathbf{j} \in D$.*

Proof. For the sake of contradiction, assume that there exist ξ and ϵ such that for every k_0 there exists $k \geq k_0$ such that for every large enough n there exists $\alpha_{n,k} \in \mathcal{E}_k$ with

$$\mathbf{z}_{n,\alpha_{n,k}} - \xi\mathbf{j} \notin D \ .$$

For a given $k \geq k_0$, let $\mathbf{z}_k$ be a limit point of the sequence $(\mathbf{z}_{n,\alpha_{n,k}})_{n\geq 1}$. Since D is closed downwards (Lemma 13 in [CP15]), then $\mathbf{z}_k - \frac{\xi}{2}\mathbf{j} \notin D$.

Moreover, by Corollary 12 in [CP15], we have $Y_{\leq k}(\mathbf{z}_k) \leq 1$. As in [CP15, Lemma 16], this implies that any limit point $\mathbf{z}_\infty$ of $(\mathbf{z}_k)_{k\geq k_0}$ satisfies $\mathbf{z}_\infty \in \overline{D}$. This is a contradiction with the fact that $\mathbf{z}_k - \frac{\xi}{2}\mathbf{j} \notin D$ for every $k \geq k_0$. $\square$

The following lemma shows that if $[\alpha]^w$ is (γ,ϵ)-good, then the evaluation of Y^u in a point close to $\mathbf{z}_{n,\alpha}$ is close to $\frac{1}{2}$ (which was shown in [CP15] to be the maximum of Y^u on D).

Lemma 3.3. *For every ρ, every ϵ and every ℓ such that $\ell < 1/\epsilon$, if $\gamma \leq \gamma_0(\rho,\ell)$, $\xi \leq \xi_0(\rho,\epsilon,\ell)$, $k_* \geq k_0(\epsilon,\xi)$ and n is large enough, then for every box $[\alpha]^w$ which is (γ,ϵ)-good the following holds for $\hat{\mathbf{z}} := \mathbf{z}_{n,\alpha} - \xi\mathbf{j}$: we have $\hat{\mathbf{z}} \in D$,*

$$Y^u(\hat{\mathbf{z}}) > \frac{1}{2} - \rho\ ,$$

and for every $U \in \mathcal{U}_{\leq\ell}$, we have

$$|\omega(U,\hat{\mathbf{z}}) - \hat{z}^U| \leq \rho\ .$$

Proof. Let $\gamma_0 := \frac{\rho}{4\ell!}$ and $\xi_0 := \frac{\rho}{2|\mathcal{U}_\epsilon|\ell!}$. Consider $\alpha \in \mathcal{E}_{k_*}$ such that the box $[\alpha]^w$ is (γ,ϵ)-good. Using the properties i) and ii) of good boxes, and (3.8), we have

$$\begin{aligned}
\tilde{Y}^u_{\mathcal{U}_\epsilon}(\mathbf{z}_{n,\alpha}) = \sum_{U\in\mathcal{U}_\epsilon} \frac{z^U_{n,\alpha}}{\mathrm{Aut}_u(U)} &= \frac{1}{|\mathcal{A}_{n,[\alpha]^w_q}|} \sum_{U\in\mathcal{U}_\epsilon} |\mathcal{B}^U_{n,[\alpha]^w}| \left(1 - \frac{|U|}{n}\right)\\
&\geq \frac{1}{|\mathcal{A}_{n,[\alpha]^w_q}|} |\mathcal{B}_{n,[\alpha]^w}|(1-\gamma)\left(1 - \frac{|U|}{n}\right)\\
&\geq \left(\frac{1}{2} - \gamma\right)(1-\gamma)\left(1 - \frac{1}{\epsilon n}\right)\\
&\geq \frac{1}{2} - 2\gamma\ ,
\end{aligned}$$

provided that n is large enough. Now, since $\tilde{Y}^u_{\mathcal{U}_\epsilon}(\hat{\mathbf{z}})$ is a finite sum, we have

$$\tilde{Y}^u_{\mathcal{U}_\epsilon}(\hat{\mathbf{z}}) \geq \tilde{Y}^u_{\mathcal{U}_\epsilon}(\mathbf{z}_{n,\alpha}) - \xi|\mathcal{U}_\epsilon|\ .$$

Together with the previous inequality and the choice of γ_0 and ξ_0, this implies

$$\tilde{Y}^u_{\mathcal{U}_\epsilon}(\hat{\mathbf{z}}) \geq \frac{1}{2} - (\xi|\mathcal{U}_\epsilon| + 2\gamma) \geq \frac{1}{2} - \frac{\rho}{\ell!}\ . \tag{3.9}$$

By definition of maximum weight, for every $U \in \mathcal{U}_\epsilon$ we have $\omega(U,\mathbf{z}) \geq z^U$, which directly implies $Y^u_{\mathcal{U}_\epsilon}(\mathbf{z}) \geq \tilde{Y}^u_{\mathcal{U}_\epsilon}(\mathbf{z})$. We thus conclude the first part of the lemma,

$$Y^u(\hat{\mathbf{z}}) \geq Y^u_{\mathcal{U}_\epsilon}(\hat{\mathbf{z}}) \geq \tilde{Y}^u_{\mathcal{U}_\epsilon}(\hat{\mathbf{z}}) \geq \frac{1}{2} - \frac{\rho}{\ell!} > \frac{1}{2} - \rho\ .$$

Observe that this is true even if $\hat{\mathbf{z}} \notin D$, since then the LHS is infinite.

By Lemma 3.2, we can choose $k_0 = k_0(\epsilon,\xi)$ such that if $k_* \geq k_0$ and n is large enough, we have $\hat{\mathbf{z}} \in D$. The choice of k_* and n is suitable *for all* vectors in $\mathcal{E}_{k_*}$. Then, Lemma 14 in [CP15] implies that $Y^u_{\mathcal{U}_\epsilon}(\hat{\mathbf{z}}) \leq Y^u(\hat{\mathbf{z}}) \leq \frac{1}{2}$. Together with (3.9), for every $U \in \mathcal{U}_\epsilon$ we have

$$\frac{\rho}{\ell!} \geq |Y^u_{\mathcal{U}_\epsilon}(\hat{\mathbf{z}}) - \tilde{Y}^u_{\mathcal{U}_\epsilon}(\hat{\mathbf{z}})| = \left|\sum_{U'\in\mathcal{U}_\epsilon} \frac{\omega(U',\hat{\mathbf{z}}) - \hat{z}^{U'}}{\mathrm{Aut}_u(U')}\right| \geq \frac{|\omega(U,\hat{\mathbf{z}}) - \hat{z}^U|}{\mathrm{Aut}_u(U)}\ ,$$

where the last inequality follows since $\omega(U',\hat{\mathbf{z}}) \geq \hat{z}^{U'}$ for each tree $U' \in \mathcal{U}_\epsilon$. Since $\mathrm{Aut}_u(U) \leq \ell!$, it follows that

$$|\omega(U,\hat{\mathbf{z}}) - \hat{z}^U| \leq \rho\ . \qquad \square$$

The next lemma states that if $\mathbf{z}$ belongs to D and $Y^u(\mathbf{z})$ is close to $\frac{1}{2}$, then $\omega(T,\mathbf{z})$ is close to $e^{-|T|}$ for every T with bounded size.

Lemma 3.4. *For every ν and every ℓ, if $\rho \le \rho_0(\nu,\ell)$, then for every ϵ, every $\mathbf{z} \in D$ that satisfies $Y^u(\mathbf{z}) > \frac{1}{2} - \rho$, and every $T \in \mathcal{T}_{\le \ell}$, we have*

$$|\omega(T,\mathbf{z}) - e^{-|T|}| < \nu \ . \tag{3.10}$$

Proof. Let $Y^e(\mathbf{z})$ be the partition function of trees rooted at an edge, where each tree is weighted by its maximal weight. As noted in [CP15], a classical trick known as the dissymmetry theorem [BLL98] implies that

$$Y^e(\mathbf{z}) = Y(\mathbf{z}) - Y^u(\mathbf{z}) \ .$$

Together with the hypothesis of the lemma and the fact that $y - 1/2 \le y^2/2$ for all $y \in \mathbb{R}$, this implies

$$Y^e(\mathbf{z}) = Y(\mathbf{z}) - Y^u(\mathbf{z}) \le Y(\mathbf{z}) - 1/2 + \rho \le \frac{1}{2}(Y(\mathbf{z}))^2 + \rho \ ,$$

For every pair of vertex rooted trees $T_1, T_2 \ \in \mathcal{T}$, let $f(T_1,T_2)$ be the edge-rooted tree obtained by adding an edge (the root) connecting the roots of T_1 and T_2. We have the following supermultiplicativity property:

$$\omega(f(T_1,T_2),\mathbf{z}) - \omega(T_1,\mathbf{z})\omega(T_2,\mathbf{z}) \ge 0 \ .$$

Also observe that the number of automorphisms of $f(T_1,T_2)$ that fix the rooted edge (as an ordered edge!), is precisely $\mathrm{Aut}_r(T_1)\mathrm{Aut}_r(T_2)$. Thus, for any pair $R_1, R_2 \ \in \mathcal{T}$, we have

$$\begin{aligned} \rho \ge Y^e(\mathbf{z}) - \frac{1}{2}(Y(\mathbf{z}))^2 &= \sum_{T_1,T_2 \in \mathcal{T}} \frac{\omega(f(T_1,T_2),\mathbf{z}) - \omega(T_1,\mathbf{z})\omega(T_2,\mathbf{z})}{\mathrm{Aut}_r(T_1)\mathrm{Aut}_r(T_2)} \\ &\ge \frac{\omega(f(R_1,R_2),\mathbf{z}) - \omega(R_1,\mathbf{z})\omega(R_2,\mathbf{z})}{|R_1|!\,|R_2|!} \ . \end{aligned} \tag{3.11}$$

Let $\bullet$ be the tree composed of a single vertex and define $x = x(\mathbf{z}) := \omega(\bullet,\mathbf{z}) = z^\bullet \in \mathbb{R}_+$. Observe that since $\mathbf{z} \in D$, we have $x \le 1$ (otherwise $Y(\mathbf{z}) = \infty$ since $\omega(T,\mathbf{z}) \ge x^{|T|}$). Using (3.11) with $R_2 = \bullet$, for every $T \in \mathcal{T}$,

$$\omega(f(T,\bullet),\mathbf{z}) \le x \cdot \omega(T,\mathbf{z}) + \rho \cdot |T|!,$$

and induction on $|T|$ implies that for every $T \in \mathcal{T}$ we have

$$x^{|T|} \le \omega(T,\mathbf{z}) \le x^{|T|} + |T|!\rho \le \left(x + (\rho|T|!)^{\frac{1}{|T|}}\right)^{|T|} \ .$$

Note that if $|T| \le \ell$, then $(\rho|T|!)^{\frac{1}{|T|}} \le c(\ell)\rho^{\frac{1}{\ell}}$, for some $c(\ell) > 0$. Consider $\mathbf{x} = (x^{|U|})_{U \in \mathcal{U}_\epsilon}$ and $\mathbf{x}_\rho = ((x + c(\ell)\rho^{\frac{1}{\ell}})^{|U|})_{U \in \mathcal{U}_\epsilon}$. By the definition of $\mathbf{x}$, note that $\omega(T,\mathbf{x}) = x^{|T|}$, therefore $\omega(T,\mathbf{x}) \le \omega(T,\mathbf{z})$ and since $\mathbf{z} \in D$, by Lemma 14 in [CP15], we have

$$Y^u(\mathbf{x}) \le Y^u(\mathbf{z}) \le \frac{1}{2} \ .$$

This implies $x \le e^{-1}$ (otherwise $Y^u(\mathbf{x})$ would not converge). Similarly $\omega(T,\mathbf{x}_\rho) = (x + c(\ell)\rho^{\frac{1}{\ell}})^{|T|}$, and using the hypothesis of the lemma, we have

$$\frac{1}{2} - \rho \le Y^u(\mathbf{z}) \le Y^u(\mathbf{x}_\rho) \ .$$

By Equation (2.6) in Lemma 2.2, this implies that $x+c(\ell)\rho^{\frac{1}{\ell}} \geq e^{-1} - O(\sqrt{c(\ell)\rho^{1/\ell}})$. Given ν and ℓ, we can now set $\rho_0(\nu,\ell)$ small enough such that for $\rho \leq \rho_0(\nu,\ell)$ we have $x > e^{-1}(1-y)$, with $y = \min\{\frac{\nu e^{\ell}}{\ell}, 1\}$, and $\rho \leq \frac{\nu}{\ell!}$. We then have, for every $T \in \mathcal{T}_{\leq \ell}$,

$$e^{-|T|}-\nu \leq e^{-|T|}(1-y|T|) \leq e^{-|T|}(1-y)^{|T|} < x^{|T|} \leq \omega(T,\mathbf{z}) \leq x^{|T|}+\rho\cdot|T|! \leq e^{-|T|}+\nu \ .$$

where we used that $(1-y)^{\ell}$ is convex for $y \in [0,1]$. □

Finally, we can prove estimates for the ratios between $|\mathcal{B}^U_{n,[\alpha]^w}|$ and $|\mathcal{A}_{n,[\alpha]^w_q}|$ for good boxes $[\alpha]^w$ and unrooted trees U with bounded size.

Proposition 3.5. *For every ϑ, every ϵ and every ℓ such that $\ell < 1/\epsilon$, if $\gamma \leq \gamma_0(\vartheta,\ell)$, $k_* \geq k_0(\vartheta,\epsilon,\ell)$ and n is large enough, then for every box $[\alpha]^w$ which is (γ,ϵ)-good and every $U \in \mathcal{U}_{\leq \ell}$*

$$\left| \frac{|\mathcal{B}^U_{n,[\alpha]^w}|}{|\mathcal{A}_{n,[\alpha]^w_q}|} - \frac{e^{-|U|}}{Aut_u(U)} \right| < \vartheta \ .$$

Proof. Let us first fix the constants that we will need in the proof. For $\nu := \vartheta/4$, we let $\rho_0 = \rho_0(\nu,\ell)$ be the value obtained from Lemma 3.4. For $\rho := \min\{\rho_0,\nu\}$, we let $\gamma_0 = \gamma_0(\rho,\ell)$, $\xi_0 = \xi_0(\rho,\epsilon,\ell)$ be the values obtained from Lemma 3.3. For $\xi := \min\{\xi_0,\nu\}$, we let $k_0 = k_0(\epsilon,\xi)(= k_0(\vartheta,\epsilon,\ell))$ be the value obtained from Lemma 3.3. Now fix $k_* \geq k_0$ and consider n large enough. Note that once k_* and n are chosen, the space $\mathcal{E}_{k_*}$ is well-determined.

Let $\hat{\mathbf{z}} = \mathbf{z}_{n,\alpha} - \xi\mathbf{j}$ as before. For a given $U \in \mathcal{U}_{\leq \ell}$, we observe

$$|z^U_{n,\alpha} - \hat{z}^U| \leq \xi \leq \vartheta/4 \ .$$

By Lemma 3.3, if $[\alpha]^w$ is (γ,ϵ)-good, we have

$$|\hat{z}^U - \omega(U,\hat{\mathbf{z}})| \leq \rho \leq \vartheta/4 \ .$$

The same lemma also implies that $\hat{\mathbf{z}} \in D$ and that $Y^u(\hat{\mathbf{z}}) > \frac{1}{2} - \rho$. Thus, $\hat{\mathbf{z}}$ satisfies the hypothesis of Lemma 3.4, which implies

$$|\omega(U,\hat{\mathbf{z}}) - e^{-|U|}| < \nu = \vartheta/4 \ .$$

Using the previous three inequalities and (3.8), we conclude

$$\left| \frac{|\mathcal{B}^U_{n,[\alpha]^w}|}{|\mathcal{A}_{n,[\alpha]^w_q}|} - \frac{e^{-|U|}}{\mathrm{Aut}_u(U)} \right| = \frac{|z^U_{n,\alpha}\left(1-\frac{|U|}{n}\right)^{-1} - e^{-|U|}|}{\mathrm{Aut}_u(U)} < \vartheta \ ,$$

provided that n is large enough. In the last inequality we used that $z^U_{n,\alpha} \leq 1$ (this can be obtained using a similar argument as the one used to obtain (2.1)). □

Proposition 3.6. *For every ϑ, every ϵ and every ℓ such $\ell < 1/\epsilon$, if $\gamma \leq \gamma_0(\vartheta,\ell)$, $k_* \geq k_0(\vartheta,\epsilon,\ell)$ and n is large enough, then for every box $[\alpha]^w$ which is (γ,ϵ)-good and every $T \in \mathcal{T}_{\leq \ell}$*

$$\left| \frac{\alpha(T)}{n} - \frac{e^{-|T|}}{Aut_r(T)} \right| < \vartheta \ .$$

Proof. Again, let us start by fixing the constants that we will need in the proof. For $\nu := \frac{\vartheta}{4|\mathcal{T}_{\leq \ell}|}$, we let $\rho_0 = \rho_0(\nu,\ell)$ be the value obtained from Lemma 3.4. For $\rho \leq \rho_0$, we let $\gamma_0 = \gamma_0(\rho,\ell)(= \gamma_0(\vartheta,\ell))$, $\xi_0 = \xi_0(\rho,\epsilon,\ell)$ be the values obtained from Lemma 3.3.

Observe that, if we fix $T \in \mathcal{T}_{\leq \ell}$, the function $\omega(T, \mathbf{z})$ is a *piecewise* polynomial in the set of variables $\{z^U : U \in \mathcal{U}_\epsilon\}$ that it is continuous at every point of $(\mathbb{R}_+)^{\mathcal{U}_\epsilon}$. Since D is bounded, there exists ξ_1 such that for every $\xi \leq \xi_1$ and every $\mathbf{z}$ at distance at most 1 from D (in the ℓ_∞ norm), we have

$$|\omega(T, \mathbf{z}) - \omega(T, \mathbf{z} - \xi \mathbf{j})| < \frac{\vartheta}{4|\mathcal{T}_{\leq \ell}|} .$$

For $\xi := \min\{\xi_0, \xi_1\}$, we let $k_0 = k_0(\epsilon, \xi)(= k_0(\vartheta, \epsilon, \ell))$ be the value obtained from Lemma 3.3. Fix $k_* \geq k_0$ and consider n large enough. By Lemma 3.3, if $[\alpha]^w$ is (γ, ϵ)-good and we write $\hat{\mathbf{z}} := \mathbf{z}_{n,\alpha} - \xi \mathbf{j}$, we have $\hat{\mathbf{z}} \in D$ and $Y^u(\hat{\mathbf{z}}) > \frac{1}{2} - \rho$. Thus, $\hat{\mathbf{z}}$ satisfies the hypothesis of Lemma 3.4 and we have

$$|\omega(T, \hat{\mathbf{z}}) - e^{-|T|}| < \nu = \frac{\vartheta}{4|\mathcal{T}_{\leq \ell}|} .$$

Using the previous inequalities, we obtain

$$|\omega(T, \mathbf{z}_{n,\alpha}) - e^{-|T|}| \leq |\omega(T, \mathbf{z}_{n,\alpha}) - \omega(T, \hat{\mathbf{z}})| + |\omega(T, \hat{\mathbf{z}}) - e^{-|T|}| < \frac{\vartheta}{2|\mathcal{T}_{\leq \ell}|} .$$

By Lemma 11 in [CP15], there exists a constant C that does not depend on n such that

$$\frac{\alpha(T)}{n} \geq \frac{\omega(T, \mathbf{z}_{n,\alpha})}{\mathrm{Aut}_r(T)} - \frac{C}{n} \geq \frac{e^{-|T|}}{\mathrm{Aut}_r(T)} - \frac{2\vartheta}{3|\mathcal{T}_{\leq \ell}|} , \tag{3.12}$$

where the last inequality holds provided n is large enough. This proves one side of the inequality in the statement.

By Lemma 2.2, if we let t be large enough with respect to ϑ, we have that

$$\sum_{T \in \mathcal{T}_{\leq t}} \frac{e^{-|T|}}{\mathrm{Aut}_r(T)} > 1 - \frac{\vartheta}{3}. \tag{3.13}$$

We can assume that $\ell \geq t$, up to increasing the value of k_* and n.

For the sake of contradiction, suppose that there exists $T_0 \in \mathcal{T}_{\leq \ell}$ such that $\frac{\alpha(T_0)}{n} > \frac{e^{-|T_0|}}{\mathrm{Aut}_r(T_0)} + \vartheta$. Then, using (3.12), (3.13) and the properties of T_0, we get

$$1 \geq \sum_{T \in \mathcal{T}_{\leq \ell}} \frac{\alpha(T)}{n} \geq \sum_{T \in \mathcal{T}_{\leq \ell}} \frac{e^{-|T|}}{\mathrm{Aut}_r(T)} - \frac{2\vartheta}{3} + \vartheta > 1 ,$$

thus obtaining a contradiction and concluding the proof of the lemma. $\square$

3.3. **Proof of Theorem 1.2 for classes of forests: the case of 1 or 2 connected components.** For every δ and every ℓ, consider the set of vectors in $\mathcal{E}_\ell$ that are δ-close to the distribution a_∞ (recall that for $T \in \mathcal{T}$, $a_\infty(T) = \frac{e^{-|T|}}{\mathrm{Aut}_r(T)}$); that is,

$$\Xi(\delta, \ell) = \left\{ \beta \in \mathcal{E}_\ell : \left| \frac{\beta(T)}{n} - a_\infty(T) \right| < \delta, \text{ for every } T \in \mathcal{T}_{\leq \ell} \right\} . \tag{3.14}$$

In what follows, for every set of graphs $\mathcal{S}_n$, every $\ell \geq 1$ and every $\beta \in \mathcal{E}_\ell$, we let $\mathcal{S}_{n,\beta}$ be the set of graphs G in $\mathcal{S}_n$ such that $\alpha^G(T) = \beta(T)$ for all $T \in \mathcal{T}_{\leq \ell}$.

Proposition 3.7. *For every θ_1 and every $U \in \mathcal{U}$, there exists ζ such that for every ζ-tight class $\mathcal{G}$ of forests and every large enough n, we have*

$$\left| \frac{|\mathcal{B}_n^U|}{|\mathcal{G}_n|} - e^{-1/2} \frac{e^{-|U|}}{Aut_u(U)} \right| < \theta_1 \; .$$

Moreover, for every θ_1, every δ, every ℓ and every $U \in \mathcal{U}$, there exists ζ such that for every ζ-tight class $\mathcal{G}$ of forests and every large enough n, we have

$$\left| \frac{\sum_{\beta \in \Xi(\delta,\ell)} \left|\mathcal{B}_{n,\beta}^U\right|}{|\mathcal{B}_n^U|} - 1 \right| < \theta_1 \; .$$

Proof. We start by fixing the constants needed in the proof. For $\vartheta := \theta_1/8$ and $\ell = |U|$, we let $\gamma_0 = \gamma_0(\vartheta, \ell)$ be the constant obtained from Proposition 3.5. Fix $\gamma \leq \gamma_0$. For $\eta := \frac{\theta_1}{4}$, we let $\epsilon_0 = \epsilon_0(\gamma, \eta)$ be the constant obtained from Lemma 3.1. For $\epsilon := \min\{\epsilon_0, 1/\ell, \theta_1/8\}$, we let $k_0(\vartheta, \epsilon, \ell)$ be the maximum of the constants obtained from Lemma 3.1 and Proposition 3.5. Fix $k_* \geq k_0$. Let ζ be the minimum between the constant obtained from Lemma 3.1 and $\theta_1/8$. Let n be large enough with respect to all the previous parameters.

Now that ϵ and k_* are fixed, we consider as before the family $\mathcal{U}_\epsilon \subset \mathcal{U}$ of unrooted trees of order at most $\lceil \epsilon^{-1} \rceil$ and the family $\mathcal{T}_* \subset \mathcal{T}$ of all rooted trees of order at most k_*. We also let w and K, and the collection of boxes $\{[\beta_i]^w, 1 \leq i \leq K\}$ be defined (relatively to the values of ϵ and k_*) as in Section 3.1. We recall that these boxes satisfy (3.2), and using (3.1) we note that $\sum_{i=1}^K |\mathcal{A}_{n,[\beta_i]_q^w}| = |\mathcal{A}_n|$.

We can write,

$$\begin{aligned} \left| \frac{|\mathcal{B}_n^U|}{|\mathcal{G}_n|} - e^{-1/2} \frac{e^{-|U|}}{\mathrm{Aut}_u(U)} \right| &\leq \left| \sum_{i=1}^K \frac{|\mathcal{B}_{n,[\beta_i]^w}^U|}{|\mathcal{G}_n|} - e^{-1/2} \frac{e^{-|U|}}{\mathrm{Aut}_u(U)} \right| + \epsilon \\ &\leq \left| \sum_{i \notin \mathrm{Good}_{\gamma,\epsilon}} \frac{|\mathcal{B}_{n,[\beta_i]^w}^U|}{|\mathcal{G}_n|} + \frac{1}{|\mathcal{G}_n|} \sum_{i \in \mathrm{Good}_{\gamma,\epsilon}} |\mathcal{B}_{n,[\beta_i]^w}^U| \right. \\ &\qquad \left. - e^{-1/2} \frac{e^{-|U|}}{\mathrm{Aut}_u(U)} \right| + \frac{\theta_1}{8} \; . \end{aligned}$$

By Proposition 3.5, for every $i \in \mathrm{Good}_{\gamma,\epsilon}$ and every $U \in \mathcal{U}_{\leq \ell}$, we have

$$\left| |\mathcal{B}_{n,[\beta_i]^w}^U| - \frac{e^{-|U|}}{\mathrm{Aut}_u(U)} \cdot |\mathcal{A}_{n,[\beta_i]_q^w}| \right| \leq \frac{\theta_1}{8} \; .$$

By Lemma 3.1, we have

$$\sum_{i \notin \mathrm{Good}_{\gamma,\epsilon}} \frac{|\mathcal{B}_{n,[\beta_i]^w}^U|}{|\mathcal{G}_n|} \leq \sum_{i \notin \mathrm{Good}_{\gamma,\epsilon}} \frac{|\mathcal{B}_{n,[\beta_i]^w}|}{|\mathcal{B}_n|} \leq \eta = \frac{\theta_1}{4} \; .$$

Let M be the number of boxes $[\beta_i]^w$ that are non-empty. Clearly, $M \leq |\mathcal{G}_n|$. Therefore,

$$\begin{aligned}
&\left| \frac{|\mathcal{B}_n^U|}{|\mathcal{G}_n|} - e^{-1/2} \frac{e^{-|U|}}{\mathrm{Aut}_u(U)} \right| \\
&\quad \leq \frac{\theta_1}{4} + \frac{\theta_1 M}{8|\mathcal{G}_n|} + \left| \frac{e^{-|U|}}{\mathrm{Aut}_u(U)|\mathcal{G}_n|} \left(\sum_{i \in \mathrm{Good}_{\gamma,\epsilon}} |\mathcal{A}_{n,[\beta_i]_q^w}| \right) - e^{-1/2} \frac{e^{-|U|}}{\mathrm{Aut}_u(U)} \right| + \frac{\theta_1}{8} \\
&\quad \leq \frac{\theta_1}{2} + \left| \frac{|\mathcal{A}_n|}{|\mathcal{G}_n|} \left(\frac{\sum_{i \in \mathrm{Good}_{\gamma,\epsilon}} |\mathcal{A}_{n,[\beta_i]_q^w}|}{|\mathcal{A}_n|} \right) - e^{-1/2} \right| \frac{e^{-|U|}}{\mathrm{Aut}_u(U)} \ .
\end{aligned}$$

Again, by Lemma 3.1 and using that $\sum_{i=1}^K |\mathcal{A}_{n,[\beta_i]_q^w}| = |\mathcal{A}_n|$, we have

$$\left| \frac{\sum_{i \in \mathrm{Good}_{\gamma,\epsilon}} |\mathcal{A}_{n,[\beta_i]_q^w}|}{|\mathcal{A}_n|} - 1 \right| = \frac{\sum_{i \notin \mathrm{Good}_{\gamma,\epsilon}} |\mathcal{A}_{n,[\beta_i]_q^w}|}{|\mathcal{A}_n|} \leq \eta = \frac{\theta_1}{4} \ .$$

Since $\mathcal{G}$ is a ζ-tight bridge-addable class, by definition, using Theorem A and provided that n is large enough, we obtain

$$(1-\zeta)e^{-1/2} \leq \frac{|\mathcal{A}_n|}{|\mathcal{G}_n|} \leq (1+\zeta)e^{-1/2} \ .$$

Since $\zeta \leq \theta_1/8$, we obtain

$$\begin{aligned}
\left| \frac{|\mathcal{B}_n^U|}{|\mathcal{G}_n|} - e^{-1/2} \frac{e^{-|U|}}{\mathrm{Aut}_u(U)} \right| &\leq \frac{\theta_1}{2} + \left(\left(1 + \frac{\theta_1}{8}\right)\left(1 + \frac{\theta_1}{4}\right) - 1 \right) e^{-1/2} \frac{e^{-|U|}}{\mathrm{Aut}_u(U)} \\
&\leq \frac{\theta_1}{2} + \frac{\theta_1}{2} \cdot e^{-1/2} \frac{e^{-|U|}}{\mathrm{Aut}_u(U)} \leq \theta_1 \ .
\end{aligned}$$

This concludes the proof of the first part of the proposition.

For the second part, let us proceed by contradiction. Suppose that there exist θ, δ, ℓ and $U \in \mathcal{U}$, such that for every ζ there exists ζ-tight class $\mathcal{G}$ and a large enough n with

$$\left| \frac{\sum_{\beta \in \Xi(\delta,\ell)} \left|\mathcal{B}_{n,\beta}^U\right|}{|\mathcal{B}_n^U|} - 1 \right| > \theta \ .$$

or equivalently,

$$\frac{\sum_{\beta \notin \Xi(\delta,\ell)} \left|\mathcal{B}_{n,\beta}^U\right|}{|\mathcal{B}_n^U|} > \theta \ . \tag{3.15}$$

Note that by the first part of the proposition with θ_1 small enough, we have that $\frac{|\mathcal{B}_n^U|}{|\mathcal{G}_n|}$ is arbitrarily close to $e^{-1/2}\frac{e^{-|U|}}{\mathrm{Aut}_u(U)}$, for ζ small and n large enough. Thus, there exists a uniform constant $c(U) > 0$ such that $\frac{|\mathcal{B}_n^U|}{|\mathcal{B}_n|} \geq c(U)$, and (3.15) is well-defined.

Let $\eta = \theta c(U)$ and let $\vartheta = \delta/2$. As in the first part of the proposition, we can choose γ, ϵ, k_*, ζ and n, such that Lemma 3.1 and Proposition 3.6 can be applied. We skip the details of this setting. We will again consider the set of boxes $\{[\beta_i]^w : 1 \leq i \leq K\}$ of $\mathcal{E}_{k_*}$ fixed in Section 3.1. For every $\alpha \in \mathcal{E}_{k_*}$, we consider its canonical projection $\pi(\alpha)$ onto $\mathcal{E}_\ell$ obtained by selecting the first $|\mathcal{E}_\ell|$ coordinates of α.

Claim. Let $\alpha \in [\beta_i]^w$, for some $i \in \mathrm{Good}_{\gamma,\epsilon}$. Then $\pi(\alpha) \in \Xi(\delta, \ell)$.

Proof of the Claim. By Proposition 3.6 and since $[\beta_i]^w$ is (γ, ϵ)-good, for every $T \in \mathcal{T}_{\leq \ell}$ we have

$$\left| \frac{\beta_i(T)}{n} - \frac{e^{-|T|}}{\mathrm{Aut}_r(T)} \right| < \vartheta \ .$$

Since $\alpha \in [\beta_i]^w$, for every $T \in \mathcal{T}_{\leq k_*}$, we have $|\beta_i(T) - \alpha(T)| \leq w$. The choice of w does not depend on n, and thus, $|\frac{\beta_i(T)}{n} - \frac{\alpha(T)}{n}| \leq \frac{\delta}{3}$, if n large enough. Since $\ell \leq k_*$, for every $T \in \mathcal{T}_{\leq \ell}$ we have

$$\left| \frac{\alpha(T)}{n} - \frac{e^{-|T|}}{\mathrm{Aut}_r(T)} \right| < \vartheta + \frac{\delta}{3} < \delta \ .$$

We conclude that $\pi(\alpha) \in \Xi(\delta, \ell)$, which proves the claim. □

As a direct corollary of the claim, we get

$$\frac{\sum_{\beta \notin \Xi(\delta,\ell)} \left|\mathcal{B}^U_{n,\beta}\right|}{|\mathcal{B}^U_n|} \leq \frac{\sum_{i \notin \mathrm{Good}_{\gamma,\epsilon}} \left|\mathcal{B}^U_{n,[\beta_i]^w}\right|}{|\mathcal{B}^U_n|} \ .$$

By Lemma 3.1, it follows that

$$\frac{\sum_{\beta \notin \Xi(\delta,\ell)} \left|\mathcal{B}^U_{n,\beta}\right|}{|\mathcal{B}^U_n|} \leq \frac{|\mathcal{B}_n|}{|\mathcal{B}^U_n|} \cdot \frac{\sum_{i \notin \mathrm{Good}_{\gamma,\epsilon}} \left|\mathcal{B}^U_{n,[\beta_i]^w}\right|}{|\mathcal{B}_n|} \leq \frac{|\mathcal{B}_n|}{|\mathcal{B}^U_n|} \cdot \eta \leq \theta \ ,$$

where we have used $|\mathcal{B}^U_{n,[\beta_i]^w}| \leq |\mathcal{B}_{n,[\beta_i]^w}|$, giving a contradiction with (3.15).

□

3.4. **Proof of Theorem 1.2 for classes of forests.** We now prove the main result of this section, Theorem 3.8, that is equivalent to our main theorem for bridge-addable classes of forests.

We say that an edge e in a graph $G \in \mathcal{G}$ is *removable* if the graph $G' = G \setminus e$ is in $\mathcal{G}$. For a subclass $\mathcal{H} \subseteq \mathcal{G}$ and a rooted tree $T \in \mathcal{T}$, we define $p(\mathcal{H}, T)$ to be the probability that given a uniformly random graph $H \in \mathcal{H}$, and a uniformly random pendant copy of T in H, the graph H' obtained by deleting the edge that connects the pendant copy of T to the rest of the graph belongs to $\mathcal{G}$ (and not only to $\mathcal{H}$). We do a slight abuse of notation by writing $p(G, T)$ for $p(\{G\}, T)$, for each $G \in \mathcal{G}$. Also, in the cases where $p(G, T)$ is not well-defined (that is, if G has no pendant copy of T), we interpret the probability as 1.

Recall the definition of $\Xi(\delta, \ell)$ given in (3.14), and recall from Section 2.2 that we use the notation $\{U_1, U_2, \ldots, U_k\}$ to denote the forest formed by a *multiset* of k unrooted trees.

Theorem 3.8. *For every $k \geq 1$, every θ_k and every $U_1, \ldots, U_k \in \mathcal{U}$, there exists ζ such that for every ζ-tight class $\mathcal{G}$ of forests and every large enough n, we have*

$$\left| \frac{\left|\mathcal{G}_n^{k+1,\{U_1,\ldots,U_k\}}\right|}{|\mathcal{G}_n|} - e^{-1/2} \frac{e^{-\sum_{i=1}^k |U_i|}}{Aut_u(U_1, \ldots, U_k)} \right| < \theta_k \ . \tag{3.16}$$

Moreover, for every k, every ℓ, every θ_k, every δ and every $U_1, \ldots, U_k \in \mathcal{U}$, there exists ζ such that for every ζ-tight class $\mathcal{G}$ of forests and every large enough n, we *have*

$$\left| \frac{\sum_{\beta\in\Xi(\delta,\ell)} \left|\mathcal{G}_{n,\beta}^{k+1,\{U_1,\dots,U_k\}}\right|}{\left|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}\right|} - 1 \right| < \theta_k \ . \tag{3.17}$$

Proof of Theorem 3.8, first part. We prove the first statement of the theorem by induction. Proposition 3.7 proves the case $k = 1$. Assume that the statement is true for $k-1$ and let us show it for k. Fix $U_1, \dots, U_k \in \mathcal{U}$ and let $u = \max|U_i|$.

We consider the following total order on the subsets of $\{1,\dots,n\}$; for every $V_1, V_2 \subseteq \{1,\dots,n\}$ we have $V_1 < V_2$ if $|V_1| < |V_2|$ or $|V_1| = |V_2|$ and V_1 precedes V_2 in lexicographical order.

Let $m(U_1,\dots,U_k)$ be the number of graphs isomorphic to U_k among $U_1,\dots,U_k$. Observe that

$$\mathrm{Aut}_u(U_1,\dots,U_k) = m(U_1,\dots,U_k)\mathrm{Aut}_u(U_k)\mathrm{Aut}_u(U_1,\dots,U_{k-1}) \ . \tag{3.18}$$

For every subset of vertices $W \subset \{1,\dots,n\}$, we use $G[W]$ to denote the graph induced by W in G. For every unlabeled graph U, the notation $G[W] \equiv U$, not only denotes graph isomorphism, but also that W induces a maximal connected component in G.

Given disjoint sets $V_1,\dots,V_{k-1} \subset \{1,\dots,n\}$, consider the graph class

$$\mathcal{H}(V_1,\dots,V_{k-1}) = \{G[\{1,\dots,n\}\backslash \cup_{i=1}^{k-1} V_i] :\ G\in\mathcal{G}_n,\, G[V_1]\equiv U_1,\dots,G[V_{k-1}]\equiv U_{k-1}\} \ .$$

In order to avoid considering the same tuple multiple times, we define the set of $(k-1)$-tuples of disjoint subsets as follows,

$$\mathcal{V} = \{(V_1,\dots,V_{k-1}),\, V_i \subset \{1,\dots,n\} \text{ disjoint; if } U_i \equiv U_j \text{ then } V_i < V_j\} \ . \tag{3.19}$$

We write $\mathcal{H} = \cup_{(V_1,\dots,V_{k-1})\in\mathcal{V}}\mathcal{H}(V_1,\dots,V_{k-1})$.

Since $\mathcal{G}_n$ is a bridge-addable class on $\{1,\dots,n\}$, then we have that $\mathcal{H}(V_1,\dots,V_{k-1})$ (for every $(V_1,\dots,V_{k-1})\in\mathcal{V}$) is also a bridge-addable class on $\{1,\dots,n\}\setminus\cup_{i=1}^{k-1}V_i$. It is worth stressing here that $|\{1,\dots,n\}\setminus\cup_{i=1}^{k-1}V_i| \geq n-(k-1)u$ is large enough (provided n is large enough), and thus, our previous results can be applied to these classes of graphs.

Consider the graphs in $\mathcal{G}_n$ with $k+1$ components such that the k smallest ones are isomorphic to $U_1,\dots,U_k$ and where one component isomorphic to U_k is marked. By counting these graphs in two ways, for n large enough, we have

$$m(U_1,\dots,U_k)\left|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}\right| = \sum_{(V_1,\dots,V_{k-1})\in\mathcal{V}} |\mathcal{H}^{2,U_k}(V_1,\dots,V_{k-1})| \ . \tag{3.20}$$

Therefore,

$$\begin{aligned}\frac{\left|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}\right|}{|\mathcal{G}_n|} = \frac{1}{m(U_1,\dots,U_k)} \sum_{(V_1,\dots,V_{k-1})\in\mathcal{V}} \Bigg(&\frac{|\mathcal{H}^{2,U_k}(V_1,\dots,V_{k-1})|}{|\mathcal{H}(V_1,\dots,V_{k-1})|} \\ &\cdot\frac{|\mathcal{H}(V_1,\dots,V_{k-1})|}{|\mathcal{H}^{(1)}(V_1,\dots,V_{k-1})|}\cdot\frac{|\mathcal{H}^{(1)}(V_1,\dots,V_{k-1})|}{|\mathcal{G}_n|}\Bigg) \ .\end{aligned} \tag{3.21}$$

Thus it suffices to estimate the three ratios in the sum above.

Let $\theta_1 := \frac{\theta_k}{8}$ and $\theta_{k-1} := \frac{\theta_k}{8}$. Let ζ_1 be the constant obtained from Proposition 3.7 with θ_1 and $U = U_k$. Let ζ_2 be the constant obtained by induction with $k-1$, θ_{k-1} and $U_1,\dots,U_{k-1}$. We set $\zeta_0 := \min\left\{\zeta_1,\zeta_2,\frac{\theta_k}{20},k^{-2}\right\}$.

Let us first show that most of graphs in $\mathcal{H}$ are in classes $\mathcal{H}(V_1,\dots,V_{k-1})$ that are close to be tight. Let $\mathcal{V}_0 \subset \mathcal{V}$ be the set of $(k-1)$-tuples such that $\mathcal{H}(V_1,\dots,V_{k-1})$ satisfies

$$\mathbf{Pr}(H \in \mathcal{H}(V_1,\dots,V_{k-1}) \text{ connected}) \geq (1+\zeta_0)e^{-1/2}, \tag{3.22}$$

and let $\mathcal{H}_0 = \cup_{(V_1,\dots,V_{k-1})\in\mathcal{V}_0}\mathcal{H}(V_1,\dots,V_{k-1})$.

Claim. There exists ζ_3 such that if $\mathcal{G}$ is ζ_3-tight and n is large enough, we have

$$|\mathcal{H}_0| \leq \zeta_0|\mathcal{H}| .$$

Proof of the Claim. For any $(k-1)$-tuple of trees $(W_1,W_2,\dots,W_{k-1})$, we define

$$\begin{aligned}&\mathcal{J}(W_1,\dots,W_{k-1};V_1,\dots,V_{k-1}) =\\ &\quad= \{G[\{1,\dots,n\}\setminus \cup_{i=1}^{k-1}V_i] :\ G\in\mathcal{G}_n,\ G[V_1]\equiv W_1,\dots,G[V_{k-1}]\equiv W_{k-1}\} .\end{aligned}$$

As in (3.19) to avoid problems of multiplicity, we define the following subsets that generalize $\mathcal{V}$,

$$\mathcal{V}(W_1,\dots,W_{k-1}) = \{(V_1,\dots,V_{k-1}),\ V_i\subset\{1,\dots,n\} \text{ disjoint; if } W_i\equiv W_j \text{ then } V_i<V_j\} .$$

We stress here that for any non-empty class $\mathcal{J}(W_1,\dots,W_{k-1};V_1,\dots,V_{k-1})$ such that $\mathcal{H}(V_1,\dots,V_{k-1})$ is non-empty, we have $|W_i|\leq u$, for every $1\leq i\leq k-1$. As before, we note that $\mathcal{J}(W_1,\dots,W_{k-1};V_1,\dots,V_{k-1})$ is bridge-addable. We will write $\mathcal{J}(W_1,\dots,W_{k-1}) = \cup_{(V_1,\dots,V_{k-1})\in\mathcal{V}(W_1,\dots,W_{k-1})}\mathcal{J}(W_1,\dots,W_{k-1};V_1,\dots,V_{k-1})$ and

$$\mathcal{J} = \bigcup_{\{W_1,\dots,W_{k-1}\}} \mathcal{J}(W_1,\dots,W_{k-1}) ,$$

where the union is taken over multisets of trees $\{W_1,\dots,W_{k-1}\}$ and where for each multiset an arbitrary ordered tuple $(W_1,\dots,W_{k-1})$ is chosen. Thus, $\mathcal{J}$ can be understood as the set of graphs in $\mathcal{G}_n$ with at least k components where exactly $k-1$ of the non-largest ones are marked. In particular,

$$|\mathcal{J}| = \sum_{j\geq 0}\binom{k+j-1}{k-1}|\mathcal{G}_n^{(k+j)}| . \tag{3.23}$$

Let $\eta=\zeta_0^3$ and m such that $\sum_{\ell\geq m-k}\frac{1}{\ell!}\leq\eta$ and $m\geq k$. By Lemma 2.1 there exists ζ_4 such that if $\mathcal{G}$ is ζ_4-tight and n is large enough, for every $1\leq i\leq m$ we have

$$\frac{|\mathcal{G}_n^{(i)}|}{|\mathcal{G}_n|} = \frac{\left(\frac{1}{2}\pm\zeta_0^3\right)^{i-1}}{(i-1)!} .$$

Moreover, using the previous bound and (2.1), if $i>m$,

$$\frac{|\mathcal{G}_n^{(i)}|}{|\mathcal{G}_n|} \leq \frac{\left(\frac{1}{2}+\zeta_0^3\right)^{m}}{(i-1)!} .$$

Therefore from (3.23) we obtain

$$
\begin{aligned}
|\mathcal{J}| &= \frac{\left(\frac{1}{2} \pm \zeta_0^3\right)^{k-1}}{(k-1)!} \left(\sum_{j=0}^{m-k} \frac{\left(\frac{1}{2} \pm \zeta_0^3\right)^j}{j!} \pm \left(\frac{1}{2} + \zeta_0^3\right)^{m-k+1} \sum_{j>m-k} \frac{1}{j!} \right) |\mathcal{G}_n| \\
&= \frac{\left(\frac{1}{2} \pm \zeta_0^3\right)^{k-1}}{(k-1)!} \left(e^{(1/2 \pm \zeta_0^3)} \pm 2\eta \right) |\mathcal{G}_n| \\
&= \left(1 \pm \frac{\zeta_0^2}{10}\right) e^{1/2} |\mathcal{G}_n^{(k)}| \,, \qquad (3.24)
\end{aligned}
$$

since $\zeta_0 \le k^{-2}$ and ζ_0 is a small constant.

Now we set $\zeta_3 := \min\left\{\frac{\zeta_0^2}{10(u^u)^k}, \zeta_4\right\}$. Fix $W_1, \dots, W_{k-1}$. Since $\mathcal{J}(W_1, \dots, W_{k-1})$ is a disjoint union of bridge-addable classes $(\mathcal{J}(W_1, \dots, W_{k-1}; V_1, \dots, V_{k-1})$, for each $(V_1, \dots, V_{k-1}))$ of graphs with $n - \sum_{j=1}^{k-1} |W_j| \ge n - (k-1)u$ vertices, if n is large enough, by Theorem A applied to each class $\mathcal{J}(W_1, \dots, W_{k-1}; V_1, \dots, V_{k-1})$, we have

$$
\begin{aligned}
|\mathcal{J}(W_1, \dots, W_{k-1})| &\le (1+\zeta_3) e^{1/2} |\mathcal{G}_n^{k,\{W_1,\dots,W_{k-1}\}}| \\
&\le \left(1 + \frac{\zeta_0^2}{10(u^u)^k}\right) e^{1/2} |\mathcal{G}_n^{k,\{W_1,\dots,W_{k-1}\}}| \,. \qquad (3.25)
\end{aligned}
$$

Since there are at most $(u^u)^k$ multisets of unrooted trees $\{W_1, \dots, W_k\}$ of order at most u, from (3.24) and (3.25), we have that for every $W_1, \dots, W_{k-1}$,

$$
|\mathcal{J}(W_1, \dots, W_{k-1})| \ge (1 - \zeta_0^2/5) e^{1/2} |\mathcal{G}_n^{k,\{W_1,\dots,W_{k-1}\}}| \,.
$$

This holds in particular for $\mathcal{H} = \mathcal{J}(U_1, \dots, U_{k-1})$, implying

$$
|\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}| \le (1 + \zeta_0^2/4) e^{-1/2} |\mathcal{H}| \,, \qquad (3.26)
$$

since ζ_0 is a small constant.

For the sake of contradiction assume now that $|\mathcal{H}_0| \ge \zeta_0 |\mathcal{H}|$.

Since $\mathcal{H} \setminus \mathcal{H}_0$ is a disjoint union of bridge-addable classes on $n - \sum_{j=1}^{k-1} |U_j| \ge n - (k-1)u$ vertices, provided that n is large enough, Theorem A implies $\mathbf{Pr}(H \in \mathcal{H} \setminus \mathcal{H}_0 \text{ connected}) \ge (1-\zeta_3) e^{-1/2}$. Moreover, by definition of $\mathcal{H}_0$, we have $\mathbf{Pr}(H \in \mathcal{H}_0 \text{ connected}) \ge (1+\zeta_0) e^{-1/2}$. We obtain

$$
\begin{aligned}
|\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}| &= \mathbf{Pr}(H \in \mathcal{H} \text{ connected}) |\mathcal{H}| \\
&= \mathbf{Pr}(H \in \mathcal{H} \setminus \mathcal{H}_0 \text{ connected}) |\mathcal{H} \setminus \mathcal{H}_0| + \mathbf{Pr}(H \in \mathcal{H}_0 \text{ connected}) |\mathcal{H}_0| \\
&\ge \left((1-\zeta_3) |\mathcal{H} \setminus \mathcal{H}_0| + (1+\zeta_0) |\mathcal{H}_0|\right) e^{-1/2} \\
&\ge \left(1 + \zeta_0^2 - \zeta_3 + \zeta_0 \zeta_3\right) e^{-1/2} |\mathcal{H}| \\
&\ge \left(1 + \zeta_0^2/2\right) e^{-1/2} |\mathcal{H}| \,,
\end{aligned}
$$

which gives a contradiction with (3.26). This concludes the proof of the claim. □

We now set $\zeta := \min\{\zeta_0, \zeta_3\}$, where ζ_3 is the one given by the previous claim.

Let $(V_1, \dots, V_{k-1}) \in \mathcal{V} \setminus \mathcal{V}_0$; that is, the class $\mathcal{H}(V_1, \dots, V_{k-1})$ is ζ_0-tight (and thus, also ζ_1-tight). By Proposition 3.7 applied to the class $\mathcal{H}(V_1, \dots, V_{k-1})$, with the chosen θ_1 and $U = U_k$, and since the class is ζ_1-tight and its elements have at least $n - \sum_{j=1}^{k-1} |V_j| \geq n - (k-1)u$ vertices, we have

$$\frac{|\mathcal{H}^{2,U_k}(V_1, \dots, V_{k-1})|}{|\mathcal{H}(V_1, \dots, V_{k-1})|} = e^{-1/2} \frac{e^{-|U_k|}}{\mathrm{Aut}_u(U_k)} \pm \frac{\theta_k}{8} . \tag{3.27}$$

Since $\mathcal{H}(V_1, \dots, V_{k-1})$ is bridge-addable and since $(V_1, \dots, V_{k-1}) \in \mathcal{V} \setminus \mathcal{V}_0$, by Theorem A and by definition of $\mathcal{V}_0$

$$\frac{|\mathcal{H}(V_1, \dots, V_{k-1})|}{|\mathcal{H}^{(1)}(V_1, \dots, V_{k-1})|} = e^{1/2}(1 \pm \zeta_0) . \tag{3.28}$$

We proceed to bound the contribution of classes indexed by $\mathcal{V}_0$. Using again the previous claim,

$$\begin{aligned} \sum_{(V_1,\dots,V_{k-1}) \in \mathcal{V}_0} |\mathcal{H}^{(1)}(V_1, \dots, V_{k-1})| &\leq |\mathcal{H}_0| \leq \zeta_0 |\mathcal{H}| \\ &\leq \zeta_0 (1-\zeta_0)^{-1} |\mathcal{H} \setminus \mathcal{H}_0| \\ &\leq 2\zeta_0 \sum_{(V_1,\dots,V_{k-1}) \in \mathcal{V} \setminus \mathcal{V}_0} |\mathcal{H}^{(1)}(V_1, \dots, V_{k-1})| , \end{aligned} \tag{3.29}$$

where the last inequality comes from (3.28) and the fact that ζ_0 is a small constant. Therefore,

$$\begin{aligned} |\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}| &= \sum_{(V_1,\dots,V_{k-1}) \in \mathcal{V}} |\mathcal{H}^{(1)}(V_1, \dots, V_{k-1})| \\ &= (1 \pm 2\zeta_0) \sum_{(V_1,\dots,V_{k-1}) \in \mathcal{V} \setminus \mathcal{V}_0} |\mathcal{H}^{(1)}(V_1, \dots, V_{k-1})| . \end{aligned}$$

Using the induction hypothesis for $k-1$, with the chosen θ_{k-1} and $U_1, \dots, U_{k-1}$, and since $\mathcal{G}$ is ζ_2-tight and its elements have at least $n - \sum_{j=1}^{k-1} |V_j| \geq n - (k-1)u$ vertices, it follows that

$$\begin{aligned} \sum_{(V_1,\dots,V_{k-1}) \in \mathcal{V} \setminus \mathcal{V}_0} \frac{|\mathcal{H}^{(1)}(V_1, \dots, V_{k-1})|}{|\mathcal{G}_n|} &= (1 \pm 2\zeta_0)^{-1} \frac{\left|\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}\right|}{|\mathcal{G}_n|} \\ &= (1 \pm 2\zeta_0)^{-1} \left(e^{-1/2} \frac{e^{-\sum_{i=1}^{k-1} |U_i|}}{\mathrm{Aut}_u(U_1, \dots, U_{k-1})} \pm \frac{\theta_k}{8} \right) \\ &= e^{-1/2} \frac{e^{-\sum_{i=1}^{k-1} |U_i|}}{\mathrm{Aut}_u(U_1, \dots, U_{k-1})} \pm \frac{\theta_k}{4} . \end{aligned} \tag{3.30}$$

We are now ready to estimate (3.21). We rewrite (3.21) as

$$\frac{\left|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}\right|}{|\mathcal{G}_n|} = \frac{1}{m(U_1, \dots, U_k)} (\Sigma_{\mathcal{V}_0} + \Sigma_{\mathcal{V} \setminus \mathcal{V}_0}) .$$

where $\Sigma_{\mathcal{V}_0}$ and $\Sigma_{\mathcal{V} \setminus \mathcal{V}_0}$ are the contribution to the sum of the elements indexed by $\mathcal{V}_0$ and by $\mathcal{V} \setminus \mathcal{V}_0$, respectively.

To estimate $\Sigma_{\mathcal{V}_0}$, we note that $|\mathcal{H}(V_1, \dots, V_{k-1})| \leq e |\mathcal{H}^{(1)}(V_1, \dots, V_{k-1})|$, since the class $\mathcal{H}(V_1, \dots, V_{k-1})$ is bridge-addable and using Theorem 2.5 in [MSW05].

Using (3.29), we obtain

$$\Sigma_{\mathcal{V}_0} \leq \frac{e\zeta_0|\mathcal{H}|}{|\mathcal{G}_n|} \leq 3\zeta_0 < \frac{\theta_k}{2} .$$

To estimate $\Sigma_{\mathcal{V}\setminus\mathcal{V}_0}$, we use (3.27), (3.28) and (3.30), to obtain that

$$\Sigma_{\mathcal{V}\setminus\mathcal{V}_0} = e^{-1/2}\frac{e^{-\sum_{i=1}^k |U_i|}}{\mathrm{Aut}_u(U_k)\mathrm{Aut}_u(U_1,\dots,U_{k-1})} \pm \frac{\theta_k}{2} .$$

Using the previous two estimates and (3.18), we get

$$\begin{aligned}\frac{\left|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}\right|}{|\mathcal{G}_n|} &= \frac{1}{m(U_1,\dots,U_k)}\cdot e^{-1/2}\frac{e^{-\sum_{i=1}^k |U_i|}}{\mathrm{Aut}_u(U_k)\mathrm{Aut}_u(U_1,\dots,U_{k-1})} \pm \theta_k \\ &= e^{-1/2}\frac{e^{-\sum_{i=1}^k |U_i|}}{\mathrm{Aut}_u(U_1,\dots,U_k)} \pm \theta_k .\end{aligned}$$

□

Proof of Theorem 3.8, second part. We use induction on k. For $k = 1$, the statement we want to prove is directly given by Proposition 3.7. Assume now that the statement is true for $k-1$.

Set $\hat{\theta}_k := e^{-1/2}\frac{e^{-ku}}{(ku)!}\theta_k$. By the induction hypothesis, for ℓ, $\theta_{k-1} := \frac{\hat{\theta}_k}{8}$, $\delta_{k-1} := 2\delta$ and $U_1,\dots,U_{k-1}$, there exists ζ_{k-1} such that if n is large enough, we have

$$\frac{\sum_{\beta\notin\Xi(\delta_{k-1},\ell)}\left|\mathcal{G}_{n,\beta}^{k,\{U_1,\dots,U_{k-1}\}}\right|}{\left|\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}\right|} < \frac{\hat{\theta}_k}{8} . \tag{3.31}$$

Since the first part of the theorem for k is already proved, we use it to estimate the ratio between $\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}$ and $\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}$. For the first one we use the first part of the theorem for k with $\theta_k := \frac{\hat{\theta}_k}{8}$ and $U_1,\dots,U_k$ and the corresponding ζ_k'. For the second one we use, as before, the first part of the theorem for $k-1$ with θ_{k-1} and $U_1,\dots,U_{k-1}$ and the corresponding ζ_{k-1}. Set $\zeta := \min\{\zeta_{k-1},\zeta_k'\}$ and let n be large enough.

Using (3.18), it follows that

$$\begin{aligned}\frac{|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}|}{|\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}|} &= \frac{e^{-1/2}\frac{e^{-\sum_{i=1}^k |U_i|}}{\mathrm{Aut}_u(U_1,\dots,U_k)} \pm \hat{\theta}_k/8}{e^{-1/2}\frac{e^{-\sum_{i=1}^{k-1} |U_i|}}{\mathrm{Aut}_u(U_1,\dots,U_{k-1})} \pm \hat{\theta}_k/8} \\ &= \frac{e^{-|U_k|}}{m(U_1,\dots,U_{k-1})\mathrm{Aut}_u(U_k)}\left(1 \pm \frac{\theta_k}{3}\right) .\end{aligned} \tag{3.32}$$

Let $T_1,\dots,T_s$ be all the possible rooted versions of the unrooted tree U_k. Observe that $|T_i| = |U_k|$ and that

$$\sum_{i=1}^s \frac{1}{\mathrm{Aut}_r(T_i)} = \frac{|U_k|}{\mathrm{Aut}_u(U_k)} . \tag{3.33}$$

Recall the definition of $p(\mathcal{H},T)$ given at the beginning of Section 3.4. We perform an exact double-counting argument between the graphs in $\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}$ and in $\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}$ using $p(G,T_i)$ with $G \in \mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}$, similar to the one used in Section 2.1. In one direction, for any such graph G, we have exactly

$\sum_{i=1}^{s}\alpha^G(T_i)p(G,T_i)$ ways to construct a graph $G'\in\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}$ by removing an edge. In the other direction, there are exactly $m(U_1,\dots,U_k)|U_k|(n-\sum_{i=1}^{k}|U_j|)$ ways to obtain a graph in $\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}$ from one in $\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}$ by adding an edge. Therefore, we have

$$\begin{aligned}&\sum_{G\in\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}}\sum_{i=1}^{s}\alpha^G(T_i)p(G,T_i)\\ &\qquad= m(U_1,\dots,U_k)|U_k|\left(n-\sum_{i=1}^{k}|U_j|\right)|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}|\ .\end{aligned}\tag{3.34}$$

Using (3.32) and (3.33), it follows that

$$\begin{aligned}&\frac{\sum_{G\in\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}}\sum_{i=1}^{s}\alpha^G(T_i)p(G,T_i)}{n|\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}|}\\ &\quad=\frac{m(U_1,\dots,U_k)|U_k|\left(n-\sum_{i=1}^{k}|U_j|\right)|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}|}{n|\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}|}\\ &\quad=\frac{n-\sum_{i=1}^{k}|U_j|}{n}\cdot\frac{|U_k|e^{-|U_k|}}{\mathrm{Aut}_u(U_k)}\left(1\pm\frac{\theta_k}{3}\right)\\ &\quad=\sum_{i=1}^{s}\frac{e^{-|T_i|}}{\mathrm{Aut}_r(T_i)}\left(1\pm\frac{\theta_k}{2}\right)\ ,\end{aligned}$$

provided that n is large enough.

Since for every $G\in\mathcal{G}_n$, $\sum_{i=1}^{s}\alpha^G(T_i)p(G,T_i)\le n$, it follows that

$$\begin{aligned}&\frac{\sum_{\beta\in\Xi(\delta_{k-1},\ell)}\left(\sum_{i=1}^{s}\beta(T_i)p(\mathcal{G}_{n,\beta}^{k,\{U_1,\dots,U_{k-1}\}},T_i)\right)\cdot|\mathcal{G}_{n,\beta}^{k,\{U_1,\dots,U_{k-1}\}}|}{n|\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}|}\\ &\quad=\frac{\sum_{G\in\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}}\sum_{i=1}^{s}\alpha^G(T_i)p(G,T_i)}{n|\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}|}\pm\frac{\sum_{\beta\notin\Xi(\delta_{k-1},\ell)}|\mathcal{G}_{n,\beta}^{k,\{U_1,\dots,U_{k-1}\}}|}{|\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}|}\\ &\quad=\sum_{i=1}^{s}\frac{e^{-|T_i|}}{\mathrm{Aut}_r(T_i)}\left(1\pm\frac{5\theta_k}{8}\right)\ .\end{aligned}\tag{3.35}$$

If G' is obtained from G by removing an edge that creates a component isomorphic to U_k, then $|\alpha^G(T)-\alpha^{G'}(T)|\le|U_k|\le u$ for every $T\in\mathcal{T}$. Therefore, if $G\in\mathcal{G}_{n,\alpha}^{k,\{U_1,\dots,U_{k-1}\}}$ for some $\beta\in\Xi(\delta_{k-1},\ell)$, then $G'\in\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}$ is such that $\alpha^{G'}\in\Xi(\delta,\ell)$ (recall that $\delta_{k-1}=\delta/2$), provided that n is large enough. We thus obtain a local version of (3.34)

$$\begin{aligned}&\sum_{\beta\in\Xi(\delta_{k-1},\ell)}\left(\sum_{i=1}^{s}\beta(T_i)p(\mathcal{G}_{n,\beta}^{k,\{U_1,\dots,U_{k-1}\}},T_i)\right)\cdot|\mathcal{G}_{n,\alpha}^{k,\{U_1,\dots,U_{k-1}\}}|\\ &\quad\le m(U_1,\dots,U_k)|U_k|\left(n-\sum_{i=1}^{k}|U_j|\right)\sum_{\beta\in\Xi(\delta,\ell)}|\mathcal{G}_{n,\beta}^{k+1,\{U_1,\dots,U_k\}}|\ .\end{aligned}$$

Using (3.35), the last inequality and (3.32), it follows that

$$\begin{aligned}
&\sum_{i=1}^{s} \frac{e^{-|T_i|}}{\mathrm{Aut}_r(T_i)}\left(1-\frac{5\theta_k}{8}\right)\\
&\leq \frac{1}{n|\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}|}\sum_{\beta\in\Xi(\delta_{k-1},\ell)}\left(\sum_{i=1}^{s}\beta(T_i)p(\mathcal{G}_{n,\beta}^{k,\{U_1,\dots,U_{k-1}\}},T_i)\right)\cdot|\mathcal{G}_{n,\beta}^{k,\{U_1,\dots,U_{k-1}\}}|\\
&\leq \frac{e^{-|U_k|}}{\mathrm{Aut}_u(U_k)|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}|}\sum_{\beta\in\Xi(\delta,\ell)}\frac{(n-\sum_{i=1}^{k}|U_j|)|U_k|}{n}|\mathcal{G}_{n,\beta}^{k+1,\{U_1,\dots,U_k\}}|\left(1+\frac{\theta_k}{3}\right)\\
&\leq \frac{|U_k|e^{-|U_k|}}{\mathrm{Aut}_u(U_k)}\cdot\frac{\sum_{\beta\in\Xi(\delta,\ell)}|\mathcal{G}_{n,\beta}^{k+1,\{U_1,\dots,U_k\}}|}{|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}|}\left(1+\frac{\theta_k}{3}\right)\\
&=\sum_{i=1}^{s}\frac{e^{-|T_i|}}{\mathrm{Aut}_r(T_i)}\cdot\frac{\sum_{\beta\in\Xi(\delta,\ell)}|\mathcal{G}_{n,\beta}^{k+1,\{U_1,\dots,U_k\}}|}{|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}|}\left(1+\frac{\theta_k}{3}\right)\ ,
\end{aligned}$$

where we used (3.33) for the last equality. We conclude,

$$\frac{\sum_{\beta\in\Xi(\delta,\ell)}|\mathcal{G}_{n,\beta}^{k+1,\{U_1,\dots,U_k\}}|}{|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}|}\geq 1-\theta_k\ ,$$

which finishes the proof of the theorem. □

4. From classes of forests to classes of graphs

In this section we extend our results from bridge-addable classes of forests to general bridge-addable classes. In 4.1 we prove that graphs in ζ-tight bridge-addable classes tend to have many removable edges, and in 4.2 we use this property and the results of Section 3 to conclude the proof of Theorem 1.2. We conclude with the proof of Corollary 1.5.

4.1. **Removable edges in tight bridge-addable classes of graphs.** A *2-block* of a graph G is a maximal 2-edge-connected graph (we assume that the graph composed of a single vertex is also 2-edge-connected). Every graph admits a unique decomposition into 2-blocks, joined by edges in a tree-like fashion.

For a graph class $\mathcal{G}_n$, we can consider the coarsest partition

$$\mathcal{G}_n = \biguplus_i \mathcal{H}_n^{[i]}\ . \tag{4.1}$$

into subclasses $\mathcal{H}_n^{[1]}, \mathcal{H}_n^{[2]}, \dots$ such that every two graphs in the same subclass have the same 2-blocks. By construction, if $\mathcal{G}_n$ is bridge-addable, then every subclass $\mathcal{H}_n^{[i]}$ is also bridge-addable.

For each such subclass $\mathcal{H}$, we assume that we have chosen, arbitrarily and once and for all, a spanning tree for each 2-block of the graphs in $\mathcal{H}$. We denote by $\mathcal{F}_{\mathcal{H}}$ the class of forests obtained by replacing each 2-block with the corresponding spanning tree in each graph in $\mathcal{H}$. This is well-defined, since, by construction, graphs in the same subclass have the same 2-blocks. Moreover, the class $\mathcal{F}_{\mathcal{H}}$ is also bridge-addable and the component structure (number and size) of each graph $H\in\mathcal{H}$ is preserved in the corresponding forest $F_H\in\mathcal{F}_{\mathcal{H}}$. This construction was introduced in [BBG08], to which we refer for more details.

The next lemma states that most graphs in a ζ-tight belong to subclasses $\mathcal{H}_n^{[i]}$ that are themselves close to be tight.

Lemma 4.1. *For every $\zeta_0 > 0$ there exists $\zeta > 0$ such that if n is large enough, for any bridge-addable class $\mathcal{G}$ that is ζ-tight, the following is true. Let $\mathcal{H}_n^{[1]}, \mathcal{H}_n^{[2]}, \dots$ be the partition of $\mathcal{G}_n$ in bridge-addable subclasses defined above and let $S_n(\zeta_0)$ be the set of values i such that*

$$\mathbf{Pr}(H_n \in \mathcal{H}_n^{[i]} \text{ connected}) \leq (1+\zeta_0)e^{-1/2}, \tag{4.2}$$

where $H_n \in \mathcal{H}_n^{[i]}$ denotes a uniformly random graph in $\mathcal{H}_n^{[i]}$. Then we have

$$\left| \biguplus_{i \in S_n(\zeta_0)} \mathcal{H}_n^{[i]} \right| \geq (1-\zeta_0)|\mathcal{G}_n| \, . \tag{4.3}$$

Proof. The proof is direct by an averaging argument in a similar way as in the claim inside the proof of Theorem 3.8. □

A vertex v in G_n is *connected to the bulk of G_n through a cut-edge*, if there is a cut-edge e incident to v such after removing e, the newly created component not containing v has size at least $3n/4$. Note that for each $v \in \{1, \dots, n\}$ there is at most one edge e with this property. The connected component containing v after removing e is called a *pendant graph.* The edge e can *a priori* be removable or not, and if it is we say that v is *connected to the bulk of G_n through a removable cut-edge.*

Lemma 4.2. *For every θ, there exist ζ and ℓ such that provided that n is large enough, for every ζ-tight bridge-addable class $\mathcal{G}$, we have that if G_n is a graph chosen uniformly at random in $\mathcal{G}_n$, and V_n is a vertex chosen uniformly at random in G_n, with probability at least $1-\theta$, V_n is connected to the bulk of G_n through a removable cut-edge and the corresponding pendant graph has order at most ℓ.*

Proof. We first prove the lemma for bridge-addable classes of forests and then we transfer it to general bridge-addable classes of graphs.

Assume that $\mathcal{G}_n$ is composed of forests. We first show that there exists ℓ such that if G_n is a graph chosen uniformly at random from $\mathcal{G}_n$, then with probability at least $(1-\theta/4)$ we have that $p(\mathcal{G}_n, T) \geq 1-\theta/4$ for every $T \in \mathcal{T}_{\leq \ell}$. Then we will prove that with probability at least $1-\theta$, most of the pendant trees in G_n have size at most ℓ.

From Lemma 2.2, we can choose ℓ large enough such that

$$\min \left\{ \sum_{T \in \mathcal{T}_{\leq \ell}} \frac{e^{-|T|}}{\mathrm{Aut}_r(T)}, \, e^{-1/2} \sum_{k=0}^{\ell} \sum_{\{U_1, \dots, U_k\} \in \mathcal{U}_{\leq \ell}} \frac{e^{-\sum_{i=1}^k |U_i|}}{\mathrm{Aut}_u(U_1, \dots, U_k)} \right\} \geq 1 - \frac{\theta}{10} \, .$$

Let $T \in \mathcal{T}_{\leq \ell}$ be a given rooted tree, we now show that $p(\mathcal{G}_n, T) \geq 1-\theta/4$. Let λ be the size of the equivalence class of the root of T (the number of vertices where T can be re-rooted giving rise to a rooted tree isomorphic to T). For every $k \leq \ell$ and every $U_1, \dots, U_k$ of order at most ℓ such that U_k is the unrooted version of T, we will write the ratio between $|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}|$ and $|\mathcal{G}_n|$ in two ways. We select ζ small enough and n large enough, such that we can apply Theorem 3.8 for every

$k \le \ell$, for $\theta_k = \tilde{\theta}$ (to be fixed later) and for every $U_1, \dots, U_k$ of size at most ℓ. If $\mathcal{G}$ is ζ-tight, we obtain

$$\frac{|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}|}{|\mathcal{G}_n|} = e^{-1/2} \frac{e^{-\sum_{i=1}^k |U_i|}}{\mathrm{Aut}_u(U_1, \dots, U_k)} \pm \tilde{\theta} .$$

As before, we perform an exact local double-counting argument with the difference that now we only count those graphs $G' \in \mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}$ that can be obtained from $G \in \mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}$ by removing an edge from where a copy of T is pendant. This can only be done if U_k is the unrooted version of T and if the edge that connects T to the rest of G is removable. Moreover, if G' is obtained from G in such a way, for every $T_0 \in \mathcal{T}_{\le \ell}$ we have $|\alpha^G(T_0) - \alpha^{G'}(T_0)| \le |T| \le \frac{\tilde{\theta} n}{2}$. In one direction, given a graph $G \in \mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}$ there are exactly $p(G,T)\alpha^G(T)$ many such ways to obtain a graph in $\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}$, and in the other one, exactly $\lambda m(U_1, \dots, U_k)(n - \sum_{j=1}^k |U_j|)$ many ones. Applying Theorem 3.8 twice with $\theta_k = \tilde{\theta}$ and $\delta = \tilde{\theta}/2$, if ζ is small enough and n is large enough, then if $\mathcal{G}$ is ζ-tight, we obtain

$$\begin{aligned}
\frac{|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}|}{|\mathcal{G}_n|} &\le \frac{1}{|\mathcal{G}_n|} \sum_{\beta \in \Xi(\delta,\ell)} |\mathcal{G}_{n,\beta}^{k+1,\{U_1,\dots,U_k\}}|(1+\tilde{\theta}) \\
&\le \frac{1}{|\mathcal{G}_n| m(U_1,\dots,U_k)} \sum_{\beta \in \Xi(\tilde{\theta},\ell)} \sum_{G \in \mathcal{G}_{n,\beta}^{k,\{U_1,\dots,U_{k-1}\}}} \frac{p(G,T)\alpha^G(T)}{(n - \sum_{j=1}^{k-1}|U_j|)\lambda}(1+\tilde{\theta}) \\
&= \frac{1}{|\mathcal{G}_n| m(U_1,\dots,U_k)} \sum_{\beta \in \Xi(\tilde{\theta},\ell)} |\mathcal{G}_{n,\beta}^{k,\{U_1,\dots,U_{k-1}\}}| \frac{p(\mathcal{G}_{n,\beta}^{k,\{U_1,\dots,U_{k-1}\}},T)\beta(T)}{(n - \sum_{j=1}^{k-1}|U_j|)\lambda}(1+\tilde{\theta}) \\
&\le \frac{1}{m(U_1,\dots,U_k)} \cdot \frac{|\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}|}{|\mathcal{G}_n|} \cdot \frac{p(\mathcal{G}'_n,T)}{\lambda} \left(\frac{e^{-|T|}}{\mathrm{Aut}_r(T)} + \tilde{\theta} \right)(1+\tilde{\theta}) \\
&\le \frac{1}{m(U_1,\dots,U_k)} \left(e^{-1/2} \frac{e^{-\sum_{i=1}^{k-1}|U_i|}}{\mathrm{Aut}_u(U_1,\dots,U_{k-1})} + \tilde{\theta} \right) p(\mathcal{G}'_n,T) \frac{e^{-|U_k|}}{\mathrm{Aut}_u(U_k)}(1+3\tilde{\theta}) \\
&\le e^{-1/2} \frac{e^{-\sum_{i=1}^k |U_i|}}{\mathrm{Aut}_u(U_1,\dots,U_k)} \cdot p(\mathcal{G}'_n,T)(1+5\tilde{\theta}) .
\end{aligned}$$

where $\mathcal{G}'_n$ is the class formed by the union of $\mathcal{G}_{n,\beta}^{k,\{U_1,\dots,U_{k-1}\}}$ for $\beta \in \Xi(\delta,\ell)$. In the previous inequalities we have used that $\mathrm{Aut}_u(U_k) = \lambda \mathrm{Aut}_r(T)$ and (3.33).

Combining these two expressions and since $|\mathcal{G}'_n| \ge (1-\tilde{\theta})|\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}}|$ (by Theorem 3.8), we obtain that for every rooted tree $T \in \mathcal{T}_{\le \ell}$,

$$p(\mathcal{G}_n^{k,\{U_1,\dots,U_{k-1}\}},T) \ge 1 - 8\tilde{\theta} . \tag{4.4}$$

Now we set $\tilde{\theta} := \theta \ell^{-(\ell^2+1)}/10$. Applying Theorem 3.8 for every $k \le \ell$, $\theta_k = \tilde{\theta}$ and $U_1, \dots, U_k$, and using the definition of ℓ

$$\begin{aligned}
\sum_{k=0}^{\ell} \sum_{U_1,\dots,U_k \in \mathcal{U}_{\le \ell}} \frac{|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}|}{|\mathcal{G}_n|} &= e^{-1/2} \sum_{k=0}^{\ell} \sum_{U_1,\dots,U_k \in \mathcal{U}_{\le \ell}} \frac{e^{-\sum_{i=1}^k |U_i|}}{\mathrm{Aut}_u(U_1,\dots,U_k)} - \tilde{\theta}\ell(\ell^\ell)^\ell \\
&\ge 1 - \frac{\theta}{5} ,
\end{aligned} \tag{4.5}$$

By averaging (4.4) over all k and $U_1, \dots, U_{k-1}$ and using the last equation, for every $T \in \mathcal{T}_{\leq \ell}$, we obtain

$$p(\mathcal{G}_n, T) \geq 1 - \theta/4 \ , \tag{4.6}$$

which proves the first part.

Let us now show that there are many removable edges that isolate a tree of size at most ℓ. Choose G_n uniformly at random from $\mathcal{G}_n$ and then choose V_n uniformly at random from $\{1, \dots, n\}$. Let A_1 be the event that V_n is connected to the bulk of G_n through a removable cut-edge and let A_2 be the event that the pendant tree rooted at V_n has order at most ℓ. We want to show that $\mathbf{Pr}(A_1 \cap A_2) \geq 1 - \theta$.

Again, by applying Theorem 3.8 for every $k \leq \ell$, $\theta_k = \tilde{\theta}$ and $U_1, \dots, U_k$, and using (4.5), we obtain

$$\begin{aligned}
\sum_{\beta \in \Xi(\delta, \ell)} |\mathcal{G}_{n,\beta}| &\geq \sum_{k=0}^{\ell} \sum_{U_1, \dots, U_k \in \mathcal{U}_{\leq \ell}} \sum_{\beta \in \Xi(\delta,\ell)} |\mathcal{G}_{n,\beta}^{k+1, \{U_1, \dots, U_k\}}| \\
&\geq \sum_{k=0}^{\ell} \sum_{U_1, \dots, U_k \in \mathcal{U}_{\leq \ell}} |\mathcal{G}_{n}^{k+1, \{U_1, \dots, U_k\}}| - \tilde{\theta}\ell(\ell^\ell)^\ell \\
&\geq (1 - \theta/4)|\mathcal{G}_n| \ .
\end{aligned}$$

Moreover, for every $\beta \in \Xi(\delta, \ell)$ and by our choice of ℓ, we have that $\sum_{T \in \mathcal{T}_{\leq \ell}} \frac{\beta(T)}{n} \geq \sum_{T \in \mathcal{T}_{\leq \ell}} \frac{e^{|T|}}{\mathrm{Aut}_r(T)} - \delta \ell^\ell \geq 1 - \theta/5$. It follows that $\mathbf{Pr}(A_2) \geq 1 - \theta/2$.

Assume that A_2 holds. Let T_n be the pendant tree rooted at V_n and note that $T_n \equiv T$, for some $T \in \mathcal{T}_{\leq \ell}$. By (4.6), the probability that the cut-edge that connects V_n to the bulk of G_n is removable is $p(\mathcal{G}_n, T) \geq 1 - \theta/4$. Thus, $\mathbf{Pr}(A_1 \mid A_2) \geq 1 - \theta/4$.

We conclude that

$$\mathbf{Pr}(A_1 \cap A_2) = 1 - \mathbf{Pr}(\overline{A_1} \cup \overline{A_2}) \geq 1 - (\mathbf{Pr}(\overline{A_2}) + \mathbf{Pr}(\overline{A_1} \mid A_2)) \geq 1 - 3\theta/4 \ , \tag{4.7}$$

which concludes the proof of the theorem when all graphs in $\mathcal{G}$ are forests.

In order to extend the result to general classes of graphs, we use the approach introduced in [BBG08]. Let $\mathcal{G}$ be a general class of graphs and let $\mathcal{H}_n^{[1]}, \mathcal{H}_n^{[2]}, \dots$ be the partition of $\mathcal{G}_n$ into subclasses defined at the beginning of this section. Given ζ_0 (to be fixed later), we let $S_n = S_n(\zeta_0)$ be the set of indices given by Lemma 4.1, and we fix an index $i \in S_n$. We let $\mathcal{H} := \mathcal{H}_n^{[i]}$ be the corresponding subclass of $\mathcal{G}_n$ and we let $\mathcal{F}_{\mathcal{H}}$ be the corresponding class of forests. We observe that $\mathcal{F}_{\mathcal{H}}$ is ζ_0-tight and bridge-addable.

Since Lemma 4.2 holds for classes of forests, we can apply it to $\mathcal{F}_{\mathcal{H}}$. Note that if a cut-edge is removable for a forest $F_H \in \mathcal{F}_{\mathcal{H}}$, then the edge does not belong to any of the 2-blocks of the corresponding graph $H \in \mathcal{H}$. This implies that this cut-edge is also removable for $H \in \mathcal{H}$. Moreover, if its removal in F_H results in a tree of size at most ℓ, then its removal in H results in a *graph* of size at most ℓ. Therefore, the result obtained in (4.7) for $\mathcal{F}_{\mathcal{H}}$ naturally transfers to the class $\mathcal{H}$, provided we change "trees" by "graphs" in what results after deleting a removable edge.

Moreover, if we choose ζ_0 small enough with respect to θ, then there exists ζ such that if $\mathcal{G}$ is ζ-tight and n is large enough, by (4.3), at least $(1-\theta/4)|\mathcal{G}_n|$ graphs in $\mathcal{G}_n$ are in subclasses $\mathcal{H}_n^{[i]}$ with $i \in S_n$. Thus, the lemma also holds for general classes of graphs $\mathcal{G}_n$. $\square$

For every class $\mathcal{G}_n$ and every $t \geq 1$, if G_n is chosen uniformly at random from $\mathcal{G}_n$ and V_n is chosen uniformly at random from $\{1, \dots, n\}$, then let $q(\mathcal{G}_n, t)$ be the probability that V_n is connected to the bulk of G_n through a removable cut-edge and the corresponding pendant graph is a tree of order at most t. Observe that if $\mathcal{G}$ is a subclass of forests, Lemma 4.2 implies that for every θ, and under some conditions, there exists ℓ such that $q(\mathcal{G}_n, \ell) \geq 1 - \theta$. Next lemma shows that the same holds for general classes of graphs.

Lemma 4.3. *For every ϑ, there exist ζ and t, such that if $\mathcal{G}$ is a ζ-tight bridge-addable class and n is large enough, then $q(\mathcal{G}_n, t) \geq 1 - \vartheta$.*

Proof. Given $G \in \mathcal{G}_n$ and a vertex $v \in \{1, \dots, n\}$ that is connected to the bulk of G through a cut-edge e, we denote by $X_G(v)$ the pendant graph (containing v) obtained when deleting e from G. Given G_n chosen uniformly at random from $\mathcal{G}_n$ and V_n chosen uniformly at random from $\{1, \dots, n\}$, as before, we define A_1 as the event that V_n is connected to the bulk of G_n through a removable cut-edge and A_2 as the event that $X_{G_n}(V_n)$ has order at most t. Also, let A_3 be the event that $X_{G_n}(V_n)$ is a tree. It is implicit in the definition of A_2 and A_3 that V_n should be connected to the bulk of G_n through a cut-edge, so in particular $X_{G_n}(V_n)$ is well-defined. Note that

$$\begin{aligned} q(\mathcal{G}_n, t) = \mathbf{Pr}(A_1 \cap A_2 \cap A_3) &= \mathbf{Pr}(A_1 \cap A_2) - \mathbf{Pr}(A_1 \cap A_2 \cap \overline{A_3}) \\ &\geq \mathbf{Pr}(A_1 \cap A_2) - \mathbf{Pr}(A_2 \cap \overline{A_3}) \; . \end{aligned} \tag{4.8}$$

so we will proceed by bounding the last two probabilities.

We again consider the partition of $\mathcal{G}_n$ into subclasses $\mathcal{H}_n^{[1]}, \mathcal{H}_n^{[2]}, \dots$ defined above. Given ζ_0 (to be fixed later), there exists ζ such that for every ζ-tight class $\mathcal{G}$, if n is large enough, we can consider $S_n = S_n(\zeta_0)$ to be the set of indices given by Lemma 4.1. We let $\mathcal{H} := \mathcal{H}_n^{[i]}$ be the corresponding subclass of $\mathcal{G}_n$, for some $i \in S_n$, and $\mathcal{F}_{\mathcal{H}}$ be corresponding class of forests.

By Lemma 4.2 with $\theta = \vartheta/3$, if ζ_0 is small enough and, n and t are large enough, since $\mathcal{H}$ is a ζ_0-tight bridge-addable class of graphs with n vertices, then the probability that a uniformly chosen vertex W_n from a uniformly chosen forest F_n in $\mathcal{F}_{\mathcal{H}}$ connects to the bulk of F_n through a removable cut-edge and that $X_{F_n}(W_n)$ is a tree of order at most t, is at least $1 - \vartheta/3$. If this is the case, as we argued before, this edge is also a removable cut-edge in the graph in $\mathcal{H}$ that corresponds to F_n. Thus, using (4.3) and provided that ζ_0 is small enough with respect to ϑ,

$$\mathbf{Pr}(A_1 \cap A_2) \geq 1 - \frac{\vartheta}{3} - \zeta_0 \geq 1 - \frac{\vartheta}{2} \; . \tag{4.9}$$

It remains to obtain an upper bound on $\mathbf{Pr}(A_2 \cap \overline{A_3})$. Using Lemma 4.2 again with $\theta_2 = \frac{\vartheta}{7t}$, and if ζ_0 is small enough and n and ℓ are large enough, since $\mathcal{H}$ is ζ_0-tight, the probability that a uniformly chosen vertex W_n from a uniformly chosen forest F_n in $\mathcal{F}_{\mathcal{H}}$ is connected to the bulk of F_n through a removable cut-edge, is at least $1 - \frac{\vartheta}{7t}$. Using (4.3) again and provided that ζ_0 is small enough with respect to ϑ and t, we obtain

$$\mathbf{Pr}(\overline{A_1}) \leq \frac{\vartheta}{7t} + \zeta_0 \leq \frac{\vartheta}{6t} \; . \tag{4.10}$$

We claim that

$$\mathbf{Pr}(A_2 \cap \overline{A_3}) \leq t\mathbf{Pr}(\overline{A_1}) \; . \tag{4.11}$$

Assuming that (4.11) holds, together with (4.9) and with (4.10), we obtain

$$q(\mathcal{G}_n, t) \geq 1 - \frac{\vartheta}{2} - \frac{\vartheta}{6} \geq 1 - \vartheta \; .$$

Thus, it only remains to prove (4.11). For this we observe that if $A_2 \cap \overline{A_3}$ holds, then $X_{G_n}(V_n)$ contains at least one vertex V_n' which is not connected to the bulk of G_n through a cut-edge (since $X_{G_n}(V_n)$ is a well-defined pendant graph, but it is not a tree). Moreover since A_2 holds, the graph distance between V_n and V_n' is less than t. Conversely, it is easy to see that given any vertex v', there are at most t vertices v at distance at less than t from v' that are connected to the bulk of G_n through a cut-edge and such that $X_{G_n}(v)$ contains v'. The inequality (4.11) thus follows by double-counting such pairs of vertices. □

4.2. **Proof of our main results.**

We finally show our main theorem.

Proof of Theorem 1.2. Let us first prove *i*). We will first prove that for every k, every θ and every $U_1, \dots, U_k$, and if ζ is small enough and n large enough, then for every ζ-tight bridge-addable class $\mathcal{G}$, we have

$$\left| \frac{\left|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}\right|}{|\mathcal{G}_n|} - e^{-1/2}\frac{e^{-\sum_{i=1}^k |U_i|}}{\mathrm{Aut}_u(U_1,\dots,U_k)} \right| < \theta \; . \tag{4.12}$$

As before we consider the partition of $\mathcal{G}_n$ into subclasses $\mathcal{H}_n^{[1]}, \mathcal{H}_n^{[2]}, \dots$. Given ζ_0 (to be fixed later), there exists ζ such that for every ζ-tight class $\mathcal{G}$, if n is large enough, we can consider the set $S_n = S_n(\zeta_0)$ given by Lemma 4.1.

Let $\mathcal{H} := \mathcal{H}_n^{[i]}$ for $i \in S_n$ and let $\mathcal{F}_\mathcal{H}$ be the corresponding ζ_0-tight class of forests. We can apply Theorem 3.8 for the given k, $\theta_k = \frac{\theta}{4}$, and the given $U_1, \dots, U_k$. If ζ_0 is small enough and n is large enough, and since $\mathcal{F}_\mathcal{H}$ is ζ_0-tight, (4.12) holds for $\mathcal{F}_\mathcal{H}$.

It follows that

$$\begin{aligned}
\left|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}\right| &= \sum_{j\in S_n} \left|(\mathcal{H}_n^{[j]})^{k+1,\{U_1,\dots,U_k\}}\right| \pm \zeta_0|\mathcal{G}_n| \\
&= \sum_{j\in S_n} \left|\mathcal{F}_{\mathcal{H}_n^{[j]}}^{k+1,\{U_1,\dots,U_k\}}\right| \pm \zeta_0|\mathcal{G}_n| \\
&= \left(e^{-1/2}\frac{e^{-\sum_{i=1}^k |U_i|}}{\mathrm{Aut}_u(U_1,\dots,U_k)} \pm \theta_k\right) \sum_{j\in S_n} |\mathcal{F}_{\mathcal{H}_n^{[j]}}| \pm \zeta_0|\mathcal{G}_n| \\
&= \left(e^{-1/2}\frac{e^{-\sum_{i=1}^k |U_i|}}{\mathrm{Aut}_u(U_1,\dots,U_k)} \pm \theta_k\right) (1\pm\zeta_0)|\mathcal{G}_n| \pm \zeta_0|\mathcal{G}_n| \\
&= \left(e^{-1/2}\frac{e^{-\sum_{i=1}^k |U_i|}}{\mathrm{Aut}_u(U_1,\dots,U_k)} \pm \theta\right) |\mathcal{G}_n| \; ,
\end{aligned}$$

provided that ζ_0 is small enough with respect to θ. This proves (4.12).

To prove the first part of the theorem, let k_* be large enough such that

$$e^{-1/2}\sum_{k=0}^{k_*}\sum_{\{U_1,\dots,U_k\}\in\mathcal{U}_{\le k_*}}\frac{e^{-\sum_{i=1}^{k}|U_i|}}{\mathrm{Aut}_u(U_1,\dots,U_k)}\ge 1-\frac{\epsilon}{4}\,. \tag{4.13}$$

The existence of such a k_* is, again, guaranteed by Lemma 2.2.

If $\mathbf{f}$ is an unrooted unlabeled forest composed of trees $U_1,\dots,U_k$, then

$$\mathbf{Pr}(Small(G_n)\equiv\mathbf{f})=\frac{\left|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}\right|}{|\mathcal{G}_n|}\,.$$

We choose $\theta := \epsilon k_*^{-k_*^2}/2$.

Let $\mathbf{f_1}$ be a forest composed of at most k_* trees of size at most k_*, then (4.12) gives that $|\mathbf{Pr}(Small(G_n)\equiv\mathbf{f_1})-p_\infty(\mathbf{f_1})|<\epsilon$.

Let $\mathbf{f_2}$ be a forest with either more than k_* trees or where at least one of the trees has size larger than k_*. Since p_∞ is a probability distribution, by (4.13) we have $p_\infty(\mathbf{f_2})\le\epsilon/4$. Since $\sum_{\mathbf{f}}\mathbf{Pr}(Small(G_n)\equiv\mathbf{f})=1$, using again (4.12) and (4.13), we have

$$\begin{aligned}|\mathbf{Pr}(Small(G_n)\equiv\mathbf{f_2})-p_\infty(\mathbf{f_2})| &\le \mathbf{Pr}(Small(G_n)\equiv\mathbf{f_2})+p_\infty(\mathbf{f_2})\\ &\le 1-\sum_{k=0}^{k_*}\sum_{\{U_1,\dots,U_k\}\in\mathcal{U}_{\le k_*}}\frac{\left|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}\right|}{|\mathcal{G}_n|}+p_\infty(\mathbf{f_2})\\ &\le \epsilon/4+\theta k_*^{k_*^2}+\epsilon/4=\epsilon\,.\end{aligned}$$

This concludes the proof of *i*).

We next prove the following property, from which *ii*) follows directly.

iii) for every ϵ,η, there exists ζ such that for every ζ-tight bridge-addable class $\mathcal{G}$ and every n large enough, if $\mathbf{f}$ is a fixed unrooted unlabeled forest,

$$\left|\mathbf{Pr}\left(Small(G_n)\equiv\mathbf{f};\ \forall T\in\mathcal{T}:\ \left|\frac{\alpha^{G_n}(T)}{n}-a_\infty(T)\right|<\eta\right)-p_\infty(\mathbf{f})\right|<\epsilon.$$

We first prove that for every θ,η,k,ℓ and $U_1,\dots,U_k$, and provided that ζ is small enough and n large enough, we have

$$\frac{\sum_{\beta\in\Xi(\eta,\ell)}\left|\mathcal{G}_{n,\beta}^{k+1,\{U_1,\dots,U_k\}}\right|}{|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}|}\ge 1-\theta\,. \tag{4.14}$$

Recall the partition of $\mathcal{G}_n$ into subclasses $\mathcal{H}_n^{[1]},\mathcal{H}_n^{[2]},\dots$. As before, let $\mathcal{H}:=\mathcal{H}_n^{[i]}$ for $i\in S_n$ and let $\mathcal{F}_\mathcal{H}$ be the corresponding class of forests. Applying the second part of Theorem 3.8 with $\theta_k=\theta/4$ and $\delta=\eta/2$ to the class $\mathcal{F}_\mathcal{H}$, we see that if ζ_0 is small enough and n large enough, then at least $(1-\theta/2)|\mathcal{H}^{k+1,\{U_1,\dots,U_k\}}|$ graphs $G\in\mathcal{H}^{k+1,\{U_1,\dots,U_k\}}$ satisfy $\alpha^{F_G}\in\Xi(\delta,\ell)$.

Theorem 3.8 also shows that there exists $c_1>0$ such that $|\mathcal{H}^{k+1,\{U_1,\dots,U_k\}}|\ge c_1|\mathcal{H}|$. By Lemma 4.3 with $\vartheta:=c_1\min\{\theta/4,\delta\}$, if ζ_0 is small enough and n large enough there exists t such that with probability at least $1-\vartheta$, a random vertex in a random graph of $\mathcal{H}$ is connected via a removable cut-edge and the corresponding pendant graph is a tree of order at most t. We can choose $t \geq \ell$. (Note that by doing so, we only increase the former probability.)

Therefore, if H_n is a random graph in $\mathcal{H}^{k+1,\{U_1,\dots,U_k\}}$, with probability at least $1-\theta/2-\vartheta/c_1 > 1-3\theta/4$, for every $T \in \mathcal{T}_{\leq \ell}$,

$$\frac{\alpha^{H_n}(T)}{n} = \frac{e^{-|T|}}{\mathrm{Aut}_r(T)} \pm \delta \pm \vartheta = \frac{e^{-|T|}}{\mathrm{Aut}_r(T)} \pm 2\delta \ .$$

In other words, with probability at least $1-3\theta/4$, we have $\alpha^{H_n} \in \Xi(2\delta,\ell) = \Xi(\eta,\ell)$.

By i), we have that $|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}| \geq c_2|\mathcal{G}_n|$, for some constant $c_2 > 0$. Therefore, there are at most $\frac{\zeta_0}{c_2}|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}|$ graphs in classes $\mathcal{H}_n^{[i]}$ that are not ζ_0-tight. We conclude that, provided ζ_0 is small enough, the probability that a graph G'_n chosen at random from $\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}$ satisfies $\alpha^{G'_n} \in \Xi(\eta,\ell)$, is at least $1-3\theta/4-\zeta_0/c_2 > 1-\theta$. This proves (4.14).

Let $A(k,\nu)$ the event that for every $T \in T_{\leq k}$ we have $\left|\frac{\alpha^{G_n}(T)}{n} - a_\infty(T)\right| < \nu$ (we might write $k=\infty$ where $\mathcal{T}_{\leq\infty} = \mathcal{T}$).

Since we have already proved i), we have that for every unrooted unlabeled forest $\mathbf{f}$ with small components $U_1,\dots,U_k$, then

$$\mathbf{Pr}\left(A(\infty,\eta), \mathit{Small}(G_n) \equiv \mathbf{f}\right) \leq \mathbf{Pr}(\mathit{Small}(G_n) \equiv \mathbf{f}) \leq p_\infty(\mathbf{f}) + \epsilon \ . \tag{4.15}$$

By Lemma 2.2, if k_* is large enough, then

$$\sum_{T \in \mathcal{T}_{\leq k_*}} \frac{e^{-|T|}}{\mathrm{Aut}_r(T)} > 1 - \frac{\eta}{4} \ .$$

Let $T' \notin \mathcal{T}_{\leq k_*}$ and choose $\rho = \eta k_*^{-k_*}/4$. As before, by the properties of k_* we have that $a_\infty(T') \leq \eta/4$ and, conditional on $A(k_*,\rho)$, $\frac{\alpha^{G_n}(T')}{n} \leq \eta/4 + \rho k_*^{k_*} = \eta/2$. This implies that, conditional on $A(k_*,\rho)$, then $A(k_*,\eta)$ implies $A(\infty,\eta)$.

If $\mathbf{f}$ is an unrooted unlabeled forest with small components $U_1,\dots,U_k$, using (4.14), we have that for every θ,

$$\begin{aligned}
&P\left(A(\infty,\eta) \mid \mathit{Small}(G_n) \equiv \mathbf{f}\right) \\
&\geq P\left(A(\infty,\eta) \mid \mathit{Small}(G_n) \equiv \mathbf{f}, A(k_*,\rho)\right) \cdot P\left(A(k_*,\rho) \mid \mathit{Small}(G_n) \equiv \mathbf{f}\right) \\
&\geq P\left(A(k_*,\eta) \mid \mathit{Small}(G_n) \equiv \mathbf{f}, A(k_*,\rho)\right) \cdot P\left(A(k_*,\rho) \mid \mathit{Small}(G_n) \equiv \mathbf{f}\right) \\
&= \frac{\sum_{\beta \in \Xi(\eta,k_*)} |\mathcal{G}_{n,\beta}^{k+1,\{U_1,\dots,U_k\}}|}{\sum_{\beta \in \Xi(\rho,k_*)} |\mathcal{G}_{n,\beta}^{k+1,\{U_1,\dots,U_k\}}|} \cdot \frac{\sum_{\beta \in \Xi(\rho,k_*)} |\mathcal{G}_{n,\beta}^{k+1,\{U_1,\dots,U_k\}}|}{|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}|} \\
&= \frac{\sum_{\beta \in \Xi(\eta,k_*)} |\mathcal{G}_{n,\beta}^{k+1,\{U_1,\dots,U_k\}}|}{|\mathcal{G}_n^{k+1,\{U_1,\dots,U_k\}}|} \\
&\geq 1-\theta \ .
\end{aligned}$$

By i), we may assume that $\mathbf{Pr}(\mathit{Small}(G_n) \equiv \mathbf{f}) \geq p_\infty(\mathbf{f}) - \epsilon/2$. Choosing $\theta := \epsilon/2$, we conclude

$$\begin{aligned}
P\left(A(\infty,\eta), \mathit{Small}(G_n) \equiv \mathbf{f}\right) &= P\left(A(\infty,\eta) \mid \mathit{Small}(G_n) \equiv \mathbf{f}\right) P\left(\mathit{Small}(G_n) \equiv \mathbf{f}\right) \\
&\geq (1-\theta)(p_\infty(\mathbf{f}) - \epsilon/2) \geq p_\infty(\mathbf{f}) - \epsilon \ .
\end{aligned}$$

Together with (4.15), this proves *iii)*, which directly proves *ii)*. This concludes the proof of Theorem 1.2. ∎

Corollary 1.5 is a simple consequence of our theorem. We conclude the paper with a detailed proof of it.

Proof of Corollary 1.5. Let G be a graph on $\{1,\dots,n\}$, $v \in \{1,\dots,n\}$ and $r \geq 1$. The *ball* of radius r centred at v, $B_{G,r}(v)$, is the graph induced in G by all vertices at distance at most r from v.

The *hull* of radius r, $H_{G,r}(v)$ is the union of $B_{G,r}(v)$ with all the connected components of $G \setminus B_{G,r}(v)$ that are of size smaller than $\frac{n}{3}$, but are not components of G. We view the hull $H_{G,r}(v)$ as a graph with a root (the vertex v) and a set, possibly empty, of *exit vertices* (the vertices to which component(s) of size larger than $\frac{n}{3}$ are attached). Note that the exit vertices are necessarily at distance r from the root. We extend the definition of hulls to infinite graphs, by replacing the condition "size smaller than $\frac{n}{3}$" by the condition "finite size".

For $k \geq 0$, let $\mathcal{T}_{r,k}$ be the set of (unlabeled) trees with a marked root, and k marked distinct vertices at distance r from the root (exit vertices). For $T \in \mathcal{T}_{r,k}$ and a rooted graph (G,v), we write $H_{G,r}(v) \equiv T$ if the hull $H_{G,r}(v)$ is isomorphic to T as an unlabeled graph, where the isomorphism preserves the root and the exit vertices (in particular this implies that $H_{G,r}(v)$ has k exit vertices). Then it is easy to see from the definition of (F_∞, V_∞) that we have, for any $r \geq 1, k \geq 0$ and $T \in \mathcal{T}_{r,k}$,

$$\mathbf{Pr}\left(H_{F_\infty,r}(V_\infty) \equiv T\right) = q_\infty(T) \, ,$$

where

$$q_\infty(T) := \begin{cases} \frac{1}{\mathrm{Aut}_{path}(T)} e^{-|T|} & \text{if } k = 1 \\ 0 & \text{if } k \neq 1 \, , \end{cases}$$

where $\mathrm{Aut}_{path}(T)$ is the number of automorphisms of T preserving the path from the root to the exit vertex. Moreover, for any $r \geq 1$ we have

$$\sum_{k \geq 0} \sum_{T \in \mathcal{T}_{r,k}} q_\infty(T) = 1 \, . \tag{4.16}$$

Let $r \geq 1$ and fix $T \in \mathcal{T}_{r,1}$, with root u and exit vertex w. Let T' be the element of $\mathcal{T}$ obtained by re-rooting the tree T at w, and let m be the number of copies of the vertex u in T'. Then, clearly, there are *at least* $m\alpha^G(T')$ vertices $v \in \{1,\dots,n\}$ such that $H_{G,r}(v) \equiv T$. We thus have,

$$\mathbf{Pr}\left(H_{G,r}(V) \equiv T\right) \geq \frac{m\alpha^G(T')}{n} \, ,$$

where V is a uniformly random vertex in G. Now let $\mathcal{G}$ be a tight bridge-addable graph class, and, for every $n \geq 1$, let G_n be a uniformly random graph in $\mathcal{G}_n$ and let V_n be a uniformly random vertex in G_n. By averaging over graphs in $\mathcal{G}_n$ and using the second part of Corollary 1.4 we obtain

$$\liminf_n \mathbf{Pr}\left(H_{G_n,r}(V_n) \equiv T\right) \geq \liminf_n \mathbf{E}\left(\frac{m\alpha^{G_n}(T')}{n}\right) \geq m a_\infty(T') = q_\infty(T) \, , \tag{4.17}$$

where for the last equality we used $m\mathrm{Aut}_{path}(T) = \mathrm{Aut}_r(T')$. Now since the events $H_{G_n,r}(V_n) \equiv T$ for $T \in \cup_{k\geq 0}\mathcal{T}_{r,k}$ are disjoint, we have

$$\sum_{k \geq 0} \sum_{T \in \mathcal{T}_{r,k}} \mathbf{Pr}\left(H_{G_n,r}(V_n) \equiv T\right) \leq 1 \, .$$

From (4.16) and (4.17) we thus get that, for any r, k and $T \in \mathcal{T}_{r,k}$ we have

$$\lim_n \mathbf{Pr}\left(H_{G_n,r}(V_n) \equiv T\right) = q_\infty(T)\ . \tag{4.18}$$

The last equation implies that, for any rooted graph B_0 of radius r (where the radius is the greatest distance from a vertex to the root), we have

$$\lim_n \mathbf{Pr}\left(B_{G_n,r}(V_n) \equiv B_0\right) = \mathbf{Pr}\left(B_{F_\infty,r}(V_\infty) \equiv B_0\right). \tag{4.19}$$

To see this, note that for every rooted graph B, we have

$$\mathbf{Pr}\left(B_{F_\infty,r}(V_\infty) \equiv B\right) = \sum_{k\geq 0} \sum_{\substack{T\in\mathcal{T}_{r,k} \\ T \rhd B}} \mathbf{Pr}\left(H_{F_\infty,r}(V_\infty) \equiv T\right),$$

where $T \rhd B$ means that $B_{T,r}(v) \equiv B$, where v is the root of T.

It follows from this equality that for any B, any $r \geq 1$ and any ϵ, we can choose a finite subset $\mathcal{T}' \subset \cup_{k\geq 0}\mathcal{T}_{r,k}$ such that $\sum_{T\in\mathcal{T}',T\rhd B} \mathbf{Pr}\left(H_{F_\infty,r}(V_\infty) \equiv T\right) \geq \mathbf{Pr}\left(B_{F_\infty,r}(V_\infty) \equiv B\right) - \epsilon$. Using (4.18), it follows that

$$\begin{aligned}
\liminf_n \mathbf{Pr}\left(B_{G_n,r}(V_n) \equiv B\right) &\geq \liminf_n \sum_{T\in\mathcal{T}',T\rhd B} \mathbf{Pr}\left(H_{G_n,r}(V_n) \equiv T\right) \\
&\geq \sum_{T\in\mathcal{T}',T\rhd B} \mathbf{Pr}\left(H_{F_\infty,r}(V_\infty) \equiv T\right) \\
&\geq \mathbf{Pr}\left(B_{F_\infty,r}(V_\infty) \equiv B\right) - \epsilon\ .
\end{aligned}$$

Since this is true for any $\epsilon > 0$, we thus have proved

$$\liminf_n \mathbf{Pr}\left(B_{G_n,r}(V_n) \equiv B\right) \geq \mathbf{Pr}\left(B_{F_\infty,r}(V_\infty) \equiv B\right)\ . \tag{4.20}$$

It follows that

$$1 \geq \liminf_n \sum_B \mathbf{Pr}\left(B_{G_n,r}(V_n) \equiv B\right) \geq \sum_B \mathbf{Pr}\left(B_{F_\infty,r}(V_\infty) \equiv B\right) = 1\ ,$$

where the sums are taken over all rooted graphs B of radius r, and using (4.20), Equation (4.19) holds for every B_0. This concludes the proof of Corollary 1.5. ☐

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Appendix A. More details on the example given in Remark 1.6

Let $\tilde{\mathcal{F}}_n$ be the class of graphs defined in Remark 1.6, and write $k_n := \lceil n^{2/3} \rceil$. In this section we prove that $\tilde{\mathcal{F}}_n$ is tight.

For $i \geq k \geq 1$ let $\tilde{a}_{i,k}$ be the number of connected graphs on $\{1, \dots, i\}$ that induce a clique on $\{1, \dots, k\}$, and such that contracting this clique gives a tree. Thus the number of *connected* graphs in our class $\tilde{\mathcal{F}}_n$ is, by definition, equal to $\tilde{a}_{n,k_n}$. Note that $\tilde{a}_{i,k}$ equals to the number of rooted forests on $\{1, \dots, i\}$ with k components rooted at $1, 2, \dots k$. Thus $\binom{i}{k} a_{i,k}$ is the number of rooted forests on $\{1, \dots, i\}$ with k components and no condition on the location of the roots, which is classically equal to $\binom{i-1}{i-k} i^{i-k}$. We thus get:

$$\tilde{a}_{i,k} = k i^{i-k-1}.$$

Observe that:

$$\frac{\tilde{a}_{i,k}}{\tilde{a}_{i+1,k}} = \frac{i^{i-k-1}}{(i+1)^{i-k}} = \frac{1}{i} \left(1 - \frac{1}{i+1}\right)^{i-k} . \tag{A.1}$$

The number g_n of all elements in the class $\tilde{\mathcal{F}}_n$ is given by:

$$\frac{g_n}{(n-k_n)!} = \sum_{\substack{i+j=n \\ j \geq 0, i \geq k_n}} \frac{\tilde{a}_{i,k_n}}{(i-k_n)!} \times \frac{f_j}{j!} , \tag{A.2}$$

where f_j counts unrooted labeled forests, with $f_0 = 1$. In the sum, i is interpreted as the number of vertices in the connected component containing the clique, and we have distributed the labeling binomial $\binom{n-k_n}{j}$ among factors.

As $F(z) = \sum_{n \geq 0} \frac{f_n}{n!} z^n$ and by Lemma 2.2, given ϵ we can choose δ small enough and j_0 large enough such that $\sum_{j \leq j_0} \frac{f_j}{j!} z^j \geq e^{1/2}(1-\epsilon)$ for any $z \geq e^{-1} - \delta$. Also, given δ and j_0, for n large enough, we have from (A.1) that for any i larger than $n - j_0$:

$$\frac{\tilde{a}_{i,k_n}/(i-k_n)!}{\tilde{a}_{i+1,k_n}/(i+1-k_n)!} \geq e^{-1} - \delta .$$

We can now lower bound the sum (A.2) by keeping the contribution of relatively small values of j. More precisely, for n large enough, we have:

$$\begin{aligned}
(\text{A.2}) &\geq \sum_{j \leq j_0} \frac{\tilde{a}_{n-j,k_n}}{(n-j-k_n)!} \frac{f_j}{j!} \\
&\geq \frac{\tilde{a}_{n,k_n}}{(n-k_n)!} \sum_{j \leq j_0} (e^{-1} - \delta)^j \frac{f_j}{j!} \\
&\geq \frac{\tilde{a}_{n,k_n}}{(n-k_n)!} e^{1/2}(1-\epsilon) .
\end{aligned}$$

Given ζ, consider $\epsilon = \zeta/2$. If n is large enough and $\tilde{F}_n$ is a uniformly random graph in $\tilde{\mathcal{F}}_n$, we thus have

$$\mathbf{Pr}(\tilde{F}_n \text{ is connected}) = \frac{\tilde{a}_{n,k_n}}{g_n} \leq e^{-1/2}(1-\epsilon)^{-1} \leq (1+\zeta)e^{-1/2}.$$

Since this is true for every ζ, the class $\tilde{\mathcal{F}}$ is tight.

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