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The Chaotic Behavior on the Unit Circle

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Abstract

Transitivity and dense periodic points are two main ingredients of Devaney chaos. There are many stronger properties than these two main ingredients that have been studied as a shortcut to chaos. In this paper, we focus on two of these, locally everywhere onto and a strong dense periodicity property, and show the implication of these properties on the unit circle.

Keywords: Devaney chaos, locally everywhere onto, exact, unit circle, strong dense periodicity property

1 Dynamical System

Dynamical systems are systems that change over time and they are very useful in modeling many different kinds of phenomena. Examples of dynamical systems include the mathematical models that describe the swinging of a clock pendulum, the flow of water in a pipe, and the number of fish in a lake. The time may be continuous ($T = [0, \infty)$) or discrete ($T = \mathbb{N}$). When the system depends on a continuous time, we call it a continuous dynamical system, otherwise it is a discrete dynamical system. A continuous dynamical is represented by ordinary differential equations ($\dot{x} = f(x)$ for one dimensional case) while a discrete dynamical system consists of a space X and a function f that acting on the space into itself. In this paper, we will consider the discrete dynamical system and denote it as (X, f).

The dynamical study of (X, f) is all about the attempt to understand the behavior of evolution of every point $x \in X$ under the iteration of function f. Therefore, it describes how one state x develops into another state $f(x)$ and so forth. Chaotic dynamics is the behavior of dynamical systems that are highly sensitive to small changes in initial conditions, so that small alterations can give rise to strikingly great consequences. Therefore, when the system is chaotic, the iteration of x under f is not approximated without perfect information of x.

1.1 Chaotic Dynamical Systems

Chaotic behavior is one of the interesting topics in the study of dynamical systems. Chaos theory is very interesting because surprisingly chaos can be found within almost trivial system. The tent map is an example of a system with simple equation but has a very complex behavior [6]. Conversely, a complex system can also exhibit a non-chaotic system. For an example there is complex biological system, which has been described as “anti-chaotic” [15]. Therefore, it is very important to define chaotic dynamical system clearly and various attempts have been made to give the notion of chaos a mathematically precise meaning, but chaos is not easy to define and there is no universally agreed definition of chaos. Li and Yorke [11] have firstly defined chaos in mathematical terms and the chaos in their sense is called Li-Yorke chaos. However, there are a number of different definitions of chaos and an incomplete list of other popular definitions includes distributional chaos [13], topological chaos [16], ω-chaos [10], P-chaos [2], Block and Coppel chaos [3] and many more. One of the most frequently used is Devaney chaos [6], which isolates two essential ingredients of a chaotic function.

Definition 1[6]: A dynamical system (X, f) is Devaney chaotic whenever it satisfies two conditions;

i) Topologically transitive, that is for any two open nonempty sets U and V, there is some $n \in \mathbb{N}$ such that $f^n(U) \cap V$ is nonempty;
ii) Has a dense set of periodic points, i.e., that every open set contains a periodic point.

Another well-known definition of chaos is topologically chaos i.e.

Definition 2 [6]: A dynamical system (X, f) is topologically chaotic whenever the system has positive entropy.

Entropy is a tool to measure dynamical behavior of a system. There are some equivalent versions of its definition. Some examples and the original definitions can be found in [1, 18]. It has been shown that any topologically chaotic dynamical system is Li-Yorke chaotic [4] and any Devaney chaotic dynamical system is also

Li-Yorke chaotic [9]. The following are well-known results on the unit interval, I and the unit circle, S^1 about positive entropy and topological chaos. However, there are some common remarkable results on the interval and on the circle, as follows;

Theorem 3 [4,12]: If f is a continuous map on the unit circle, then the following statements are equivalent:
1. the entropy of f is positive,
2. there exists a closed invariant subinterval $D \subseteq S^1$ such that $f|\_D$ is Devaney chaotic,
3. f has a periodic point of period q^2p for an odd q and integer p.

In addition, if f is a function on the interval, the statements 1,2 and 3 are also equivalent.

The next remarkable result only holds on the unit interval.

Theorem 4 [7]: On the unit interval, transitivity implies Devaney chaos.

The surprising equivalence on the interval is because transitivity implies dense periodic points, but the converse is not necessarily true [7]. The result cannot be generalized to higher dimensions or the unit circle because the proof of this result uses the ordering in $\mathbb{R}$ in an essential way. Furthermore, an irrational rotation on a circle is a transitive map but does not have any periodic points, which is a counterexample.

1.3 Some Other Chaotic Characterizations

In this paper, we also consider some other strong chaotic concepts that relate closely to two main ingredients of Devaney chaos i.e. locally everywhere onto and a strong dense periodicity property, which are defined as follows;

Definition 5 [8]: Let $f: X \to X$ be a continuous map on a compact metric space X. The function f is said to be locally everywhere onto or simply l.e.o or exact, if for every open subset $U \subset X$ there exists a positive integer n such that $f^n(U) = X$. It is obvious from the definition that l.e.o implies transitivity. Since transitivity is equal to Devaney chaos on the interval, then we directly have the following

Theorem 6: On the interval, locally everywhere onto implies Devaney chaos.

On the unit circle, transitivity is not equal to Devaney chaos, so the above implication is not straightforward. We are going to clarify this in the next section.

Definition 7 [7]: Let $f: X \to X$ be a continuous map on a compact metric space X. The function is said to poses a strong dense periodicity property whenever the set P_n is dense for all integer n and P_n is a collection of periodic points of prime period larger than n.

As far as we are concerned, the property of strong dense periodic points were firstly introduced in 2015 [7] with following result on the interval.

Theorem 8 [7]: If f is Devaney chaotic on a compact metric space with no isolated points, then the set of periodic points with prime period at least n is dense for each n.

Conversely, if f is a continuous function from a closed interval to itself, for which the set of points with prime period at least n is dense for each n, then there is a decomposition of the interval into closed subintervals on which either f or f^2 is Devaney Chaotic.

Corollary 9: If f is a continuous function from a closed interval to itself, for which the set of points with prime period at least n is dense for each n, then there is a closed subinterval on which f is Devaney Chaotic.

This notion was firstly introduced in 2015 [7] and as far as we know, there is no other stronger dense periodicity property introduced to describe chaos. The identity map has dense periodic points but is not transitive. Having only fixed points become an obstacle for the system to be chaotic. Motivated by this example, it is important to highlight the stronger dense periodicity property, P_n dense for all n. They show that for a system without any isolated point, this property is equivalent to Devaney chaos. On the interval they shown that if f is a continuous function from a closed interval to itself, has P_n dense for all n, then there is a decomposition of the interval into closed subintervals on which either f or f^2 is Devaney Chaotic.

2 The Unit Circle

The unit circle can be represented in many different ways. For this work, we let S^1 denote the unit circle in the plane, i.e. $S^1 = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} | x_1^2 + x_2^2 = 1\}$. By using basic trigonometry, for every $x \in S^1$, there exists a unique $\theta \in [0, 2\pi)$ such that $x = (x_1, x_2) = (cos(\theta + 2\pi k), sin(\theta + 2\pi k))$ for all natural numbers k. Therefore, every element of the unit circle x may sometimes be referred to by its angle $\theta \in [0, 2\pi)$ measured in radians, in the standard manner. For every $x, y \in S^1$, the metric d of x and y will be the length of the minor arc between x and y. So, if $x = (cos(\theta_1 + 2\pi k), sin(\theta_1 + 2\pi k))$ and $y = (cos(\theta_2 + 2\pi k), sin(\theta_2 + 2\pi k))$ and $\theta_1 < \theta_2$ then $d(x, y) = min\{\theta_1 - \theta_2, \theta_2 - \theta_1\}$ in radians. Therefore we denote $(x, y) = (y, x)$ as the collection of all elements in S^1 in the minor arc between x and y and call it a subarc of S^1. The collection of subarcs (x, y) forms a basis of open sets for the topology of S^1.

2.1 Representation of the Circle Maps

Analogous to the interval map, we then call a function $f: S^1 \to S^1$ a circle map. When dealing with a circle map, we can think the unit circle S^1 as the interval $[0, 2\pi)$ with 0 and 2π identified. Therefore a continuous map f on S^1 will be associated with an interval map F on $[0, 2\pi)$ which satisfies some properties.

Definition 10: Let f be a circle map on S^1. Corresponding to f, we define an interval map F on $[0,2\pi] \subset \mathbb{R}$ by $F(\theta) = f(\theta)$ for every $\theta \in [0,2\pi)$ and $F(2\pi) = f(0)$.

For example, the Doubling Map $g: S^1 \to S^1$ defined by the expression $g(\theta) = 2\theta$, the interval map $G: [0,2\pi) \to [0,2\pi)$ which can be associated to g is a piecewise linear defined by $G(x) = 2x (mod\ 2\pi)$ where its graph is given in Figure 1.

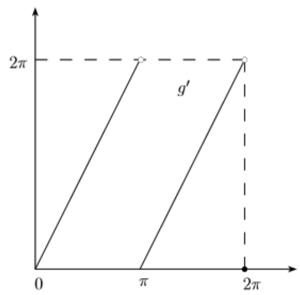

Figure 1: The graph of G where g is the Doubling Map on the unit circle

Referring to the graph of G in Figure 1, G is discontinuous at π and $G(\pi) = g(\pi) = 0$. In general, F is discontinuous at $\theta \in [0,2\pi)$ whenever $F(\pi) = F(\pi) = 0$ since it is either $\lim_{x \to \theta^-} F(x) \neq \lim_{x \to \theta^+} F(x)$ or $\lim_{x \to \theta} F(x) \neq F(\theta)$. So for any continuous circle map f, the corresponding interval map F is continuous at θ if $F(\theta) \neq 0$. Therefore there are subintervals of $[0,2\pi]$ where F is continuous in the real sense whenever we restricted to the subinterval.

Lemma 11: Let f be a continuous circle map and F be the interval map corresponding to f. If $(\theta_1, \theta_2) \subseteq [0,2\pi]$ is strongly invariant under F and $0 \notin (\theta_1, \theta_2)$ then $F|_{(\theta_1,\theta_2)}$ is continuous.

Proof. Since $F(\theta_1, \theta_2) = (\theta_1, \theta_2)$ and $0 \notin (\theta_1 \theta_2)$, then for every $\theta \in (\theta_1, \theta_2)$, $f(\theta) \neq 0$. Therefore F is continuous at θ. ■

Finally we will give some remarks on the relationship between a circle map f and its corresponding interval map F.

Remark 12: The following are facts about f and F for a continuous circle map f;

1. If f has dense periodic points then so does F.

2. For any $\theta \in (0,2\pi)$, θ is a periodic point of period n under f iff θ is a periodic point of period n under F.
3. If F is transitive, then so is f.
4. For any subset $(\theta_1, \theta_2) \subset (0,2\pi)$, (θ_1, θ_2) is invariant under f iff (θ_1, θ_2) is invariant under F.

This representation of the circle map is useful for our further discussion on the dynamics of the circle maps later.

2.2 The Chaotic Behavior of Circle Maps

Even though transitivity implies Devaney chaos on the interval, the same is not true on the circle. However Silverman [14] proved that transitivity almost implies Devaney chaos, as follows:

Theorem 13 [14]: Let $f: S^1 \to S^1$ be a transitive continuous circle map. If f is not one-to-one, then f is Devaney chaotic.

Using this result, we will show the significant implication of locally everywhere onto on the circle map.

Theorem 14: Let $f: S^1 \to S^1$ be a continuous map on the circle. If f is l.e.o. then f is Devaney chaotic.
Proof. Let U and V be two disjoint open sets. Since f is l.e.o, there exists k such that $f^k(U) = f^k(V) = S^1$. Therefore there exists $\theta_1 \in V$ and $\theta_2 \in U$ such that $f^k(\theta_2) = f^k(\theta_1)$. Hence f is not one-to-one and by Theorem 13, f is Devaney chaotic. ■

Nevertheless, dense periodic points in the definition of Devaney chaos may be weaken to the existence of two periodic points.

Theorem 15: Suppose that $f: S^1 \to S^1$ is a continuous circle map with periodic points of periods $m < n$. If f is transitive, then f is Devaney chaotic.

Proof. We claim that f is not one-to-one. Let $x_1, \ldots, x_m$ be period m orbit and let $y_1, \ldots, y_n$ be a period n orbit written so that they run anti-clockwise around S^1. If we do not have $f(y_i) = y_{\{i+1\}}$ for each $i\ (mod\ n)$, then some interval $[y_i, y_{\{i+1\}}]$ will have image that contains some y_j in its interior, so that f is not 1-1.
So assume that we have $f(y_i) = y_{\{i+1\}}$ for each $i\ (mod\ n)$. Since m < n, there is an interval $[y_i, y_{\{i+1\}}]$ which contains some x_j but with the property that $f[y_i, y_{\{i+1\}}] = [y_{\{i+1\}}, y_{\{i+2\}}]$ does not contain any x_k. Since x_j is in $[y_i, y_{\{i+1\}}]$ but $f(x_j)$ is not in $f[y_i, y_{\{i+1\}}] = [y_{\{i+1\}}, y_{\{i+2\}}]$, some point in the interior of $[y_i, y_{\{i+1\}}]$ must map to $y_{\{i+1\}}$, so that f is not 1-1. ■

Our next result is the implication of having the strong dense periodicity property.

Proposition 16: Let $f: S^1 \to S^1$ be a continuous map on the unit circle S^1 with P_n dense for all n. If there exists a closed subarc $A \subset S^1$ which is strongly invariant under f and $0 \notin A$, then there exists a closed subarc $B \subset S^1$ such that $f|_B$ is transitive.
Proof. Let $A \subset S^1$ such that it is strongly invariant under f and $0 \notin A$. By Remark 12, A is invariant under F. Since $0 \notin A$, Lemma 11 then gives that $F|_A$ is continuous. We can then apply Corollary 9 to obtain a closed subarc B of A on which $F|_B$ is transitive. But then $f|_B$is also transitive. ■

The next question is to ask whether P_n dense for all n can assure the existence of the invariant open subarc of S^1 so that the system will have a chaotic subsystem (in sense of Devaney) and have positive entropy (topologically chaotic). We give the positive answer in the main theorem in this section as follows;

Theorem 17: Let $f: S^1 \to S^1$ be a continuous map on the unit circle S^1. If f satisfies P_n dense for all n, then the entropy of f is positive.
Proof. Suppose f has dense P_n for all n but is not transitive on S^1. We firstly claim that there exists a subarc $(\theta_1, \theta_2) \subset [0,2\pi)$ such that $f^m(\theta_1, \theta_2) = (\theta_1, \theta_2)$ for some integer m. Since f is not transitive, there exists an open subarc A such that $\overline{\cup_{\iota \in \mathbb{N}} f^\iota(A)} \neq [0,2\pi)$.

Let $\theta \in A$ such that $f^n(\theta) = \theta$ with the smallest period in A. Therefore for every $m \in \mathbb{N}$ and for every $i = 0,1,2,\cdots,n-1$, $f^{mn+i}(A)$ contain $f^i(\theta)$. Since f has dense periodic points in $[0, 2\pi)$, for every i, $f^{mn+i}\,(A)$ is a non-degenerate subarc i.e. it is neither empty nor reduced to a single point. Hence for every $i = 0,1,2,\cdots,n-1$, $K_i = \cup_{m\in\mathbb{N}} f^{mn+i}(A)$ is a non-degenerate subarc and $\cup_{i\in\mathbb{N}} f^i\,(A) = \cup_{i=0}^{n-1} K_i$. Therefore $\cup_{i=0}^{n-1} K_i$ is strongly invariant and $f^m(K_i) = K_i$ for every $i = 0,1,2,\cdots,n-1$ and for some $m \leq n-1$.
Suppose $(\theta_1, \theta_2) + [0, 2\pi)$ such that $f^m(\theta_1, \theta_2) = (\theta_1, \theta_2)$ for some integer m. Choose $\phi \in [0,2\pi])$ such that $0 \notin [\theta_1 + \phi, \theta_2 + \phi]$ and define a continuous circle map $g: S^1 \to S^1$ such that $g(\theta) = f^m(\theta - \phi) + \phi$. We then claim that $[\theta_1 + \phi, \theta_2 + \phi]$ is strongly invariant under g. By Remark 4.8, it is sufficient to show that the closed subinterval $[\theta_1 + \phi, \theta_2 + \phi]$ is strongly invariant under the interval map G. Since $f^m(\theta_1, \theta_2) = (\theta_1, \theta_2)$ then $g(\theta_1 + \phi, \theta_2 + \phi) = (\theta_1 + \phi, \theta_2 + \phi)$ i.e. $G(\theta_1 + \phi, \theta_2 + \phi) = (\theta_1 + \phi, \theta_2 + \phi)$. Since $0 \notin [\theta_1 + \phi, \theta_2 + \phi]$ then the interval map G is continuous on the interval $[\theta_1 + \phi, \theta_2 + \phi]$. This gives $G[\theta_1 + \phi, \theta_2 + \phi] = [\theta_1 + \phi, \theta_2 + \phi]$. By Proposition 16, there exists $B = [\Theta_1, \Theta_2] \subseteq [\theta_1 + \phi, \theta_2 + \phi]$ such that $G|_B$ is transitive.

For $B = [\Theta_1, \Theta_2]$, we claim that $f|_{[\Theta_1 - \phi, \Theta_2 - \phi]}$ is transitive. So let (α_1, α_2) and (β_1, β_2) be any subarcs of $[\Theta_1 - \phi, \Theta_2 \phi]$. Therefore $(\alpha_1 + \phi, \alpha_1 + \phi)$ and $(\beta_1 + \phi, \beta_2 + \phi)$ are subarcs of $B = [\Theta_1, \Theta_2]$. Since $G|_B$ is transitive, there exists an integer l such that $g^l(x) \in (\beta_1 + \phi, \beta_2 + \phi)$ for some $x \in (\alpha_1 + \phi, \alpha_2 + \phi)$. Therefore $f^{ml}(x) \in (\beta_1, \beta_2)$ for some $x - \phi \in (\alpha_1, \alpha_2)$. Hence $f|_{[\Theta_1 - \phi, \Theta_2 - \phi]}$ is transitive. By Theorem 3 f has positive entropy. ∎

3 Conclusion

It is interesting to highlight the implication of another chaos characterization, locally everywhere onto which is stronger than transitivity. Locally everywhere onto implies Devaney chaos on two spaces, the interval and the unit circle. This is a surprise result since transitivity is equal to Devaney chaos on the interval but not on the unit circle.

We end by highlighting the difference between the implication of P_n dense for all n on the interval and on the unit circle. On the interval, P_n dense for all n implies that the whole system can be decomposed into subsystems where every subsystem is Devaney chaotic. Therefore, the system with no invariant proper subinterval is Devaney chaotic whenever it satisfies this strong dense periodicity property. In fact, this strong property implies that the system has positive entropy. That also happens on the unit circle. P_n dense for all n can guarantee that the system on the unit circle has positive entropy i.e. is behaving topologically chaotic. Unlike what happens on the interval, this stronger density property cannot guarantee that the whole system is chaotic in the sense of Devaney even if it does not have any invariant proper subset. This is because the only invariant subinterval under F is the whole interval $[0, 2\pi)$ which contains 0 and therefore discontinuity of F at some points is possibly occurs. Hence its prevent us to use the same argument to show that the whole system is Devaney chaotic. However, it is an interesting fact that this strong property can guarantee that a Devaney chaotic subsystem exists which means the stronger property is more significant than the property of dense periodic points since dense periodic points does not implies any sort of chaos.

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