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CLASSIFICATION OF CURTIS-TITS AND PHAN AMALGAMS WITH 3-SPHERICAL DIAGRAM

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ABSTRACT

We classify all non-collapsing Curtis-Tits and Phan amalgams with 3-spherical diagram over all fields. In particular, we show that amalgams with spherical diagram are unique, a result required by the classification of finite simple groups. We give a simple condition on the amalgam which is necessary and sufficient for it to arise from a group of Kac-Moody type. This also yields a definition of a large class of groups of Kac-Moody type in terms of a finite presentation.

1. Introduction

Local recognition results play an important role in various parts of mathematics. A key example comes from the monumental classification of finite simple groups. Local analysis of the unknown finite simple group G yields a local datum consisting of a small collection of subgroups fitting together in a particular way, called an amalgam. The Curtis-Tits theorem [7, 16, 17, 18, 19] and the Phan (-type) theorems [25, 26, 27] describe amalgams appearing in known groups of Lie type. Once the amalgam in G is identified as one of the amalgams given by these theorems, G is known.

The present paper was partly motivated by a question posed by R. Solomon and R. Lyons about this identification step, arising from their work on the classification [9, 10, 11, 12, 13, 14]: Are Curtis-Tits and Phan type amalgams uniquely determined by their subgroups? More precisely is there a way of fitting these subgroups together so that the amalgam gives rise to a different group? In many cases it is known that, indeed, depending on how one fits the subgroups together, either the resulting amalgam arises from these theorems, or it does not occur in any non-trivial group. This is due to various results of Bennett and Shpectorov [1], Gramlich [15], Dunlap [8], and R. Gramlich, M. Horn, and W. Nickel [23]. However, all of these results use, in essence, a crucial observation by Bennett and Shpectorov about tori in rank-3 groups of Lie type, which fails to hold for small fields. In the present paper we replace the condition on tori by a more effective condition on root subgroups, which holds for all fields. This condition is obtained by a careful analysis of maximal subgroups of groups of Lie type. Thus the identification step can now be made for all possible fields. A useful consequence of the identification of the group G, together with the Curtis-Tits and Phan type theorems, is that it yields a simplified version of the Steinberg presentation for G.

Note that this solves the - generally much harder - existence problem: "how can we tell if a given amalgam appears in any non-trivial group?"

The unified approach in the present paper not only extends the various results on Curtis-Tits and Phan amalgams occurring in groups of Lie type to arbitrary fields, but in fact also applies to a much larger class of Curtis-Tits and Phan type amalgams, similar to those occurring in groups of Kac-Moody type. Here, both the uniqueness and the existence problem become significantly more involved.

Groups of Kac-Moody type were introduced by J. Tits as automorphism groups of certain geometric objects called twin-buildings [29]. In the same paper J. Tits conjectured that these groups are uniquely determined by the fact that the group acts on some twin-building, together with local geometric data called Moufang foundations. As an example he sketched an approach towards classifying such foundations in the case of simply-laced diagrams. This conjecture was subsequently proved for Moufang foundations built from locally split and locally finite rank-2 residues by B. Mühlherr in [24] and refined by P. E. Caprace in [6]. All these results produce a classification of groups of Kac-Moody type using local data in the form of an amalgam, together with a global geometric assumption stipulating the existence of a twin-building on which the group acts.

Ideally, one would use the generalizations of the Curtis-Tits and Phan type theorems to describe the groups of Kac-Moody type in terms of a simplified Steinberg type presentation. However, the geometric assumption is unsatisfactory for this purpose as it is impossible to verify directly from the presentation itself.

In our unified approach we consider all possible amalgams whose local structure is any one of those appearing in the above problems. There is no condition on the field. Then, we classify those amalgams that satisfy our condition on root groups and show that in the spherical case they are unique. This explains why groups of Lie type can uniquely be recognized by their amalgam. By contrast, in the non-spherical case the amalgams are not necessarily unique and, indeed, not all such amalgams give rise to groups of Kac-Moody type. This is a consequence of the fact that we impose no global geometric condition. Nevertheless, we give a simple condition on the amalgam itself which decides whether it comes from a group of Kac-Moody type or not. As a result, we obtain a purely group theoretic definition of a large class of groups of Kac-Moody type just in terms of a finite presentation.

Finally, we note that an amalgam must satisfy the root subgroup condition to occur in a non-trivial group. A subsequent study generalizing [3, 5] shows that in fact all amalgams satisfying the root group condition do occur in non-trivial groups. Thus, in this much wider context the existence problem is also solved.

We shall now give an overview of the results in the present paper. Recall that a Dynkin diagram Γ is an oriented edge-labelled graph. We say that Γ is *connected* if the underlying (unlabelled) graph is connected in the usual sense. Moreover, we use topological notions such as spanning tree and homotopy rank of Γ referring to the underlying graph.

For Phan amalgams we prove the following (for the precise statement see Theorem 5.21).

THEOREM A: *Let q be any prime power and let Γ be a connected 3-spherical diagram with homotopy rank r. Then, there is a bijection between the elements of $\prod_{s=1}^{r} \mathrm{Aut}(\mathbb{F}_{q^2})$ and the type preserving isomorphism classes of Phan amalgams with diagram Γ over $\mathbb{F}_q$.*

For Curtis-Tits amalgams the situation is slightly more complicated (for the precise statement see Theorem 4.24).

THEOREM B: *Let q be a prime power and let Γ be a connected 3-spherical diagram with homotopy rank r. Then there exists a set of positive integers $\{e_1, \ldots, e_r\}$ so that there is a bijection between the elements of $\prod_{s=1}^{r} \mathrm{Aut}(\mathbb{F}_{q^{e_s}}) \times \mathbb{Z}/2\mathbb{Z}$ and the type preserving isomorphism classes of Curtis-Tits amalgams with diagram Γ over $\mathbb{F}_q$.*

COROLLARY C: *Let q be a prime power and let Γ be a 3-spherical tree. Then, up to type preserving isomorphism, there is a unique Curtis-Tits and a unique Phan amalgam over $\mathbb{F}_q$ with diagram Γ.*

Note that Corollary C includes all spherical diagrams of rank ≥ 3. Several special cases of the above results were proved elsewhere. Indeed, Theorem B was proved for simply-laced diagrams and $q \geq 4$ in [4]. Corollary C was proved for Phan amalgams with $\Gamma = A_n$ in [1], for general simply-laced tree diagram in [8], and for $\Gamma = C_n$ for $q \geq 3$ in [20, 22].

The classification of Curtis-Tits amalgams will be done along the following lines. Note if $(\mathbf{G}, \overline{\mathbf{G}}_i, \overline{\mathbf{G}}_j)$ is a Curtis-Tits standard of type different from $A_1 \times A_1$, and $\overline{\mathbf{X}}$ is any Sylow p-subgroup in one of the vertex groups, say $\overline{\mathbf{G}}_i$, then generically it generates $\mathbf{G}$ together with $\overline{\mathbf{G}}_j$. In Subsection 4.1 we show that there is a unique pair $(\overline{\mathbf{X}}_i^+, \overline{\mathbf{X}}_i^-)$ of Sylow p-subgroups in $\overline{\mathbf{G}}_i$ whose members do not have this property. Moreover, each member commutes with a unique member in the other vertex group.

In Subsection 4.2 we show that in a non-collapsing Curtis-Tits amalgam $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i, j \in I\}$ for each i there exists a pair $(\mathbf{X}_i^+, \mathbf{X}_i^-)$ of Sylow subgroups in $\mathbf{G}_i$ such that for any edge $\{i,j\}$ $(\mathbf{g}_{i,j}(\mathbf{X}_i^+,\mathbf{X}_i^-)$ is the pair for $(\mathbf{G}_{i,j},\overline{\mathbf{G}}_i,\overline{\mathbf{G}}_j)$ as above. The collection $\mathcal{X}=\{\mathbf{X}_i^+,\mathbf{X}_i^-: i\in I\}$ is called a *weak system of fundamental root groups.* Without loss of generality one can assume that any amalgam with the same diagram has the exact same weak system $\mathcal{X}$. As a consequence all amalgams with the same diagram can be determined up to isomorphism by studying the coefficient system associated to $\mathcal{X}$, that is, the graph of groups consisting of automorphisms of the vertex and edge groups preserving $\mathcal{X}$. In Subsection 4.3 we determine the coefficient system associated to $\mathcal{X}$. In Subsection 4.4, we pick a spanning tree Σ for Γ and use precise information about the coefficient system to create a standard form of a Curtis-Tits amalgam in which all vertex-edge inclusion maps are trivial except for the edges in Σ. In particular this shows that if Γ is a tree, then the amalgam is unique up to isomorphism. Finally in Subsection 4.5 we show that for a suitable choice of Σ, the remaining non-trivial inclusion maps uniquely determine the amalgam.

The classification of Phan amalgams in Section 5 follows the same pattern. However in this case the role of the weak system of fundamental root groups is replaced by a system of tori in the vertex groups, whose images in the edge groups must form a torus there.

As shown here, the existence of a weak system of fundamental root groups is a necessary condition for the existence of a non-trivial completion. A natural question of course is whether it is also sufficient. In the spherical cases, the amalgams are unique and the Curtis-Tits and Phan theorems identify universal completions of these amalgams. In [5] it is shown that any Curtis-Tits amalgam with 3-spherical simply-laced diagram over a field with at least four elements having property (D) has a non-trivial universal completion, which is identified up to a rather precisely described central extension. In the present paper we will not study completions of the Curtis-Tits and Phan amalgams classified here, but merely note that similar arguments yield non-trivial completions for all amalgams. In particular, the conditions mentioned above are indeed sufficient for the existence of these completions. In general we don't know of a direct way of giving conditions on an amalgam ensuring the existence of a non-trivial completion.

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2. Curtis-Tits and Phan amalgams and their diagrams

2.1. Diagrams

In order to fix some notation, we start with some definitions.

Definition 2.1: A *Coxeter matrix over the set* $I = \{1, 2, \ldots, n\}$ of finite cardinality n is a symmetric matrix $M = (m_{ij})_{i,j\in I}$ with entries in $\mathbb{N}_{\geq 1} \cup \{\infty\}$ such that, for all $i, j \in I$ distinct we have $m_{ii} = 1$ and $m_{ij} \geq 2$.

A *Coxeter diagram with Coxeter matrix* M is an edge-labelled graph $\Delta = (I, E)$ with vertex set $I = \mathrm{V}_\Delta$ and edge-set $E = \mathrm{E}_\Delta$ without loops such that for any distinct $i, j \in I$, there is an edge labelled m_{ij} between i and j whenever $m_{ij} > 2$; if $m_{i,j} = 2$, there is no such edge. Thus, M and Δ determine each other uniquely. For any subset $J \subseteq I$, we let Δ_J denote the diagram induced on vertex set J. We say that Δ is *connected* if the underlying (unlabelled) graph is connected in the usual sense. Moreover, we use topological notions such as spanning tree and homotopy rank of Δ referring to the underlying graph.

A *Coxeter system with Coxeter matrix* M is a pair (W, S), where W is a group generated by the set $S = \{s_i \colon i \in I\}$ subject to the relations $(s_i s_j)^{m_{ij}} = 1$ for all $i, j \in I$. For each subset $J \subseteq I$, we let $W_J = \langle s_j \colon j \in J\rangle_W$. We call M and (W, S) m-spherical if every subgroup W_J with $|J| = m$ is finite ($m \in \mathbb{N}_{\geq 2}$). Call (W, S) spherical if it is n-spherical.

In order to describe Curtis-Tits and Phan amalgams more precisely, we also introduce a Lie diagram.

Definition 2.2: Let $\Delta = (I, E)$ be a Coxeter diagram. A *Lie diagram of Coxeter type* Δ is an untwisted or twisted Dynkin diagram Γ whose edge labels l_{ij} do not specify the orientation. In this paper we shall only be concerned with Lie diagrams of Coxeter type A_n, B_n, D_n, E_6, E_7, E_8, and F_4. For these, we have the following correspondence

Δ	A_n	B_n	D_n,	$E_n (n = 6, 7, 8)$	F_4
Γ	A_n	$B_n, C_n, {}^2D_{n+1}, {}^2A_{2n-1}, {}^2A_{2n}$	D_n	E_n	$F_4, F_4^*, {}^2E_6, {}^2E_6^*$

Here F_4 and ${}^2E_6$ (resp. F_4^* and ${}^2E_6^*$) denote the diagrams where node 1 corresponds to the long (resp. short) root (Bourbaki labeling).

Let us introduce some more notation. We shall denote the Frobenius automorphism of order 2 of $\mathbb{F}_{q^2}$ by σ. Below we will consider sesquilinear forms h on an $\mathbb{F}_{q^2}$-vector space V. By convention, all these forms are linear in the first coordinate, that is $\mathsf{h}(\lambda u, \mu v) = \lambda \mathsf{h}(u, v)\mu^\sigma$ for $u, v \in V$ and $\lambda, \mu \in \mathbb{F}_{q^2}$. Recall that h is hermitian if $\mathsf{h}(v, u) = \mathsf{h}(u, v)^\sigma$ for all $u, v \in V$.

2.2. Standard pairs of Curtis-Tits type

Let Γ be a Lie diagram of type A_2, B_2/C_2, ${}^2D_3/{}^2A_3$ and $q = p^e$ for some prime $p \in \mathbb{Z}$ and $e \in \mathbb{Z}_{\geq 1}$. Then a *Curtis-Tits standard pair of type* $\Gamma(q)$ is a triple $(\mathbf{G}, \mathbf{G}_1, \mathbf{G}_2)$ of groups such that one of the following occurs (each case is marked by an underlined header in boldface):

$\underline{\mathbf{\Gamma = A_1 \times A_1}}$ Now $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$ and $\mathbf{G}_1 \cong \mathbf{G}_2 \cong \mathrm{SL}_2(q)$.

$\underline{\mathbf{\Gamma = A_2}}$ Now $\mathbf{G} = \mathrm{SL}_3(q) = \mathrm{SL}(V)$ for some $\mathbb{F}_q$-vector space with basis $\{e_1, e_2, e_3\}$ and $\mathbf{G}_1$ (resp. $\mathbf{G}_2$) is the stabilizer of the subspace $\langle e_1, e_2\rangle$ (resp. $\langle e_2, e_3\rangle$) and the vector e_3 (resp. e_1).

Explicitly we have

$$\mathbf{G}_1 = \left\{ \begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix} : a, b, c, d \in \mathbb{F}_q \text{ with } ad - bc = 1 \right\},$$
$$\mathbf{G}_2 = \left\{ \begin{pmatrix} 1 & & \\ & a & b \\ & c & d \end{pmatrix} : a, b, c, d \in \mathbb{F}_q \text{ with } ad - bc = 1 \right\}.$$

$\underline{\mathbf{\Gamma = C_2}}$ Now $\mathbf{G} = \mathrm{Sp}_4(q) = \mathrm{Sp}(V, \beta)$, where V is an $\mathbb{F}_q$-vector space with basis $\{e_1, e_2, e_3, e_4\}$ and β is the symplectic form with Gram matrix

$$M = \begin{pmatrix} & & 1 & 0 \\ & & 0 & 1 \\ -1 & 0 & & \\ 0 & -1 & & \end{pmatrix}.$$

$\mathbf{G}_1 \cong \mathrm{SL}_2(q)$ is the derived subgroup of $\mathrm{Stab}_{\mathbf{G}}(\langle e_1, e_2\rangle) \cap \mathrm{Stab}_{\mathbf{G}}(\langle e_3, e_4\rangle)$ and $\mathbf{G}_2 = \mathrm{Stab}_{\mathbf{G}}(e_1) \cap \mathrm{Stab}_{\mathbf{G}}(e_3) \cong \mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$. Explicitly we have

$$\mathbf{G}_1 = \left\{ \begin{pmatrix} a & b & & \\ c & d & & \\ & & d & -c \\ & & -b & a \end{pmatrix} : a, b, c, d \in \mathbb{F}_q \text{ with } ad - bc = 1 \right\},$$

$$\mathbf{G}_2 = \left\{ \begin{pmatrix} 1 & & & \\ & a & & b \\ & & 1 & \\ & c & & d \end{pmatrix} : a, b, c, d \in \mathbb{F}_q \text{ with } ad - bc = 1 \right\}.$$

Remark 2.3: We are only interested in Curtis-Tits standard pairs of type B_2 for q odd. However, in that case we have $\mathrm{Spin}_5(q) \cong \mathrm{Sp}_4(q)$ is the unique central extension of the simple group $\Omega_5(q) \cong \mathrm{PSp}_4(q)$. Therefore, we can also describe the Curtis-Tits standard pair for B_2 as a Curtis-Tits standard pair for C_2 with $\mathbf{G}_1$ and $\mathbf{G}_2$ interchanged.

$\underline{\mathbf{\Gamma = {}^2A_3}}$ Now $\mathbf{G} = \mathrm{SU}_4(q) = \mathrm{SU}(V)$ for some $\mathbb{F}_{q^2}$-vector space V with basis $\{e_1, e_2, e_3, e_4\}$ equipped with a non-degenerate hermitian form h for which this basis is hyperbolic with Gram matrix

$$M = \begin{pmatrix} & & 1 & 0 \\ & & 0 & 1 \\ 1 & 0 & & \\ 0 & 1 & & \end{pmatrix}.$$

Now $\mathbf{G}_1$ is the derived subgroup of the simultaneous stabilizer of the subspaces $\langle e_1, e_2\rangle$ and $\langle e_3, e_4\rangle$ and $\mathbf{G}_2$ is the stabilizer of the vectors e_1 and e_3 and the hyperbolic line

$\langle e_2, e_4\rangle$. We have $\mathbf{G}_2 \cong \mathrm{SU}_2(q) \cong \mathrm{SL}_2(q)$ and $\mathbf{G}_1 \cong \mathrm{SL}_2(q^2)$. Explicitly we have

$$\mathbf{G}_1 = \left\{ \begin{pmatrix} a & b & & \\ c & d & & \\ & & d^\sigma & -c^\sigma \\ & & -b^\sigma & a^\sigma \end{pmatrix} : a,b,c,d \in \mathbb{F}_{q^2} \text{ with } ad-bc=1 \right\},$$

$$\mathbf{G}_2 = \left\{ \begin{pmatrix} 1 & & & \\ & a & & b\eta \\ & & 1 & \\ & c\eta^{-1} & & d \end{pmatrix} : a,b,c,d \in \mathbb{F}_{q} \text{ with } ad-bc=1 \right\},$$

where $\eta \in \mathbb{F}_{q^2}$ has $\eta + \eta^q = 0$.

Remark 2.4: For completeness we also define a standard Curtis-Tits pair $(\mathbf{H}, \mathbf{H}_1, \mathbf{H}_2)$ of type ${}^2D_3(q)$. Take $\mathbf{H} = \Omega_6^-(q) = \Omega^-(V, \mathcal{Q})$, where V is an $\mathbb{F}_q$-vector space with basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $\mathcal{Q}(\sum_{i=1}^5 x_i e_i) = x_1x_3 + x_2x_4 + f(x_5, x_6)$, for some quadratic polynomial $f(x, 1)$ that is irreducible over $\mathbb{F}_q$. Here, $\mathbf{H}_1 \cong \mathrm{SL}_2(q)$ is the derived subgroup of $\mathrm{Stab}_{\mathbf{H}}(\langle e_1, e_2\rangle) \cap \mathrm{Stab}_{\mathbf{H}}(\langle e_3, e_4\rangle)$ if $\mathrm{SL}_2(q)$ is perfect that is $q > 2$, and it is the subgroup $\mathrm{Stab}_{\mathbf{H}}(\langle e_1, e_2\rangle) \cap \mathrm{Stab}_{\mathbf{H}}(\langle e_3, e_4\rangle) \cap \mathrm{Stab}(v)$ for some non-singular vector $v \in \langle e_5, e_6\rangle$ if $q = 2$, and $\mathbf{H}_2 = \mathrm{Stab}_{\mathbf{H}}(e_1) \cap \mathrm{Stab}_{\mathbf{H}}(e_3) \cong \Omega_4^-(q) \cong \mathrm{PSL}_2(q^2)$.

Now note that in all standard pairs of type A_2, B_2/C_2 and ${}^2A_3$ the vertex/rank-1 groups are isomorphic to $\mathrm{SL}_2(q)$ or $\mathrm{SL}_2(q^2)$, whereas this is not the case for type ${}^2D_3$. However, there exists a unique standard Curtis-Tits pair $(\mathbf{G}, \mathbf{G}_1, \mathbf{G}_2)$ of type ${}^2A_3(q)$ and a surjective homomorphism $\pi\colon \mathbf{G} \to \mathbf{H}$ with $\ker \pi = \{\pm 1\} = Z(\mathbf{G}_1) \le Z(\mathbf{G})$. It induces $\pi\colon \mathbf{G}_1 \cong \mathrm{SL}_2(q^2) \to \Omega_4^-(q) = \mathbf{H}_1 \cong \mathrm{PSL}_2(q^2)$ and $\pi\colon \mathbf{G}_2 \cong \mathrm{SU}_2(q) \to \mathrm{SL}_2(q) = \mathbf{H}_2$. In other words, the standard Curtis-Tits pair of type ${}^2D_3(q)$ is a central quotient of a standard Curtis-Tits pair of type ${}^2A_3(q)$.

Definition 2.5: For Curtis-Tits amalgams, the *standard identification map* will be the isomorphism $\mathbf{g}\colon\ \mathrm{SL}_2(q^e) \to \mathbf{G}_i$ sending

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to the corresponding matrix of $\mathbf{G}_i$ as described above. Here $e = 1$ unless $\Gamma(q) = {}^2A_3(q)$ and $i = 1$ or $\Gamma(q) = {}^2D_3$ and $i = 2$, in which case $e = 2$.

2.3. Standard pairs of Phan type

Let Γ be as above. Then a *Phan standard pair of type* $\Gamma(q)$ is a triple $(\mathbf{G}, \mathbf{G}_1, \mathbf{G}_2)$ such that one of the following occurs (each case is marked by an underlined header in boldface):

$\underline{\mathbf{\Gamma = A_1 \times A_1}}$ Now $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$ and $\mathbf{G}_1 \cong \mathbf{G}_2 \cong \mathrm{SU}_2(q) = \mathrm{SU}(V)$ for some $\mathbb{F}_{q^2}$-vector space V with basis $\{e_1, e_2\}$ equipped with a non-degenerate hermitian form h for which this basis is orthonormal.

$\underline{\mathbf{\Gamma = A_2}}$ Now $\mathbf{G} = \mathrm{SU}_3(q) = \mathrm{SU}(V)$ for some $\mathbb{F}_{q^2}$-vector space V with basis $\{e_1, e_2, e_3\}$ equipped with a non-degenerate hermitian form h for which this basis is orthonormal. As in the Curtis-Tits case, $\mathbf{G}_1$ (resp. $\mathbf{G}_2$) is the stabilizer of the subspace $\langle e_1, e_2\rangle$ (resp. $\langle e_2, e_3\rangle$) and the vector e_3 (resp. e_1). We have $\mathbf{G}_1 \cong \mathbf{G}_2 \cong \mathrm{SU}_2(q)$.

Explicitly we have

$$\mathbf{G}_1 = \left\{ \begin{pmatrix} a & b & \\ -b^\sigma & a^\sigma & \\ & & 1 \end{pmatrix} : a, b \in \mathbb{F}_{q^2} \text{ with } aa^\sigma + bb^\sigma = 1 \right\},$$

$$\mathbf{G}_2 = \left\{ \begin{pmatrix} 1 & & \\ & a & b \\ & -b^\sigma & a^\sigma \end{pmatrix} : a, b \in \mathbb{F}_{q^2} \text{ with } aa^\sigma + bb^\sigma = 1 \right\}.$$

$\underline{\mathbf{\Gamma = C_2}}$ Let V be an $\mathbb{F}_{q^2}$-vector space with basis $\{e_1,\, e_2,\, e_3,\, e_4\}$ and let β be the symplectic form with Gram matrix

$$M = \begin{pmatrix} & & 1 & 0 \\ & & 0 & 1 \\ -1 & 0 & & \\ 0 & -1 & & \end{pmatrix}.$$

Moreover, let h be the (non-degenerate) hermitian form for which this basis is orthonormal.

Now $\mathbf{G} = \mathrm{Sp}(V, \beta) \cap \mathrm{SU}(V, \mathsf{h}) \cong \mathrm{Sp}_4(q)$, $\mathbf{G}_1 \cong \mathrm{SU}_2(q)$ is the derived subgroup of $\mathrm{Stab}_{\mathbf{G}}(\langle e_1, e_2\rangle) \cap \mathrm{Stab}_{\mathbf{G}}(\langle e_3, e_4\rangle)$ and $\mathbf{G}_2 = \mathrm{Stab}_{\mathbf{G}}(e_1) \cap \mathrm{Stab}_{\mathbf{G}}(e_3) \cong \mathrm{Sp}_2(q) \cong \mathrm{SU}_2(q)$. Note that $Z(\mathbf{G}) = Z(\mathbf{G}_1)$ and $Z(\mathbf{G}) \cap \mathbf{G}_2 = \{1\}$.

Explicitly we have

$$\mathbf{G}_1 = \left\{ \begin{pmatrix} a & b & & \\ -b^\sigma & a^\sigma & & \\ & & a^\sigma & b^\sigma \\ & & -b & a \end{pmatrix} : a, b \in \mathbb{F}_{q^2} \text{ with } aa^\sigma + bb^\sigma = 1 \right\},$$

$$\mathbf{G}_2 = \left\{ \begin{pmatrix} 1 & & & \\ & a & & b \\ & & 1 & \\ & -b^\sigma & & a^\sigma \end{pmatrix} : a, b \in \mathbb{F}_{q^2} \text{ with } aa^\sigma + bb^\sigma = 1 \right\}.$$

Definition 2.6: For Phan amalgams, the *standard identification map* will be the isomorphism $\mathbf{g}\colon \mathrm{SU}_2(q) \to \mathbf{G}_i$ sending

$$\begin{pmatrix} a & b \\ -b^\sigma & a^\sigma \end{pmatrix}$$

to the corresponding matrix of $\mathbf{G}_i$ as described above.

2.4. Amalgams of Curtis-Tits and Phan type

Definition 2.7: An *amalgam* over a poset $(\mathcal{P}, \prec)$ is a collection $\mathscr{A} = \{\mathbf{A}_x \mid x \in \mathcal{P}\}$ of groups, together with a collection $\mathbf{a}_\bullet = \{\mathbf{a}_x^y \mid x \prec y, x, y \in \mathcal{P}\}$ of monomorphisms $\mathbf{a}_x^y\colon \mathbf{A}_x \hookrightarrow \mathbf{A}_y$, called *inclusion maps* such that whenever $x \prec y \prec z$, we have $\mathbf{a}_x^z = \mathbf{a}_y^z \circ \mathbf{a}_x^y$; we shall write $\overline{\mathbf{A}}_x = \mathbf{a}_x^y(\mathbf{A}_x) \leq \mathbf{A}_y$. A *completion* of $\mathscr{A}$ is a group A together with a collection $\alpha_\bullet = \{\alpha_x \mid x \in \mathcal{P}\}$ of homomorphisms $\alpha_x\colon \mathbf{A}_x \to A$, whose images - often denoted $A_x = \alpha_x(\mathbf{A}_x)$ - generate A, such that for any $x, y \in \mathcal{P}$ with $x \prec y$ we have $\alpha_y \circ \alpha_x^y = \alpha_x$. The amalgam $\mathscr{A}$ is *non-collapsing* if it has a non-trivial completion. As a convention, for any subgroup $\mathbf{H} \leq \mathbf{A}_J$, let $\mathrm{H} = \alpha(\mathbf{H}) \leq A$.

A completion $(\tilde{A}, \tilde{\alpha}_\bullet)$ is called *universal* if for any completion $(A, \alpha_\bullet)$ there is a unique surjective group homomorphism $\pi\colon \tilde{A} \to A$ such that $\alpha_\bullet = \pi \circ \tilde{\alpha}_\bullet$. A universal completion always exists.

Definition 2.8: Let $\Gamma = (I, E)$ be a Lie diagram. A *Curtis-Tits (resp. Phan) amalgam with Lie diagram* Γ *over* $\mathbb{F}_q$ is an amalgam $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i, j \in I\}$ over $\mathcal{P} = \{J \mid \emptyset \neq J \subseteq I \text{ with } |J| \leq 2\}$ ordered by inclusion such that for every $i, j \in I$, $(\mathbf{G}_{i,j}, \overline{\mathbf{G}}_i, \overline{\mathbf{G}}_j)$ is a Curtis-Tits/Phan standard pair of type $\Gamma_{i,j}(q^e)$, for some $e \geq 1$ as defined in Subsection 2.2 and 2.3. Moreover $e = 1$ is realized for some $i, j \in I$. Note that in fact e is always a power of 2. This follows immediately from connectedness of the diagram and the definition of the standard pairs of type A_2, C_2, and ${}^2A_3$. For any subset $K \subseteq I$, we let

$$\mathscr{G}_K = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i, j \in K\}.$$

Definition 2.9: Suppose $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i, j \in I\}$ and $\mathscr{G}^+ = \{\mathbf{G}_i^+, \mathbf{G}_{i,j}^+, \mathbf{g}_{i,j}^+ \mid i, j \in I\}$ are two Curtis-Tits (or Phan) amalgams over $\mathbb{F}_q$ with the same diagram Γ. Then a type preserving isomorphism $\phi\colon \mathscr{G} \to \mathscr{G}^+$ is a collection $\phi = \{\phi_i, \phi_{i,j}\colon i, j \in I\}$ of group isomorphisms such that, for all $i, j \in I$, we have

$$\phi_{i,j} \circ \mathbf{g}_{i,j} = \mathbf{g}_{i,j}^+ \circ \phi_i$$
$$\phi_{i,j} \circ \mathbf{g}_{j,i} = \mathbf{g}_{j,i}^+ \circ \phi_j.$$

It is also possible to consider type permuting isomorphisms, defined in a similar way via a permutation of the diagram. More precisely if σ is a permutation of the diagram, the corresponding maps $\phi_I : G_I \to G_{\sigma(i)}^+$ respectively $\phi_{i,j} : G_{i,j} \to G_{\sigma(i)\,\sigma(j)}$. However, henceforth all isomorphisms that we consider shall be type-preserving, except when clearly stated otherwise.

Remark 2.10: Suppose that one considers an amalgam $\mathscr{H} = \{\mathbf{H}_i, \mathbf{H}_{i,j}, \mathbf{h}_{i,j} \mid i,j \in I\}$ over $\mathbb{F}_q$ with diagram Γ, such that for any $i,j \in I$, the triple $(\mathbf{H}_{i,j}, \overline{\mathbf{H}}_i, \overline{\mathbf{H}}_j)$ is not a standard pair, but there is a standard pair $(\mathbf{G}_{i,j}, \overline{\mathbf{G}}_i, \overline{\mathbf{G}}_j)$ such that the respective $\mathbf{H}$'s are central quotients of the corresponding $\mathbf{G}$'s. Note that the standard Curtis-Tits amalgams of type $A_1 \times A_1$, A_2, C_2, and ${}^2A_3$ are universal. Therefore the $\mathbf{G}$'s are unique given the $\mathbf{H}$'s. Note that vertex groups are either $\mathrm{SL}_2(q)$ or $\mathrm{PSL}_2(q)$ for some q. By examining the cases we see that the vertex groups $\overline{\mathbf{G}}_i$ are independent of the standard pair considered. Thus, $\mathscr{H}$ is the quotient of a unique Curtis-Tits amalgam over $\mathbb{F}_q$ with diagram Γ. Hence for classification purposes it suffices to consider standard Curtis-Tits amalgams. In particular, in view of Remark 2.4, we can restrict ourselves to Curtis-Tits amalgams in which the only rank-2 subdiagrams are of type $A_1 \times A_1$, A_2, C_2, and ${}^2A_3$.

3. Background on groups of Lie type

3.1. Automorphisms of groups of Lie type of small rank

Automorphisms of groups of Lie type are all known. In this subsection we collect some facts that we will need later on. We shall use the notation from [30].
Automorphisms of $\mathrm{SL}_n(q)$. Define automorphisms of $\mathrm{SL}_n(q)$ as follows (where $x = (x_{ij})_{i,j=1}^n \in \mathrm{SL}_n(q)$):

$$\begin{aligned} c_g\colon x &\mapsto x^g = g^{-1}xg && (g \in \mathrm{PGL}_n(q)),\\ \alpha\colon x &\mapsto x^\alpha = (x_{ij}^\alpha)_{i,j=1}^n && (\alpha \in \mathrm{Aut}(\mathbb{F}_q)),\\ \tau\colon x &\mapsto x^\tau = {}^t x^{-1} && (\text{transpose-inverse}). \end{aligned}$$

We note that for $n = 2$, τ coincides with the map $x \mapsto x^\mu$, where $\mu = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We let $\mathrm{P\Gamma L}_n(q) = \mathrm{PGL}_n(q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$.
Automorphisms of $\mathrm{Sp}_{2n}(q)$. Outer automorphisms of $\mathrm{Sp}_{2n}(q)$ are of the form $\mathrm{Aut}(\mathbb{F}_q)$ as for $\mathrm{SL}_{2n}(q)$, defined with respect to a symplectic basis, or come from the group of linear similarities of the symplectic form $\mathrm{GSp}_{2n}(q) \cong \mathrm{Sp}_{2n}(q).(\mathbb{F}_q^*/(\mathbb{F}_q^*)^2)$, where $\mathbb{F}_q^*$ acts as conjugation by

$$\delta(\lambda) = \begin{pmatrix} \lambda I_n & 0_n \\ 0_n & I_n \end{pmatrix} \qquad (\lambda \in \mathbb{F}_q^*)$$

This only provides a true outer automorphism if λ is not a square and we find that $\mathrm{PGSp}_{2n}(q) \cong \mathrm{PSp}_{2n}(q).2$ if q is odd and $\mathrm{PGSp}_{2n}(q) = \mathrm{PSp}_{2n}(q)$ if q is even. We define

$$\begin{aligned}\Gamma\mathrm{Sp}_{2n}(q) &= \mathrm{GSp}_{2n}(q) \rtimes \mathrm{Aut}(\mathbb{F}_q)\\ \mathrm{P}\Gamma\mathrm{Sp}_{2n}(q) &= \mathrm{PGSp}_{2n}(q) \rtimes \mathrm{Aut}(\mathbb{F}_q).\end{aligned}$$

Note that, as in $\mathrm{SL}_2(q)$, the map $\tau\colon A \to {}^tA^{-1}$ is the inner automorphism given by

$$M = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$

Automorphisms of $\mathrm{SU}_n(q)$. All linear outer automorphisms of $\mathrm{SU}_n(q)$ are induced by $\mathrm{GU}_n(q)$ the group of linear isometries of the hermitian form, or are induced by $\mathrm{Aut}(\mathbb{F}_{q^2})$ as for $\mathrm{SL}_n(q^2)$ with respect to an orthonormal basis. The group $\mathrm{Aut}(\mathbb{F}_{q^2})$ has order $2e$, where $q = p^e$, p prime. We let $\Gamma\mathrm{U}_n(q) = \mathrm{GU}_n \rtimes \mathrm{Aut}(\mathbb{F}_{q^2})$ and let $\mathrm{P}\Gamma\mathrm{U}_n(q)$ denote its quotient over the center (consisting of the scalar matrices). In this case, the transpose-inverse map τ with respect to a hyperbolic basis is the composition of the inner automorphism given by

$$M = \begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix}.$$

and the field automorphism $x \mapsto \overline{x} = x^q$ (with respect to the hyperbolic basis).
The group $\widehat{\mathrm{Aut}}(\mathbb{F}_{q^2})$ of field automorphisms of $\mathrm{SU}_n(q)$ on a hyperbolic basis. For $\Gamma\mathrm{U}_{2n}(q)$ note that $\mathrm{Aut}(\mathbb{F}_{q^2}) = \langle\alpha\rangle$ acts with respect to an orthonormal basis $\mathcal{U} = \{u_1, \ldots, u_{2n}\}$ for the $\mathbb{F}_{q^2}$-vector space V with σ-hermitian form h preserved by the group (see [30]). . We now identify a complement $\widehat{\mathrm{Aut}}(\mathbb{F}_{q^2})$ of semilinear automorphisms of $\mathrm{GU}_{2n}(q)$ in $\Gamma\mathrm{U}_{2n}(q)$ with respect to a hyperbolic basis. Fix the standard hyperbolic basis $\mathcal{H} = \{e_i, f_i \colon i = 1, 2, \ldots, n\}$ so that the elements in $\mathrm{GU}(V, \mathsf{h})$ are represented by a matrix in $\mathrm{GU}_{2n}(q)$ with respect to $\mathcal{H}$. Let $\alpha \in \mathrm{Aut}(\mathbb{F}_{q^2})$ act on V via $\mathcal{U}$. Then, $\mathcal{H}^\alpha = \{e_i^\alpha, f_i^\alpha \colon i = 1, 2 \ldots, n\}$ is also a hyperbolic basis for V, so for some $A \in \mathrm{GU}_{2n}(q)$, we have $A\mathcal{H} = \mathcal{H}^\alpha$. Now the composition $\widehat{\alpha} = A^{-1} \circ \alpha$ is an α-semilinear map that fixes $\mathcal{H}$. The corresponding automorphism of $\mathrm{GU}_{2n}(q)$ acts by applying α to the matrix entries.

Remark 3.1: The following special case will be of particular interest when considering a Curtis-Tits standard pairs of type ${}^2A_3(q)$. In this case the action of $\widehat{\alpha}$ as above on $\mathrm{SU}_4(q)$ translates via the standard identification maps (see Definition 2.5) to actions on $\mathrm{SL}_2(q)$ and $\mathrm{SL}_2(q^2)$ as follows. The action on $\mathrm{SL}_2(q^2)$ is the natural entry-wise field automorphism action. The action on $\mathrm{SL}_2(q)$ will be a product of the natural entrywise action of $\widehat{\alpha}$ and a diagonal automorphism $\mathrm{diag}(f, 1)$, where $f \in \mathbb{F}_q$ is such that $\widehat{\alpha}(\eta) = f\eta$. Note that $N_{\mathbb{F}_q/\mathbb{F}_p}(f) = -1$, so in particular $\sigma = \widehat{\alpha}^e$ translates to (left) conjugation by $\mathrm{diag}(-1, 1)$ only.

Definition 3.2: Since the norm is surjective, there exists $\zeta \in \mathbb{F}_{q^2}$ such that $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\zeta) = f^{-1}$. We then have that $\mathrm{diag}(\zeta,\zeta,\zeta^{-q},\zeta^{-q}) \in \mathrm{GU}_4(q)$ acts trivially on $\mathrm{SL}_2(q^2)$ and acts as left conjugation by $\mathrm{diag}(f^{-1},1)$ on $\mathrm{SL}_2(q)$. It follows that the composition $\widetilde{\alpha}$ of $\widehat{\alpha}$ and this diagonal automorphism acts entrywise as α on both $\mathrm{SL}_2(q)$ and $\mathrm{SL}_2(q^2)$. We now define

$$\widetilde{\mathrm{Aut}}(\mathbb{F}_{q^2}) = \langle \widetilde{\alpha} \rangle \leq \mathrm{Aut}(\mathrm{SU}_4(q)).$$

Lemma 3.3: *(See* [28, 30]*.)*

(1) *As* $\mathrm{Sp}_2(q) = \mathrm{SL}_2(q) \cong \mathrm{SU}_2(q)$, *we have*

$$\mathrm{Aut}(\mathrm{Sp}_2(q)) = \mathrm{Aut}(\mathrm{SL}_2(q)) = \mathrm{P\Gamma L}_2(q) \cong \mathrm{P\Gamma U}_2(q) = \mathrm{Aut}(\mathrm{SU}_2(q)).$$

(2) *In higher rank we have*

$$\begin{aligned}\mathrm{Aut}(\mathrm{SL}_n(q)) &= \mathrm{P\Gamma L}_n(q) \rtimes \langle \tau \rangle\\ \mathrm{Aut}(\mathrm{Sp}_{2n}(q)) &= \mathrm{P\Gamma Sp}_{2n}(q)\\ \mathrm{Aut}(\mathrm{SU}_n(q)) &= \mathrm{P\Gamma U}_n(q)\end{aligned}$$

3.1.1. *Some normalizers and centralizers.*

Corollary 3.4: *Let* $G = \mathrm{SL}_3(q)$. *Let* $\varphi\colon \mathrm{SL}_2(q) \to G$ *given by* $A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ *and let* $L = \mathrm{im}\,\varphi$. *Then,*

$$C_{\mathrm{Aut}(G)}(L) = \langle \mathrm{diag}(a,b,b)\colon a,b \in \mathbb{F}_q^*\rangle \rtimes \langle \theta \rangle.$$

where $\theta = \tau \circ c_\nu\colon X^\theta \mapsto {}^t(\nu^{-1}X\nu)^{-1}$ *and* $\nu = \begin{pmatrix} 1 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}$.

Proof This follows by an easy computation from the fact that $\mathrm{Aut}(G) \cong \mathrm{P\Gamma L}_3(q)$. Let $\tau^i \circ \alpha \circ c_g$, where c_g denotes conjugation by $g \in \mathrm{GL}_3(q)$ and $\alpha \in \mathrm{Aut}(\mathbb{F}_q)$. Using transvection matrices from L over the fixed field $\mathbb{F}_q^\alpha$ one sees that if $i = 0$, then g must be of the form $\mathrm{diag}(a,b,b)$, and if $i = 1$, then it must be of the form $\mathrm{diag}(a,b,b)\nu$, for some $a, b \in \mathbb{F}_q^*$. Then, if $\alpha \neq \mathrm{id}$, picking transvections from L with a few entries in $\mathbb{F}_q - \mathbb{F}_q^\alpha$ one verifies that α must be the identity. □

4. Classification of Curtis-Tits amalgams

4.1. Fundamental root groups in Curtis-Tits standard pairs

Lemma 4.1: *Let q be a power of the prime p. Suppose that* $(\mathbf{G}, \mathbf{G}_1, \mathbf{G}_2)$ *is a Curtis-Tits standard pair of type* $\Gamma(q)$ *as in Subsection 2.2. For* $\{i,j\} = \{1,2\}$, *let* $\mathcal{S}_j = \mathrm{Syl}_p(\mathbf{G}_j)$.

(1) *There exist two groups* $\mathbf{X}_j^{i,\varepsilon} \in \mathcal{S}_j$ $(\varepsilon = +,-)$ *such that for any* $\mathbf{X} \in \mathcal{S}_j$ *we have*

$$\langle \mathbf{G}_i, \mathbf{X}\rangle \le \mathbf{P}_i^\varepsilon \text{ if and only if } \mathbf{X} = \mathbf{X}_j^{i,\varepsilon}$$

where $\mathbf{P}_i^+$ *and* $\mathbf{P}_i^-$ *are the two parabolic subgroups of* $\mathbf{G}$ *containing* $\mathbf{G}_i$. *If* $\mathbf{X} \ne \mathbf{X}_j^{i,\varepsilon}$, *then*

$$\langle \mathbf{G}_i, \mathbf{X}\rangle = \begin{cases} (\mathbf{G}_i \times \mathbf{G}_i^x) \rtimes \langle x\rangle & \text{if } \Gamma(q) = C_2(2), \\ \mathbf{G} & \text{else.} \end{cases}$$

where in the $C_2(2)$ *case* $\mathbf{X} = \langle x\rangle$.

(2) *We can select the signs* ε *so that* $\mathbf{X}_i^{j,\varepsilon}$ *commutes with* $\mathbf{X}_j^{i,-\varepsilon}$, *but not with* $\mathbf{X}_j^{i,\varepsilon}$ *and, in fact* $\langle \mathbf{X}_i^{j,\varepsilon}, \mathbf{X}_j^{i,\varepsilon}\rangle$ *is contained in the unipotent radical* $\mathbf{U}_{i,j}^\varepsilon$ *of a unique Borel subgroup of* $\mathbf{G}_{i,j}$, *namely* $\mathbf{B}_{i,j}^\varepsilon = \mathbf{P}_i^\varepsilon \cap \mathbf{P}_j^\varepsilon$.

Proof We first prove part 1. by considering all cases (each case is marked by an underlined header in boldface).

$\underline{\mathbf{A_2(q), q \ge 3}}$ View $\mathbf{G} = \mathrm{SL}_3(q) = \mathrm{SL}(V)$ for some $\mathbb{F}_q$-vector space with basis $\{e_1, e_2, e_3\}$. By symmetry we may assume that $i = 1$ and $j = 2$. Let $\mathbf{G}_1$ (resp. $\mathbf{G}_2$) stabilize $\langle e_1, e_2\rangle$ and fix e_3 (resp. stabilize $\langle e_2, e_3\rangle$) and fix e_1). A root group in $\mathbf{G}_2$ is of the form $\mathbf{X}_v = \mathrm{Stab}_{\mathbf{G}_2}(v)$ for some $v \in \langle e_2, e_3\rangle$. We let $\mathbf{X}_2^+ = \mathbf{X}_{e_2}$ and $\mathbf{X}_2^- = \mathbf{X}_{e_3}$. It clear that for $\varepsilon = +$ (resp. $\varepsilon = -$) $\langle \mathbf{G}_1, \mathbf{X}_2^\varepsilon\rangle = \mathbf{P}^\varepsilon$ is contained in (but not equal to) the parabolic subgroup stabilizing $\langle e_1, e_2\rangle$ (resp. $\langle e_3\rangle$). Now suppose that $\mathbf{X} \in \mathcal{S}_2$ is different from $\mathbf{X}_2^\varepsilon$ $(\varepsilon = +,-)$ and $\mathbf{X} = \mathbf{X}_{\lambda e_2 + e_3}$ for some $\lambda \in \mathbb{F}_q^*$. Consider the action of a torus element $d = \mathrm{diag}(\mu, \mu^{-1}, 1) \in \mathbf{G}_1$ by conjugation on $\mathbf{G}_2$. Then $\mathbf{X}^d = \mathbf{X}_{\mu\lambda e_2 + e_3}$. Since $|\mathbb{F}_q| \ge 3$, $\mathbf{X}^d \ne \mathbf{X}$ for some d and so we have

$$\langle \mathbf{G}_i, \mathbf{X}\rangle \ge \langle \mathbf{G}_i, \mathbf{X}, \mathbf{X}^d\rangle = \langle \mathbf{G}_i, \mathbf{G}_j\rangle = \mathbf{G}. \tag{4.1}$$

$\underline{\mathbf{A_2(2)}}$ In this case $\mathcal{S}_2 = \{\mathbf{X}_2^+, \mathbf{X} = \langle r\rangle, \mathbf{X}_2^-\}$, where r is the Coxeter element fixing e_1 and interchanging e_2 and e_3. It follows that $\mathbf{G}_1^r$ is the stabilizer of the subspace decomposition $\langle e_2\rangle \oplus \langle e_1, e_3\rangle$ and hence $\langle \mathbf{G}_1, \mathbf{X}\rangle = \mathbf{G}$.

$\underline{\mathbf{C_2(q), q \ge 3, X}\text{ short root}}$ We use the notation of Subsection 2.2. First, let $i = 2$, $j = 1$, let $\mathbf{G}_2 \cong \mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$ be the stabilizer of e_1 and e_3 and let $\mathbf{G}_1 \cong \mathrm{SL}_2(q)$ be the derived subgroup of the stabilizer of the isotropic 2-spaces $\langle e_1, e_2\rangle$ and $\langle e_3, e_4\rangle$. Root groups in $\mathbf{G}_1$ are of the form $\mathbf{X}_{u,v} = \mathrm{Stab}_{\mathbf{G}_1}(u) \cap \mathrm{Stab}_{\mathbf{G}_1}(v)$, where $u \in \langle e_1, e_2\rangle$ and $v \in \langle e_3, e_4\rangle$ are orthogonal. Let $\mathbf{X}_1^+ = \mathbf{X}_{e_1,e_4}$ and $\mathbf{X}_1^- = \mathbf{X}_{e_2,e_3}$. It is easy to verify that for $\varepsilon = +$ (resp. $\varepsilon = -$) $\langle \mathbf{G}_2, \mathbf{X}_1^\varepsilon\rangle = \mathbf{P}^\varepsilon$ is contained in the parabolic subgroup stabilizing $\langle e_1\rangle$ (resp. $\langle e_3\rangle$). Now let $\mathbf{X} = \mathbf{X}_{e_1+\lambda e_2, e_3-\lambda^{-1}e_4}$ for some $\lambda \in \mathbb{F}_q^*$. Consider the action of a torus element $d = \mathrm{diag}(1, \mu^{-1}, 1, \mu) \in \mathbf{G}_2$ by conjugation on $\mathbf{G}_1$. Then $\mathbf{X}^d = \mathbf{X}_{\langle e_1+\lambda\mu e_2\rangle, e_3-\lambda^{-1}\mu^{-1}e_4}$. Since $q \geq 3$, $\mathbf{X}^d \neq \mathbf{X}$ for some d and so, for $i = 1$, and these $\mathbf{G}_2$, $\mathbf{X}$ and d, we have (4.1) again.

$\underline{\mathbf{C_2(q), q \geq 4, X} \text{ long root}}$ Now, we let $i = 1$ and $j = 2$. Root groups in $\mathbf{G}_2$ are of the form $\mathbf{X}_u = \mathrm{Stab}_{\mathbf{G}_2}(u)$ where $u \in \langle e_2, e_4\rangle$. Let $\mathbf{X}_2^+ = \mathbf{X}_{e_2}$ and $\mathbf{X}_2^- = \mathbf{X}_{e_4}$. It is easy to verify that for $\varepsilon = +$ (resp. $\varepsilon = -$) $\langle \mathbf{G}_1, \mathbf{X}_2^\varepsilon\rangle = \mathbf{P}^\varepsilon$ is contained in the parabolic subgroup stabilizing $\langle e_1, e_2\rangle$ (resp. $\langle e_1, e_4\rangle$). Now let $\mathbf{X} = \mathbf{X}_{e_2+\lambda e_4}$ for some $\lambda \in \mathbb{F}_q^*$.

Consider the action of a torus element $d = \mathrm{diag}(\mu, \mu^{-1}, \mu^{-1}, \mu) \in \mathbf{G}_1$ by conjugation on $\mathbf{G}_2$. Then $\mathbf{X}^d = \mathbf{X}_{\mu e_2+\mu^{-1}\lambda e_4}$. Now if $q \geq 4$, then $\mathbf{X}^d \neq \mathbf{X}$ for some d and so so, for these $\mathbf{G}_1$, $\mathbf{X}$ and d, we have (4.1) again.

$\underline{\mathbf{C_2(q), q = 3, X} \text{ long root}}$ The proof for the case $q \geq 4$ does not yield the result since, for $q = 3$, the element d centralizes $\mathbf{G}_2$. A direct computation in GAP shows that the conclusion still holds, though. Let $x \in \mathbf{X} = \mathbf{X}_{e_2+e_4}$ send e_2 to e_4. Then $\mathbf{G}_1$ and $\mathbf{G}_1^x$ contains two short root groups fixing e_1 and e_3. Their commutators generate a long root group fixing e_1, e_2, and e_4, while being transitive on the points $\langle e_3 + \lambda e_1\rangle$. Further conjugation with an element in $\mathbf{G}_1$ interchanging the points $\langle e_1\rangle$ and $\langle e_2\rangle$ yields a long root group in $\mathbf{G}_2$ different from $\mathbf{X}$ and we obtain an equation like (4.1) again.

$\underline{\mathbf{C_2(2)}}$ First note that $\mathbf{G} \cong \mathrm{Sp}_4(2) \cong O_5(2)$ is self point-line dual, so we only need to consider the case where $\mathbf{G}_2 = \mathrm{Stab}_{\mathbf{G}}(e_1) \cap \mathrm{Stab}_{\mathbf{G}}(e_3)$ and $\mathbf{G}_1 = \mathrm{Stab}_{\mathbf{G}}(\langle e_1, e_2\rangle) \cap \mathrm{Stab}_{\mathbf{G}}(\langle e_3, e_4\rangle)$. Now $\mathcal{S}_1 = \{\mathbf{X}_1^+, \mathbf{X}_1^-, \langle x\rangle\}$, where x is the permutation matrix of $(1,2)(3,4)$. The conclusion follows easily.

$\underline{\mathbf{{}^2A_3(q)}}$ We use the notation of Subsection 2.2. First, let $i = 2$, $j = 1$, let $\mathbf{G}_2 \cong \mathrm{SU}_2(q) \cong \mathrm{SL}_2(q)$ be the stabilizer of e_1 and e_3 and let $\mathbf{G}_1 \cong \mathrm{SL}_2(q^2)$ be the derived subgroup of the simultaneous stabilizer in $\mathbf{G} = \mathrm{SU}_4(q)$ of the isotropic 2-spaces $\langle e_1, e_2\rangle$ and $\langle e_3, e_4\rangle$. Root groups in $\mathbf{G}_1$ are of the form $\mathbf{X}_{u,v} = \mathrm{Stab}_{\mathbf{G}_1}(u) \cap \mathrm{Stab}_{\mathbf{G}_1}(v)$, where $u \in \langle e_1, e_2\rangle$ and $v \in \langle e_3, e_4\rangle$ are orthogonal. Let $\mathbf{X}_1^+ = \mathbf{X}_{e_1,e_4}$ and $\mathbf{X}_1^- = \mathbf{X}_{e_2,e_3}$. It is easy to verify that for $\varepsilon = +$ (resp. $\varepsilon = -$) $\langle \mathbf{G}_2, \mathbf{X}_1^\varepsilon\rangle = \mathbf{P}^\varepsilon$ is contained in the parabolic subgroup stabilizing $\langle e_1\rangle$ (resp. $\langle e_3\rangle$). Now let $\mathbf{X} = \mathbf{X}_{e_1+\lambda e_2, e_3-\lambda^{-\sigma}e_4}$ for some $\lambda \in \mathbb{F}_{q^2}^*$. Consider the action of a torus element $d = \mathrm{diag}(1, \mu^{-1}, 1, \mu) \in \mathbf{G}_2$ (with $\mu \in \mathbb{F}_q^*$) by conjugation on $\mathbf{G}_1$. Then $\mathbf{X}^d = \mathbf{X}_{e_1+\lambda\mu e_2, e_3-\lambda^{-\sigma}\mu^{-1}e_4}$. There are

$q-1$ choices for μ, so if $q \geq 3$, then $\mathbf{X}^d \neq \mathbf{X}$ for some d. Hence, for $i=1$, and these $\mathbf{G}_2$, $\mathbf{X}$ and d, we have (4.1) again. In the case $q=2$, (4.1) is verified by a simple calculation in GAP.

Now, we let $i=1$ and $j=2$. Root groups in $\mathbf{G}_2$ are of the form $\mathbf{X}_u = \mathrm{Stab}_{\mathbf{G}_2}(u)$ where $u \in \langle e_2, e_4\rangle$ is isotropic. Let $\mathbf{X}_2^+ = \mathbf{X}_{e_2}$ and $\mathbf{X}_2^- = \mathbf{X}_{e_4}$. It is easy to verify that for $\varepsilon = +$ (resp. $\varepsilon = -$) $\langle \mathbf{G}_1, \mathbf{X}_2^\varepsilon\rangle = \mathbf{P}^\varepsilon$ is contained in the parabolic subgroup stabilizing $\langle e_1, e_2\rangle$ (resp. $\langle e_1, e_4\rangle$). Now let $\mathbf{X} = \mathbf{X}_{e_2+\lambda e_4}$ for some $\lambda \in \mathbb{F}_{q^2}^*$ where $\mathrm{Tr}(\lambda) = \lambda + \lambda^\sigma = 0$. Consider the action of a torus element $d = \mathrm{diag}(\mu, \mu^{-1}, \mu^{-\sigma}, \mu^\sigma) \in \mathbf{G}_1$ (for some $\mu \in \mathbb{F}_{q^2}^*$) by conjugation on $\mathbf{G}_2$. Then $\mathbf{X}^d = \mathbf{X}_{\mu e_2 + \mu^{-\sigma}\lambda e_4}$. The q^2-1 choices for μ result in $q-1$ different conjugates. Thus, if $q-1 \geq 2$, then $\mathbf{X}^d \neq \mathbf{X}$ for some d and so so, for these $\mathbf{G}_1$, $\mathbf{X}$ and d, we have (4.1) again. In the case where $q=2$, (4.1) is verified using a simple calculation in GAP. Namely, in this case, $\mathbf{X} = \langle x\rangle$, where x is the only element of order 2 in $\mathbf{G}_2 \cong S_3$ that does not belong to $\mathbf{X}_1^+ \cup \mathbf{X}_1^-$; it is the Coxeter element that fixes e_1 and e_3 and interchanges e_2 and e_4. Now $\langle \mathbf{G}_1, \mathbf{G}_1^\tau\rangle$ contains the long root group generated by the commutators of the short root group fixing e_1 in $\mathbf{G}_1$ and $\mathbf{G}_1^\tau$, and likewise for e_2, e_3, and e_4. In particular, we have

$$\langle \mathbf{G}_1, \mathbf{X}\rangle \geq \langle \mathbf{G}_1, \mathbf{G}_1^\tau\rangle \geq \langle \mathbf{G}_1, \mathbf{G}_2\rangle = \mathbf{G} \tag{4.2}$$

We now address part 2. Note that the positive and negative fundamental root groups with respect to the torus $\mathbf{B}_{i,j}^+ \cap \mathbf{B}_{i,j}^-$ satisfy the properties of $\mathbf{X}_i^{j,\varepsilon}$ and $\mathbf{X}_j^{i,\varepsilon}$ so by the uniqueness statement in 1. they must be equal. Now the claims in part 2. are the consequences of the Chevalley commutator relations. □

Remark 4.2:

Explicitly, the groups $\{\mathbf{X}_i^+, \mathbf{X}_i^-\}$ $(i=1,2)$, possibly up to a switch of signs, for the Curtis-Tits standard pairs are as follows (each case is marked by an underlined header in boldface).

$\underline{\mathbf{\Gamma = A_2}}$ In this case, we have

$$\mathbf{X}_1^+ = \left\{\begin{pmatrix}1 & b & \\ 0 & 1 & \\ & & 1\end{pmatrix} : b \in \mathbb{F}_q\right\}, \text{ and } \mathbf{X}_1^- = \left\{\begin{pmatrix}1 & 0 & \\ c & 1 & \\ & & 1\end{pmatrix} : c \in \mathbb{F}_q\right\},$$
$$\mathbf{X}_2^+ = \left\{\begin{pmatrix}1 & & \\ & 1 & b \\ & 0 & 1\end{pmatrix} : b \in \mathbb{F}_q\right\}, \text{ and } \mathbf{X}_2^- = \left\{\begin{pmatrix}1 & & \\ & 1 & 0 \\ & c & 1\end{pmatrix} : c \in \mathbb{F}_q\right\}.$$

$\underline{\mathbf{\Gamma = C_2}}$ In this case, we have

$$\mathbf{X}_1^+ = \left\{ \begin{pmatrix} 1 & b & & \\ 0 & 1 & & \\ & & 1 & 0 \\ & & -b & 1 \end{pmatrix} : b \in \mathbb{F}_q \right\}, \text{ and } \mathbf{X}_1^- = \left\{ \begin{pmatrix} 1 & 0 & & \\ c & 1 & & \\ & & 1 & -c \\ & & 0 & 1 \end{pmatrix} : c \in \mathbb{F}_q \right\},$$

$$\mathbf{X}_2^+ = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & b \\ & & 1 & \\ & 0 & & 1 \end{pmatrix} : b \in \mathbb{F}_q \right\}, \text{ and } \mathbf{X}_2^- = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ & & 1 & \\ & c & & 1 \end{pmatrix} : c \in \mathbb{F}_q \right\}.$$

$\underline{\mathbf{\Gamma = {}^2A_3}}$ In this case, we have (with $\eta \in \mathbb{F}_{q^2}$ of trace 0),

$$\mathbf{X}_1^+ = \left\{ \begin{pmatrix} 1 & b & & \\ 0 & 1 & & \\ & & 1 & 0 \\ & & -b & 1 \end{pmatrix} : b \in \mathbb{F}_{q^2} \right\}, \text{ and } \mathbf{X}_1^- = \left\{ \begin{pmatrix} 1 & 0 & & \\ c & 1 & & \\ & & 1 & -c \\ & & 0 & 1 \end{pmatrix} : c \in \mathbb{F}_{q^2} \right\},$$

$$\mathbf{X}_2^+ = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & b\eta \\ & & 1 & \\ & 0 & & 1 \end{pmatrix} : b \in \mathbb{F}_q \right\}, \text{ and } \mathbf{X}_2^- = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ & & 1 & \\ & c\eta^{-1} & & 1 \end{pmatrix} : c \in \mathbb{F}_q \right\}.$$

4.2. Weak systems of fundamental groups

In this subsection we show that a Curtis-Tits amalgam with 3-spherical diagram determines a collection of subgroups of the vertex groups, called a weak system of fundamental root groups. We then use this to determine the coefficient system of the amalgam in the sense of [2], which, in turn is applied to classify these amalgams up to isomorphism.

Definition 4.3: Suppose that $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i,j \in I\}$ is a CT amalgam. For each $i \in I$ let $\mathbf{X}_i^+, \mathbf{X}_i^- \leq \mathbf{G}_i$ be a pair of opposite root groups. We say that $\{\mathbf{X}_i^+, \mathbf{X}_i^- \mid i \in I\}$ is a *weak system of fundamental root groups* if, for any edge $\{i,j\} \in \mathrm{E}$ there are opposite Borel groups $\mathbf{B}_{i,j}^+$ and $\mathbf{B}_{ij}^-$ in $\mathbf{G}_{i,j}$, each of which contains exactly one of $\{\overline{\mathbf{X}}_i^+, \overline{\mathbf{X}}_i^-\}$.

We call $\mathscr{G}$ *orientable* if we can select $\mathbf{X}_i^\varepsilon$, $\mathbf{B}_{ij}^\varepsilon$ $(\varepsilon = +,-)$ for all $i,j \in \mathrm{V}$ such that $\overline{\mathbf{X}}_i^\varepsilon, \overline{\mathbf{X}}_j^\varepsilon \leq \mathbf{B}_{ij}^\varepsilon$. If this is not possible, we call $\mathscr{G}$ *non-orientable.*

The relation between root groups and Borel groups is given by the following well-known fact.

LEMMA 4.4: *Let q be a power of the prime p. Let $\mathbf{G}$ be a universal group of Lie type $\Gamma(q)$ and let $\mathbf{X}$ be a Sylow p-subgroup. Then, $N_{\mathbf{G}}(\mathbf{X})$ is the unique Borel group $\mathbf{B}$ of $\mathbf{G}$ containing $\mathbf{X}$.*

PROPOSITION 4.5: *Suppose that $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i, j \in I\}$ is a CT amalgam with connected 3-spherical diagram Γ. If $\mathscr{G}$ has a non-trivial completion (G, γ), then it has a unique weak system of fundamental root groups.*

Proof We first show that there is some weak system of fundamental root groups. For every edge $\{i, j\}$, let $\mathbf{X}_j^{i,\varepsilon}$ be the groups of Lemma 4.1. Suppose that there is some subdiagram Γ_J with $J = \{i, j, k\}$ in which j is connected to both i and k, such that $\{\mathbf{X}_j^{i,+}, \mathbf{X}_j^{i,-}\} \neq \{\mathbf{X}_j^{k,+}, \mathbf{X}_j^{k,-}\}$ as sets. Without loss of generality assume that $\Gamma_{i,j} = A_2$ (by 3-sphericity) and moreover, that $\mathbf{X}_j^{k,+} \not\in \{\mathbf{X}_j^{i,+}, \mathbf{X}_j^{i,-}\}$. For any subgroup $\mathbf{H}$ of a group in $\mathscr{G}$, write $H = \gamma(\mathbf{H})$. Now note that $X_k^{j,-}$ commutes with $X_j^{k,+}$ and since Γ contains no triangles it also commutes with G_i. But then $X_k^{j,-}$ commutes with $\langle X_j^{k,+}, G_i\rangle$ which, by Lemma 4.1, equals $G_{i,j}$ (this is where we use that $\Gamma_{i,j} = A_2$), contradicting that $X_k^{j,-}$ does not commute with $X_j^{k,-} \leq G_{i,j}$. Thus, if there is a completion, then by connectedness of Γ, for each $i \in I$ we can pick a $j \in I$ so that $\{i, j\} \in E$ and set $\mathbf{X}_i^{\pm} = \mathbf{X}_i^{j,\pm}$ and drop the superscript. We claim that $\{\mathbf{X}_i^{\pm} \mid i \in I\}$ is a weak system of fundamental root groups. But this follows from part 2 of Lemma 4.1.

The uniqueness derives immediately from the fact that by Lemma 4.1 $\mathbf{g}_{i,j}(\mathbf{X}_j^{i,+})$ and $\mathbf{g}_{j,i}(\mathbf{X}_j^{i,-})$ are the only two Sylow p-subgroups in $\mathbf{g}_{j,i}(\mathbf{G}_j)$ which do not generate $\mathbf{G}_{i,j}$ with $\mathbf{g}_{i,j}(\mathbf{G}_i)$. □

An immediate consequence of the results above is the following observation.

COROLLARY 4.6: *Suppose that $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i, j \in I\}$ is a CT amalgam with connected 3-spherical diagram Γ. Then, an element of $N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\overline{\mathbf{G}}_i, \overline{\mathbf{G}}_j)$ either fixes each of the pairs $(\overline{\mathbf{X}}_i^+, \overline{\mathbf{X}}_i^-)$, $(\overline{\mathbf{X}}_j^+, \overline{\mathbf{X}}_j^-)$, and $(\mathbf{B}_{i,j}^+, \mathbf{B}_{i,j}^-)$ or it reverses each of them. In particular,*

$$N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\overline{\mathbf{G}}_i, \overline{\mathbf{G}}_j) = N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\{\overline{\mathbf{X}}_i^+, \overline{\mathbf{X}}_i^-\}) \cap N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\{\overline{\mathbf{X}}_j^+, \overline{\mathbf{X}}_j^-\}).$$

4.3. The coefficient system of a Curtis-Tits amalgam

The automorphisms of a Curtis-Tits standard pair will be crucial in the classification of Curtis-Tits amalgams and we will need some detailed description of them.

We now fix a Curtis-Tits amalgam $\underline{\mathscr{G}} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \underline{\mathbf{g}}_{i,j} \mid i, j \in I\}$ of type $\Gamma(q)$, where for every $i, j \in I$, $\underline{\mathbf{g}}_{i,j}$ is the standard identification map of Definition 2.5. Then, $\underline{\mathscr{G}}$ has a weak system of fundamental root groups $\mathcal{X} = \{\{\mathbf{X}_i^+, \mathbf{X}_i^-\} \colon i \in I\}$ as in Subsection 4.1.

Remark 4.7: Let $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i,j \in I\}$ be a Curtis-Tits amalgam over $\mathbb{F}_q$ with given diagram Γ. Next suppose that Γ is connected 3-spherical, and that $\underline{\mathscr{G}}$ and $\mathscr{G}$ are non-collapsing. Then, by Proposition 4.5, $\underline{\mathscr{G}}$ and $\mathscr{G}$ each have a weak system of fundamental root groups. Now note that for each $i \in I$, $\mathrm{Aut}(\mathbf{G}_i)$ is 2-transitive on the set of Sylow p-subgroups. Thus, for each $i \in I$ and all $j \in I - \{i\}$, we can replace $\mathbf{g}_{i,j}$ by $\mathbf{g}_{i,j} \circ \alpha_i$, to form a new amalgam isomorphic to $\mathscr{G}$, whose weak system of fundamental root groups is exactly $\mathcal{X}$. Thus, in order to classify non-collapsing Curtis-Tits amalgams over $\mathbb{F}_q$ with diagram Γ up to isomorphism, it suffices to classify those whose weak system of fundamental root groups is exactly $\mathcal{X}$.

Definition 4.8: Suppose that $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i,j \in I\}$ is a Curtis-Tits amalgam over $\mathbb{F}_q$ with connected 3-spherical diagram Γ. Let $\mathcal{X} = \{\{\mathbf{X}_i^+, \mathbf{X}_i^-\} \colon i \in I\}$ be the associated weak system of fundamental root groups. The *coefficient system associated to* $\mathscr{G}$ is the collection $\mathscr{A} = \{\mathbf{A}_i, \mathbf{A}_{i,j}, \mathbf{a}_{i,j} \mid i,j \in I\}$ where, for any $i,j \in I$ we set

$$\begin{aligned}
\mathbf{A}_i &= N_{\mathrm{Aut}(\mathbf{G}_i)}(\{\mathbf{X}_i^+, \mathbf{X}_i^-\}),\\
\mathbf{A}_{i,j} &= N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\{\overline{\mathbf{X}}_i^\varepsilon \colon \varepsilon = +,-\}) \cap N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\{\overline{\mathbf{X}}_j^\varepsilon \colon \varepsilon = +,-\}),\\
\mathbf{a}_{i,j} &\colon \mathbf{A}_{i,j} \to \mathbf{A}_j \text{ is given by restriction: } \varphi \mapsto \mathbf{g}_{j,i}^{-1} \circ \rho_{i,j}(\varphi) \circ \mathbf{g}_{j,i}.
\end{aligned}$$

where $\rho_{i,j}(\varphi)$ is the restriction of φ to $\overline{\mathbf{G}}_j \leq \mathbf{G}_{i,j}$.

From now on we let $\mathscr{A}$ be the coefficient system associated to $\underline{\mathscr{G}}$. The significance for the classification of Curtis-Tits amalgams with weak system of fundamental root groups is as follows:

Proposition 4.9: *Suppose that $\mathscr{G}$ and $\mathscr{G}^+$ are Curtis-Tits amalgams with diagram Γ over $\mathbb{F}_q$ with weak system of fundamental root groups $\mathcal{X}$.*

(1) *For all $i,j \in I$, we have $\mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j} \circ \delta_{i,j}$ and $\mathbf{g}_{i,j}^+ = \underline{\mathbf{g}}_{i,j} \circ \delta_{i,j}^+$ for some $\delta_{i,j}, \delta_{i,j}^+ \in \mathbf{A}_i$,*
(2) *For any isomorphism $\phi \colon \mathscr{G} \to \mathscr{G}^+$ and $i,j \in I$, we have $\phi_i \in \mathbf{A}_i$, $\phi_{\{i,j\}} \in \mathbf{A}_{i,j}$, and $\mathbf{a}_{i,j}(\phi_{\{i,j\}}) = \delta_{i,j}^+ \circ \phi_i \circ \delta_{i,j}^{-1}$.*

Proof Part 1. follows since, for any $i,j \in I$ we have $\mathbf{g}_{i,j}^{-1} \circ \underline{\mathbf{g}}_{i,j} \in \mathrm{Aut}(\mathbf{G}_1)$ and

$$\{\mathbf{g}_{i,j}(\mathbf{X}_i^+), \mathbf{g}_{i,j}(\mathbf{X}_i^-)\} = \{\underline{\mathbf{g}}_{i,j}(\mathbf{X}_i^+), \underline{\mathbf{g}}_{i,j}(\mathbf{X}_i^-)\}.$$

Part 2. follows from Corollary 4.6 since, for any $i,j \in I$,

$$(\mathbf{G}_{i,j}, \mathbf{g}_{i,j}(\mathbf{G}_i), \mathbf{g}_{j,i}(\mathbf{G}_j)) = (\mathbf{G}_{i,j}, \underline{\mathbf{g}}_{i,j}(\mathbf{G}_i), \underline{\mathbf{g}}_{j,i}(\mathbf{G}_j)) = (\mathbf{G}_{i,j}, \mathbf{g}_{i,j}^+(\mathbf{G}_i), \mathbf{g}_{j,i}^+(\mathbf{G}_j)).$$

□

We now determine the groups appearing in the coefficient system $\mathscr{A}$ associated to $\underline{\mathscr{G}}$.

Lemma 4.10: *Fix $i \in I$ and let q be such that $\mathbf{G}_i \cong \mathrm{SL}_2(q)$. Then,*

$$\mathbf{A}_i = \mathbf{T}_i \rtimes \mathbf{C}_i,$$

where $\mathbf{T}_i$ is the subgroup of diagonal automorphisms in $\mathrm{PGL}_2(q)$ and $\mathbf{C}_i = \langle \tau, \mathrm{Aut}(\mathbb{F}_q)\rangle$.

Proof This follows from the fact that via the standard embedding map $\underline{\mathbf{g}}_{i,j}$ the groups $\mathbf{X}_i^+$ and $\mathbf{X}_i^-$ of the weak system of fundamental root groups are the subgroups of unipotent upper and lower triangular matrices in $\mathrm{SL}_2(q)$. □

Lemma 4.11: *Let $\mathscr{A}$ be the coefficient system associated to the standard Curtis-Tits amalgam $\underline{\mathscr{G}}$ of type $\Gamma(q)$ and the weak system of fundamental root groups $\mathcal{X}$.*

If $\Gamma = A_1 \times A_1$, we have $\mathbf{G}_{i,j} = \mathbf{G}_i \times \mathbf{G}_j$, $\underline{\mathbf{g}}_{i,j}$ and $\underline{\mathbf{g}}_{j,i}$ are identity maps, and

$$\mathbf{A}_{i,j} = \mathbf{A}_i \times \mathbf{A}_j \cong \mathbf{T}_{i,j} \rtimes \mathbf{C}_{i,j}. \tag{4.3}$$

where $\mathbf{T}_{i,j} = \mathbf{T}_i \times \mathbf{T}_j$ and $\mathbf{C}_{i,j} = \mathbf{C}_i \times \mathbf{C}_j$. Otherwise,

$$\mathbf{A}_{i,j} = \mathbf{T}_{i,j} \rtimes \mathbf{C}_{i,j},$$

where

$$\mathbf{C}_{i,j} = \begin{cases} \mathrm{Aut}(\mathbb{F}_q) \times \langle \tau \rangle & \text{for } \Gamma = A_2, C_2 \\ \widetilde{\mathrm{Aut}}(\mathbb{F}_{q^2}) \times \langle \tau \rangle & \text{for } \Gamma = {}^2A_3 \end{cases} \tag{4.4}$$

and $\mathbf{T}_{i,j}$ denotes the image of the standard torus T in $\mathrm{Aut}(\mathbf{G}_{i,j})$. Note that

$$\mathrm{T} = \begin{cases} \langle \mathrm{diag}(a,b,c)\colon a,b,c \in \mathbb{F}_q^*\rangle \le \mathrm{GL}_3(q) & \text{if } \Gamma = A_2 \\ \langle \mathrm{diag}(ab, a^{-1}b, a^{-1}, a)\colon a,b \in \mathbb{F}_q^*\rangle \le \mathrm{GSp}_4(q) & \text{if } \Gamma = C_2 \\ \langle \mathrm{diag}(a,b,a^{-q},b^{-q})\colon a,b \in \mathbb{F}_{q^2}^*\rangle \le \mathrm{GU}_4(q) & \text{if } \Gamma = {}^2A_3. \end{cases}$$

Remark 4.12: Consider Lemma 4.11.

1. (1) We view $\mathrm{Sp}_{2n}(q)$ and $\mathrm{SU}_{2n}(q)$ as a matrix group with respect to a symplectic (resp. hyperbolic) basis for the $2n$-dimensional vector space V and $\mathrm{Aut}(\mathbb{F}_q)$ (resp. $\widehat{\mathrm{Aut}}(\mathbb{F}_{q^2})$) acts entrywise on the matrices.
2. (2) The map τ is the transpose-inverse map of Subsection 3.1.
3. (3) Recall that in the ${}^2A_3$ case, Remark 3.1 and Definition 3.2 describe the actions of $\widetilde{\mathrm{Aut}}(\mathbb{F}_{q^2}) \le \mathbf{C}_{i,j}$ on $\mathbf{G}_i$ and $\mathbf{G}_j$ via the standard identification maps.

Proof We first consider the $A_1 \times A_1$ case of (4.3). When $\Gamma = A_1 \times A_1$, then $\mathbf{G}_{i,j} = \overline{\mathbf{G}}_i \times \overline{\mathbf{G}}_j$ and since the standard root groups $\overline{\mathbf{X}}_i^{\pm}$ generate $\overline{\mathbf{G}}_i$ ($i = 1,2$), their simultaneous normalizer must also normalize $\overline{\mathbf{G}}_i$ and $\overline{\mathbf{G}}_j$. Thus the claim follows from Lemma 4.10.

We now deal with all remaining cases simultaneously. In the ${}^2A_3$ case we note that from Remark 3.1 and Definition 3.2 we see that $\widetilde{\mathrm{Aut}}(\mathbb{F}_{q^2}) \le \mathbf{T}_{i,j} \rtimes \widehat{\mathrm{Aut}}(\mathbb{F}_{q^2})$ is simply a different complement to $\mathbf{T}_{i,j}$, so it suffices to prove the claim with $\widetilde{\mathrm{Aut}}(\mathbb{F}_{q^2})$ replaced by $\widehat{\mathrm{Aut}}(\mathbb{F}_{q^2})$.

Consider the descriptions of the set $\{\mathbf{X}_i^+, \mathbf{X}_i^-\}$ in all cases from Subsection 4.1. We see that since τ acts by transpose-inverse, it interchanges $\mathbf{X}_i^+$ and $\mathbf{X}_i^-$ for $i = 1, 2$ in all cases, hence it also interchanges positive and negative Borel groups (see Corollary 4.6). Thus it suffices to consider those automorphisms that normalize the positive and negative fundamental root groups. Since all field automorphisms (of $\mathrm{Aut}(\mathbb{F}_q)$ and $\widehat{\mathrm{Aut}}(\mathbb{F}_{q^2})$) act entrywise, these automorphisms do so. Clearly so does T. Thus we have established $\supseteq$.

We now turn to the reverse inclusion. By Lemma 3.3 and the description of the automorphism groups in Subsection 3.1 any automorphism of $\mathbf{G}_{i,j}$ is a product of the form $g\alpha\tau^i$ where g is linear, α is a field automorphism (from $\widehat{\mathrm{Aut}}(\mathbb{F}_{q^2})$ in the ${}^2A_3$ case) and $i = 0, 1$. As we saw above τ and α preserve the root groups, so it suffices to describe g in case it preserves the sets of opposite root groups. A direct computation shows that g must be in T. □

Next we describe the connecting maps $\mathbf{a}_{i,j}$ of $\mathscr{A}$.

Lemma 4.13: *Let $\mathscr{A}$ be the coefficient system of the standard Curtis-Tits amalgam $\underline{\mathscr{G}}$ over $\mathbb{F}_q$ with diagram Γ and weak system of fundamental root groups $\mathcal{X}$. Fix $i, j \in I$ and let $(\mathbf{G}_{i,j}, \overline{\mathbf{G}}_i, \overline{\mathbf{G}}_j)$ be a Curtis-Tits standard pair in $\underline{\mathscr{G}}$ with diagram $\Gamma_{i,j}$. Denote $\mathbf{a} = (\mathbf{a}_{j,i}, \mathbf{a}_{i,j})\colon \mathbf{A}_{i,j} \to \mathbf{A}_i \times \mathbf{A}_j$. Then, we have the following:*

1. *If $\Gamma_{i,j} = A_1 \times A_1$, then $\mathbf{a}$ is an isomorphism inducing $\mathbf{T}_{i,j} \cong \mathbf{T}_i \times \mathbf{T}_j$ and $\mathbf{C}_{i,j} \cong \mathbf{C}_i \times \mathbf{C}_j$.*
2. *If $\Gamma_{i,j} = A_2$, or ${}^2A_3$, then $\mathbf{a}\colon \mathbf{T}_{i,j} \to \mathbf{T}_i \times \mathbf{T}_j$ is bijective.*
3. *If $\Gamma_{i,j} = C_2$, then $\mathbf{a}\colon \mathbf{T}_{i,j} \overset{\cong}{\to} \mathbf{T}_i^2 \times \mathbf{T}_j$ has index 1 or 2 in $\mathbf{T}_i \times \mathbf{T}_j$ depending on whether q is even or odd.*
4. *If $\Gamma_{i,j} = A_2$ or C_2, then $\mathbf{a}\colon \mathbf{C}_{i,j} \to \mathbf{C}_i \times \mathbf{C}_j$ is given by $\tau^s\alpha \mapsto (\tau^s\alpha, \tau^s\alpha)$ (for $s \in \{0,1\}$ and $\alpha \in \mathrm{Aut}(\mathbb{F}_q)$) which is a diagonal embedding.*
5. *If $\Gamma = {}^2A_3$, then $\mathbf{a}\colon \mathbf{C}_{i,j} \to \mathbf{C}_i \times \mathbf{C}_j$, is given by $\tau^s\widetilde{\alpha}^r \mapsto (\tau^s\alpha^r, \tau^s\alpha^r)$ (for $s \in \{0,1\}$, $r \in \mathbb{N}$, and $\alpha\colon x \mapsto x^p$ for $x \in \mathbb{F}_{q^2}$. Here $\widetilde{\sigma} \mapsto (\sigma, \mathrm{id})$.*

Remark 4.14: (1) In 4. τ acts as transpose-inverse and α acts entry-wise on $\mathbf{G}_{i,j}$, $\mathbf{G}_i$ and $\mathbf{G}_j$.

Proof (1) This is immediate from Lemma 4.11.

For the remaining cases, recall that for any $\varphi \in \mathbf{A}_{i,j}$, we have $\mathbf{a}_{i,j}\colon \varphi \mapsto \underline{\mathbf{g}}_{j,i}^{-1} \circ \rho_{i,j}(\varphi) \circ \underline{\mathbf{g}}_{j,i}$, where $\rho_{i,j}(\varphi)$ is the restriction of φ to $\overline{\mathbf{G}}_j \leq \mathbf{G}_{i,j}$ (Definition 4.8) and $\underline{\mathbf{g}}_{i,j}$ is the standard identification map of Definition 2.5. Note that for $\Gamma_{i,j} = A_2, C_2$ the standard identification map transforms the automorphism $\rho_{j,i}(\varphi)$ of $\overline{\mathbf{G}}_i$ essentially to the "same" automorphism φ of $\mathbf{G}_i$, whereas for $\Gamma_{i,j} = {}^2A_3$, we must take Remark 3.1 into account.

(2) Let $\Gamma_{i,j} = A_2$. Every element of $\mathbf{T}_{i,j}$ ($\mathbf{T}_i$, and $\mathbf{T}_j$ respectively) is given by a unique matrix of the form $\mathrm{diag}(a,1,c)$ ($\mathrm{diag}(a,1)$, and $\mathrm{diag}(1,c)$), and we have

$$(\mathbf{a}_{j,i}, \mathbf{a}_{i,j})\colon \ \mathrm{diag}(a,1,c) \mapsto (\mathrm{diag}(a,1), \mathrm{diag}(1,c)) \qquad (a, c \in \mathbb{F}_q^*),$$

which is clearly bijective. In the ${}^2A_3$ case, every element of $\mathbf{T}_{i,j}$ ($\mathbf{T}_i$, and $\mathbf{T}_j$ respectively) is given by a unique matrix of the form $\mathrm{diag}(ab^{-1}, 1, a^{-q}b^{-1}, b^{-(q+1)})$ ($\mathrm{diag}(1, b^{-(q+1)})$, and $\mathrm{diag}(ab^{-1}, 1)$), and we have

$$\mathbf{a}\colon \ \mathrm{diag}(ab^{-1}, 1, a^{-q}b^{-1}, b^{-(q+1)}) \mapsto (\mathrm{diag}(1, b^{-(q+1)}), \mathrm{diag}(ab^{-1}, 1)).$$

This map is onto since $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}\colon b \mapsto b^{q+1}$ is onto. Its kernel is trivial, as it is given by pairs $(a,b) \in \mathbb{F}_{q^2}$, with $a = b$ and $b^{q+1} = 1$ so that also $a^{-q}b^{-1} = 1$.

(3) In the C_2-case, every element of $\mathbf{T}_{i,j}$ is given by a unique matrix $\mathrm{diag}(a^2b, b, 1, a^2)$ ($a, b \in \mathbb{F}_q$). Every element of $\mathbf{T}_i$ (resp. $\mathbf{T}_j$) is given by a unique $\mathrm{diag}(c,1)$ (resp. $\mathrm{diag}(d,1)$). Now we have

$$\mathbf{a}\colon \ \mathrm{diag}(a^2b, b, 1, a^2) \mapsto (\mathrm{diag}(a^2,1), \mathrm{diag}(ba^{-2},1)).$$

It follows that $\mathbf{a}$ is injective and has image $\mathbf{T}_i^2 \times \mathbf{T}_j$. The rest of the claim follows.

(4) The field automorphism $\alpha \in \mathrm{Aut}(\mathbb{F}_q)$ acts entrywise on the matrices in $\mathbf{G}_{i,j} = \mathrm{SL}_3(q)$, or $\mathrm{Sp}_4(q)$, and $\mathbf{G}_i = \mathbf{G}_j = \mathrm{SL}_2(q)$. In the case $\mathbf{G} = \mathrm{Sp}_4(q)$, we saw in Subsection 3.1 that τ is inner and coincides with conjugation by M. This clearly restricts to conjugation by μ on both $\overline{\mathbf{G}}_2 = \mathrm{Sp}_2(q)$ and $\overline{\mathbf{G}}_1 = \mathrm{SL}_2(q)$, which is again τ. Clearly these actions correspond to each other via the standard identification maps $\underline{\mathbf{g}}_{i,j}$ and $\underline{\mathbf{g}}_{j,i}$.

(5) The action of $\widetilde{\mathrm{Aut}}(\mathbb{F}_{q^2}) \le \mathbf{C}_{i,j}$ on $\mathbf{G}_i$ and $\mathbf{G}_j$ via $\mathbf{a}$ was explained in Remark 3.1 and Definition 3.2. In case $\mathbf{G}_{i,j} = \mathrm{SU}_4(q)$, τ is given by conjugation by M composed with the field automorphism $\widehat{\sigma}$, where $\sigma\colon x \mapsto x^q$ for $x \in \mathbb{F}_{q^2}$. The same holds for $\overline{\mathbf{G}}_j = \mathrm{SU}_2(q)$ and τ restricts to $\overline{\mathbf{G}}_i = \mathrm{SL}_2(q^2)$ as transpose-inverse. In view of Remark 3.1 we see that via the standard identification map each restricts to transpose inverse on $\mathbf{G}_i$ and $\mathbf{G}_j$. $\square$

4.4. A standard form for Curtis-Tits amalgams

Suppose that $\underline{\mathscr{G}} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \underline{\mathbf{g}}_{i,j} \mid i,j \in I\}$ is a Curtis-Tits amalgam over $\mathbb{F}_q$ with 3-spherical diagram Γ. Without loss of generality we will assume that all inclusion maps $\underline{\mathbf{g}}_{i,j}$ are the standard identification maps of Definition 2.5.

By Proposition 4.5 it possesses a weak system of fundamental root groups

$$\mathcal{X} = \{\{\mathbf{X}_i^+, \mathbf{X}_i^-\}\colon i \in I\},$$

which via the standard embeddings $\underline{\mathbf{g}}_{i,j}$ can be identified with those given in Subsection 4.1 (note that orienting $\mathcal{X}$ may involve changing some signs). Let $\mathscr{A} = \{\mathbf{A}_i, \mathbf{A}_{i,j}, \mathbf{a}_{i,j} \mid i,j \in I\}$ be the coefficient system associated to $\underline{\mathscr{G}}$ and $\mathcal{X}$.

We wish to classify all Curtis-Tits amalgams $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i,j \in I\}$ over $\mathbb{F}_q$ with the same diagram as $\underline{\mathscr{G}}$ with weak system of fundamental root groups $\mathcal{X}$ up to isomorphism of Curtis-Tits amalgams. By Proposition 4.9 we may restrict to those amalgams whose connecting maps are of the form $\mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j} \circ \delta_{i,j}$ for $\delta_{i,j} \in \mathbf{A}_i$ for all $i \in I$.

Definition 4.15: The *trivial support* of $\mathscr{G}$ (with respect to $\underline{\mathscr{G}}$) is the set $\{(i,j) \in I \times I \mid \mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j}\}$ (that is, $\delta_{i,j} = \mathrm{id}_{\mathbf{G}_i}$ in the notation of Proposition 4.9). The word "trivial" derives from the assumption that the $\underline{\mathbf{g}}_{i,j}$'s are the standard identification maps of Definition 2.5.

Fix some spanning tree $\Sigma \subseteq \Gamma$ and suppose that $\mathrm{E} - \mathrm{E}_\Sigma = \{\{i_s, j_s\} \colon s = 1, 2, \ldots, r\}$ so that $H_1(\Gamma, \mathbb{Z}) \cong \mathbb{Z}^r$.

PROPOSITION 4.16: *There is a Curtis-Tits amalgam $\mathscr{G}(\Sigma)$ over $\mathbb{F}_q$ with the same diagram as $\underline{\mathscr{G}}$ and the same $\mathcal{X}$, which is isomorphic to $\mathscr{G}$ and has the following properties:*

(1) *$\mathscr{G}$ has trivial support $S = \{(i,j) \in I \times I \mid \{i,j\} \in \mathrm{E}_\Sigma\} \cup \{(i_s, j_s) \colon s = 1, 2 \ldots, r\}$.*
(2) *for each $s = 1, 2, \ldots, r$, we have $\mathbf{g}_{j_s,i_s} = \underline{\mathbf{g}}_{j_s,i_s} \circ \gamma_{j_s,i_s}$, where $\gamma_{j_s,i_s} \in \mathbf{C}_{j_s}$.*

Before proving 4.16, we will state and prove a useful Lemma and Corollary.

LEMMA 4.17: *There is a Curtis-Tits amalgam $\mathscr{G}^+$ over $\mathbb{F}_q$ with the same diagram as $\underline{\mathscr{G}}$ and the same $\mathcal{X}$, which is isomorphic to $\mathscr{G}$ and has the following properties: For any $u, v \in I$, if $\mathbf{g}_{u,v} = \underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v} \circ d_{u,v}$, for some $\gamma_{u,v} \in \mathbf{C}_u$ and $d_{u,v} \in \mathbf{T}_u$, then $\mathbf{g}^+_{u,v} = \underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v}$.*

Proof Note that we have $|I| \geq 2$ and that Γ is connected. Fix $u \in I$. Since Γ is 3-spherical, there is at most one $w \in I$ such that $(\mathbf{G}_{u,w}, \overline{\mathbf{G}}_u, \overline{\mathbf{G}}_w)$ is a Curtis-Tits standard pair of type B_2 or C_2. If there is no such w, let w be an arbitrary vertex such that $\{u, v\} \in \mathrm{E}_\Gamma$. We define $\mathscr{G}^+$ by setting $\mathbf{g}^+_{u,v} = \underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v}$ for all $v \neq u$.

Next we define $\phi \colon \mathscr{G} \to \mathscr{G}^+$ setting $\phi_u = d_{u,w}$ and $\phi_v = \mathrm{id}_{\mathbf{G}_v}$ for all $v \neq u$. Now note that setting $\phi_{u,w} = \mathrm{id}_{\mathbf{G}_{u,v}}$, $\{\phi_{u,w}, \phi_u, \phi_w\}$ is an isomorphism of the subamalgams of $\mathscr{G}_{\{u,w\}}$ and $\mathscr{G}^+_{\{u,w\}}$. As for $\phi_{u,v}$ for $v \neq w$, note that in order for $\{\phi_{u,v}, \phi_u, \phi_v\}$ to be an isomorphism of the subamalgams of $\mathscr{G}_{\{u,v\}}$ and $\mathscr{G}^+_{\{u,v\}}$, we must have

$$\mathbf{g}^+_{u,v} \phi_u = \phi_{u,v} \circ \mathbf{g}_{u,v}$$
$$\mathbf{g}^+_{v,u} \phi_v = \phi_{u,v} \circ \mathbf{g}_{v,u}$$

which translates as

$$\begin{aligned}\underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v} \circ d_{u,w} &= \phi_{u,v} \circ \underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v} \circ d_{u,v} \\ \underline{\mathbf{g}}_{v,u} \circ \delta_{v,u} &= \phi_{u,v} \circ \underline{\mathbf{g}}_{v,u} \circ \delta_{v,u}\end{aligned}$$

or in other words

$$\begin{aligned}\gamma_{u,v} \circ d_{u,w} \circ d_{u,v}^{-1} \circ \gamma_{u,v}^{-1} &= \mathbf{a}_{v,u}(\phi_{u,v}) \\ \mathrm{id}_{\mathbf{G}_v} &= \mathbf{a}_{u,v}(\phi_{u,v}).\end{aligned}$$

Note that $\gamma_{u,v} \circ d_{u,w} \circ \circ d_{u,v}^{-1} \circ \gamma_{u,v}^{-1} \in \mathbf{T}_u \lhd \mathbf{A}_u$. Now by Lemma 4.13 as $(\mathbf{G}_{u,v}, \overline{\mathbf{G}}_u, \overline{\mathbf{G}}_v)$ is not of type B_2 or C_2 the map $(\mathbf{a}_{j,i}, \mathbf{a}_{i,j})\colon \mathbf{T}_{i,j} \to \mathbf{T}_i \times \mathbf{T}_j$ is onto. In particular, the required $\phi_{u,v} \in \mathbf{T}_{u,v}$ can be found. This completes the proof. □

By Lemma 4.17 in order to prove Proposition 4.16 we may now assume that $\mathbf{g}_{u,v} = \underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v}$ for some $\gamma_{u,v} \in \mathbf{C}_u$ for all $u, v \in I$.

Let $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_j, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j} \circ \gamma_{i,j} \mid i, j \in I\}$ be a Curtis-Tits amalgam over $\mathbb{F}_q$ with $|I| = 2$ and $\gamma_{i,j} \in \mathbf{C}_i$ and $\gamma_{j,i} \in \mathbf{C}_j$. We will describe all possible amalgams $\mathscr{G}^+ = \{\mathbf{G}_i, \mathbf{G}_j, \mathbf{G}_{i,j}, \mathbf{g}_{i,j}^+ = \underline{\mathbf{g}}_{i,j} \circ \gamma_{i,j}^+ \mid i, j \in I\}$ with $\gamma_{i,j}^+ \in \mathbf{C}_i$ and $\gamma_{j,i}^+ \in \mathbf{C}_j$, isomorphic to $\mathscr{G}$ via an isomorphism ϕ with $\phi_i \in \mathbf{C}_i$, $\phi_j \in \mathbf{C}_j$ and $\phi_{i,j} \in \mathbf{C}_{i,j}$.

$$\begin{tikzcd}
 & \mathbf{G}_{i,j} \arrow[ddd, dotted, "\phi_{i,j}"] & \\
\mathbf{G}_i \arrow[ur, hook, "\mathbf{g}_{i,j}"] \arrow[d, "\phi_i"'] & & \mathbf{G}_j \arrow[ul, hook', "\mathbf{g}_{j,i}"'] \arrow[d, "\phi_j"] \\
\mathbf{G}_i \arrow[dr, hook, "\mathbf{g}_{i,j}^+"'] & & \mathbf{G}_j \arrow[dl, hook', "\mathbf{g}_{j,i}^+"] \\
 & \mathbf{G}_{i,j} &
\end{tikzcd}$$

Figure 1. The commuting hexagon of Corollary 4.18.

Corollary 4.18: *With the notation introduced above, fix the maps $\gamma_{i,j}, \gamma_{i,j}^+, \phi_i \in \mathbf{C}_i$ as well as $\gamma_{j,i} \in \mathbf{C}_j$. Then for any one of $\gamma_{j,i}^+, \phi_j \in \mathbf{C}_j$, there exists a choice $\gamma \in \mathbf{C}_i$ for the remaining map in $\mathbf{C}_j$ so that there exists $\phi_{i,j}$ making the diagram in Figure 1 commute. Moreover, if $\Gamma_{i,j}$ is one of A_2, B_2, C_2, ${}^2A_3$, then γ is unique, whereas if $\Gamma_{i,j} = {}^2D_3$, then there are exactly two choices for γ.*

Proof The first claim follows immediately from the fact that the maps $\mathbf{a}_{j,i}\colon \mathbf{C}_{i,j} \to \mathbf{C}_i$ and $\mathbf{a}_{i,j}\colon \mathbf{C}_{i,j} \to \mathbf{C}_j$ in part 4. and 5. of Lemma 4.13 are both surjective. The second claim follows from the fact that $\mathbf{a}_{j,i}\colon \mathbf{C}_{i,j} \to \mathbf{C}_i$ is injective except if $\Gamma_{i,j} = {}^2D_3$ in which case it has a kernel of order 2. $\square$

Proof (of Proposition 4.16) By Lemma 4.17 we may assume that $\mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j} \circ \gamma_{i,j}$ for some $\gamma_{i,j} \in \mathbf{C}_i$ for all $i, j \in I$.

For any (possibly empty) subset $T \subseteq \mathrm{V}$ let $S(T)$ be the set of pairs $(i,j) \in S$ such that $i \in T$. Clearly the trivial support of $\mathscr{G}$ contains $S(\emptyset)$.

We now show that if T is the vertex set of a (possibly empty) proper subtree of Σ, and u is a vertex such that $T \cup \{u\}$ is also the vertex set of a subtree of Σ, then for any Curtis-Tits amalgam $\mathscr{G}$ whose trivial support contains $S(T)$, there is a Curtis-Tits amalgam $\mathscr{G}^+$ isomorphic to $\mathscr{G}$, whose trivial support contains $S(T \cup \{u\})$.

Once this is proved, Claim 1. follows since we can start with $T = \emptyset$ and end with a Curtis-Tits amalgam, still isomorphic to $\mathscr{G}$, whose trivial support contains S.

Now let T and u be as above. We first deal with the case where $T \neq \emptyset$. Let t be the unique neighbor of u in the subtree of Σ with vertex set $T \cup \{u\}$. We shall define an amalgam $\mathscr{G}^+ = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}^+_{i,j} = \underline{\mathbf{g}}_{i,j} \circ \gamma^+_{i,j} \mid i,j \in I\}$ and an isomorphism $\phi\colon \mathscr{G} \to \mathscr{G}^+$, where $\gamma^+_{i,j}, \phi_i \in \mathbf{C}_i$ and $\phi_{\{i,j\}} \in \mathbf{C}_{i,j}$ for all $i,j \in I$. First note that it suffices to define $\mathbf{g}^+_{i,j}$, ϕ_i and $\phi_{\{i,j\}}$ for $\{i,j\} \in \mathrm{E}$: given this data, by the $A_1 \times A_1$ case in Lemma 4.11 and Lemma 4.18, for any non-edge $\{k,l\}$ there is a unique $\phi_{\{k,l\}} \in \mathbf{C}_{k,l}$ such that $(\phi_{k,l}, \phi_k, \phi_l)$ is an isomorphism between $\mathscr{G}_{\{k,l\}}$ and $\mathscr{G}^+_{\{k,l\}}$.

Before defining inclusion maps on edges, note that since Γ is 3-spherical, no two neighbors of u in Γ are connected by an edge. Therefore we can unambiguously set

$$\mathbf{g}^+_{i,j} = \mathbf{g}_{i,j} \text{ for } u \notin \{i,j\} \in \mathrm{E}_\Gamma\,.$$

Note that both maps $\mathbf{g}^+_{t,u}$ and $\mathbf{g}^+_{u,t}$ are forced upon us, but at this point for any other neighbor v of u, only one of $\mathbf{g}^+_{u,v}$ and $\mathbf{g}^+_{v,u}$ is forced upon us. We set

$$\begin{aligned}
&\mathbf{g}^+_{t,u} = \mathbf{g}_{t,u}, \text{ and}\\
&\mathbf{g}^+_{v,u} = \mathbf{g}_{v,u} \text{ for } v \in I \text{ with } (u,v) \notin S \text{ and } (v,u) \in S.
\end{aligned}$$

To extend the trivial support as required, we set

$$\mathbf{g}^+_{u,v} = \underline{\mathbf{g}}_{u,v} \text{ for } v \in I \text{ with } (u,v) \in S.$$

We can already specify part of ϕ: Set

$$\begin{aligned}
\phi_i &= \mathrm{id}_{\mathbf{G}_i} \text{ for } i \in I - \{u\},\\
\phi_{\{i,j\}} &= \mathrm{id}_{\mathbf{G}_{i,j}} \text{ for } u \notin \{i,j\} \in \mathrm{E}_\Gamma\,.
\end{aligned}$$

Thus, what is left to specify is the following: ϕ_u and $\phi_{\{u,t\}}$ and, for all neighbors $v \neq t$ of u we must specify $\phi_{\{u,v\}}$ as well as

$$\mathbf{g}^+_{u,v} \text{ if } (u,v) \notin S,$$
$$\mathbf{g}^+_{v,u} \text{ if } (u,v) \in S.$$

Figures 2 and 3 describe the amalgam $\mathscr{G}$ (top half) and $\mathscr{G}^+$ (bottom half) at the vertex u, where $t \in \mathrm{V}_\Sigma$, $v \in \mathrm{V}_\Gamma$, and $\{u,t\}, \{u,v\} \in \mathrm{E}_\Gamma$. Inclusion maps from $\mathscr{G}^+$ forced upon us are indicated in bold, the dotted arrows are those we must define so as to make the diagram commute.

$$\begin{tikzcd}
 & \mathbf{G}_{t,u} \arrow[ddd, dotted, "\phi_{u,t}"'] & & \mathbf{G}_{u,v} \arrow[ddd, dotted, "\phi_{u,v}"] & \\
\mathbf{G}_t \arrow[ur, hook, "\mathbf{g}_{t,u}"] \arrow[d, "\phi_t := \mathrm{id}_{\mathbf{G}_t}"'] & & \mathbf{G}_u \arrow[ul, hook', "\mathbf{g}_{u,t}"'] \arrow[ur, hook, "\mathbf{g}_{u,v}"] \arrow[d, dotted, "\phi_u"] & & \mathbf{G}_v \arrow[ul, hook', "\mathbf{g}_{v,u}"'] \arrow[d, "\phi_v := \mathrm{id}_{\mathbf{G}_v}"] \\
\mathbf{G}_t^+ \arrow[dr, hook, thick, "\mathbf{g}_{t,u}"'] & & \mathbf{G}_u^+ \arrow[dl, hook', thick, "\mathbf{g}_{u,t}"] \arrow[dr, hook, thick, "\mathbf{g}_{u,v}"'] & & \mathbf{G}_v^+ \arrow[dl, hook', dotted, "\mathbf{g}^+_{v,u}"] \\
 & \mathbf{G}^+_{t,u} & & \mathbf{G}^+_{u,v} &
\end{tikzcd}$$

Figure 2. The case $(u,v) \in S$ and $(v,u) \notin S$.

$$\begin{tikzcd}
 & \mathbf{G}_{t,u} \arrow[ddd, dotted, "\phi_{u,t}"'] & & \mathbf{G}_{u,v} \arrow[ddd, dotted, "\phi_{u,v}"] & \\
\mathbf{G}_t \arrow[ur, hook, "\mathbf{g}_{t,u}"] \arrow[d, "\phi_t := \mathrm{id}_{\mathbf{G}_t}"'] & & \mathbf{G}_u \arrow[ul, hook', "\mathbf{g}_{u,t}"'] \arrow[ur, hook, "\mathbf{g}_{u,v}"] \arrow[d, dotted, "\phi_u"] & & \mathbf{G}_v \arrow[ul, hook', "\mathbf{g}_{v,u}"'] \arrow[d, "\phi_v := \mathrm{id}_{\mathbf{G}_v}"] \\
\mathbf{G}_t^+ \arrow[dr, hook, thick, "\mathbf{g}_{t,u}"'] & & \mathbf{G}_u^+ \arrow[dl, hook', thick, "\mathbf{g}_{u,t}"] \arrow[dr, hook, dotted, "\mathbf{g}^+_{u,v}"'] & & \mathbf{G}_v^+ \arrow[dl, hook', thick, "\mathbf{g}_{v,u}"] \\
 & \mathbf{G}^+_{t,u} & & \mathbf{G}^+_{u,v} &
\end{tikzcd}$$

Figure 3. The case $(u,v) \notin S$ and $(v,u) \in S$.

In these figures all non-dotted maps are of the form $\underline{\mathbf{g}}_{i,j} \circ \gamma_{i,j}$ for some $\gamma_{i,j} \in \mathbf{C}_i$ hence we can find the desired maps using Corollary 4.18.

In case $T = \emptyset$, the situation is as described in Figures 2 and 3 after removing the $\{u, t\}$-hexagon and any conditions it may impose on ϕ_u, and letting v run over all neighbors of u. That is, we must now define ϕ_u, and for any neighbor v of u, we must find $\phi_{u,v}$ as well as

$$\mathbf{g}^+_{u,v} \text{ if } (u,v) \notin S,$$
$$\mathbf{g}^+_{v,u} \text{ if } (u,v) \in S.$$

To do so we let $\phi_u = \mathrm{id}_{\mathbf{G}_u} \in \mathbf{C}_u$. Finally, for each neighbor v of u we simply let $\mathbf{g}^+_{u,v} = \mathbf{g}_{u,v}$ (so that $\phi_{u,v} = \mathrm{id}_{\mathbf{G}_{i,j}} \in \mathbf{C}_{i,j}$) if $(u,v) \notin S$, and we obtain $\mathbf{g}^+_{v,u}$ and $\phi_{u,v} \in \mathbf{C}_{u,v}$ using Corollary 4.18 if $(u,v) \in S$. □

4.5. Classification of Curtis-Tits amalgams with 3-spherical diagram

In the case where $\mathscr{G}$ is a Curtis-Tits amalgam over $\mathbb{F}_q$ whose diagram is a 3-spherical tree, Proposition 4.16 says that $\mathscr{G} \cong \underline{\mathscr{G}}$.

Theorem 4.19: *Suppose that $\mathscr{G}$ is a Curtis-Tits amalgam with a diagram that is a 3-spherical tree. Then, $\mathscr{G}$ is unique up to isomorphism. In particular any Curtis-Tits amalgam with spherical diagram is unique.*

Lemma 4.20: *Given a Curtis-Tits amalgam over $\mathbb{F}_q$ with connected 3-spherical diagram Γ there is a spanning tree Σ such that the set of edges in $\mathrm{E}_\Gamma - \mathrm{E}_\Sigma = \{\{i_s, j_s\} : s = 1, 2, \ldots, r\}$ has the property that*

1. *$(\mathbf{G}_{\{i_s,j_s\}}, \mathbf{g}_{i_s,j_s}(\mathbf{G}_{i_s}), \mathbf{g}_{i_s,j_s}(\mathbf{G}_{j_s}))$ has type $A_2(q^{e_s})$, where e_s is some power of 2.*
2. *There is a loop Λ_s containing $\{i_s, j_s\}$ such that any vertex group of Λ_s is isomorphic to $\mathrm{SL}_2(q^{e_s 2^l})$ for some $l \geq 0$.*

Proof Induction on the rank r of $H^1(\Gamma, \mathbb{Z})$. If $r = 0$, then there is no loop at all and we are done.

Consider the collection of all edges $\{i, j\}$ of Γ such that $\Gamma_{\{i,j\}}$ has type A_2 and $H^1(\Gamma - \{i, j\}, \mathbb{Z})$ has rank $r - 1$, and choose one such that $\mathbf{G}_i \cong \mathrm{SL}_2(q^{e_1})$ where e_1 is minimal among all these edges. Next replace Γ by $\Gamma - \{i, j\}$ and use induction. Suppose $\{i_s, j_s\} \mid s = 1, 2, \ldots, r\}$ is the resulting selection of edges so that $\Sigma = \Gamma - \{i_s, j_s\} \mid s = 1, 2, \ldots, r\}$ is a spanning tree and condition (1) is satisfied. Note that by choice of these edges, also condition (2) is satisfied by at least one of the loops of $\Gamma - \{\{i_t, j_t\} : t = 1, 2, \ldots, s-1\}$ that contains $\{i_s, j_s\}$. Note that this uses the fact that by 3-sphericity every vertex belongs to at least one subdiagram of type A_2. □

Definition 4.21: Fix a connected 3-spherical diagram Γ and a prime power q. Let Σ be a spanning tree and let the set of edges $\mathrm{E}_\Gamma - \mathrm{E}_\Sigma = \{\{i_s, j_s\}: s = 1, 2, \dots, r\}$ together with the integers $\{e_s: s = 1, 2, \dots, r\}$ satisfy the conclusions of Lemma 4.20. Let $\mathsf{CT}(\Gamma, q)$ be the collection of isomorphism classes of Curtis-Tits amalgams of type $\Gamma(q)$ and let $\underline{\mathscr{G}} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \underline{\mathbf{g}}_{i,j} \mid i, j \in I\}$ be the standard Curtis-Tits amalgam over $\mathbb{F}_q$ with diagram Γ as in Subsection 4.4.

Consider the following map:

$$\kappa\colon \prod_{s=1}^{r} \mathrm{Aut}(\mathbb{F}_{q^{e_s}}) \times \langle \tau \rangle \to \mathsf{CT}(\Gamma).$$

where $\kappa((\alpha_s)_{s=1}^r)$ is the isomorphism class of the amalgam $\mathscr{G}^+ = \mathscr{G}((\alpha_s)_{s=1}^r)$ given by setting $\mathbf{g}^+_{j_s,i_s} = \underline{\mathbf{g}}_{j_s,i_s} \circ \alpha_s$ for all $s = 1, 2, \dots, r$.

We now have

Corollary 4.22: *The map κ is onto.*

Proof Note that, for each $s = 1, 2, \dots, r$, the Curtis-Tits standard pair $(\mathbf{G}_{\{i_s,j_s\}}$, $\mathbf{g}_{i_s,j_s}(\mathbf{G}_{i_s})$, $\mathbf{g}_{i_s,j_s}(\mathbf{G}_{j_s}))$ has type $A_2(q^{e_s})$ and so $\mathbf{C}_{j_s} = \mathrm{Aut}(\mathbb{F}_{q^{e_s}})$. Thus the claim is an immediate consequence of Proposition 4.16. $\square$

We note that if we select Σ differently, the map κ will still be onto. However, the "minimal" choice made in Lemma 4.20 ensures that κ is injective as well, as we will see.

Lemma 4.23: *Suppose $\Gamma(q)$ is a 3-spherical diagram Γ that is a simple loop. Then, κ is injective.*

Proof Suppose there is an isomorphism $\kappa(\alpha) = \mathscr{G} \xrightarrow{\phi} \mathscr{G}^+ = \kappa(\beta)$, for some $\alpha, \beta \in \mathrm{Aut}(\mathbb{F}_q) \times \langle \tau \rangle$. Write $I = \{0, 1, \dots, n-1\}$ so that $\{i, i+1\} \in \mathrm{E}_\Gamma$ for all $i \in I$ (subscripts modulo n). Without loss of generality assume that $(i_1, j_1) = (1, 0)$ so that by Proposition 4.16 we may assume that $\mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j} = \mathbf{g}^+_{i,j}$ for all $(i, j) \neq (1, 0)$. This means that $\mathbf{a}\colon \mathbf{C}_{i,i+1} \to \mathbf{C}_i \times \mathbf{C}_{i+1}$ sends $\phi_{i,i+1}$ to (ϕ_i, ϕ_{i+1}) for any edge $\{i, i+1\} \neq \{0, 1\}$. Now note that by minimality of q, $\mathbf{C}_i$ (and $\mathbf{C}_{i,i+1}$) has a quotient $\overline{\mathbf{C}}_i$ (and $\overline{\mathbf{C}}_{i,i+1}$) isomorphic to $\langle \mathrm{Aut}(\mathbb{F}_q) \rangle \times \langle \tau \rangle$ for every $i \in I$, by considering the action of $\mathbf{C}_i$ on the subgroup of $\mathbf{G}_i$ isomorphic to $\mathrm{SL}_2(q)$. By Part 4 and 5 of Lemma 4.13 the maps $\mathbf{a}^{-1}_{i+1,i}$ and $\mathbf{a}_{i,i+1}$ induce isomorphisms $\overline{\mathbf{C}}_i \to \overline{\mathbf{C}}_{i,i+1}$ and $\overline{\mathbf{C}}_{i,i+1} \to \overline{\mathbf{C}}_{i+1}$, which compose to an isomorphism

$$\phi_i \mapsto \underline{\mathbf{g}}^{-1}_{i+1,i} \circ \underline{\mathbf{g}}_{i,i+1} \circ \phi_i \circ \underline{\mathbf{g}}^{-1}_{i,i+1} \circ \underline{\mathbf{g}}_{i+1,i},$$

sending the image of τ and α in $\overline{\mathbf{C}}_i$ to the image of τ (and α respectively) in $\overline{\mathbf{C}}_{i+1}$, where $\alpha\colon x \mapsto x^p$ for x in the appropriate extension of $\mathbb{F}_q$ defining $\mathbf{G}_{i,i+1}$. Concatenating these maps along the path $\{1, 2, \dots, n-1, 0\}$ and considering the edge $\{0, 1\}$

we see that the images of $\beta^{-1}\phi_1\alpha$ and ϕ_1 in $\overline{\mathbf{C}}_1 = \mathbf{C}_1$ coincide. Since $\mathbf{C}_1$ is abelian this means that $\beta = \alpha$. □

THEOREM 4.24: *Let Γ be a connected 3-spherical diagram with spanning tree Σ and set of edges $\mathrm{E}_\Gamma - \mathrm{E}_\Sigma = \{\{i_s, j_s\} : s = 1, 2, \ldots, r\}$ together with the integers $\{e_s : s = 1, 2, \ldots, r\}$ satisfying the conclusions of Lemma 4.20. Then κ is a bijection between the elements of $\prod_{s=1}^r \mathrm{Aut}(\mathbb{F}_{q^{e_s}}) \times \langle\tau\rangle$ and the type preserving isomorphism classes of Curtis-Tits amalgams with diagram Γ over $\mathbb{F}_q$.*

Proof Again, it suffices to show that κ is injective. This in turn follows from Lemma 4.23, for if two amalgams are isomorphic (via a type preserving isomorphism), then the amalgams induced on subgraphs of Γ must be isomorphic and Lemma 4.23 shows that κ is injective on the subamalgams supported by the loops Λ_s $(s = 1, 2, \ldots, r)$. □

5. Classification of Phan amalgams

5.1. Introduction

The classification problem is formulated as follows: Determine, up to isomorphism of amalgams, all Phan amalgams $\mathscr{G}$ with given diagram Γ possessing a non-trivial (universal) completion.

5.2. Classification of Phan amalgams with 3-spherical diagram

5.2.1. *Tori in Phan standard pairs.* Let $\mathscr{G} = \{\mathbf{G}_{i,j}, \mathbf{G}_i, \mathbf{g}_{i,j} \mid i, j \in I\}$ be a Phan amalgam over $\mathbb{F}_q$ with 3-spherical diagram $\Gamma = (I, E)$. This means that the subdiagram of Γ induced on any set of three vertices is spherical. This is equivalent to Γ not containing triangles of any kind and such that no vertex is on more than one C_2-edge.

Definition 5.1: For any $i, j \in I$ with $\{i, j\} \in \mathrm{E}_\Gamma$, let

$$\mathbf{D}_i^j = N_{\mathbf{G}_{i,j}}(\mathbf{g}_{j,i}(\mathbf{G}_j)) \cap \mathbf{g}_{i,j}(\mathbf{G}_i)$$

LEMMA 5.2: *Suppose that $(\mathbf{G}, \mathbf{G}_1, \mathbf{G}_2)$ is a Phan standard pair of type $\Gamma(q)$ as in Subsection 2.3.*

(1) *If $\Gamma(q) = A_2(q)$, then $\langle \mathbf{D}_1^2, \mathbf{D}_2^1 \rangle$ is the standard torus stabilizing the orthonormal basis $\{e_1, e_2, e_3\}$. Here $\mathbf{D}_1^2$ (resp. $\mathbf{D}_2^1$) is the stabilizer in this torus of e_1 (resp. e_3).*
(2) *If $\Gamma(q) = C_2(q)$, then $\langle \mathbf{D}_1^2, \mathbf{D}_2^1 \rangle$ is the standard torus stabilizing the basis $\{e_1, e_2, e_3 = f_1, e_4 = f_2\}$ which is hyperbolic for the symplectic form of $\mathrm{Sp}_4(q^2)$ and orthonormal for the unitary form of $\mathrm{SU}_4(q)$; Here $\mathbf{D}_2^1$ (resp. $\mathbf{D}_1^2$) is the stabilizer of $\langle e_2 \rangle$ and $\langle f_2 \rangle$ (resp. the pointwise stabilizer of both*

$\langle e_1, f_1\rangle$ and $\langle e_2, f_2\rangle$). Thus,

$$\mathbf{D}_2^1 = \langle \mathrm{diag}(1, a, 1, a^\sigma) \colon a \in \mathbb{F}_{q^2} \textit{ with } aa^\sigma = a^{q+1} = 1\rangle,$$
$$\mathbf{D}_1^2 = \langle \mathrm{diag}(a, a^\sigma, a^\sigma, a) \colon a \in \mathbb{F}_{q^2} \textit{ with } aa^\sigma = a^{q+1} = 1\rangle.$$

(3) *In either case, for $\{i, j\} = \{1, 2\}$, $\mathbf{D}_i^j = C_{\mathbf{G}_{i,j}}(\mathbf{D}_j^i) \cap \mathbf{G}_i$ and $\mathbf{D}_j^i$ is the unique torus of $\mathbf{G}_j$ normalized by $\mathbf{D}_i^j$.*

Proof Parts 1 and 2 as well as the first claim of Part 3 are straightforward matrix calculations. As for the last claim note that in both cases, $\mathbf{D}_i^j$ acts diagonally on $\mathbf{G}_j$ viewed as $\mathrm{SU}_2(q)$ in it natural representation V via the standard identification map; in fact (in the case C_2, in fact $\mathbf{D}_2^1$ acts as a group of inner automorphisms on $\mathbf{G}_1$). If $\mathbf{D}_i^j$ normalizes a torus $\mathbf{D}'$ in $\mathbf{G}_j$ then it will have to stabilize its eigenspaces. Since $q + 1 \geq 3$, the eigenspaces of $\mathbf{D}_i^j$ in its action on V have dimension 1, so $\mathbf{D}_i^j$ and $\mathbf{D}'$ must share these eigenspaces. This means that $\mathbf{D}' = \mathbf{D}_j^i$. □

5.2.2. *Property (D) for Phan amalgams.* We state Property (D) for 3-spherical Phan amalgams, extending the definition from [4] which was given for Curtis-Tits amalgams under the assumption that $q \geq 4$.

Definition 5.3: (property (D)) We say that $\mathscr{G}$ *has property (D)* if there is a *system of tori* $\mathcal{D} = \{\mathbf{D}_i \colon i \in I\}$ such that for all edges $\{i, j\} \in \mathrm{E}_\Gamma$ we have $\mathbf{g}_{i,j}(\mathbf{D}_i) = \mathbf{D}_i^j$.

Lemma 5.4: *Suppose that $\mathscr{G}$ has a completion (G, γ) so that γ_i is non-trivial for all $i \in I$. Then, for any $i, j, k \in I$ such that $\{i, j\}, \{j, k\} \in \mathrm{E}_\Gamma$, there is a torus $\mathbf{D}_j \leq \mathbf{G}_j$ such that $\mathbf{g}_{j,i}(\mathbf{D}_j) = \mathbf{D}_j^i$ and $\mathbf{g}_{j,k}(\mathbf{D}_j) = \mathbf{D}_j^k$. In particular, $\mathscr{G}$ has property (D).*

Proof First note that in case $q = 2$, the conclusion of the lemma is trivially true as, for all $i \in I$, $\mathbf{G}_i \cong S_3$ has a unique Phan torus.

We now consider the general case. For $\Gamma(q) = A_3(q)$, this was proved by Bennett and Shpectorov in [1] (see also [4]). For completeness we recall the argument, which applies in this more general case as well. We shall prove that

$$\gamma(\mathbf{D}_j^i) = \gamma(\mathbf{D}_j^k)$$

and then let $\mathbf{D}_j \leq \mathbf{G}_j$ be such that $\gamma(\mathbf{D}_j) = \gamma(\mathbf{D}_j^i) = \gamma(\mathbf{D}_j^k)$. Note that since γ_j is non-trivial, it now follows that $\mathbf{g}_{j,i}(\mathbf{D}_j) = \mathbf{D}_j^i$ and $\mathbf{g}_{j,k}(\mathbf{D}_j) = \mathbf{D}_j^k$.

Recall that for any subgroup $\mathbf{H}$ of a group in $\mathscr{G}$ we'll write $H = \gamma(\mathbf{H})$. We show that D_j^i is normalized by D_k^j and use Lemma 5.2 to conclude that $D_j^i = D_j^k$. To that end we let $h \in D_k^j$ and prove that $hD_j^i h^{-1} = D_j^i$. To achieve this we show that $hD_j^i h^{-1}$ is normalized by D_i^j and again use Lemma 5.2. So now let $g \in D_i^j$ and note that since Γ is 3-spherical, $\{i, k\} \notin \mathrm{E}_\Gamma$ so that g and h commute. In addition note that by Lemma 5.2, $gD^i_jg^{-1} = D^i_j$. Therefore we have

$$ghD^i_jh^{-1}g^{-1} = hgD^i_jg^{-1}h^{-1} = hD^i_jh^{-1},$$

as required. □

5.2.3. *The coefficient system of a Phan amalgam.*

Definition 5.5: We now fix a standard Phan amalgam $\underline{\mathscr{G}} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \underline{\mathbf{g}}_{i,j} \mid i,j \in I\}$ over $\mathbb{F}_q$ with diagram $\Gamma(q)$, where for every $i,j \in I$, $\underline{\mathbf{g}}_{i,j}$ is the standard identification map of Definition 2.6. Then, $\underline{\mathscr{G}}$ has property (D) with system of tori $\mathcal{D} = \{\mathbf{D}_i : i \in I\}$ as in Lemma 5.2.

If $\mathscr{G}$ is any other non-collapsing Phan amalgam over $\mathbb{F}_q$ with diagram Γ, then since all tori of $\mathbf{G}_i$ are conjugate under $\mathrm{Aut}(\mathbf{G}_i)$, by adjusting the inclusion maps $\mathbf{g}_{i,j}$ we can replace $\mathscr{G}$ by an isomorphic amalgam whose system of tori is exactly $\mathcal{D}$.

From now on we assume that $\underline{\mathscr{G}}$, $\mathcal{D} = \{\mathbf{D}_i : i \in I\}$ and $\mathscr{G}$ are as in Definition 5.5

Definition 5.6: Suppose that $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i,j \in I\}$ is a Phan amalgam with connected 3-spherical diagram Γ having property (D). Let $\mathcal{D} = \{\mathbf{D}_i : i \in I\}$ be the associated system of tori. The *coefficient system associated to* $\mathscr{G}$ is the collection $\mathscr{A} = \{\mathbf{A}_i, \mathbf{A}_{i,j}, \mathbf{a}_{i,j} \mid i,j \in I\}$ where, for any $i,j \in I$ we set

$$\begin{aligned}
\mathbf{A}_i &= N_{\mathrm{Aut}(\mathbf{G}_i)}(\mathbf{D}_i),\\
\mathbf{A}_{i,j} &= N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\mathbf{g}_{i,j}(\mathbf{G}_i)) \cap N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\mathbf{g}_{j,i}(\mathbf{G}_j)),\\
\mathbf{a}_{i,j} &: \mathbf{A}_{i,j} \to \mathbf{A}_j \text{ is given by restriction: } \varphi \mapsto \mathbf{g}_{j,i}^{-1} \circ \rho_{i,j}(\varphi) \circ \mathbf{g}_{j,i}.
\end{aligned}$$

where $\rho_{i,j}(\varphi)$ is the restriction of φ to $\overline{\mathbf{G}}_j \leq \mathbf{G}_{i,j}$.

From now on we let $\mathscr{A}$ be the coefficient system associated to $\underline{\mathscr{G}}$ with respect to the system of tori $\mathcal{D}$. The fact that the $\mathbf{a}_{i,j}$ are well-defined follows from the following simple observation.

Lemma 5.7: *For any $i,j \in I$ with $\{i,j\} \in \mathrm{E}_\Gamma$, we have*

$$\mathbf{A}_{i,j} \leq N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\mathbf{g}_{i,j}(\mathbf{D}_i)) \cap N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\mathbf{g}_{j,i}(\mathbf{D}_j)).$$

Proof The inclusion $\leq$ is immediate from the definitions. □

The significance for the classification of Phan amalgams with the same system of tori is as follows:

Proposition 5.8: *Suppose that $\mathscr{G}$ and $\mathscr{G}^+$ are Phan amalgams of type $\underline{\mathscr{G}}$ with the same system of tori $\mathcal{D} = \{\mathbf{D}_i : i \in I\}$.*

(1) *For all $i,j \in I$, we have $\mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j} \circ \delta_{i,j}$ and $\mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j} \circ \delta^+_{i,j}$ for some $\delta_{i,j}, \delta^+_{i,j} \in \mathbf{A}_i$,*

(2) *For any isomorphism* $\phi\colon \mathscr{G} \to \mathscr{G}^+$ *and* $i, j \in I$, *we have* $\phi_i \in \mathbf{A}_i$, $\phi_{\{i,j\}} \in \mathbf{A}_{i,j}$, *and* $\mathbf{a}_{i,j}(\phi_{\{i,j\}}) = \delta^+_{i,j} \circ \phi_i \circ \delta^{-1}_{i,j}$.

Proof Part 1. follows since, for any $i, j \in I$ we have $\mathbf{g}^{-1}_{i,j} \circ \underline{\mathbf{g}}_{i,j} \in \mathrm{Aut}(\mathbf{G}_i)$ and

$$\mathbf{g}_{i,j}(\mathbf{D}_i) = \underline{\mathbf{g}}_{i,j}(\mathbf{D}_i).$$

Part 2. follows from Lemma 5.7 since, for any $i, j \in I$,

$$(\mathbf{G}_{i,j}, \mathbf{g}_{i,j}(\mathbf{G}_i), \mathbf{g}_{j,i}(\mathbf{G}_j)) = (\mathbf{G}_{i,j}, \underline{\mathbf{g}}_{i,j}(\mathbf{G}_i), \underline{\mathbf{g}}_{j,i}(\mathbf{G}_j)) = (\mathbf{G}_{i,j}, \mathbf{g}^+_{i,j}(\mathbf{G}_i), \mathbf{g}^+_{j,i}(\mathbf{G}_j)).$$

□

We now determine the groups appearing in a coefficient system by looking at standard pairs.

Lemma 5.9: *Fix* $i \in I$ *and let* q *be such that* $\mathbf{G}_i \cong \mathrm{SU}_2(q)$. *Then,*

$$\mathbf{A}_i = \mathbf{T}_i \rtimes \mathbf{C}_i,$$

where $\mathbf{T}_i$ *is the subgroup of diagonal automorphisms in* $\mathrm{PGU}_2(q)$ *and* $\mathbf{C}_i = \mathrm{Aut}(\mathbb{F}_{q^2})$.

Proof This follows from the fact that via the standard embedding map $\underline{\mathbf{g}}_{i,j}$ the groups $\mathbf{D}_i$ of the system of tori are the subgroups of standard diagonal matrices in $\mathrm{SU}_2(q)$.

To see this note that $\mathbf{G}_i \cong \mathrm{SU}_2(q)$ and that $\mathrm{Aut}(\mathbf{G}_i) \cong \mathrm{PGU}_2(q) \rtimes \mathrm{Aut}(\mathbb{F}_{q^2})$. Also, $\mathbf{D}_i = \langle d\rangle$ for some $d = \mathrm{diag}(\zeta, \zeta^q)$ and ζ a primitive $q + 1$-th root of 1 in $\mathbb{F}_{q^2}$. A quick calculation now shows that τ and σ are the same in their action, which is inner and one verifies that $N_{\mathrm{GU}_2(q)}(\mathbf{D}_i) = \langle \tau, \mathrm{diag}(a, b)\colon a, b \in \mathbb{F}^2_q\rangle$. □

Lemma 5.10: *Let* $\mathscr{A}$ *be the coefficient system associated to the standard Phan amalgam* $\underline{\mathscr{G}}$ *of type* $\Gamma(q)$ *and the system of tori* $\mathcal{D}$.

If $\Gamma = A_1 \times A_1$, *we have* $\mathbf{G}_{i,j} = \mathbf{G}_i \times \mathbf{G}_j$, $\underline{\mathbf{g}}_{i,j}$ *and* $\underline{\mathbf{g}}_{j,i}$ *are identity maps, and*

$$\mathbf{A}_{i,j} = \mathbf{A}_i \times \mathbf{A}_j \cong \mathbf{T}_{i,j} \rtimes \mathbf{C}_{i,j}. \tag{5.1}$$

where $\mathbf{T}_{i,j} = \mathbf{T}_i \times \mathbf{T}_j$ *and* $\mathbf{C}_{i,j} = \mathbf{C}_i \times \mathbf{C}_j$. *Otherwise,*

$$\mathbf{A}_{i,j} = \mathbf{T}_{i,j} \rtimes \mathbf{C}_{i,j},$$

where $\mathbf{C}_{i,j} = \mathrm{Aut}(\mathbb{F}_{q^2})$ *and* $\mathbf{T}_{i,j}$ *denotes the image of the standard torus* T *in* $\mathrm{Aut}(\mathbf{G}_{i,j})$. *Note that* T *is as follows*

$$\langle \mathrm{diag}(a, b, c)\colon a, b, c \in \mathbb{F}_{q^2} \text{ with } aa^\sigma = bb^\sigma = cc^\sigma = 1\rangle \text{ if } \Gamma = A_2,$$
$$\langle \mathrm{diag}(c^\sigma b, ab, c, a^\sigma)\colon a, b, c \in \mathbb{F}_{q^2} \text{ with } aa^\sigma = bb^\sigma = cc^\sigma = 1\rangle \text{ if } \Gamma = C_2.$$

Before proving Lemma 5.10 make the following observations regarding the precise action of τ.:

Remark 5.11:

(1) In case $\Gamma = C_2$, $\mathbf{G} \cong \mathrm{Sp}_4(q)$ is realized as $\mathrm{Sp}_4(q^2) \cap \mathrm{SU}_4(q)$ with respect to a basis that is hyperbolic for the symplectic form and orthonormal for the unitary form, and $\mathrm{Aut}(\mathbb{F}_{q^2})$ acts entry-wise on these matrices. Moreover, τ acts as transpose-inverse on these matrices.
(2) In all cases τ coincides with σ.

Proof The $A_1 \times A_1$ case is self evident. Now consider the case $\Gamma = A_2$. As in the proof of Lemma 5.9, $\mathrm{Aut}(\mathbb{F}_{q^2}) \leq N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\mathbf{G}_i) \cap N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\mathbf{G}_j)$, $A^\tau = {}^tA^{-1} = A^\sigma$ and $\mathrm{Aut}(\mathbf{G}_{i,j}) \cong \mathrm{PGU}_3(q) \rtimes \mathrm{Aut}(\mathbb{F}_{q^2})$, so it suffices to consider linear automorphisms. As before this is an uncomplicated calculation.

Now consider the case $\Gamma = C_2$. Writing $\Gamma\mathrm{L}(V) \cong \mathrm{GL}_4(q^2) \rtimes \mathrm{Aut}(\mathbb{F}_{q^2})$ with respect to the basis $\mathcal{E} = \{e_1, e_2, e_3 = f_1, e_4 = f_2\}$, which is hyperbolic for the symplectic form of $\mathrm{Sp}_4(q^2)$ and orthonormal for the unitary form of $\mathrm{SU}_4(q)$, we have $\mathbf{G}_{i,j} = \mathrm{Sp}_4(q^2) \cap \mathrm{SU}_4(q)$.

There is an isomorphism $\Phi\colon \mathbf{G}_{i,j} \to \mathrm{Sp}_4(q)$ as in [21]. Abstractly $\mathrm{Aut}(\mathrm{Sp}_4(q)) = \mathrm{GSp}_4(q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$ (with respect to a suitable basis E for V). Since the embedding of $\mathrm{Sp}_4(q)$ into $\mathrm{Sp}_4(q^2)$ is non-standard, we are reconstructing the automorphism group here.

We first note that changing bases just replaces $\mathrm{Aut}(\mathbb{F}_{q^2})$ with a different complement to the linear automorphism group. As for linear automorphisms we claim that

$$\mathrm{GSp}_4(q^2) \cap \mathrm{GU}_4(q) = \mathrm{GSp}_4(q)$$

(viewing the latter as a matrix group with respect to E). Clearly, up to a center, we have $\mathbf{G}_{i,j} \leq \mathrm{GSp}_4(q^2) \cap \mathrm{GU}_4(q) \leq \mathrm{GSp}_4(q)$ and we note that $\mathrm{GSp}_4(q)/\mathrm{Sp}_4(q) \cong (\mathbb{F}_q^*)^2/(\mathbb{F}_q^*)$. Thus for q even, the claim follows. For q odd, let $\mathbb{F}_{q^2}^* = \langle\zeta\rangle$ and define $\beta = \mathrm{diag}(\zeta^{q-1}, \zeta^{q-1}, 1, 1)$. Then $\beta \in \mathrm{GSp}_4(q^2) \cap \mathrm{GU}_4(q)$ acts on $\mathbf{G}_{i,j}$ as $\mathrm{diag}(\zeta^q, \zeta^q, \zeta, \zeta)$, which scales the symplectic form of $\mathrm{Sp}_4(q^2)$ by ζ^{q+1}. By [21] the form of $\mathrm{Sp}_4(q)$ is proportional and since ζ^{q+1} is a non-square in $\mathbb{F}_q$, β is a linear outer automorphism of $\mathrm{Sp}_4(q)$. Thus, $\mathrm{GSp}_4(q) = \langle \mathrm{Sp}_4(q), \beta\rangle$ and the claim follows.

We now determine $\mathbf{A}_{i,j}$. First we note that β, as well as the group $\mathrm{Aut}(\mathbb{F}_{q^2})$ with respect to the basis $\mathcal{E}$, clearly normalize $\mathbf{G}_i$ and $\mathbf{G}_j$ hence by Lemma 5.7, $\mathrm{Aut}(\mathbb{F}_{q^2}) \leq \mathbf{A}_{i,j}$. So it suffices to determine inner automorphisms of $\mathrm{Sp}_4(q)$ normalizing $\mathbf{D}_i^j$ and $\mathbf{D}_j^i$.

Any inner automorphism in $\mathrm{Sp}_4(q)$ is induced by an inner automorphism of $\mathrm{Sp}_4(q^2)$. So now the claim reduces to a matrix calculation in the group $\mathrm{Sp}_4(q^2)$. $\square$

Next we describe the restriction maps $\mathbf{a}_{i,j}$ for Phan amalgams made up of a single standard pair with trivial inclusion maps.

Lemma 5.12: *Let $\mathscr{A}$ be the coefficient system of the standard Phan amalgam $\underline{\mathscr{G}}$ over $\mathbb{F}_q$ with diagram Γ and system of tori $\mathcal{D}$. Fix $i, j \in I$ and let $(\mathbf{G}_{i,j}, \overline{\mathbf{G}}_i, \overline{\mathbf{G}}_j)$ be a Phan* *standard pair in $\mathscr{G}$ with diagram $\Gamma_{i,j}$. Denote $\mathbf{a} = (\mathbf{a}_{j,i}, \mathbf{a}_{i,j})\colon \mathbf{A}_{i,j} \to \mathbf{A}_i \times \mathbf{A}_j$. Then, we have the following:*

(1) *If $\Gamma_{i,j} = A_1 \times A_1$, then $\mathbf{a}$ is an isomorphism inducing $\mathbf{T}_{i,j} \cong \mathbf{T}_i \times \mathbf{T}_j$ and $\mathbf{C}_{i,j} \cong \mathbf{C}_i \times \mathbf{C}_j$.*

(2) *If $\Gamma_{i,j} = A_2$ or C_2, then $\mathbf{a}$ induces an isomorphism $\mathbf{T}_{i,j} \to \mathbf{T}_i \times \mathbf{T}_j$.*

(3) *If $\Gamma(q) = A_2(q)$ or $\Gamma(q) = C_2(q)$, then $\mathbf{a}\colon \mathbf{C}_{i,j} \to \mathbf{C}_i \times \mathbf{C}_j$ is given by $\alpha \mapsto (\alpha, \alpha)$ (for $\alpha \in \mathrm{Aut}(\mathbb{F}_{q^2})$) which is a diagonal embedding.*

Proof 1. This is immediate from Lemma 5.10.

For the remaining cases, recall that for any $\varphi \in \mathbf{A}_{i,j}$, we have $\mathbf{a}_{i,j}\colon \varphi \mapsto \underline{\mathbf{g}}_{j,i}^{-1} \circ \rho_{i,j}(\varphi) \circ \underline{\mathbf{g}}_{j,i}$, where $\rho_{i,j}(\varphi)$ is the restriction of φ to $\overline{\mathbf{G}}_j \le \mathbf{G}_{i,j}$ (Definition 5.6) and $\underline{\mathbf{g}}_{i,j}$ is the standard identification map of Definition 2.6. Note that the standard identification map transforms the automorphism $\rho_{j,i}(\varphi)$ of $\overline{\mathbf{G}}_i$ essentially to the "same" automorphism φ of $\mathbf{G}_i$.

First let $\Gamma(q) = A_2(q)$. The map $\mathbf{a}$ is well-defined. On T, it is induced by the homomorphism

$$\mathrm{diag}(ac, c, ec) \mapsto (\mathrm{diag}(1, e), \mathrm{diag}(a, 1)),$$

where $a, c, e \in \mathbb{F}_{q^2}$ are such that $aa^\sigma = cc^\sigma = ee^\sigma = 1$. Note that the kernel is $Z(\mathrm{T})$ so that $\mathbf{a}$ is injective. The map is obviously surjective, so we are done. Thus if we factor $\mathbf{a}$ by $\mathbf{T}_{i,j}$ and $\mathbf{T}_i \times \mathbf{T}_j$, we get

$$\mathbf{C}_{i,j} \hookrightarrow \mathbf{C}_i \times \mathbf{C}_j \tag{5.2}$$

which is a diagonal embedding given by $\alpha^r \mapsto (\alpha^r, \alpha^r)$, where $r \in \mathbb{N}$ and $\alpha\colon x \mapsto x^p$ for $x \in \mathbb{F}_{q^2}$.

Next let $\Gamma(q) = C_2(q)$. We can rewrite the elements of T as $\mathrm{diag}(xyz, xz, zy^{-1}, z)$, by taking $z = a^\sigma$, $y = (ac)^{-1}$, $x = a^2 b$. On T the map $\mathbf{a}$ is induced by the homomorphism

$$\mathrm{diag}(xyz, xz, y^{-1}z, z) \mapsto (\mathrm{diag}(y, 1), \mathrm{diag}(x, 1))$$

with kernel $\{\mathrm{diag}(z, z, z, z)\colon z \in \mathbb{F}_{q^2} \text{ with } zz^\sigma = 1\} = Z(\mathrm{GU}_4(q))$. Clearly $\mathbf{a}\colon \mathbf{T}_{i,j} \to \mathbf{T}_i \times \mathbf{T}_j$ is an isomorphism. Taking the quotient over these groups, $\mathbf{a}$ induces a diagonal embedding as in (5.2), where we now interpret it in the $C_2(q)$ setting. $\square$

5.2.4. *A standard form for Phan amalgams.* Suppose that $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \underline{\mathbf{g}}_{i,j} \mid i, j \in I\}$ is a Phan amalgam over $\mathbb{F}_q$ with 3-spherical diagram Γ. Without loss of generality we will assume that all inclusion maps $\underline{\mathbf{g}}_{i,j}$ are the standard identification maps of Definition 2.6. By Lemma 5.4, $\mathscr{G}$ has Property (D) and possesses a system $\mathcal{D} = \{\mathbf{D}_i\colon i \in I\}$ of tori, which, as noted in Definition 5.5 via the standard embeddings $\underline{\mathbf{g}}_{i,j}$ can be identified with those given in Lemma 5.2.

We wish to classify all Phan amalgams $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i,j \in I\}$ over $\mathbb{F}_q$ with the same diagram as $\underline{\mathscr{G}}$. As noted in Definition 5.5 we may assume that all such amalgams share $\mathcal{D}$. Let $\mathscr{A} = \{\mathbf{A}_i, \mathbf{A}_{i,j}, \mathbf{a}_{i,j} \mid i,j \in I\}$ be the coefficient system of $\underline{\mathscr{G}}$ associated to $\mathcal{D}$. By Proposition 5.8, we may restrict to those amalgams whose connecting maps are of the form $\mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j} \circ \delta_{i,j}$ for $\delta_{i,j} \in \mathbf{A}_i$ for all $i \in I$.

Definition 5.13: The *trivial support* of $\mathscr{G}$ (with respect to $\underline{\mathscr{G}}$) is the set $\{(i,j) \in I \times I \mid \mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j}\}$ (that is, $\delta_{i,j} = \mathrm{id}_{\mathbf{G}_i}$ in the notation of Proposition 5.8). The word "trivial" derives from the assumption that the $\underline{\mathbf{g}}_{i,j}$'s are the standard identification maps of Definition 2.6.

Fix some spanning tree $\Sigma \subseteq \Gamma$ and suppose that $\mathrm{E} - \mathrm{E}_\Sigma = \{\{i_s, j_s\} \colon s = 1, 2, \ldots, r\}$ so that $H_1(\Gamma, \mathbb{Z}) \cong \mathbb{Z}^r$. We now have

PROPOSITION 5.14: *There is a Phan amalgam $\mathscr{G}(\Sigma)$ with the same diagram as $\underline{\mathscr{G}}$ and the same $\mathcal{D}$, which is isomorphic to $\mathscr{G}$ and has the following properties:*

(1) *$\mathscr{G}$ has trivial support $S = \{(i,j) \in I \times I \mid \{i,j\} \in \mathrm{E}\,\Sigma\} \cup \{(i_s, j_s) \colon s = 1, 2 \ldots, r\}$.*
(2) *for each $s = 1, 2, \ldots, r$, we have $\mathbf{g}_{j_s,i_s} = \underline{\mathbf{g}}_{j_s,i_s} \circ \gamma_{j_s,i_s}$, where $\gamma_{j_s,i_s} \in \mathbf{C}_{j_s}$.*

LEMMA 5.15: *There is a Phan amalgam $\mathscr{G}^+$ over $\mathbb{F}_q$ with the same diagram as $\underline{\mathscr{G}}$ and the same $\mathcal{D}$, which is isomorphic to $\mathscr{G}$ and has the following properties: For any $u, v \in I$, if $\mathbf{g}_{u,v} = \underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v} \circ d_{u,v}$, for some $\gamma_{u,v} \in \mathbf{C}_u$ and $d_{u,v} \in \mathbf{T}_u$, then $\mathbf{g}^+_{u,v} = \underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v}$.*

Proof The proof follows the same steps as that of Lemma 4.17 using Lemma 5.12 Part 2 instead of Lemma 4.13 Part 2. □

By Lemma 5.15 in order to prove Proposition 5.14 we may now assume that $\mathbf{g}_{u,v} = \underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v}$ for some $\gamma_{u,v} \in \mathbf{C}_u$ for all $u, v \in I$.

We now prove a Corollary for Phan amalgams analogous to, but stronger than Corollary 4.18. To this end consider the situation of Figure 1 interpreted in the Phan setting.

COROLLARY 5.16: *With the notation introduced in Figure 1, fix the maps $\gamma_{i,j}, \gamma^+_{i,j}, \phi_i \in \mathbf{C}_i$ as well as $\gamma_{j,i} \in \mathbf{C}_j$. Then for any one of $\gamma^+_{j,i}, \phi_j \in \mathbf{C}_j$, there exists a unique choice $\gamma \in \mathbf{C}_i$ for the remaining map in $\mathbf{C}_j$ so that there exists $\phi_{i,j}$ making the diagram in Figure 1 commute.*

Proof This follows immediately from the fact that the maps $\mathbf{a}_{j,i} \colon \mathbf{C}_{i,j} \to \mathbf{C}_i$ and $\mathbf{a}_{i,j} \colon \mathbf{C}_{i,j} \to \mathbf{C}_j$ in part 3. of Lemma 5.12 are isomorphisms. □

Proof (of Proposition 5.14) The proof follows the same steps as that of Proposition 4.16, replacing Lemma 4.17 and Corollary 4.18 by Lemma 5.15 and Corollary 5.16. □

5.2.5. *Classification of Phan amalgams with* 3*-spherical diagram.* In the case where $\mathscr{G}$ is a Phan amalgam over $\mathbb{F}_q$ whose diagram is a 3-spherical tree, Proposition 5.14 says that $\mathscr{G} \cong \underline{\mathscr{G}}$.

THEOREM 5.17: *Suppose that $\mathscr{G}$ is a Curtis-Tits amalgam with a diagram that is a 3-spherical tree. Then, $\mathscr{G}$ is unique up to isomorphism. In particular any Phan amalgam with spherical diagram is unique.*

Definition 5.18: Fix a connected 3-spherical diagram Γ and a prime power q. Let Σ be a spanning tree and let the set of edges $\mathrm{E}_\Gamma - \mathrm{E}_\Sigma = \{\{i_s, j_s\}\colon s = 1, 2, \ldots, r\}$ together with the integers $\{e_s\colon s = 1, 2, \ldots, r\}$ satisfy the conclusions of Lemma 4.20. Note that since in the Phan case we do not have subdiagrams of type ${}^2A_3(q)$, we have $e_s = 1$ for all $s = \{1, 2, \ldots, r\}$.

Let $\mathsf{Ph}(\Gamma, q)$ be the collection of isomorphism classes of Phan amalgams of type $\Gamma(q)$ and let $\underline{\mathscr{G}} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \underline{\mathbf{g}}_{i,j} \mid i, j \in I\}$ be a Phan amalgam over $\mathbb{F}_q$ with diagram Γ.

Consider the following map:

$$\kappa\colon \prod_{s=1}^{r} \mathrm{Aut}(\mathbb{F}_{q^2}) \to \mathsf{Ph}(\Gamma).$$

where $\kappa((\alpha_s)_{s=1}^r)$ is the isomorphism class of the amalgam $\mathscr{G}^+ = \mathscr{G}((\alpha_s)_{s=1}^r)$ given by setting $\mathbf{g}^+_{j_s,i_s} = \mathbf{g}_{j_s,i_s} \circ \alpha_s$ for all $s = 1, 2, \ldots, r$.

As for Curtis-Tits amalgams, one shows the following.

COROLLARY 5.19: *The map κ is onto.*

LEMMA 5.20: *Suppose $\Gamma(q)$ is a 3-spherical diagram Γ that is a simple loop. Then, κ is injective.*

Proof The proof is identical to that of Lemma 4.23 replacing Proposition 4.16 by Proposition 5.14 and Lemma 4.13 by Lemma 5.12, and noting that in the Phan case, we can consider the group $\mathbf{C}_i$ and $\mathbf{C}_{i,j}$ themselves rather than some suitably chosen quotient. □

THEOREM 5.21: *Let Γ be a connected 3-spherical diagram with spanning tree Σ and set of edges $\mathrm{E}_\Gamma - \mathrm{E}_\Sigma = \{\{i_s, j_s\}\colon s = 1, 2, \ldots, r\}$. Then κ is a bijection between the elements of $\prod_{s=1}^r \mathrm{Aut}(\mathbb{F}_{q^2})$ and the isomorphism classes of Curtis-Tits amalgams with diagram Γ over $\mathbb{F}_q$.*

Proof This follows from Lemma 5.20 as Theorem 4.24 follows from Lemma 4.23. □

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